

Phase structure and dynamics of QCD at high densities

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Universität Heidelberg & ExtreMe Matter Institute

Informal talk, Hefei, September 25 2026



fqcd-collaboration.github.io



STRUCTURES
CLUSTER OF
EXCELLENCE

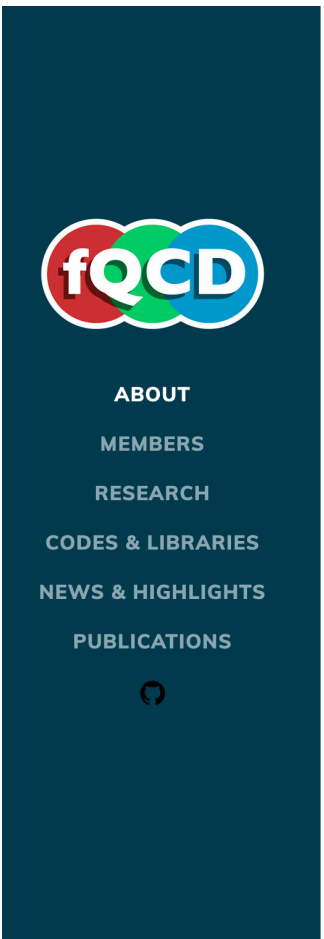


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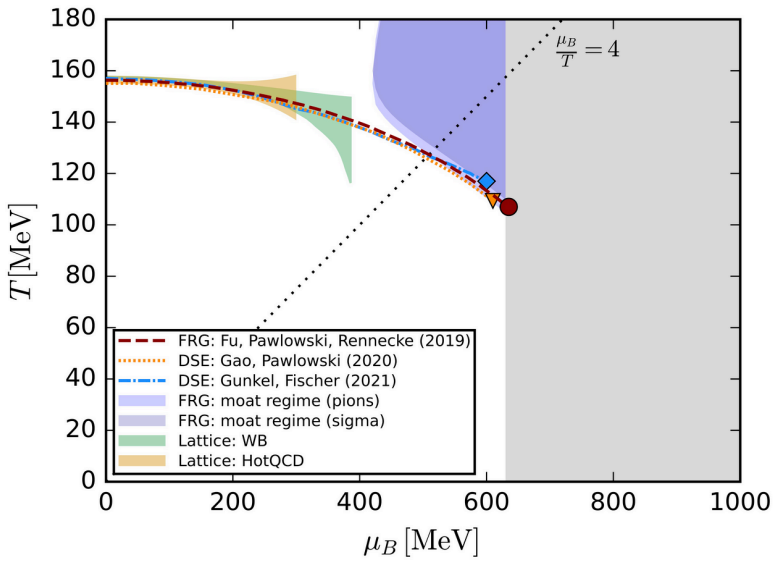


fQCD collaboration



fQCD COLLABORATION

We study the phase structure, thermodynamics and transport properties of strongly interacting matter. In order to access the notoriously difficult regimes of QCD, finite density and real time, we develop first-principles functional approaches with a focus on the functional renormalisation group and Dyson-Schwinger equations.

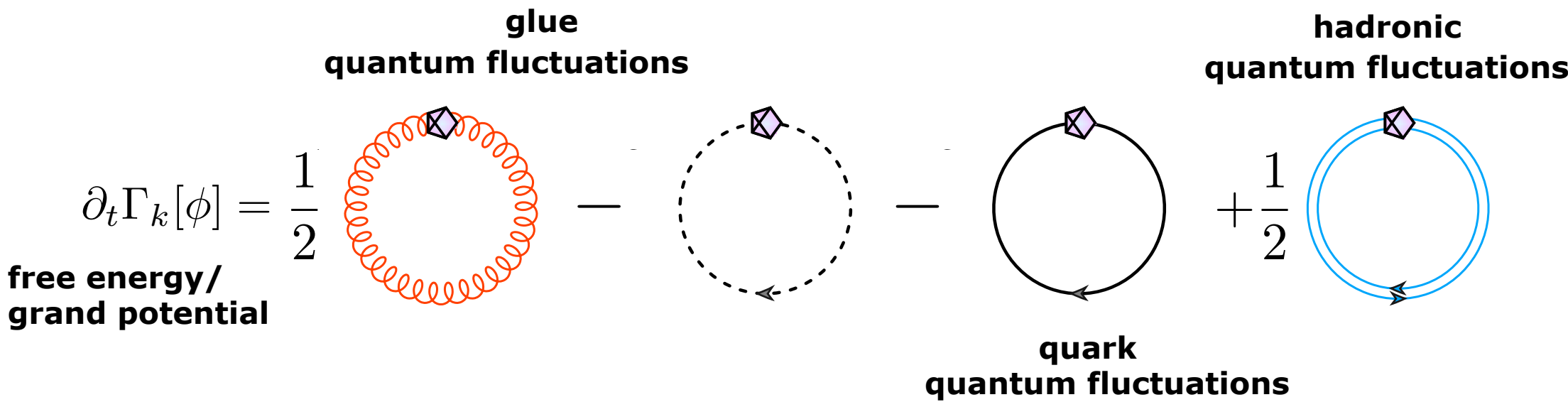


github

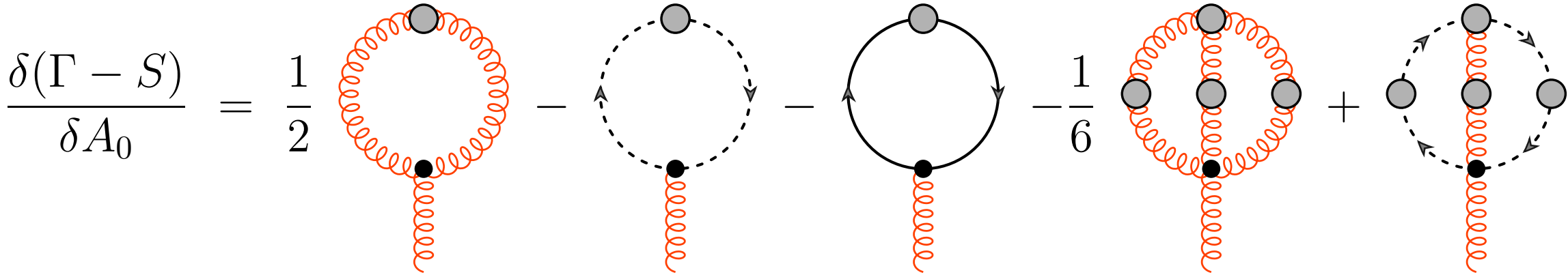
Dalian, Beijing, Darmstadt, Heidelberg, Gießen

Braun, Chen, Fu, Gao, Huang, Ihssen, Kockler, Lu, Pawłowski, Rennecke, Sattler, Stoll, Tan, Wang, Wen, Wessely, Yin, Zheng, Zorbach

Functional renormalisation group



Dyson-Schwinger equations



Material

Collection of talks, colloquia & slides from schools

Topical reviews

Collection of reviews & lecture notes on the FRG & DSE

The nonperturbative functional renormalization group and its applications

Dupuis et al, Phys.Rep. 910 (2021) 1-114

QCD at finite temperature and density within the fRG approach: An overview

Fu, Commun.Theor.Phys. 74 (2022) 9, 097304

QCD phase structure & equation of state: A functional perspective

Rennecke, QM plenary talk, arxiv:2510.11270

Phase structure and observables at high densities from first principles QCD

Fischer, JMP, arxiv: 2603.11135, to be published in EPJST

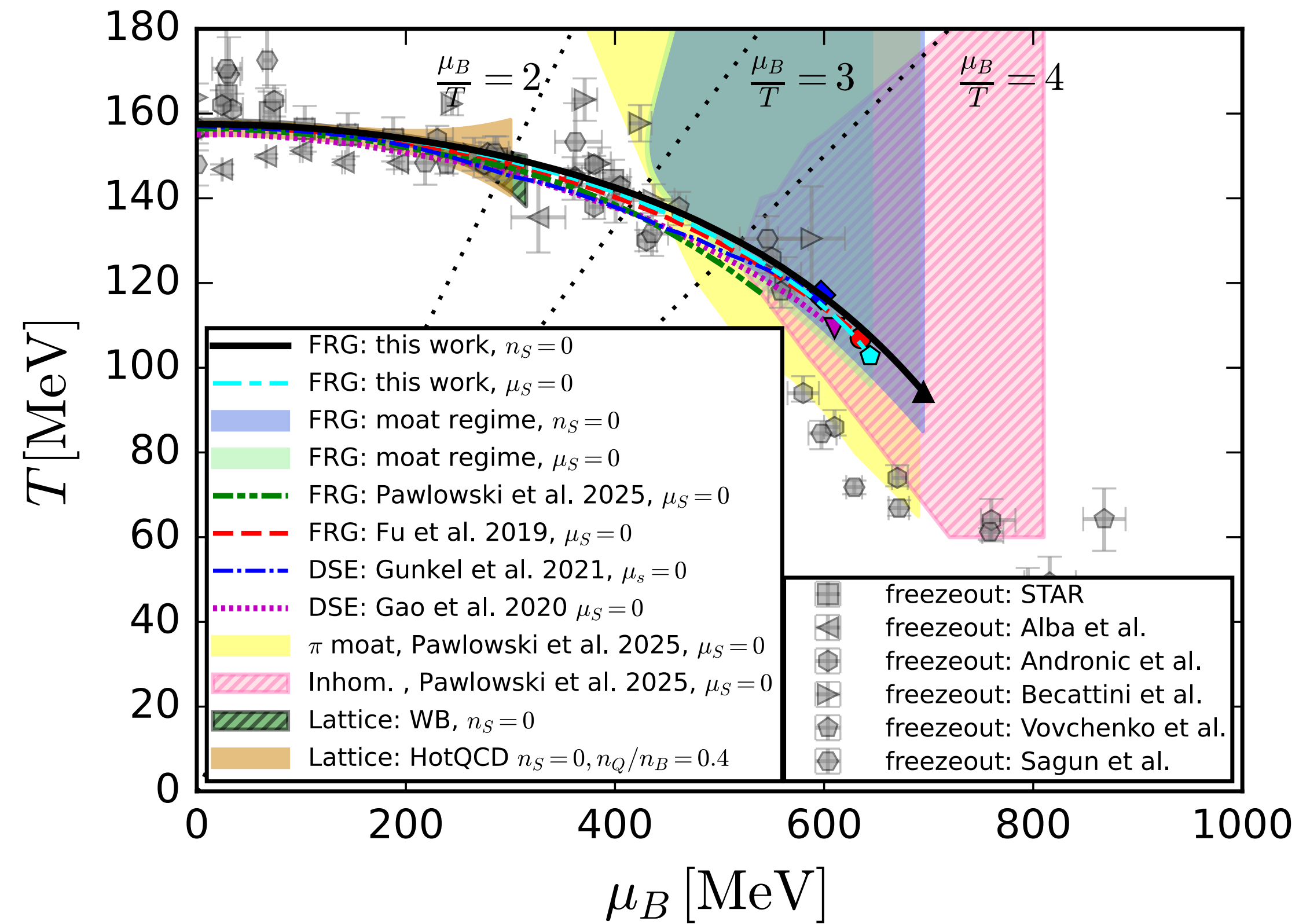
Phase structure of strong interaction matter from Functional QCD

Fischer, JMP, arxiv: 2606.03703, contribution to Encyclopedia of Nuclear Physics

Phase structure and dynamics of QCD at high densities

- **Critical end point and new phases: where do we stand?**
- **Strong correlations above the chiral crossover**
- **Ripples of the critical end point and new phases**
- **QCD moat and inhomogeneous phases**

Critical end point and new phases: where do we stand?

