N-particle irreducible actions for stochastic fluids

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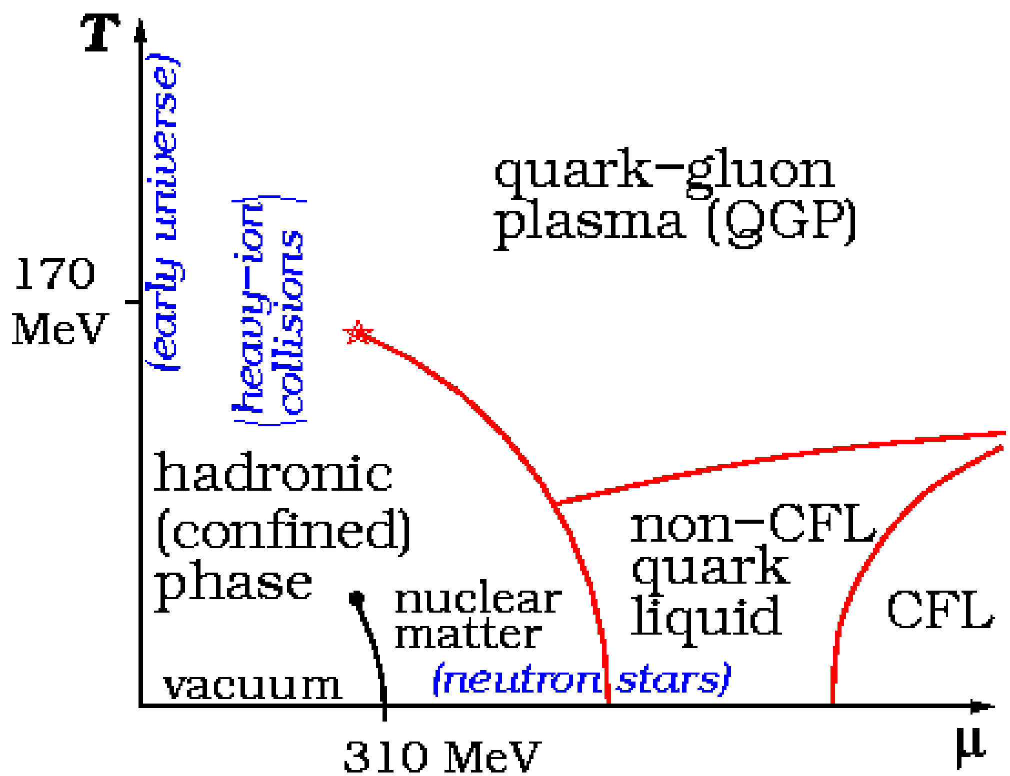
OUTLINE AND MOTIVATIONS

- To provide a field-theoretical justification for stochastic hydrodynamics
- To present the equations of motion for n-particle correlators
- To reveal some unusual behavior (critical phenomena) through fluctuations

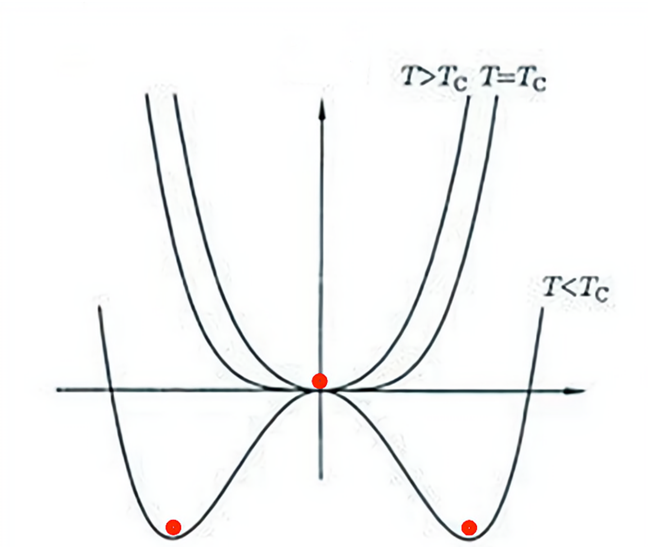
QCD Phase Diagram

The main topics related to QCD phase transition are: 1) detecting a critical point, and 2) identifying a first-order transition.



SIGNATURES OF THE CRITICAL ENDPOINT

In terms of the scalar order parameter ϕ, and so on, and its functional free energy, one predicts a critical equation of state and correlation length ξ:



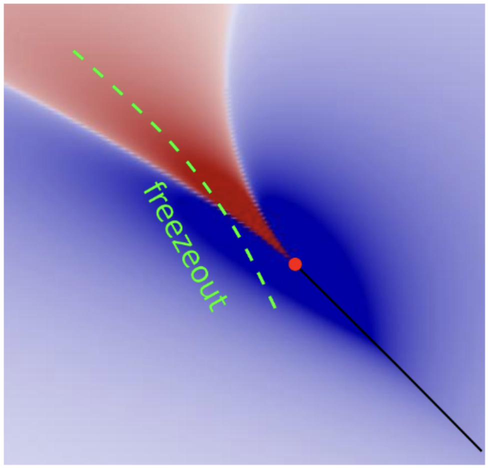
$$\xi \sim t^{-\nu}$$
$$\nu = \frac{3}{4}\epsilon + \mathcal{O}(\epsilon^2)$$
$$t = \frac{T - T_c}{T_c}$$

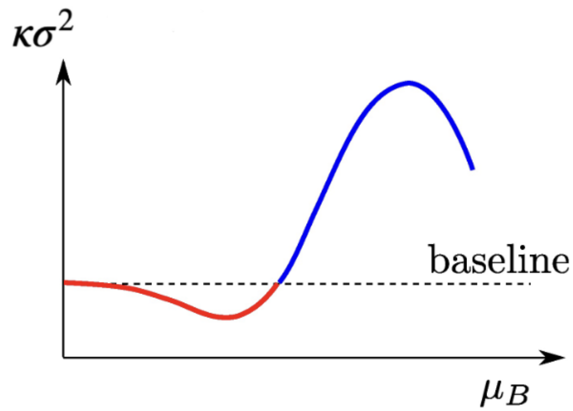
More precisely, the QCD fluid belongs to the universality class of the 3D Ising model.

NON-GAUSSIAN CUMULANTS

It expects a non-monotonic variation of the 4th-order cumulant:

🔗 Luo, Shi, Xu and Zhang, Particles.3:278(2020)





Consider the kurtosis: $\kappa_4 = \langle\phi^4\rangle - 3\langle\phi^2\rangle$

There is a stronger divergence near the critical point: $\kappa_4/\kappa_2^2 \sim \xi^3$

Has a non-trivial dependence on t (=beam energy)

Fluid Dynamics for Relativistic QCD Matter

Fluid dynamics is a universal effective field theory (EFT) of non-equilibrium many-body systems with a stable equation of state and

- Conservation of charge: $\partial_\mu J^\mu = 0$
- Conservation of energy and momentum: $\partial_\mu T^{\mu\nu} = 0$

$$
\begin{aligned}
J^\mu &= n\, u^\mu + \nu^\mu \\
T^{\mu\nu} &= \varepsilon\, u^\mu u^\nu + p\Delta^{\mu\nu} + \pi^{\mu\nu} \\
\Delta^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu \qquad \nu^\mu = -\kappa T \Delta^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) \\
\pi^{ij} &= -\eta\left(\partial^i u^j + \partial^j u^i - \frac{2}{3}\delta^{ij}\nabla\cdot\mathbf{u}\right) - \zeta\delta^{ij}\nabla\cdot\mathbf{u}
\end{aligned}
$$

The dissipation terms are described by the shear viscosity η, bulk viscosity ζ and charge conductivity κ

Dynamical Model in RHIC Beam Energy Scan

The real world is more complicated than the predictions in the first order. Additional factors must be considered, such as:

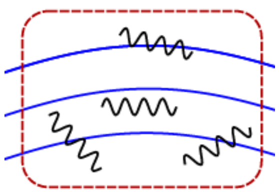
- Finite size and finite expansion rate effects
- Freeze-out, resonances, global charge conservation, and others
- Non-dissipation effects, including memory and critical slowing down
  - The role of fluctuations is enhanced in nearly perfect fluids
  - Fluctuations are dominant near critical points

Fluctuations in Hydro

- The deterministic hydro equations do not lead to spontaneous fluctuations
- The occurrence of fluctuations is a consequence of the microscopic dynamics and must persist at the coarse-grained hydro-level

Introducing non-linear dissipation with temperature-dependent transport coefficients and random noises:

$$J^\mu \quad \rightarrow \quad J^\mu + \theta^\mu$$
$$T^{\mu\nu} \quad \rightarrow \quad T^{\mu\nu} + \theta^{\mu\nu}$$



$$\langle \theta^\mu \rangle = 0 \quad \left\langle (\theta^\mu)^2 \right\rangle \sim \delta(x - x')(t - t')$$
$$\langle \theta^{\mu\nu} \rangle = 0 \quad \left\langle (\theta^{\mu\nu})^2 \right\rangle \sim \delta(x - x')(t - t')$$

Representation in MSRJD Field Theory

In terms of the slow variable (a conserved density), the free energy of the fluid:

$$\mathcal{F}[\psi] = \int d^3x \left\{\frac{1}{2}(\vec{\nabla}\psi)^2 + \frac{r}{2}\psi(x,t)^2 + \frac{\lambda}{3!}\psi(x,t)^3 + \ldots + h(x,t)\psi(x,t)\right\}$$

The diffusion equation:

$$\partial_t\psi(x,t) = \vec{\nabla}\left\{\kappa(\psi)\,\vec{\nabla}\left(\frac{\delta\mathcal{F}[\psi]}{\delta\psi}\right)\right\} + \theta(x,t)$$

where the Gaussian noise term $\theta(x,t)$ has a distribution

$$P[\theta] \sim \exp\left(-\frac{1}{4}\int d^3x\,dt\,\theta(x,t)L(\psi)^{-1}\theta(x,t)\right)$$

Representation in MSRJD Field Theory, cont.