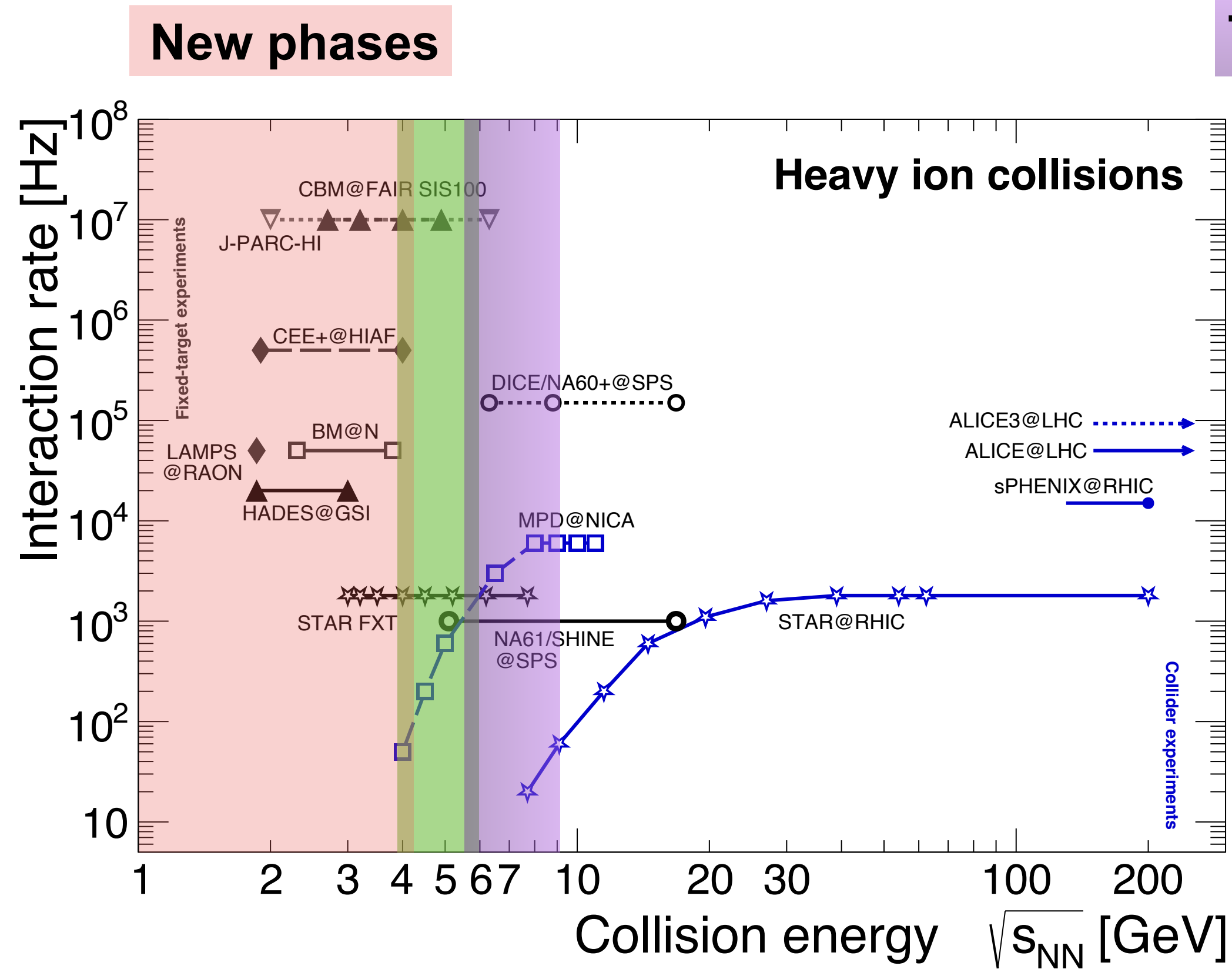


Experimental & Theoretical Landscape

Experimental landscape

Theoretical landscape

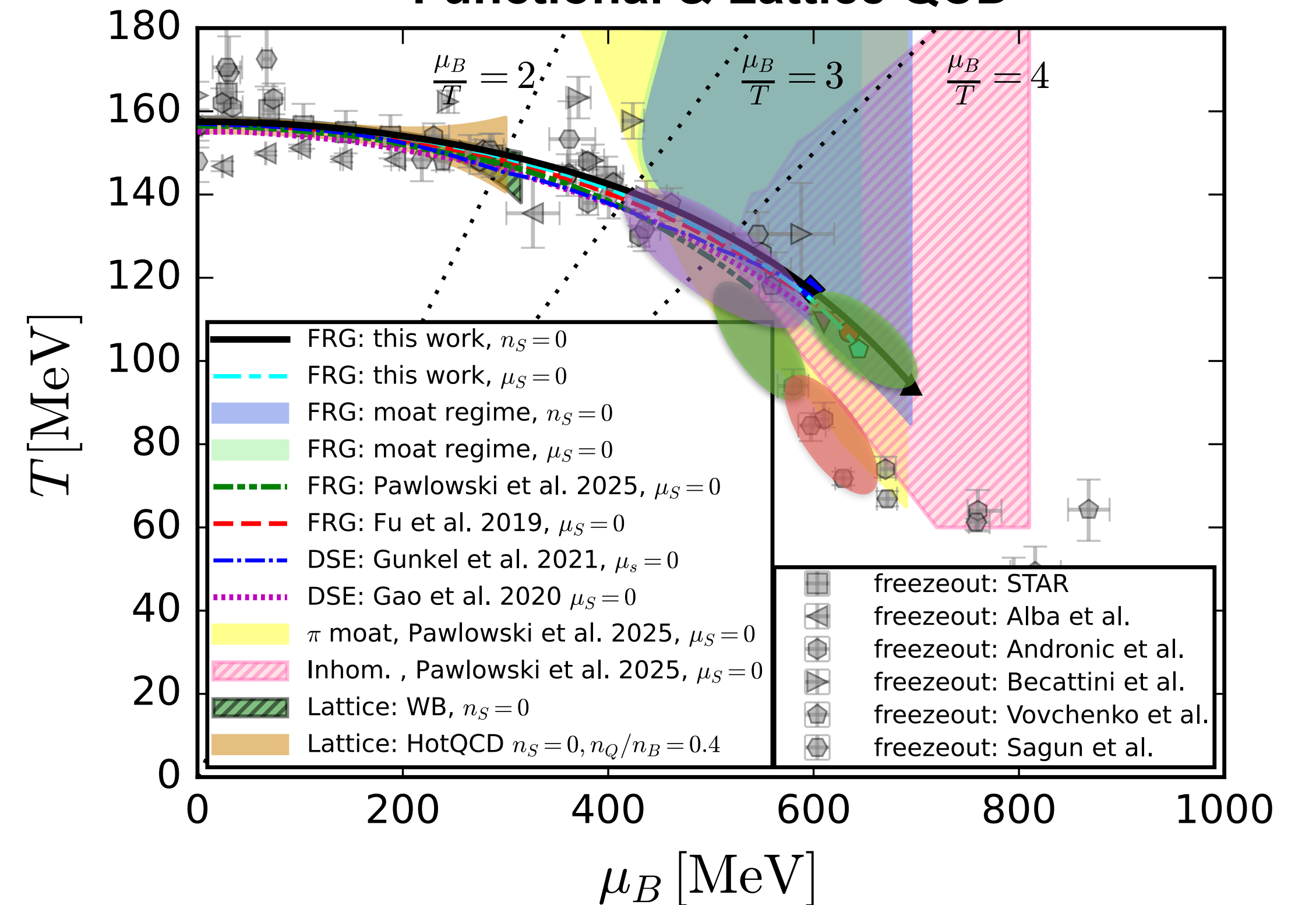
Onset of new phases on the chiral transition line (CEP)



Peak of kurtosis on the freeze-out line

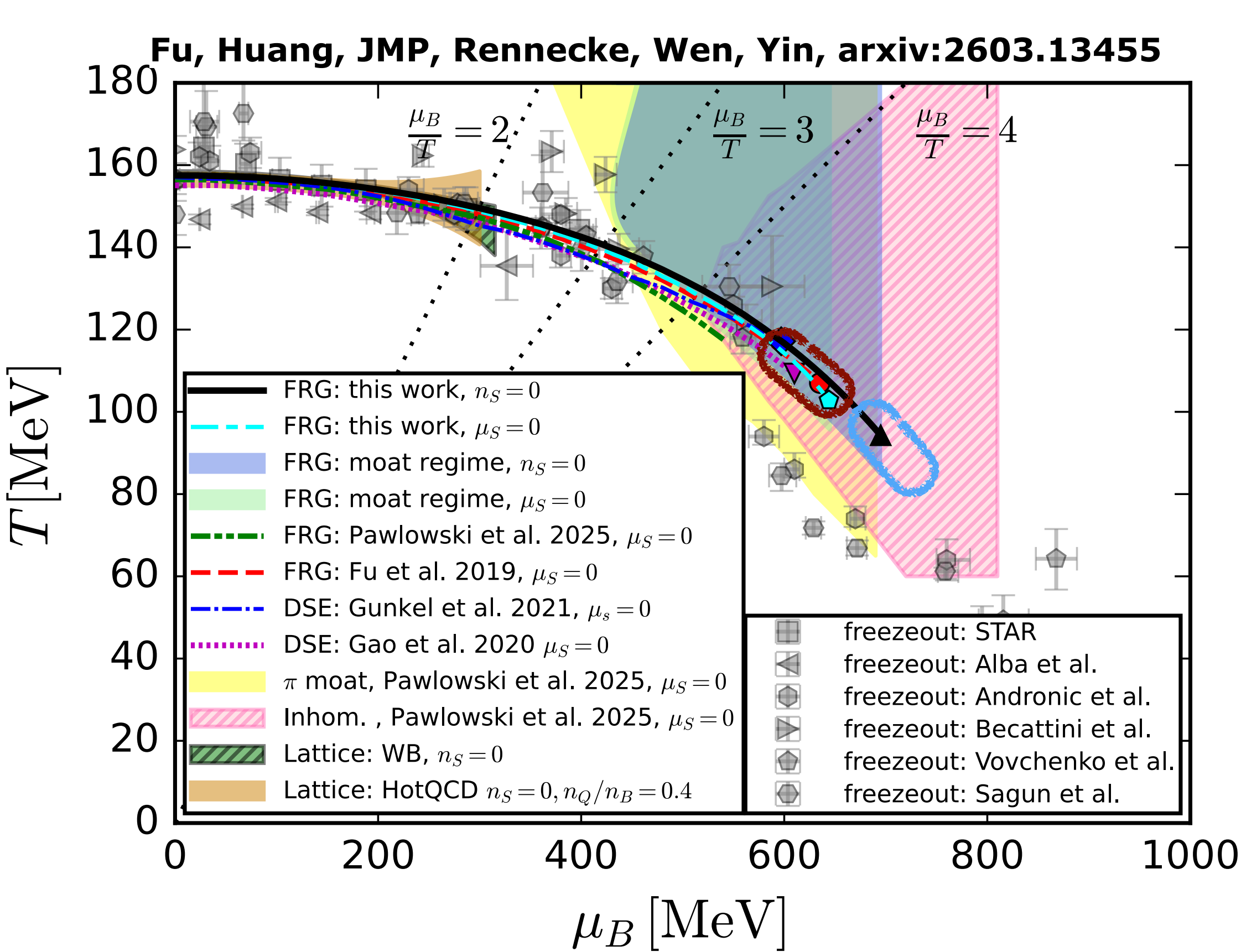
Transition regime

Functional & Lattice QCD



Critical end point and new phases: where do we stand?

Quantitative functional QCD



$$\frac{\mu_B}{T} \lesssim 4.5$$

Functional QCD: CEP estimate

$$\mu_u = \mu_d = \mu_s$$

fQCD Fu, JMP, Rennecke, PRD 101 (2020) 054032 (fRG)

fQCD Gao, JMP, PLB 820 (2021) 136584 (DSE)

$$\mu_u = \mu_d \quad \mu_s = 0$$

Gunkel, Fischer, PRD 104 (2021) 054022 (DSE)

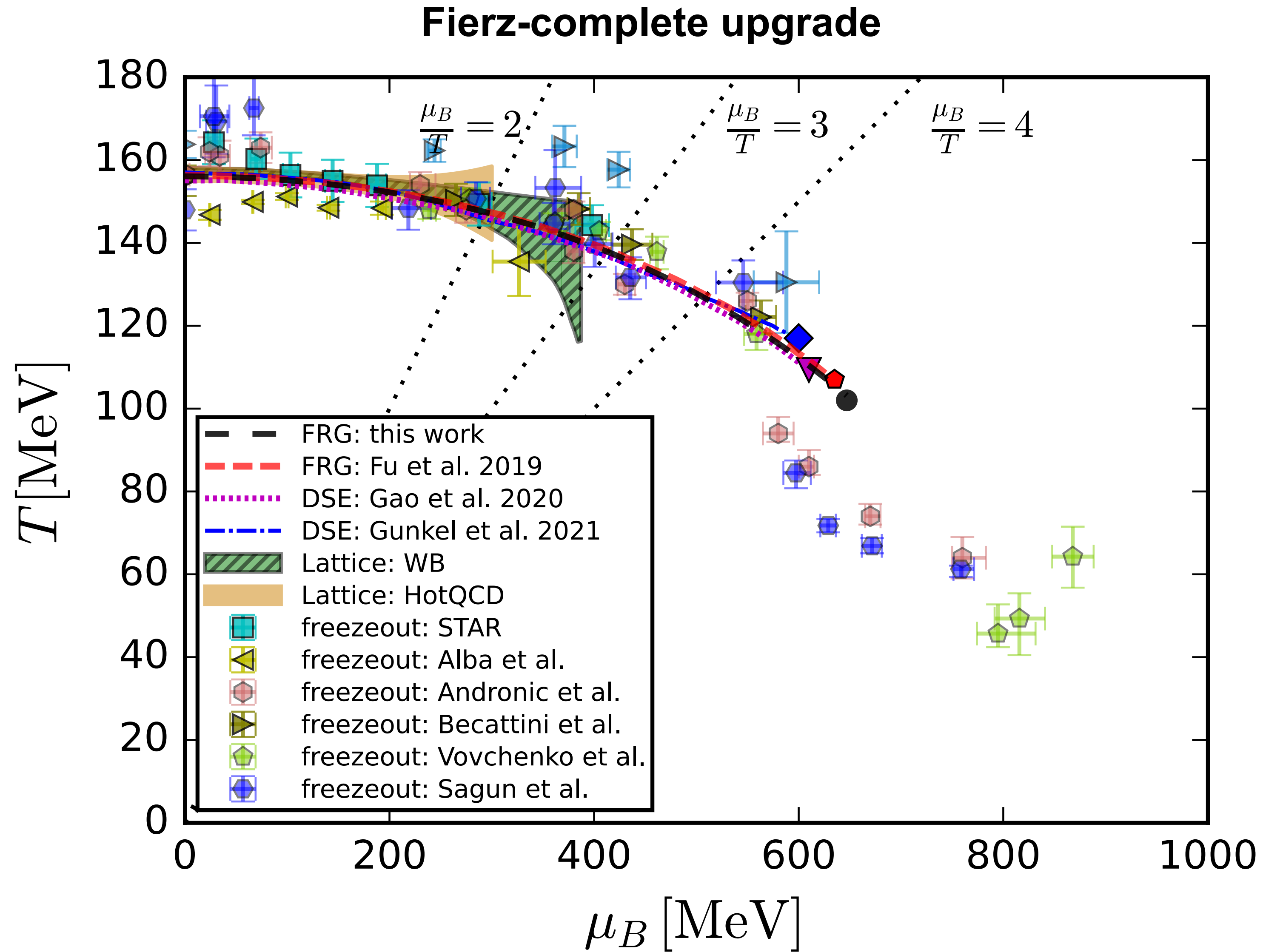
$$(T, \mu_B)_{\text{CEP}} \sim (105 - 115, 600 - 650) \text{ MeV}$$

Strangeness neutrality

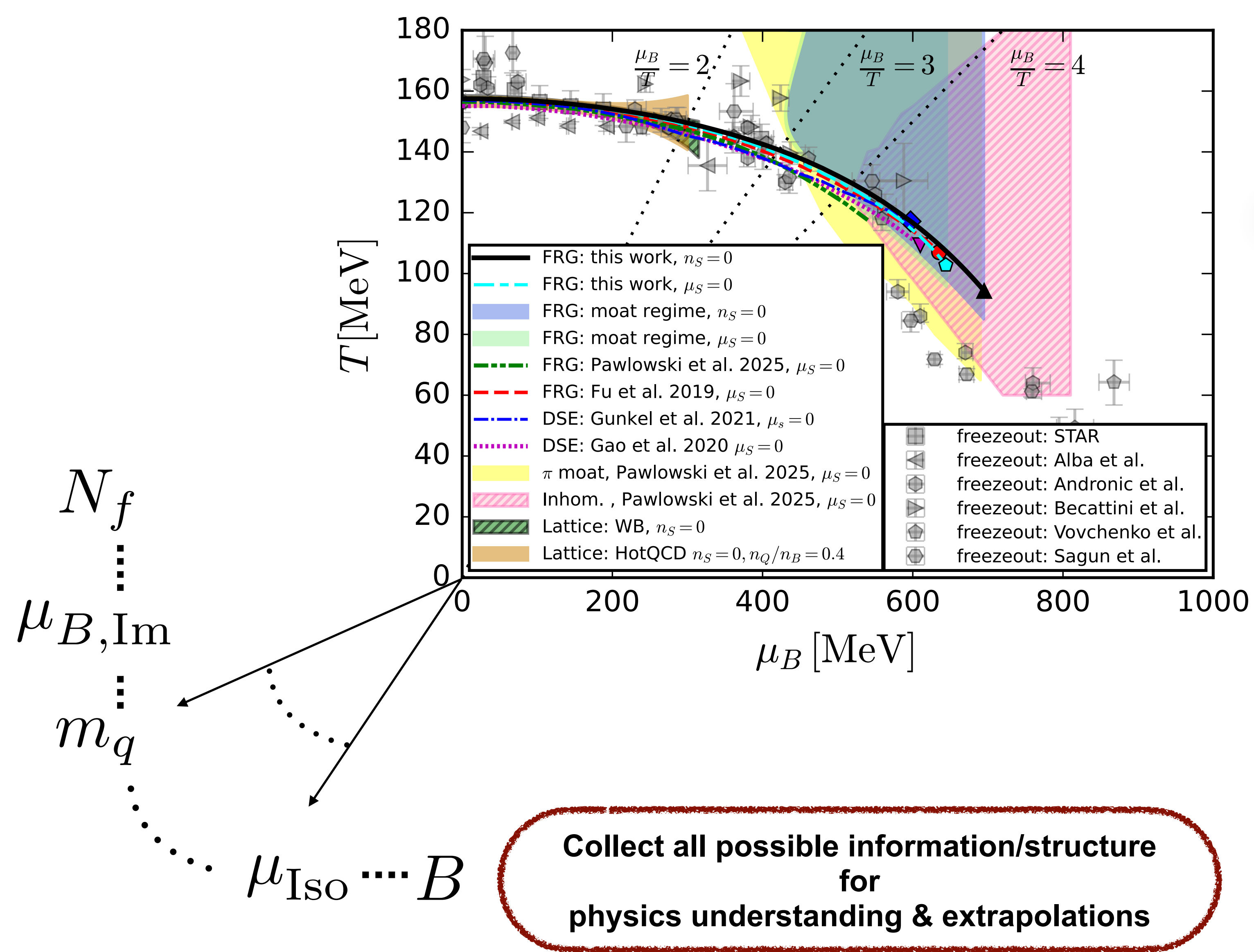
$$(T, \mu_B)_{\text{CEP}} = (92, 696) \text{ MeV}$$

$$\frac{\kappa_{n_s=0}}{\kappa_{\mu_i=\mu_q}} = 0.897$$

Critical end point and new phases: where do we stand?