The conductivity, $\kappa(\psi)$, is field-dependent: $\kappa(\psi) = \kappa_0 (1 + \lambda_D \psi)$

The partition function is given as: MSR, PhysRevA.8:423(1973)

$$\begin{aligned} Z &= \int \mathcal{D}\psi \, P[\theta] \exp\left(-i\tilde{\psi} \, (\text{e.o.m}[\psi, \theta])\right) \\ &= \int \mathcal{D}\psi \, \mathcal{D}\tilde{\psi} \, \exp\left(-\int d^3x \, dt \, \mathcal{L}(\psi, \tilde{\psi})\right) \end{aligned}$$

The effective Lagrangian of this theory is:

$$\mathcal{L}(\psi, \tilde{\psi}) = \tilde{\psi} \left(\partial_t - D_0 \nabla^2\right) \psi - \frac{D_0 \lambda'}{2} \left(\nabla^2 \tilde{\psi}\right) \psi^2 - \tilde{\psi} L(\psi) \tilde{\psi}$$

Note: $D_0 = r\kappa_0$ and $\lambda' = \lambda/r + \lambda_D$.

The noise kernel is chosen as $L(\psi) = \overleftarrow{\nabla} \left[k_B T \kappa(\psi)\right] \overrightarrow{\nabla}$

Time Reversal Symmetry

Stochastic theories must describe the detailed balance condition:

$$\frac{P\left(\psi_1 \longrightarrow \psi_2\right)}{P\left(\psi_2 \longrightarrow \psi_1\right)} = e^{-\Delta \mathscr{F}/k_B T}$$

which is related to time-reversal symmetry:

$$\Psi(t) \longrightarrow \psi(-t)$$
$$\tilde{\Psi}(t) \longrightarrow -\left[\tilde{\psi}(-t) + \frac{\delta F}{\delta \psi}\right]$$
$$\mathcal{L} \rightarrow \mathcal{L} + \frac{d}{dt} F$$

Janssen, ZPhyB.23:377(1976)

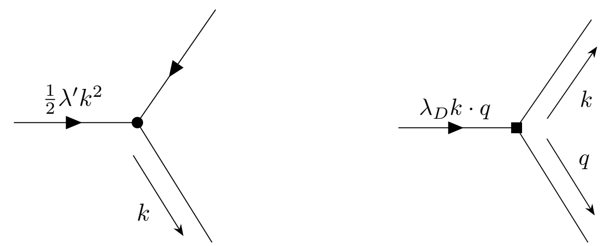
The Ward identity is revised to

$$\left\langle \psi(x_1, t_1)\left[\overleftarrow{\nabla}\kappa(\psi)\vec{\nabla}\tilde{\psi}\right](x_2, t_2)\right\rangle = \Theta(t_2 - t_1)\left\langle \psi(x_1, t_1)\dot{\psi}(x_2, t_2)\right\rangle$$

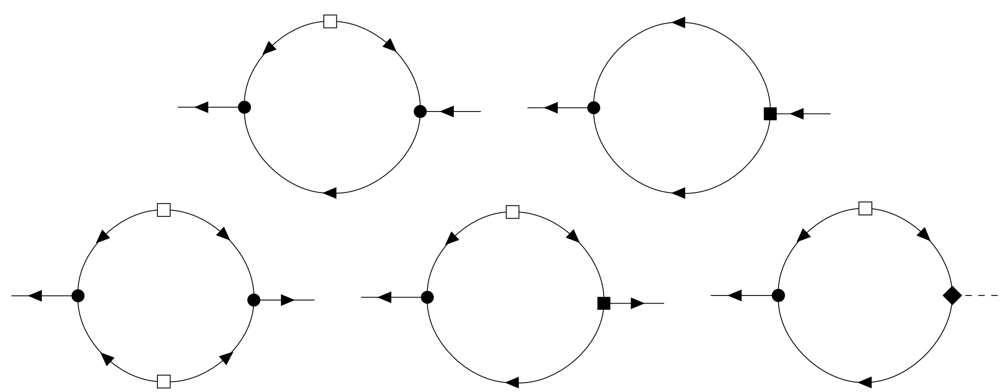
Simpler Example of Model B

- Linearized propagator:

- Vertex and new vertices:



- Loop contributions:



Analytical Results of One-Loop

The retarded function

$$G^{-1}(\omega,k) = \frac{1}{-i\omega + D_0 k^2 + \Sigma(\omega,k)}$$

with

$$\Sigma(\omega,k) = \frac{\lambda'}{32\pi}\left(i\lambda'\omega k^2 + \lambda_D\left[i\omega - D_0k^2\right]k^2\right)\sqrt{k^2 - \frac{2i\omega}{D_0}}$$

The charge (thermal) conductivity in this system becomes a scale-dependent term in the low-energy effective hydro theory

The vertex function of composite operator $\lambda_D[\psi\vec{\nabla}\tilde{\psi}]$ is given by

$$\Gamma_D(\omega,k) \equiv (-i\omega + D_0k^2)\left\langle D_0\lambda_D[\psi\vec{\nabla}\tilde{\psi}]\vec{\nabla}\psi\right\rangle_{\omega,k}$$

The field-dependent fluctuation-dissipation relation becomes:

$$2\,\mathrm{Im}\left\{G(\omega,k)\left[D_0k^2 + \Gamma_D(\omega,k)\right]\right\} = \omega C(\omega,k)$$

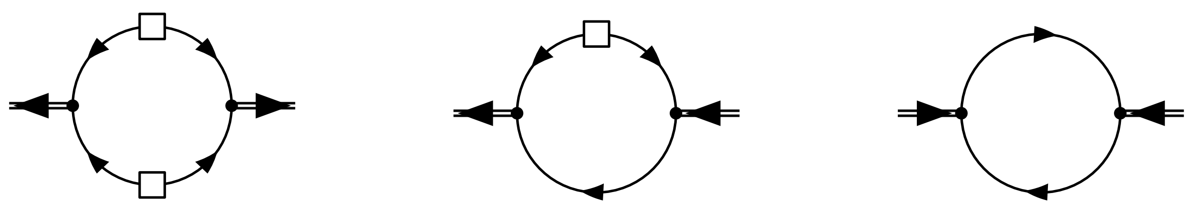
1PI Effective Action

Consider the generating functional with local source J, $\tilde{J}$:

$$W[J,\tilde{J}] = -\ln \int \mathscr{D}\psi \mathscr{D}\tilde{\psi}\; e^{-\int dt\, d^3x \left\{\mathscr{L} + J\psi + \tilde{J}\tilde{\psi}\right\}}$$

Performing a Legendre transform to the 1PI effective action via background field method with $\psi = \Psi + \delta\psi$:

$$\Gamma[\Psi,\tilde{\Psi}] = W[J,\tilde{J}] - \int dt\, d^3x \left(J\Psi + \tilde{J}\tilde{\Psi}\right)$$



Taking the derivative of the 1PI effective action w.r.t. the classical field Ψ yields the e.o.m. encoded the fluctuation effects:

$$(\partial_t - D\nabla^2)\Psi - \frac{\kappa\lambda_3^2}{2}\nabla^2\Psi^2 + \int d^3x'\, dt'\, \Psi(x',t')\Sigma(x,t;x',t') = 0$$

Double Legendre Transformation

☞ nPI effective action $\Longrightarrow$ e.o.m. for n-point functions

✓ Couple a bi-local source $\frac{1}{2}\psi_a K_{ab} \psi_b$ to the system 🔗 Cornwall, Jackiw and Tomboulis, PhysRevD.10:2428 (1974)

✓ Plug in the 1-loop 1PI effective action

✓ Sum beyond 1-loop terms

✓ Apply the stationary conditions:

$$\frac{\delta W}{\delta J_a} = \langle \psi_a \rangle = \Psi_a \,, \qquad \frac{\delta W}{\delta K_{ab}} = \frac{1}{2}\langle \psi_a \psi_b \rangle = \frac{1}{2}\left[\Psi_a \Psi_b + G_{ab}\right]$$

✓ Perform a Legendre transform to yield the 2PI effective action:

$$\Gamma[\Psi_a, G_{ab}] = W[J_a, K_{ab}] - J_A \Psi_A - \frac{1}{2} K_{AB} \left[\Psi_A \Psi_B + G_{AB}\right]$$

2PI Effective Action

The 2PI effective action is given by:

$$\Gamma[\Psi_a, G_{ab}] = S[\Psi_a] + \frac{1}{2}\frac{\delta^2 S}{\delta\Psi_A \delta\Psi_B} G_{AB} - \frac{1}{2}\mathrm{Tr}\left[\log(G)\right] + \Gamma_F[\Psi_a, G_{ab}]$$

The higher order fluctuations are:

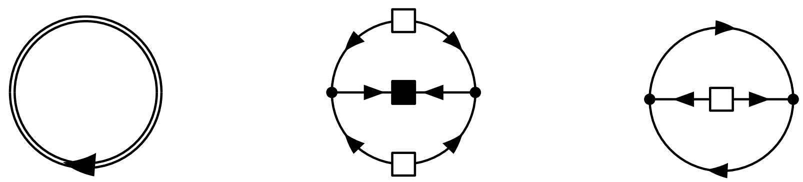
$$\begin{aligned}\exp(-\Gamma_F[\Psi_a, G_{ab}]) \quad = \quad & \frac{1}{\sqrt{\det(G)}}\int D(\delta\psi_a)\, \exp\Big\{ -\frac{1}{2}\delta\psi_A (G^{-1})_{AB}\delta\psi_B \\ & - \Big[S_3[\Psi_a, \delta\psi_a] - \bar{J}_A \delta\psi_A - \bar{K}_{AB}(\delta\psi_A\delta\psi_B - G_{AB}) \Big] \Big\}\end{aligned}$$

with

$$\bar{J}_a = \frac{1}{2}\frac{\delta^3 S}{\delta\Psi_a\delta\Psi_B\delta\Psi_C} G_{BC} + \frac{\delta\Gamma_F}{\delta\Psi_a}, \quad \bar{K}_{ab} = \frac{\delta\Gamma_F}{\delta G_{ab}}$$

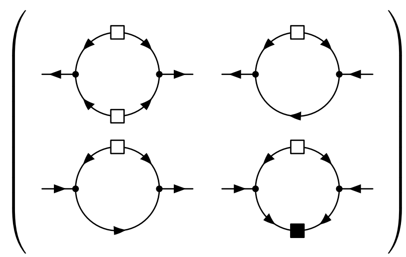
DSE in Mixed Representation

The loop diagrams generated by Γ_F use the full propagator G_{ab}:



Taking the derivative w.r.t G, obtain the Dyson-Schwinger equation:

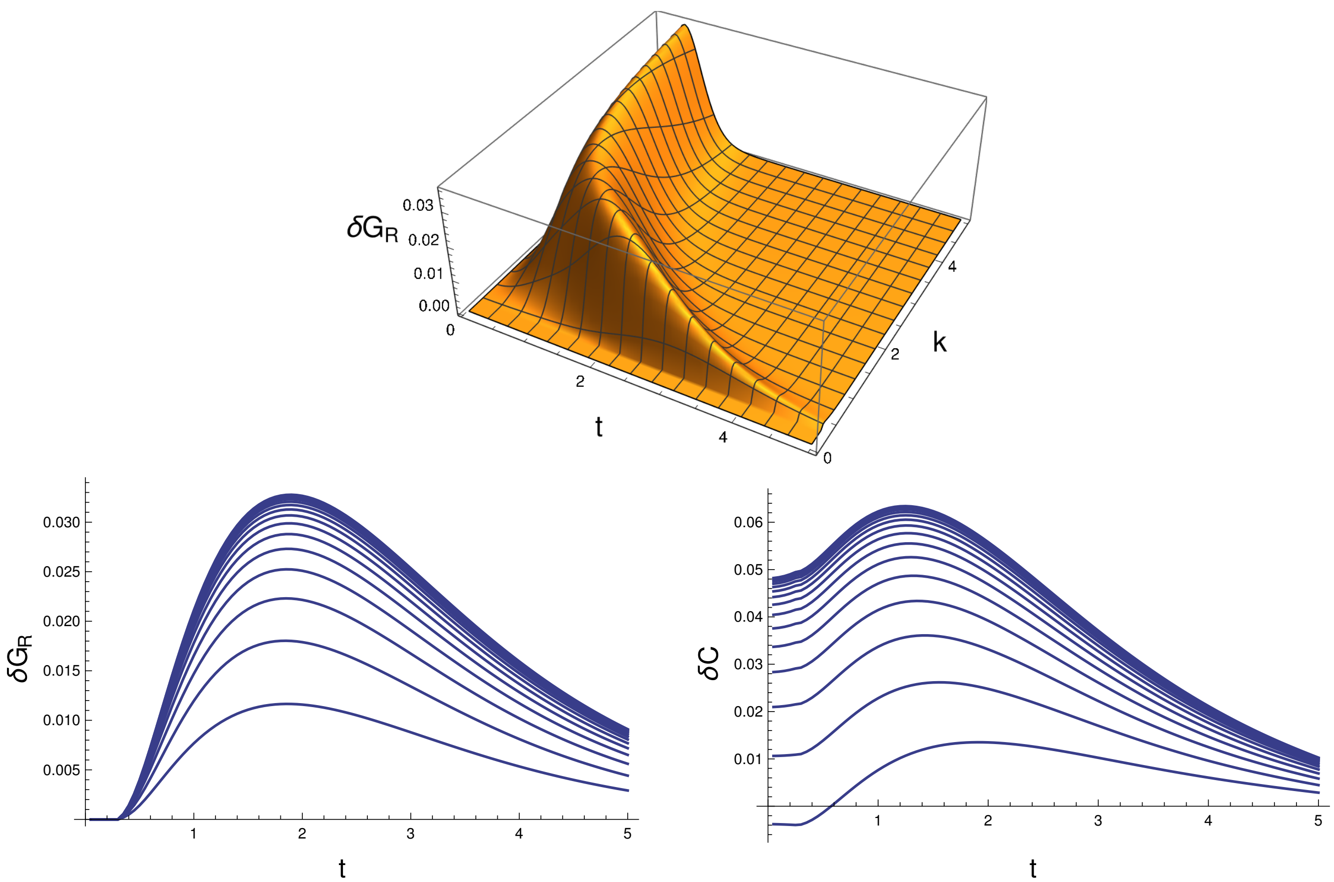
$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} =$$



In time-momentum mixed representation

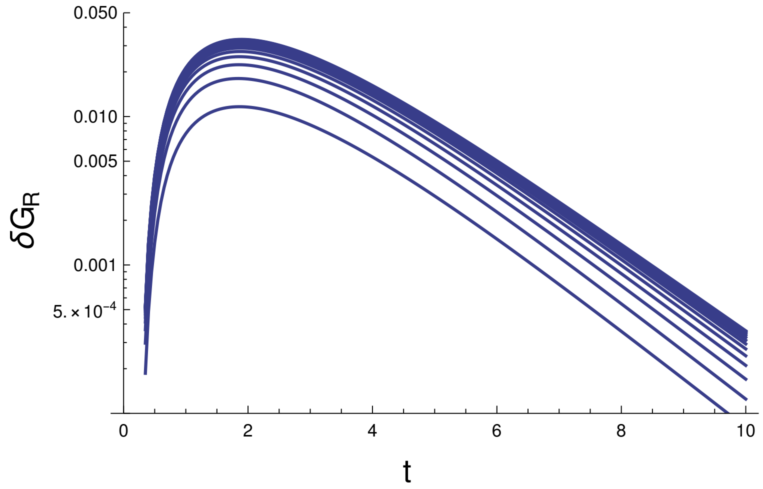
$$\begin{aligned} \Sigma(t,k^2) &= (\kappa\lambda_3)^2 \int d^3k'\, k^2 (k+k')^2\, C(t,k')\, G_R(t,k+k')\,, \\ \delta D(t,k^2) &= \frac{(\kappa\lambda_3)^2}{2} \int d^3k'\, k^4 C(t,k')\, C(t,k+k') \end{aligned}$$