Critical end point and new phases: where do we stand?



$$\mu_u = \mu_d = \mu_s$$

$$\mu_u = \mu_d \quad \mu_s = 0$$

$$(T, \mu_B)_{CEP} \sim (105 - 115, 600 - 650) \text{ MeV}$$

Strangeness neutrality

$$(T, \mu_B)_{CEP} = (92, 696) \text{ MeV}$$

Lattice

HotQCD:

Bazavov, Ding, Hegde, Kaczmarek, Karsch, Karthik, Laermann, Lahiri, Larsen, Li, Mukherjee, Ohno, Petreczky, Sandmeyer, Schmidt, Sharma, Steinbrecher, PLB 795 (2019)

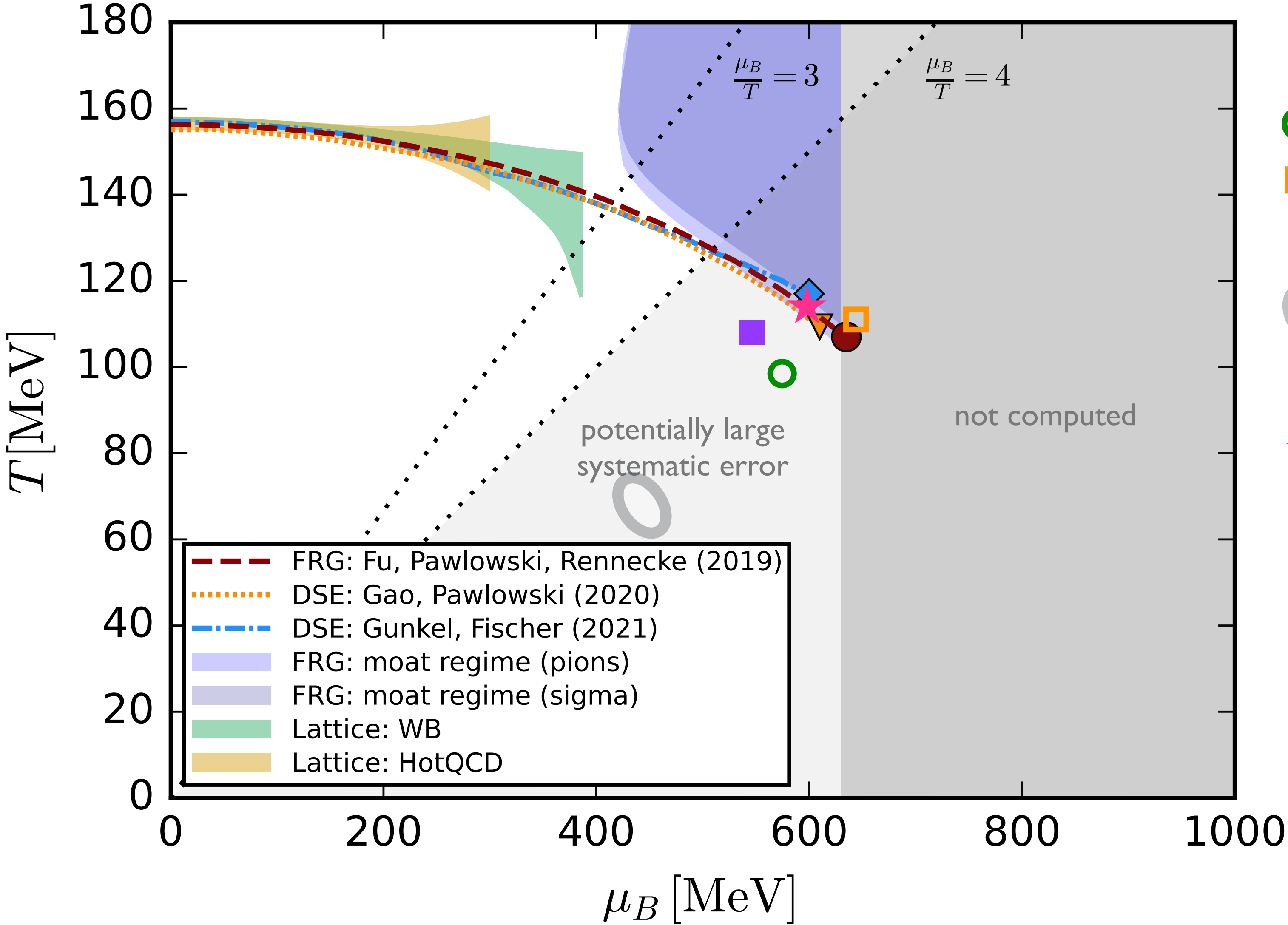
WB:

Borsányi, Fodor, Guenther, Kara, Katz, Parotto, Pasztor, Ratti, Szabo, PRL 125 (2020) 052001

QCD PHASE DIAGRAM & THE CEP

Modified from Fabian Rennecke's
plenary talk @ QM 2025

FRG & DSE results corroborated by subsequent "extrapolations" of lattice data



using Yang-Lee edge singularities:

- conformal Padé [Basar, 2312.06952] (green circle)
- multi-point Padé [Clarke et al., 2405.10196] (orange square)
 $N_\tau = 6, 8$ results + continuum estimate [Schmidt, 2504.00629]
- very recent lattice extrapolation ($16^3 \times 8$) [Adam et al, 2507.13254] (grey circle)

using thermodynamics:

- constant entropy density [Shah et al. 2410.16206] (pink star)
- See talk of Volodymyr [Shah et al. 2601.08823] (brown text)

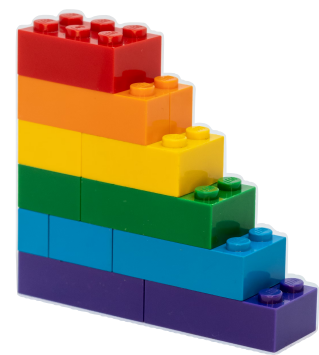
model-based extrapolation (tiny selection):

- holography [Hippert et al., 2309.00579] (purple square)
(agrees with [Cai et al., 2201.02004])

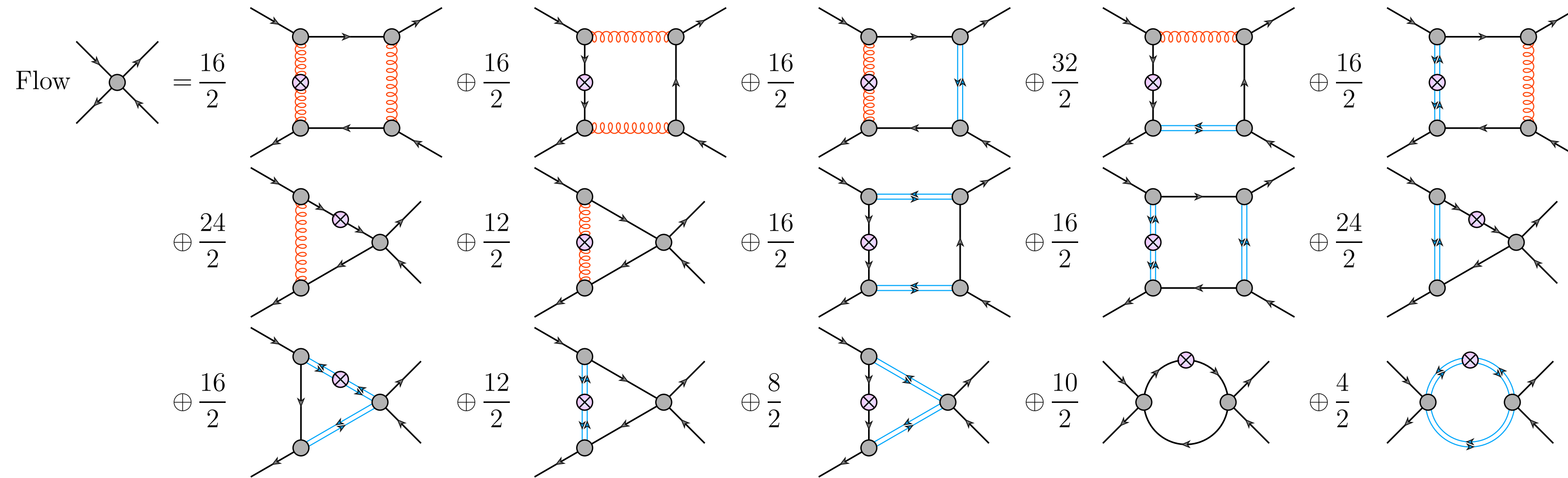
CEP location well constrained
by now. And it's in **FAIR** range!

Jump forward: Functional flows in QCD

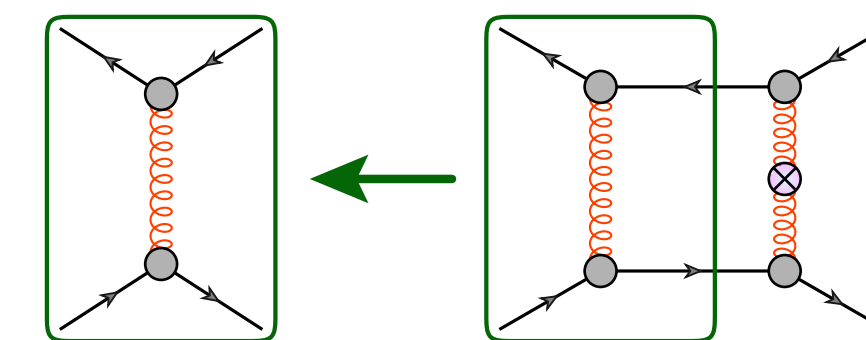
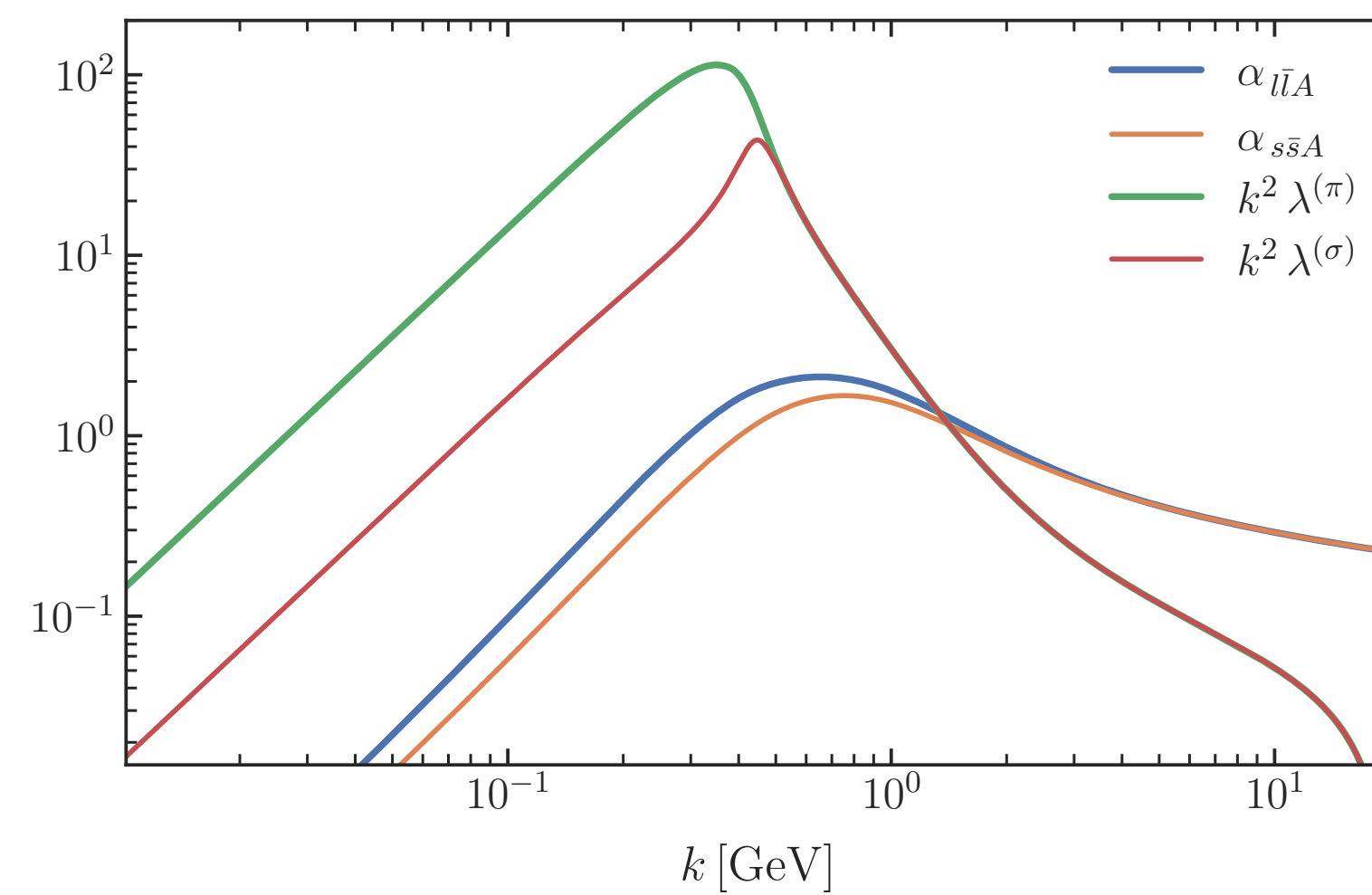
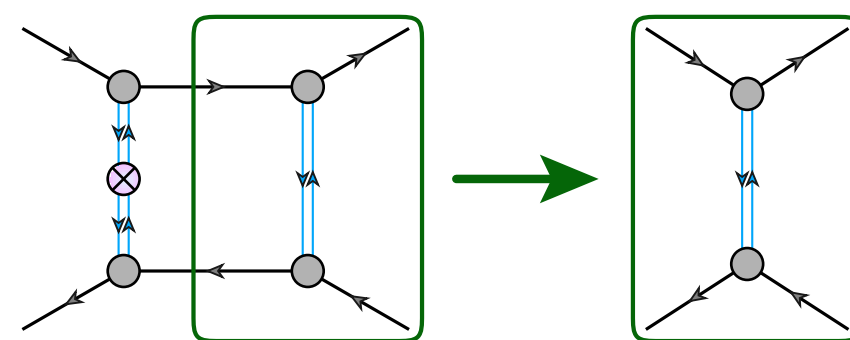
Phase structure from functional QCD: how to

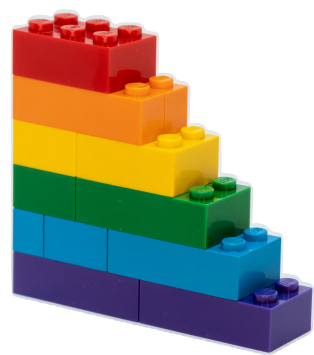


How to: systematic error estimates & the LEGO[®] principle

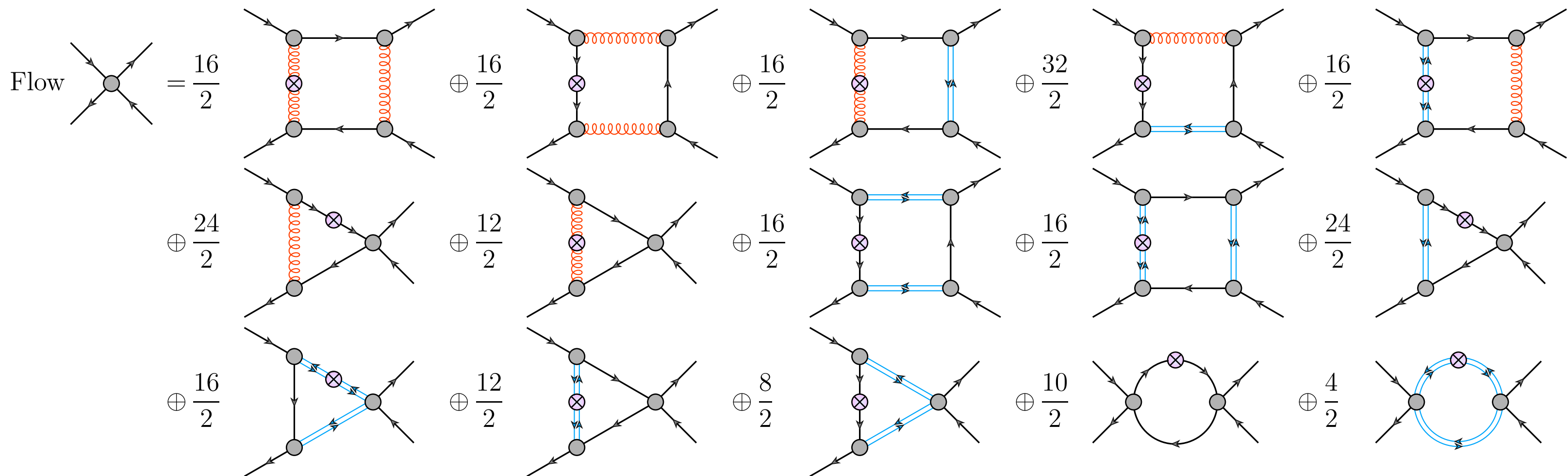


Example: 4-quark scattering vertex

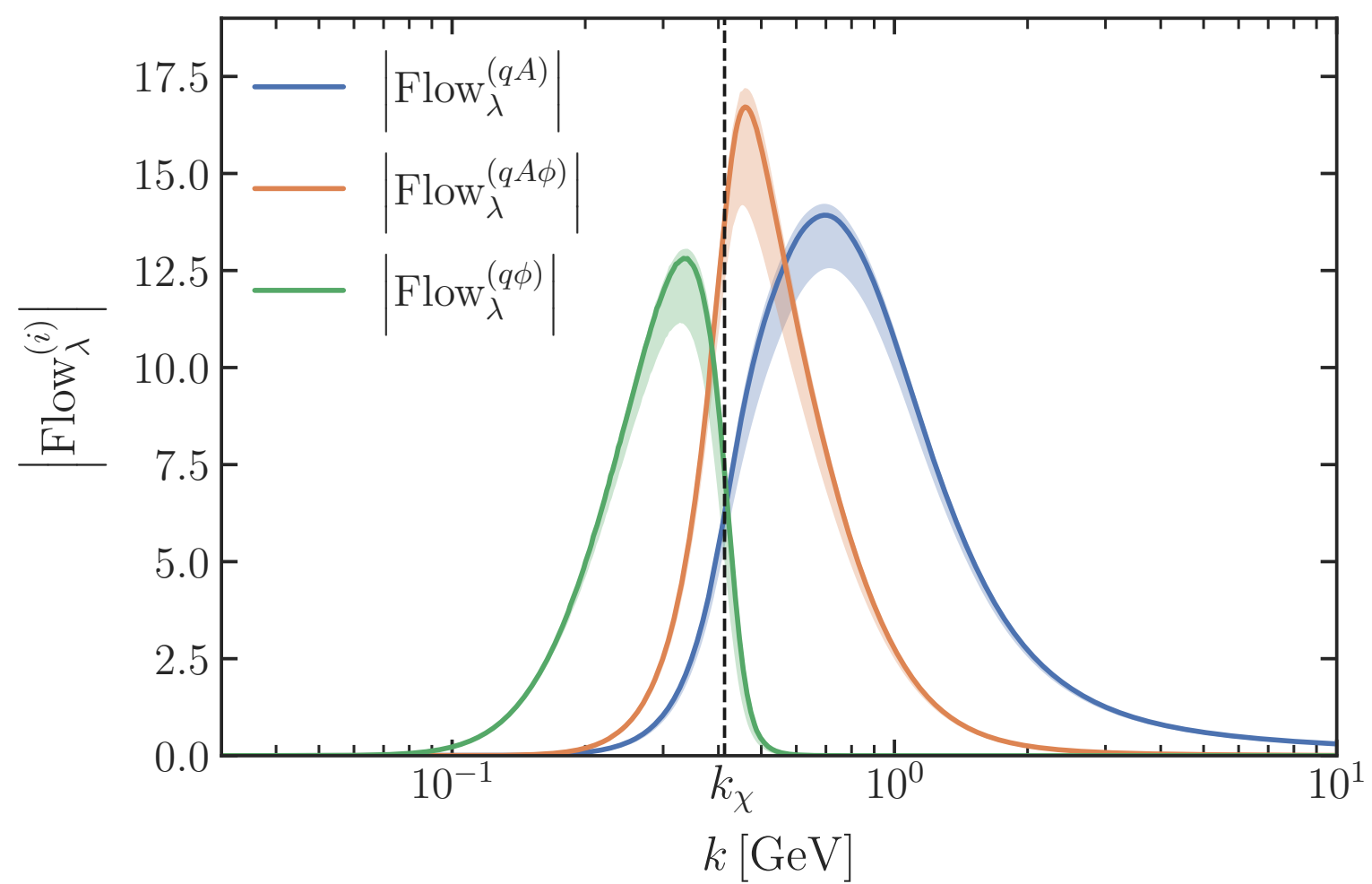


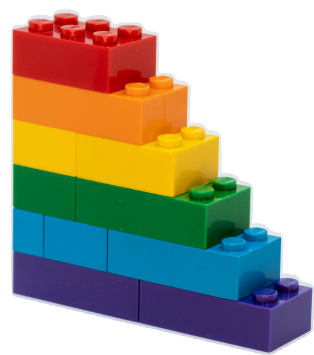


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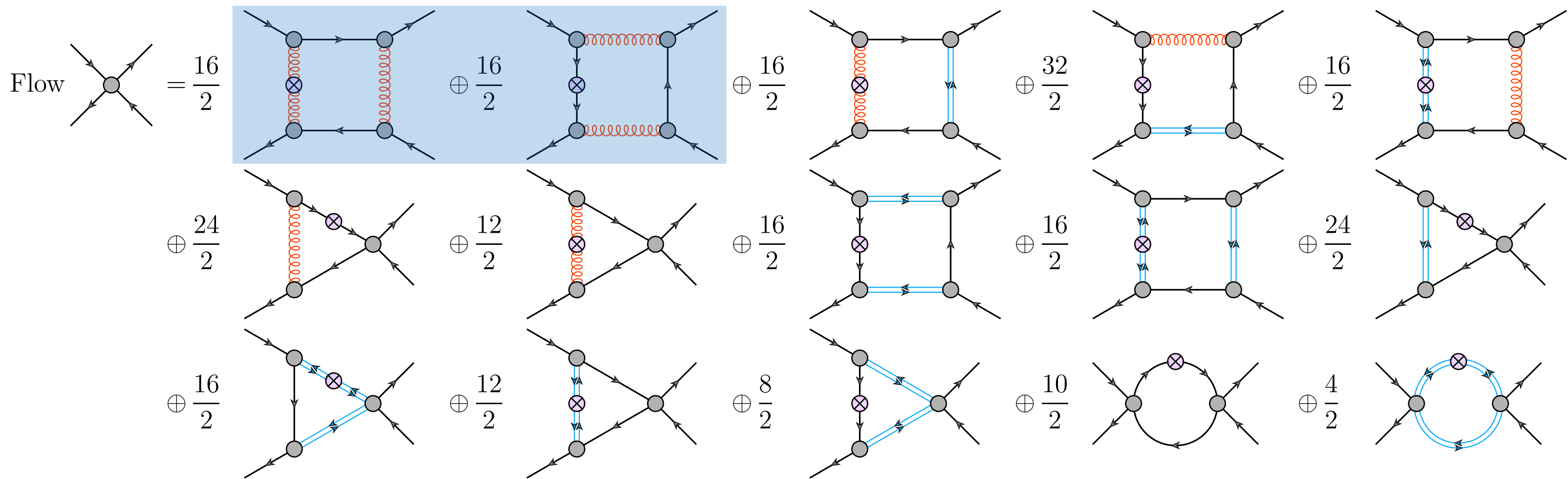


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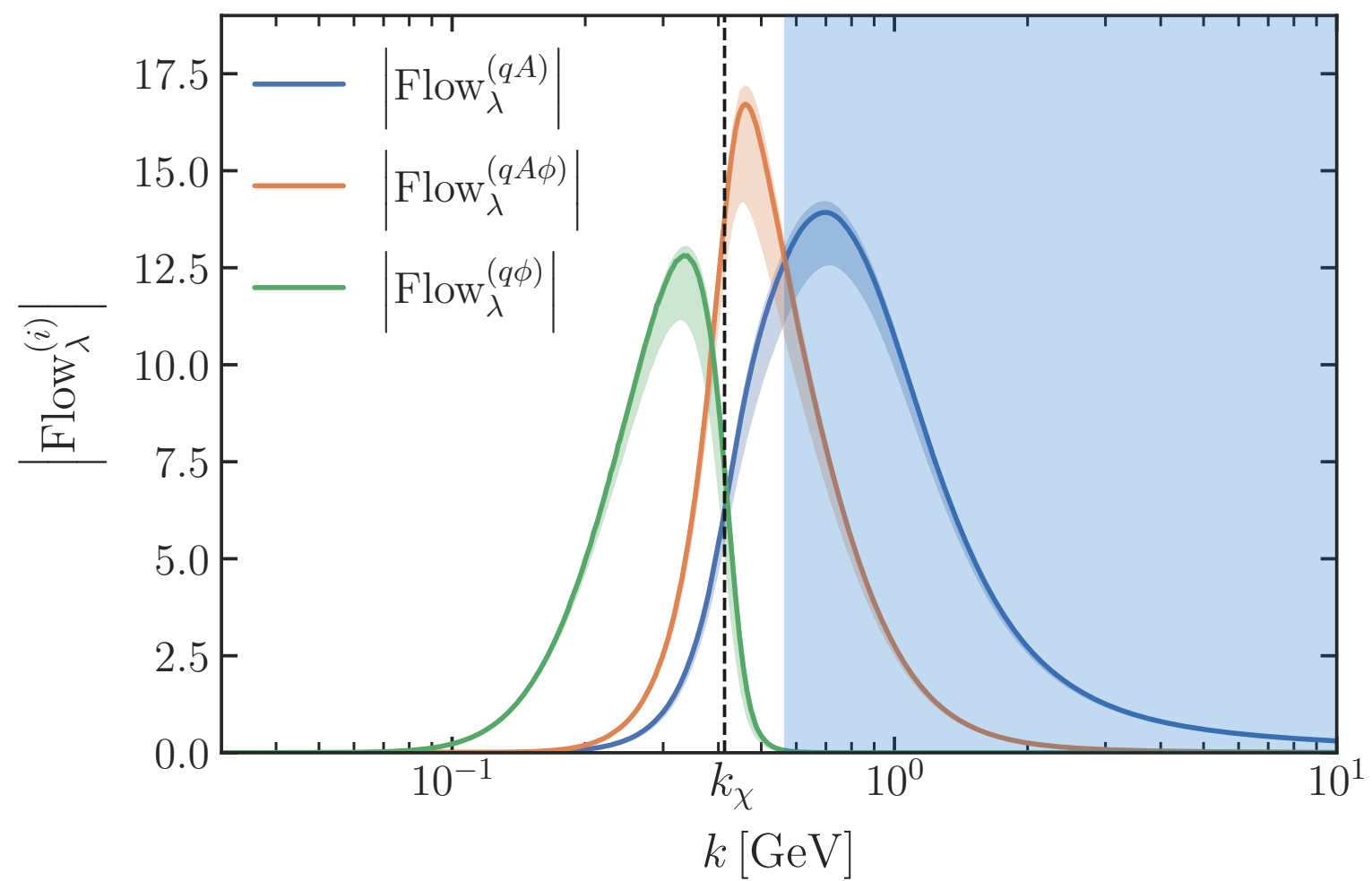