Naive Numerical Simulations

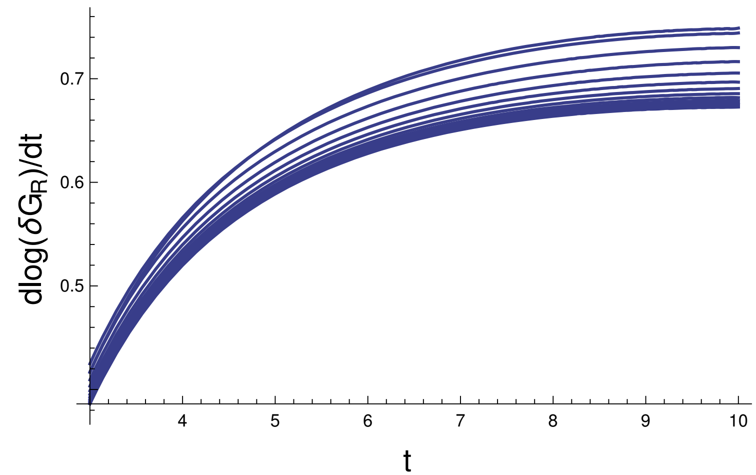


3D curve of $\delta G(t, k)$ and the iterative solutions of DSE taking advantage of convergence

Long-Time Behavior



A logarithmic plot of the loop corrections to the retarded $\delta G_R(k, t)$



The logarithmic derivative of $\delta G_R(k, t)$ w.r.t t

The long-time behavior of the diffusion cascade is conjectured to be $\sim n! \exp(-Dk^2 t/n)$ because of the n-loop terms. Delacretaz, SciPostPhys.9:034(2020)

Mode Coupling Theory

- For non-critical fluids, use the gradient expansion method where $k\xi \ll 1$
- For critical fluids, their behaviors are characterized by the transport coefficients in the MCT

By applying an uncontrolled approximation within the MCT, the retarded function for the diffusion mode:

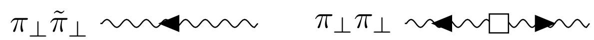
$$G^{-1}(\omega, k) = i\omega - \Gamma_k$$

$$\Gamma_k = \frac{T}{6\pi\eta_0\xi^3}K(k\xi) \qquad K(k\xi = x) = \frac{3}{4}\left[1 + x^2 + (x^3 - x^{-1})\arctan(x)\right]$$

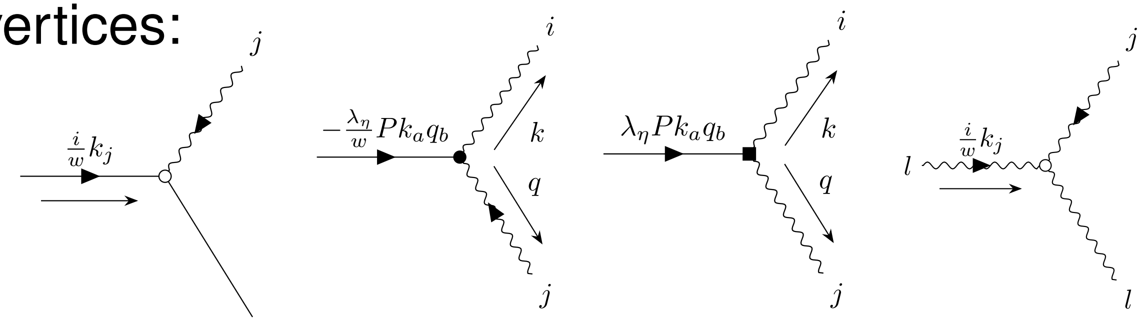
η_0 is the bare shear viscosity. 🔗 Kawasaki, AnnPhys.61:1(1970); JC and T. Schaefer, work-in-process

MODEL H

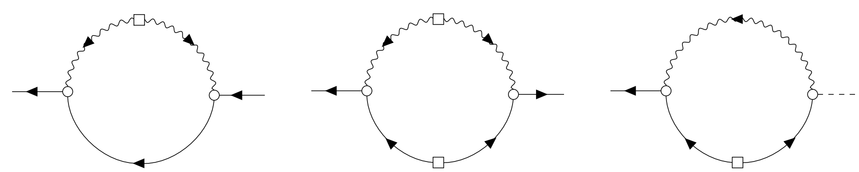
- Linearized propagator:



- Vertices and new vertices:

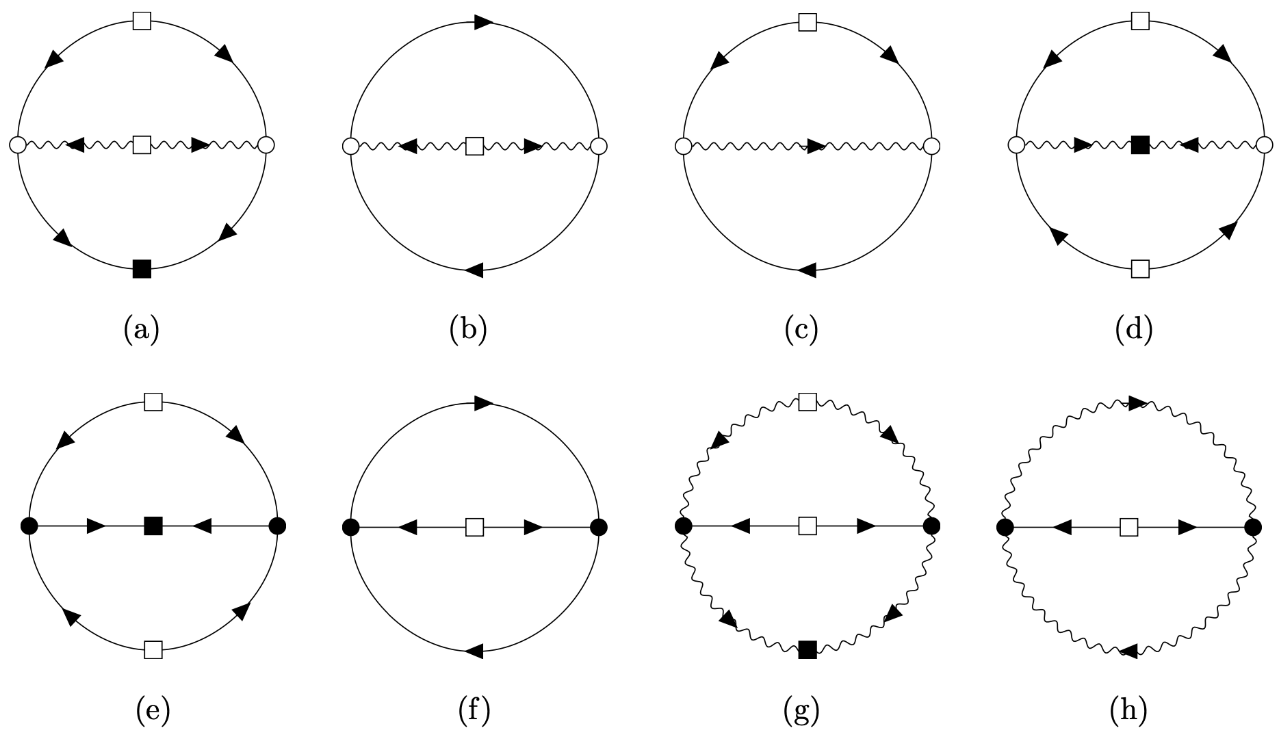


- Mode-coupling loop contributions:



The multiplicative noise contribution to the tails is subleading compared to the contributions induced by mode couplings in hydro limit

2PI Effective Action in Model H



Above: The traditional contribution, which originates from the vertex of the Poisson bracket, is illustrated within the MCT
Below: Additional contributions are derived from the newer vertex

Scaling Forms of the Transport Coefficients

The modified critical transport coefficients:

$$
\begin{aligned}
D \to D^c(\omega, k, \xi) &= D\,(k\xi)^{x_D} F_D\,(\omega\xi^z, k\xi) \\
\kappa \to \kappa^c(\omega, k, \xi) &= \kappa\,(k\xi)^{x_\kappa} F_\kappa\,(\omega\xi^z, k\xi) \\
\eta \to \eta^c(\omega, k, \xi) &= \eta\,(k\xi)^{x_\eta} F_\eta\,(\omega\xi^z, k\xi) \\
\gamma \to \gamma^c(\omega, k, \xi) &= \gamma\,(k\xi)^{x_\gamma} F_\gamma\,(\omega\xi^z, k\xi)
\end{aligned}
$$

- Contrary to the hydrodynamic limit, where $D = \kappa m^2$, $\eta = \gamma w$ and w is enthalpy
- The relaxation frequency scales as $\omega \sim k^z$
- The dynamical exponent z is determined as $z = 4 - \eta + x_D$ for the diffusion mode in the regime where $k \gg \xi^{-1}$

Critical Matrix

The Ornstein-Zernike form is utilized, expressed as $\chi^{-1}(x) = g(x) = 1 + x^2$, with the static critical exponent set to $\eta = 0$. And then,

$$\begin{aligned}
\Sigma^c_{12}(s,x) &= D\,\xi^{z-2}\,x^{2+x_D} g(x) F_D(s,x)\,,\\
\Sigma^c_{11}(s,x) &= \kappa\,\xi^{2z-2}\,x^{2+x_\kappa} F_\kappa(s,x)\,,\\
\Delta^c_{12}(s,x) &= \gamma\,\xi^{z-2}\,x^{2+x_\gamma} F_\gamma(s,x)\,,\\
\Delta^c_{11}(s,x) &= \eta\,\xi^{2z-2}\,x^{2+x_\eta} F_\eta(s,x)\,.
\end{aligned}$$

where

$$F_i(s=0, x\to\infty) = F_i^\infty = \text{constant}$$

and

$$F_i(s=0, x\to 0) = F_i^0\, x^{-x_i}$$

with $i = D, \kappa, \gamma, \eta$.