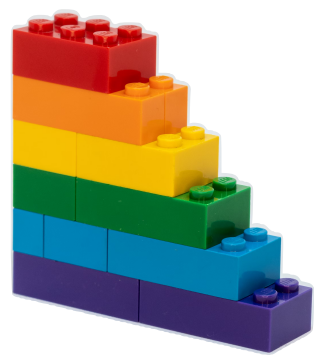


How to: systematic error estimates & the LEGO[®] principle

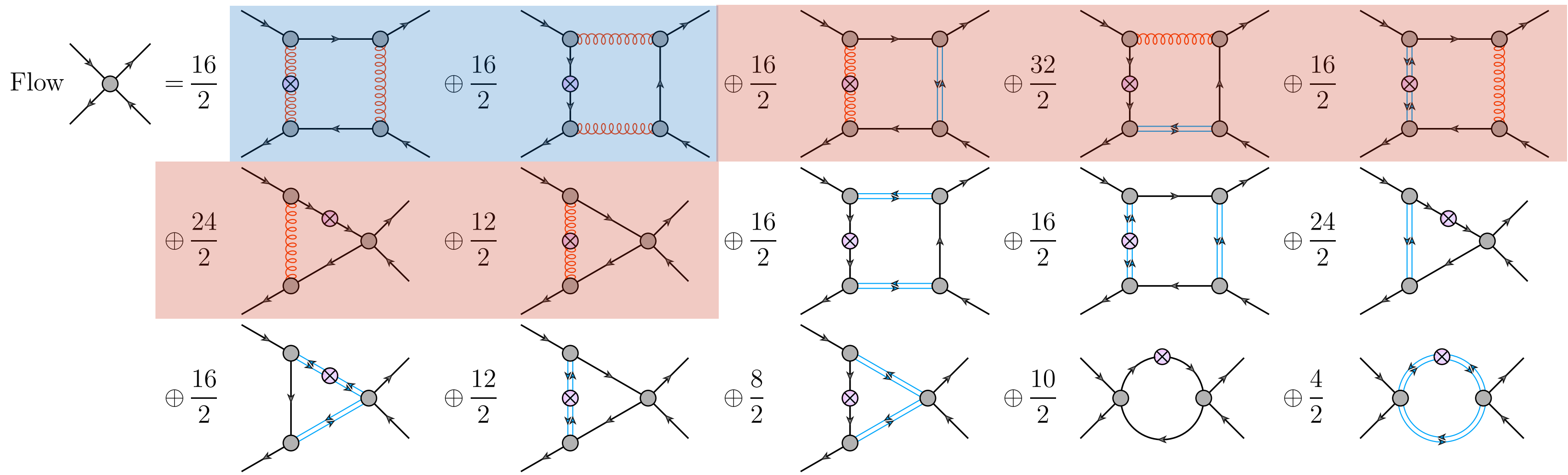


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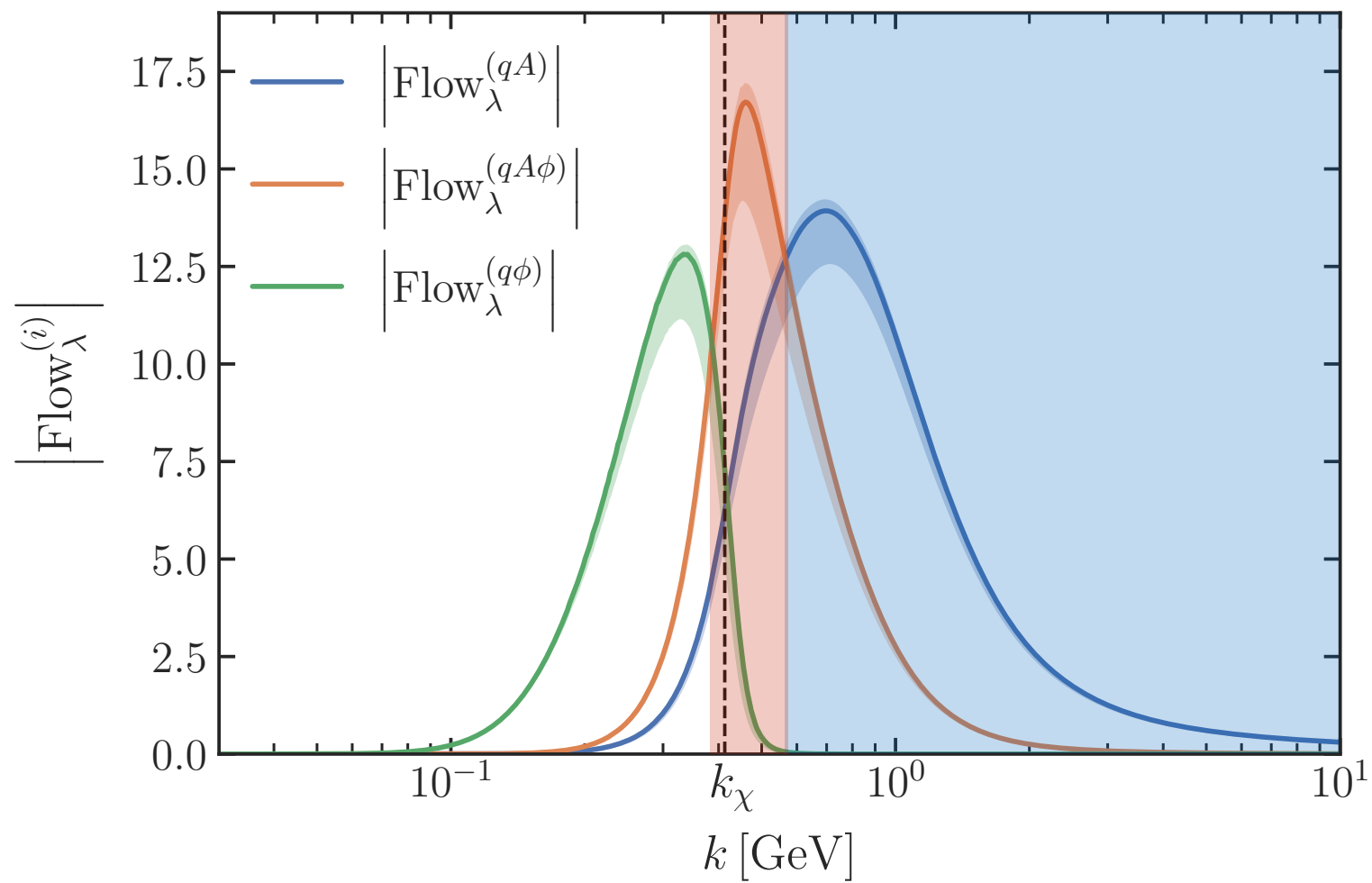


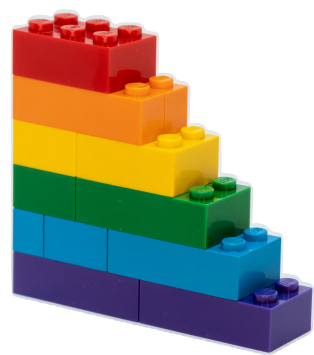


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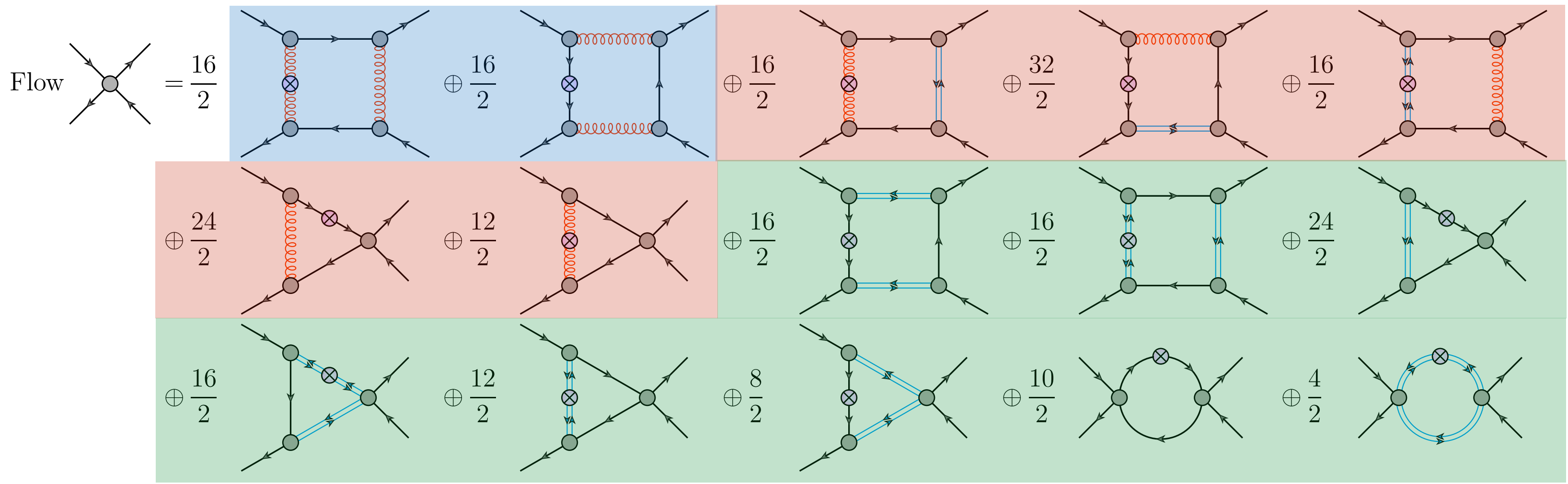


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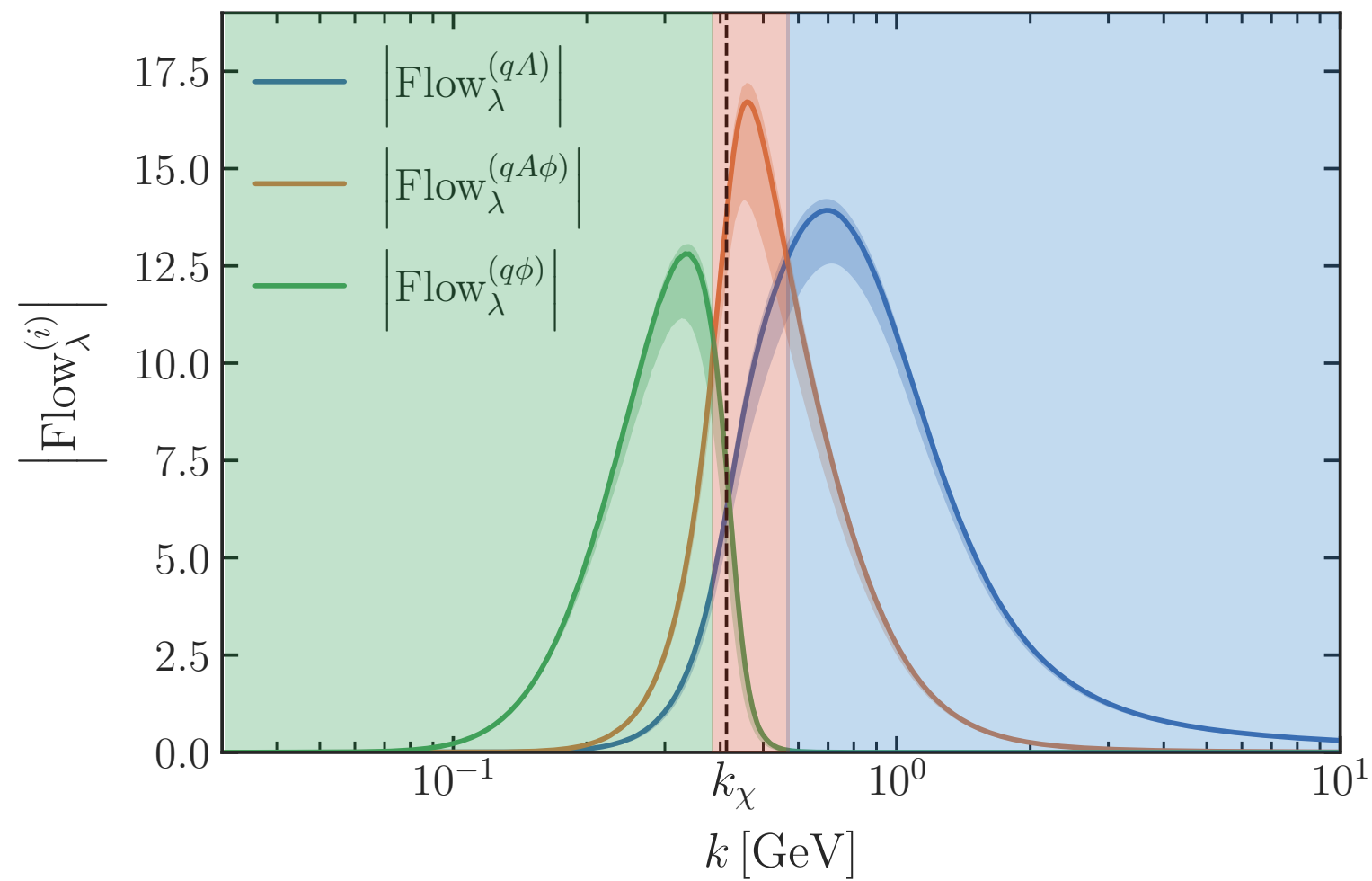