UV FINITE SELF-CONSISTENT EQUATIONS

Re-scale the frequency and the momentum as $(s,r) = (\omega\xi^z, \omega'\xi^z)$ and $(x,y) = (k\xi, k'\xi)$, the self-energies are:

$$\Sigma^c_{12}(s,x) = \xi^{z-7}\int_{r,y}\left[\frac{\Sigma^c_{11}(r,y)}{r^2+|\Sigma^c_{12}(r,y)|^2}\frac{(\kappa\lambda_3)^2 x^2(\vec{x}+\vec{y})^2}{i(s+r)+\Sigma^c_{12}(-s-r,x+y)}\right.$$
$$-\frac{\Delta^c_{11}(r,y)}{r^2+|\Delta^c_{12}(r,y)|^2}\frac{\xi^2}{w^2y^2}\frac{x^2y^2-(\vec{x}\cdot\vec{y})^2}{i(s+r)+\Sigma^c_{12}(-s-r,x+y)}$$
$$\left.-\frac{\Sigma^c_{11}(r,y)}{r^2+|\Sigma^c_{12}(r,y)|^2}\frac{x^2-y^2}{w(\vec{x}+\vec{y})^2}\frac{x^2(\vec{x}+\vec{y})^2-(x^2+\vec{x}\cdot\vec{y})^2}{i(s+r)+\Delta^c_{12}(-s-r,x+y)}\right],$$

$$\Sigma^c_{11}(s,x) = \xi^{z-7}\int_{r,y}\left\{\frac{\Sigma^c_{11}(r,y)}{r^2+|\Sigma^c_{12}(r,y)|^2}\frac{(\kappa\lambda_3)^2}{2}\frac{x^4\,\Sigma^c_{11}(s+r,x+y)}{(s+r)^2+|\Sigma^c_{12}(s+r,x+y)|^2}\right.$$
$$\left.+\frac{\Delta^c_{11}(r,y)}{r^2+|\Delta^c_{12}(r,y)|^2}\frac{\xi^2}{w^2y^2}\frac{\left(x^2y^2-(\vec{x}\cdot\vec{y})^2\right)\Sigma^c_{11}(s+r,x+y)}{(s+r)^2+|\Sigma^c_{12}(s+r,x+y)|^2}\right\},$$

$$\Delta_{12}^c(s,x) = \xi^{z-7}\int_{r,y}\Bigg\{\frac{\Sigma_{11}^c(r,y)}{r^2+|\Sigma_{12}^c(r,y)|^2}\frac{(\gamma\lambda_\eta)^2\mathscr{P}_t(x,y)(x^2+\vec{x}\cdot\vec{y})^2}{i(s+r)+\Delta_{12}^c(-s-r,x+y)}$$
$$-\frac{\Sigma_{11}^c(r,y)}{r^2+|\Sigma_{12}^c(r,y)|^2}\frac{x^2y^2-(\vec{x}\cdot\vec{y})^2}{wx^2}\frac{\left(x^2+2\,\vec{x}\cdot\vec{y}\right)}{i(s+r)+\Sigma_{12}^c(-s-r,x+y)}\Bigg\}\,,$$

$$\Delta_{11}^c(s,x) = \xi^{z-7}\int_{r,y}\Bigg\{\frac{\Sigma_{11}^c(r,y)}{r^2+|\Sigma_{12}^c(r,y)|^2}\frac{(\gamma\lambda_\eta)^2\mathscr{P}_t(x,y)(x^2+\vec{x}\cdot\vec{y})^2\Delta_{11}^c(s+r,x+y)}{(s+r)^2+|\Delta_{12}^c(s+r,x+y)|^2}$$
$$+\frac{\Sigma_{11}^c(r,y)}{r^2+|\Sigma_{12}^c(r,y)|^2}\frac{x^2y^2-(\vec{x}\cdot\vec{y})^2}{\xi^2x^2}\frac{(\vec{x}+\vec{y})^2\left(x^2+2\,\vec{x}\cdot\vec{y}\right)\Sigma_{11}^c(s+r,x+y)}{(s+r)^2+|\Sigma_{12}^c(s+r,x+y)|^2}\Bigg\}\,,$$

where $\mathscr{P}_t(x,y)=1+(x^2+\vec{x}\cdot\vec{y})^2\,x^{-2}(\vec{x}+\vec{y})^{-2}$

stay tune for the numerical results!

Summary and Outlooks

- To complete the numerical simulation of the mode coupling approximation, as it can provide important insights into the critical region
- To consider extending our model to include expanding systems, which can help us gain a deeper understanding of the dynamical nature of the phase transitions
- To investigate how our approach can be connected to kinetic theory to provide more valuable insights into the microscopic behavior of QCD matter

Thank You for Your Attention!