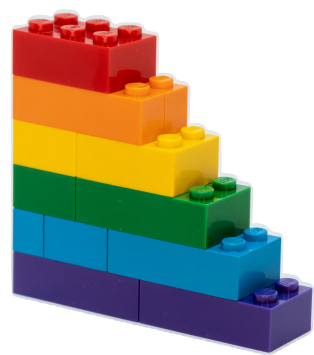


How to: systematic error estimates & the LEGO[®] principle

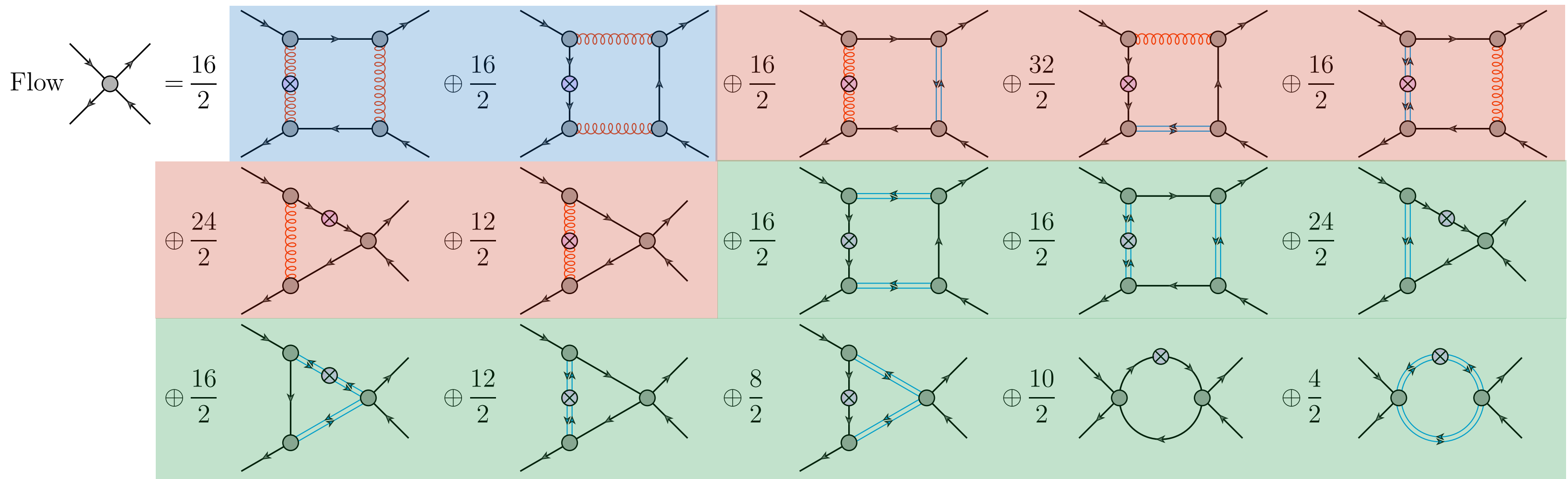


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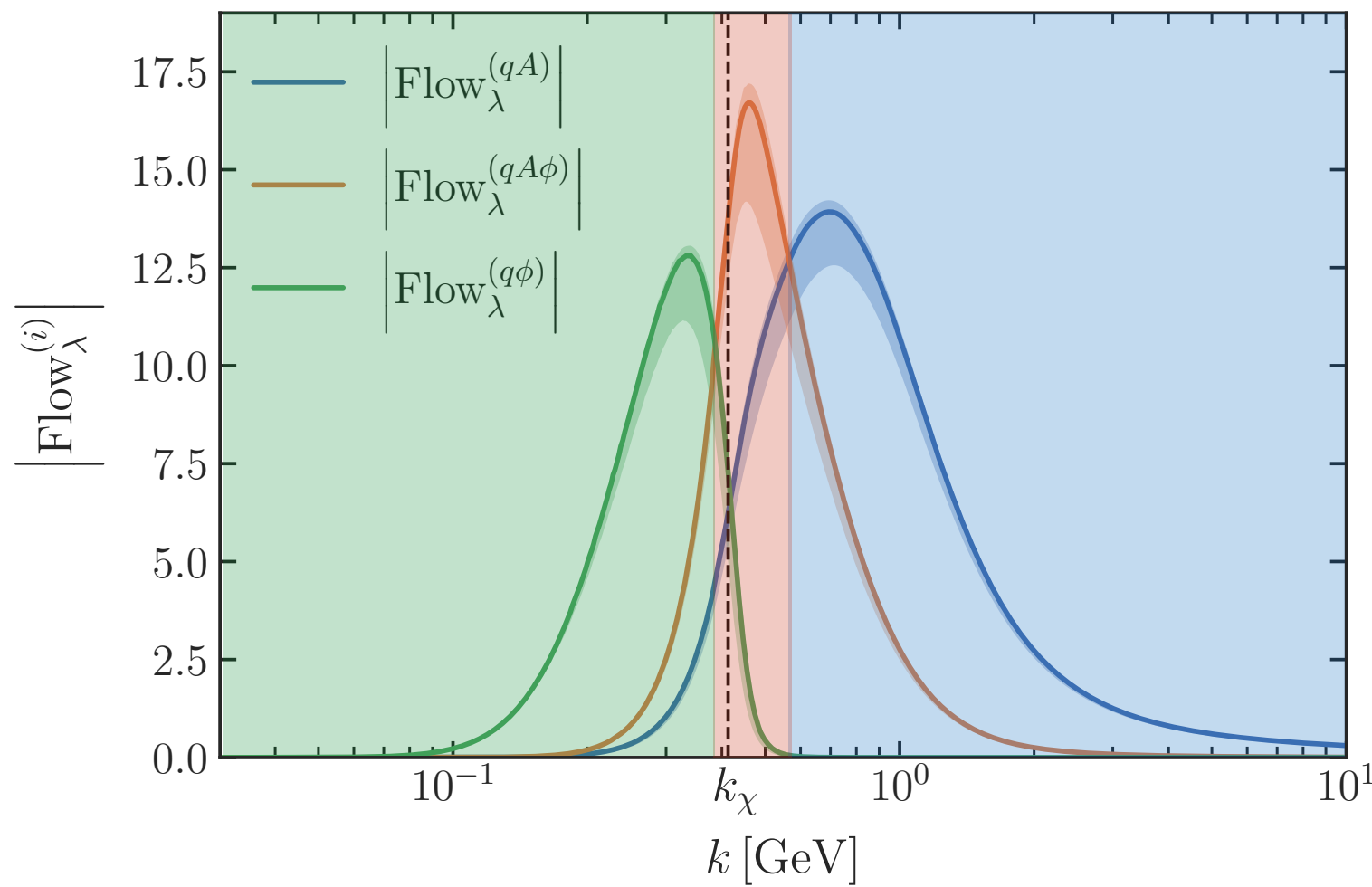




How to: systematic error estimates & the LEGO[®] principle



The unreasonable effectiveness of low energy effective theories



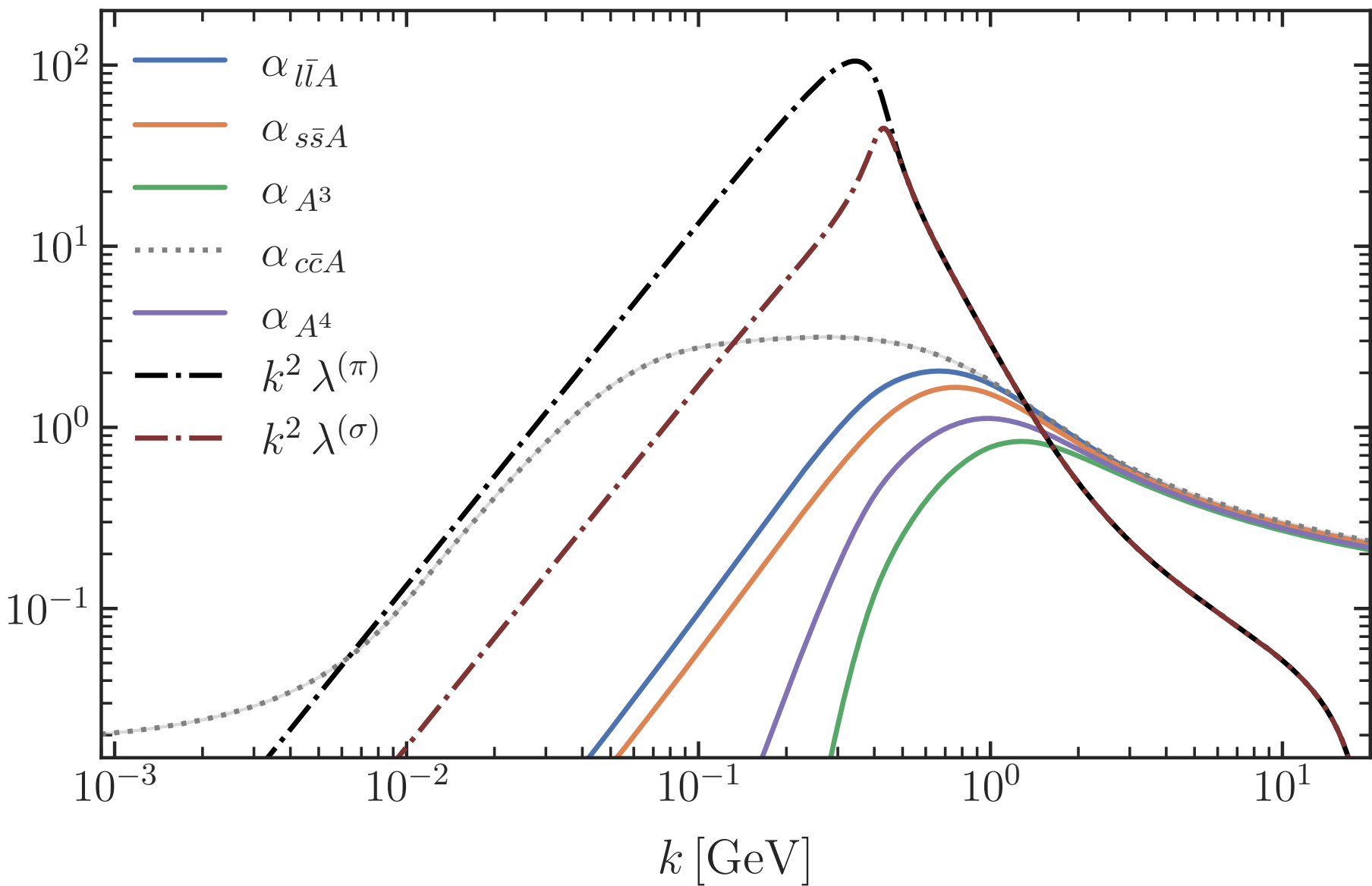
Access and combined use of
error estimates
from functional QCD & LEFTs

How to: the unreasonable effectiveness of low energy effective theories

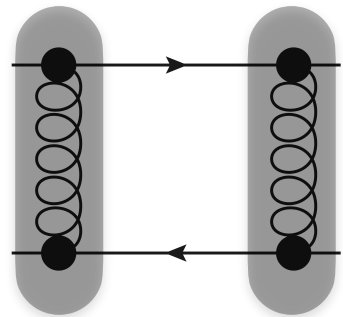
$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \left(\text{orange loop} - \text{dashed loop} - \text{solid loop} + \frac{1}{2} \text{blue loop} \right)$$

fQCD

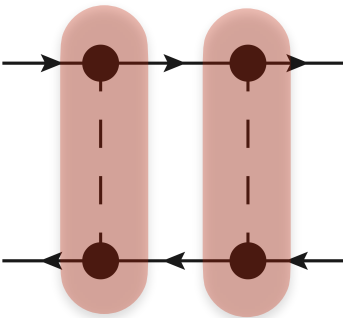
Sequential decoupling of gluon, quark, sigma, pion fluctuations



$$\frac{g_{lAl}^2}{g_{\bar{s}As}^2}$$



$$k^2 \lambda^{(\pi, \sigma)}$$



Vacuum: Ihssen, JMP, Sattler, Wink, arXiv:2408.08413

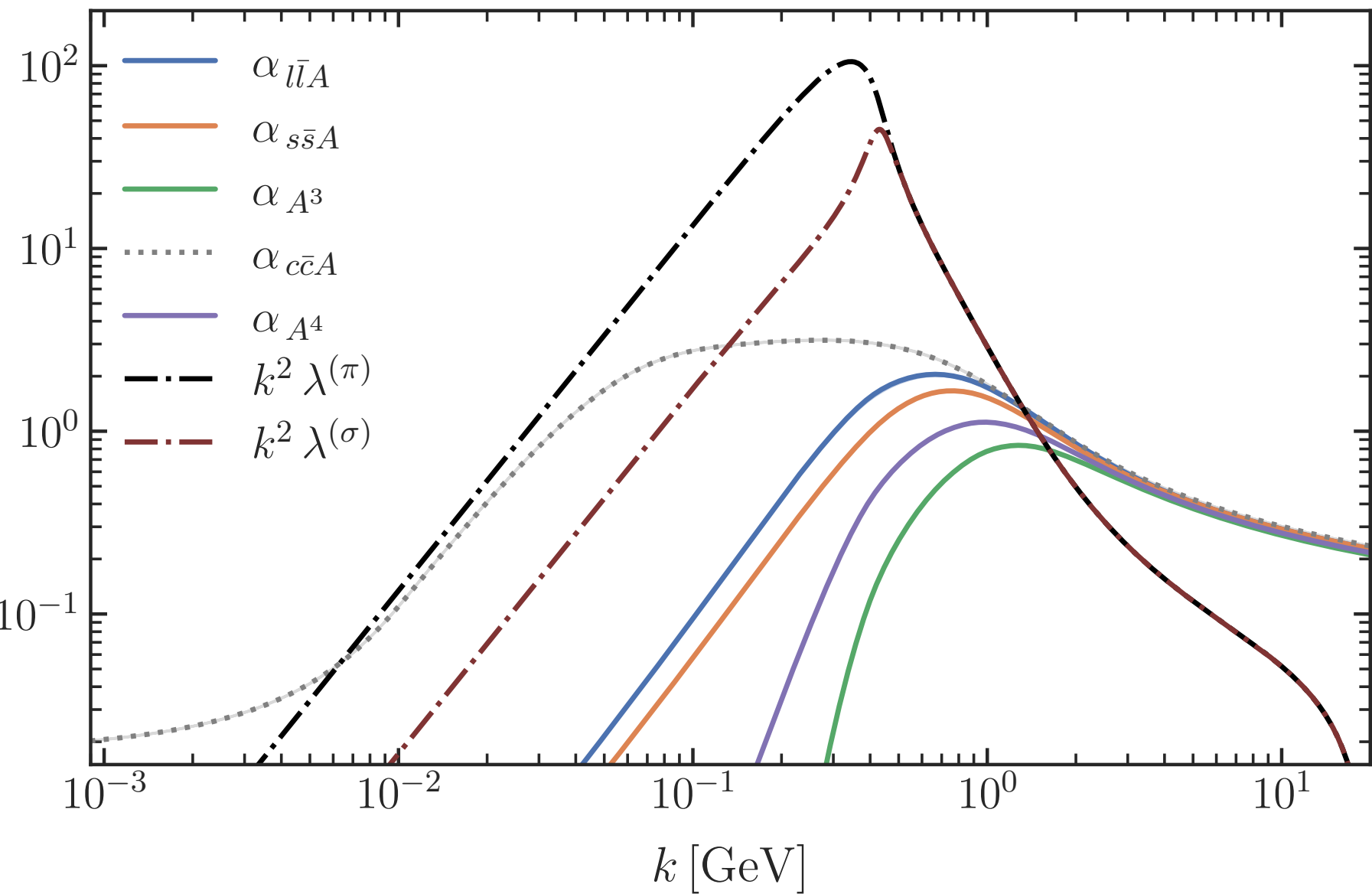
Phase structure: Fu, JMP, Rennecke, PRD 101, (2020) 054032

How to: the unreasonable effectiveness of low energy effective theories

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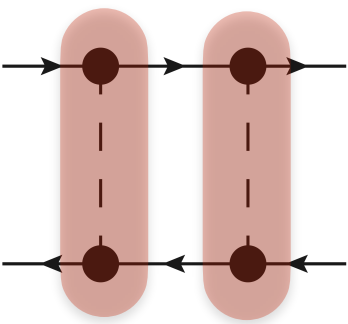
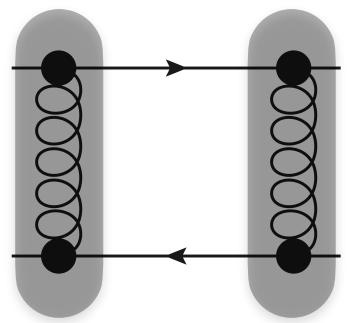
fQCD

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Phase structure: Fu, JMP, Rennecke, PRD 101, (2020) 054032

PQM-model

$$\frac{1}{2} \left(\text{orange loop} - \text{dashed loop} + \text{solid loop} + \frac{1}{2} \text{blue loop} \right)$$

PNJL-model

$$\frac{1}{2} \left(\text{orange loop} - \text{dashed loop} + \text{solid loop} \right)$$

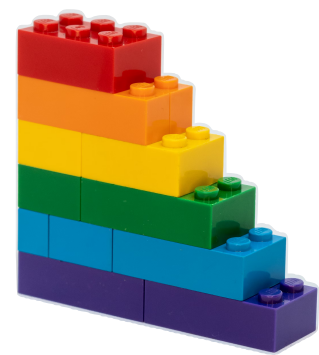
QM-model

$$- \text{solid loop} + \frac{1}{2} \text{blue loop}$$

NJL-model

$$- \text{solid loop}$$

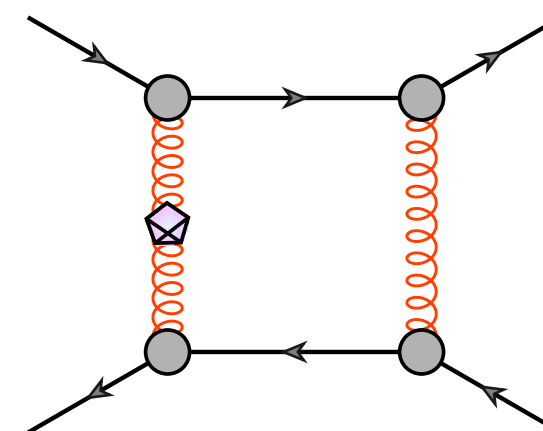
QCD-assisted low energy effective theories

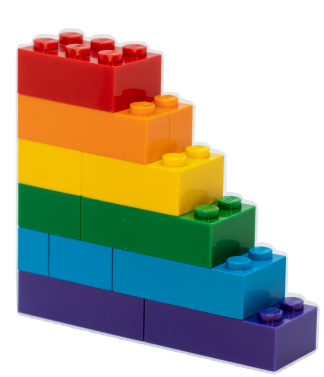


How to: systematic error estimates & the LEGO[®] principle

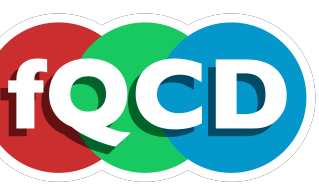


Example: Disect quark-gluon box

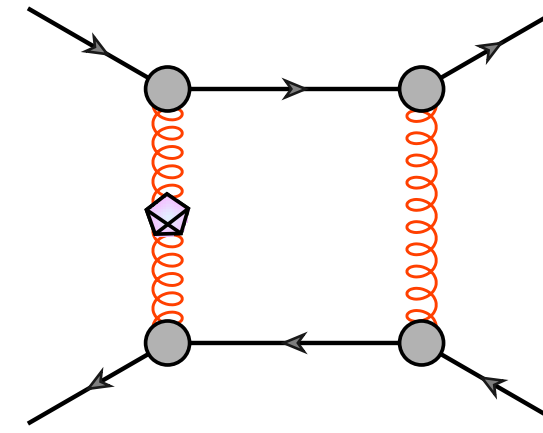




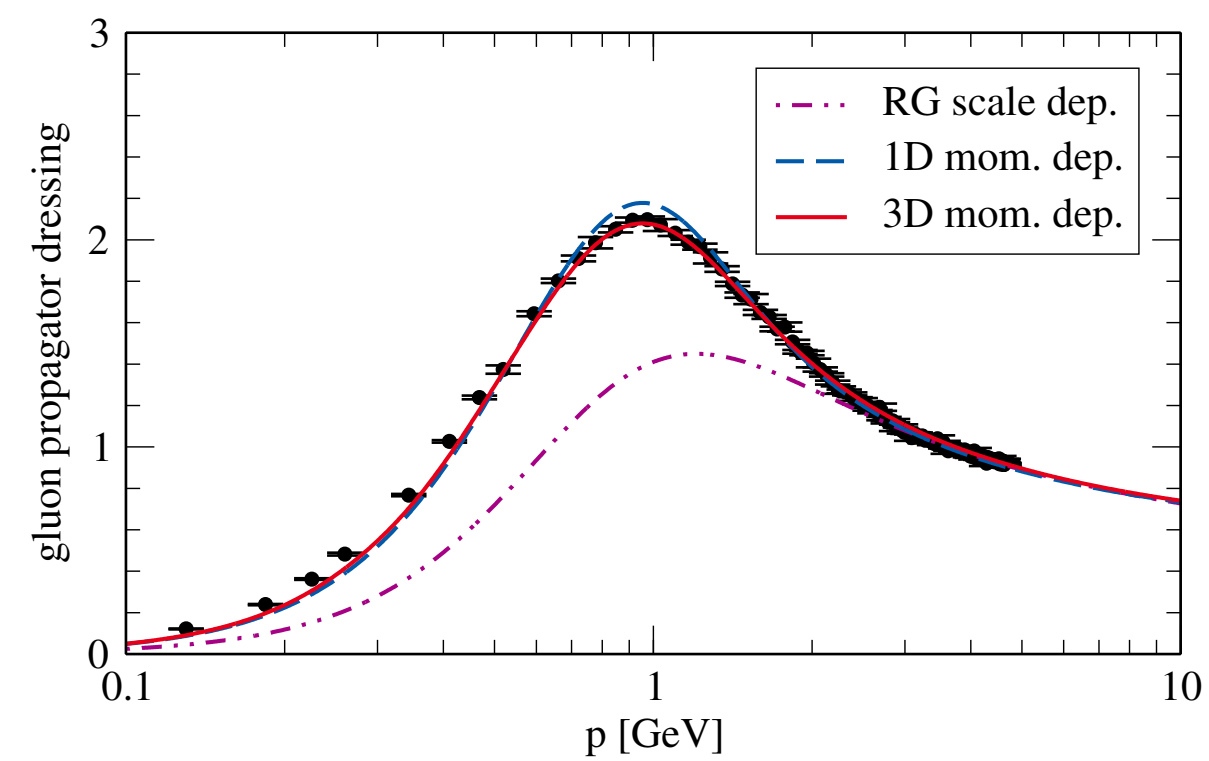
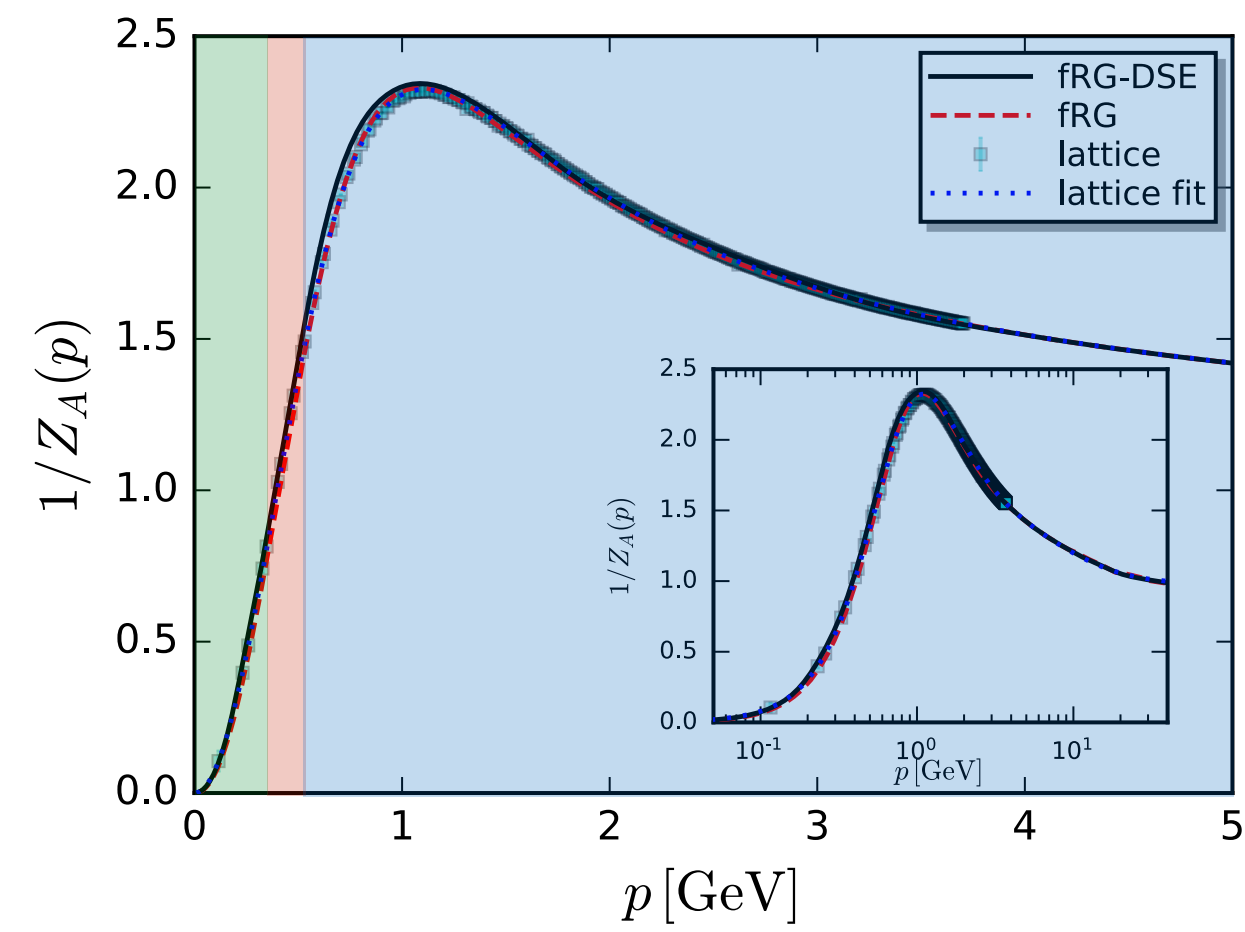
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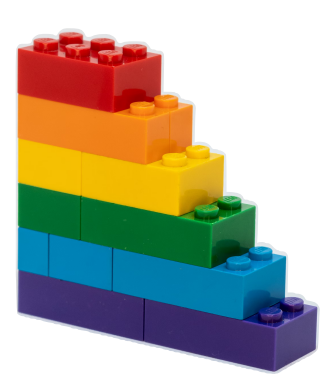


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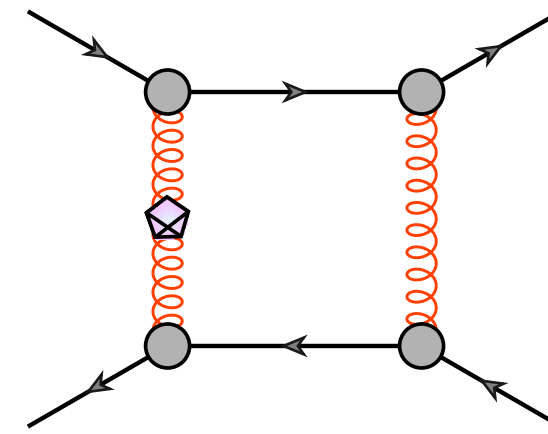
gluon two-point correlator 



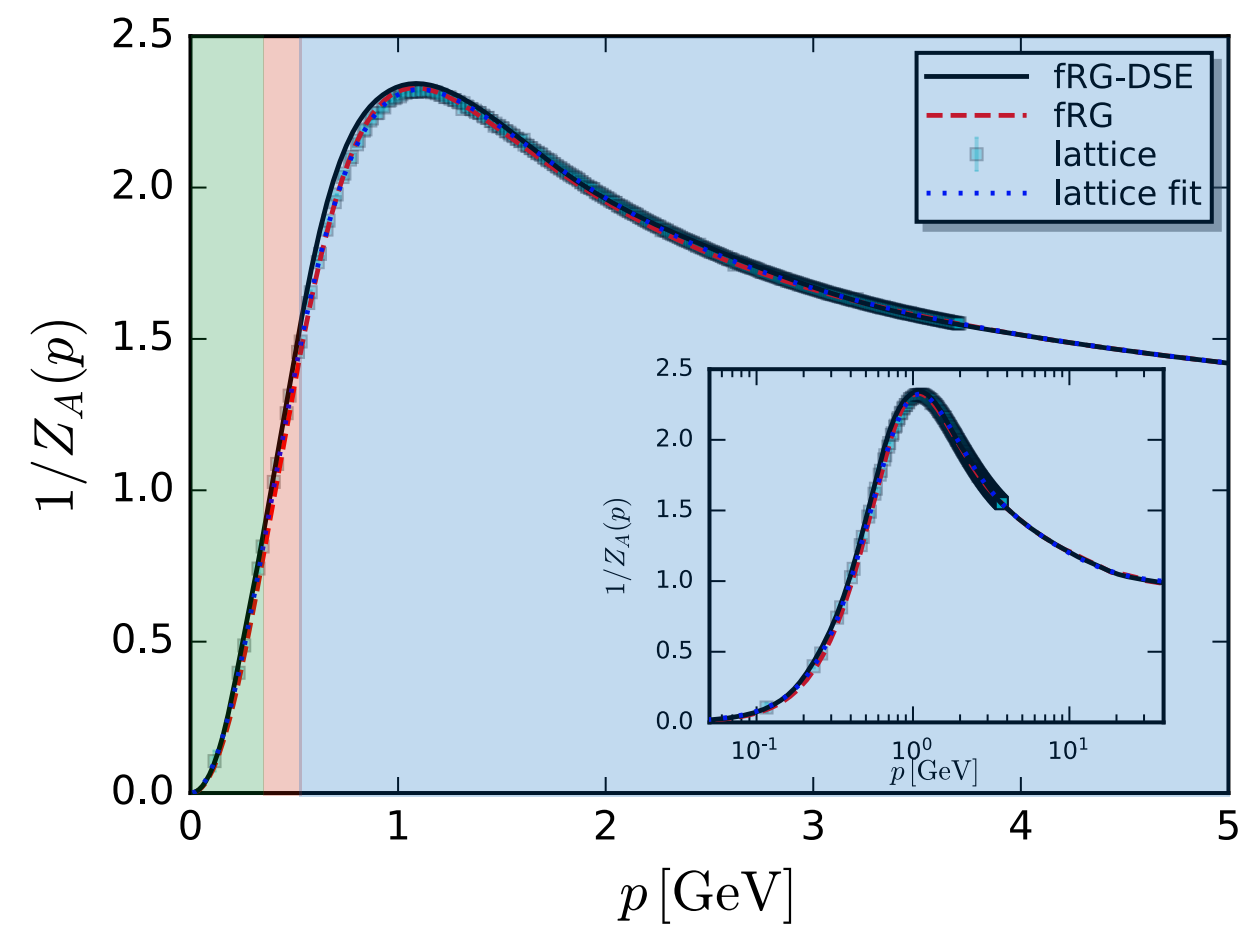


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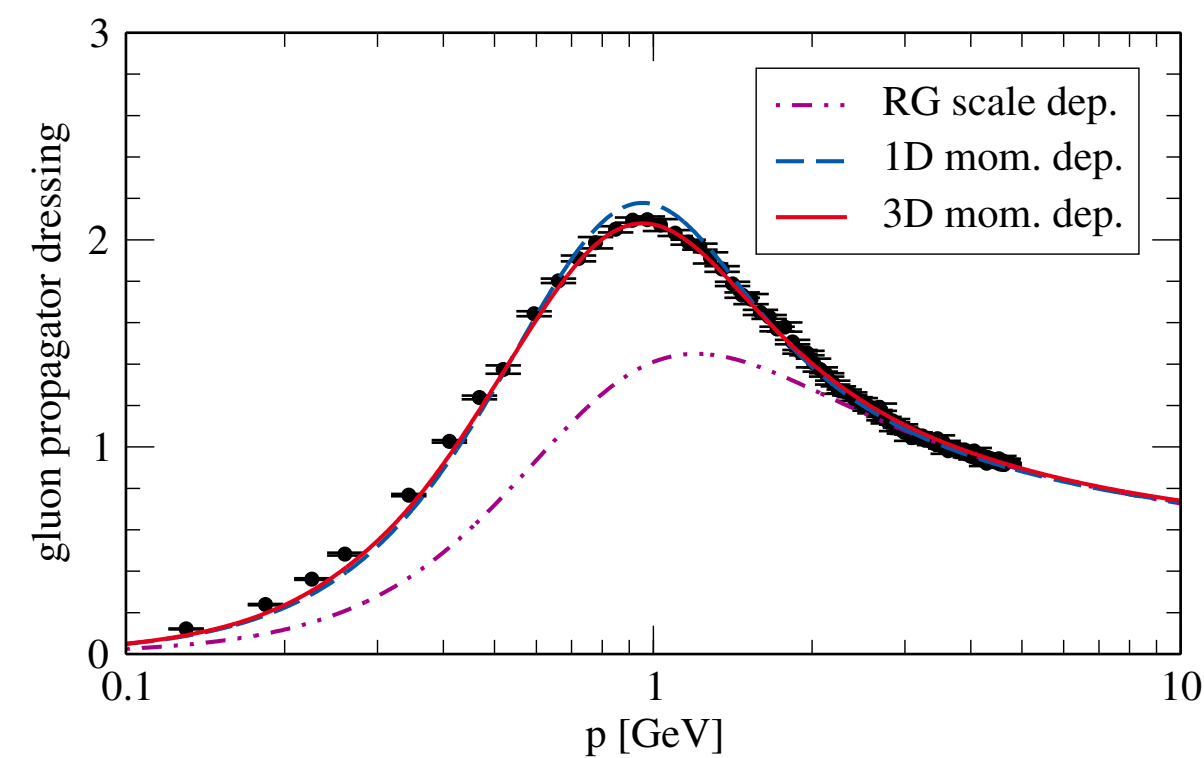
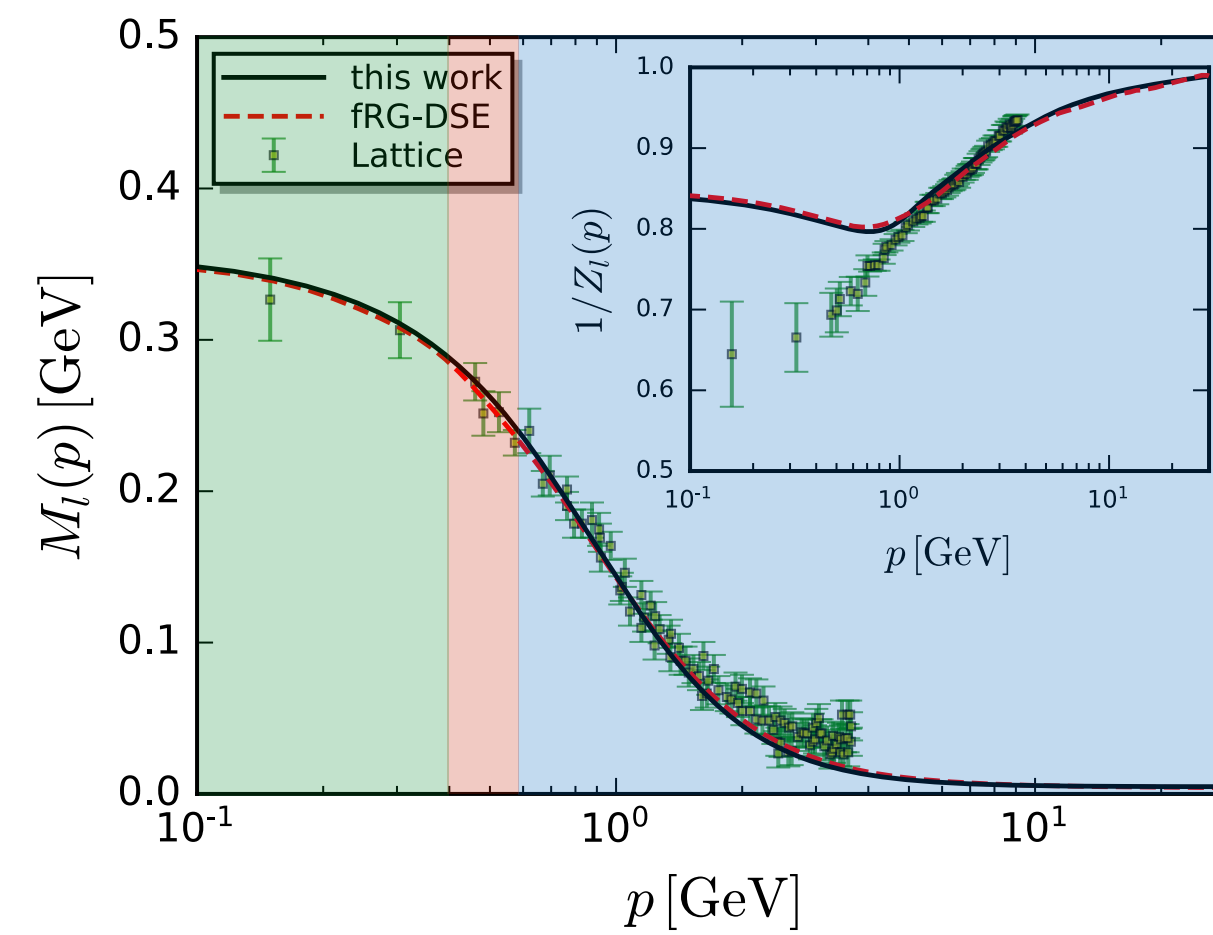
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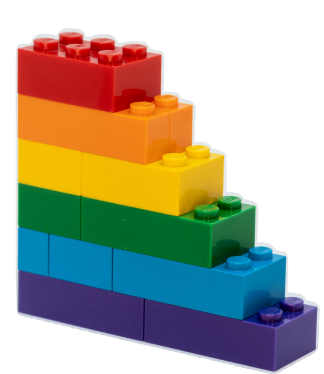


gluon two-point correlator



quark two-point correlator

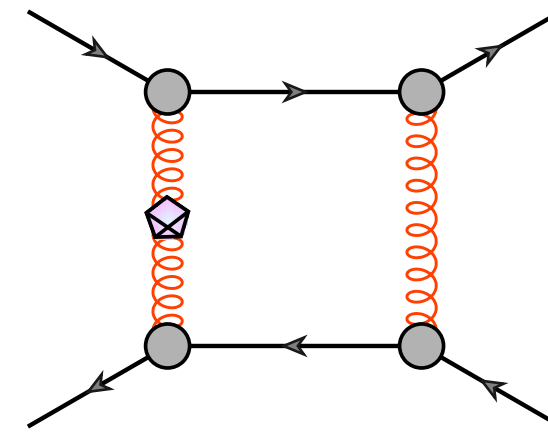




How to: systematic error estimates & the LEGO[®] principle

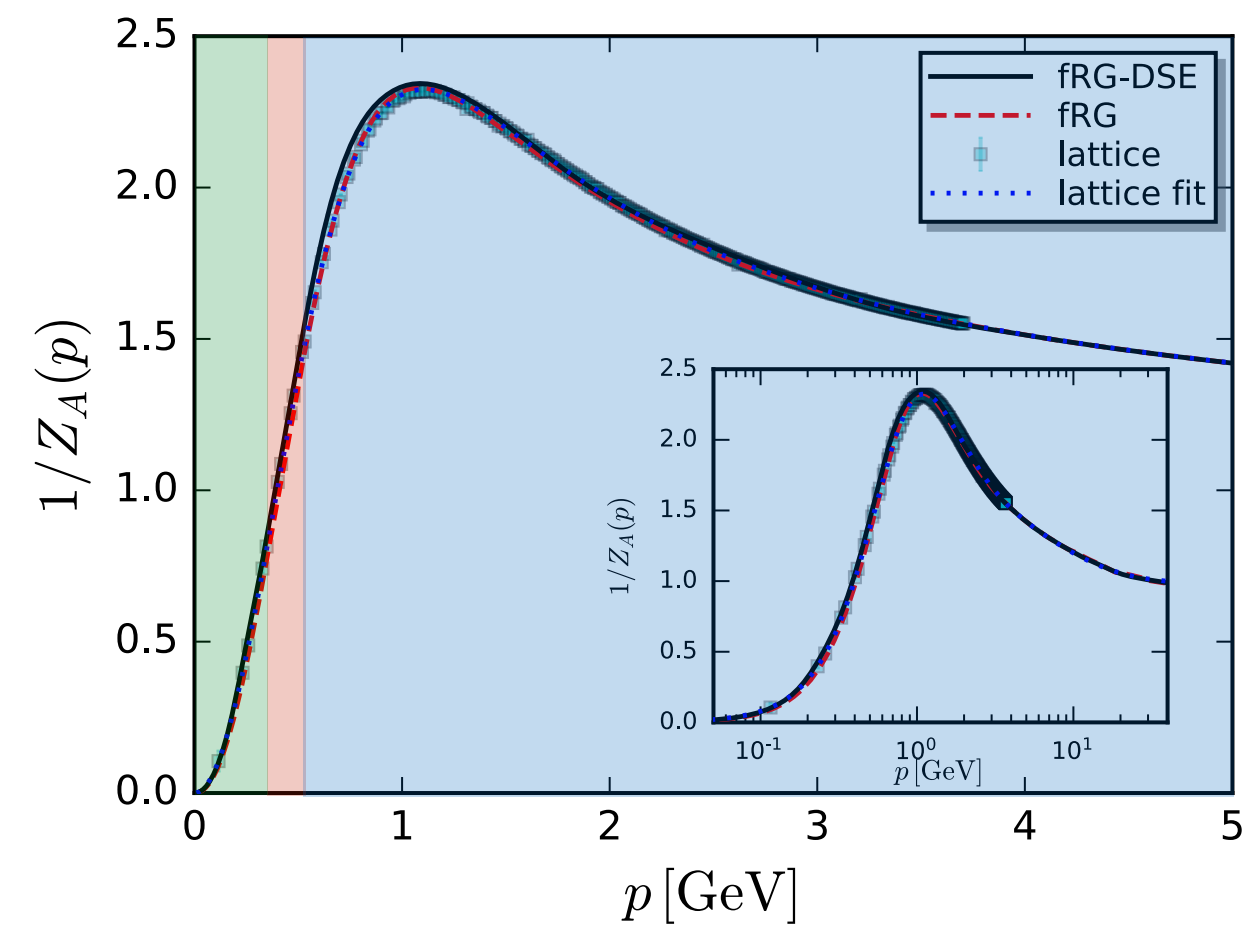


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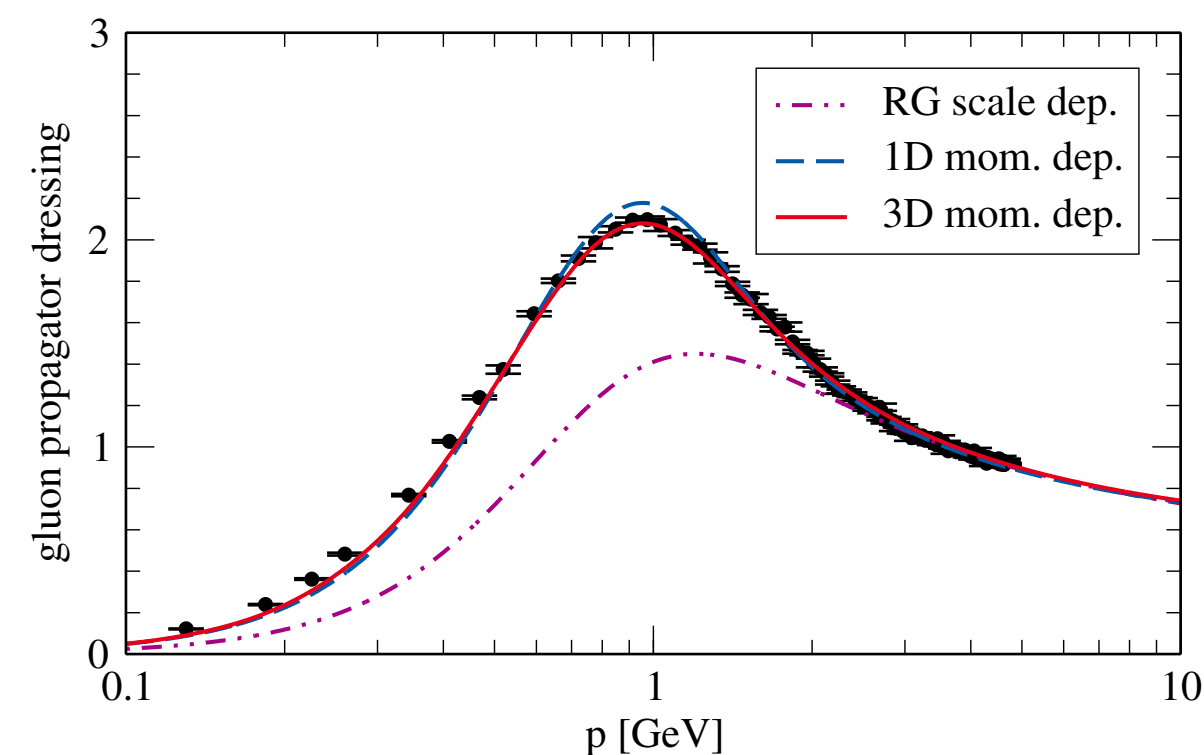
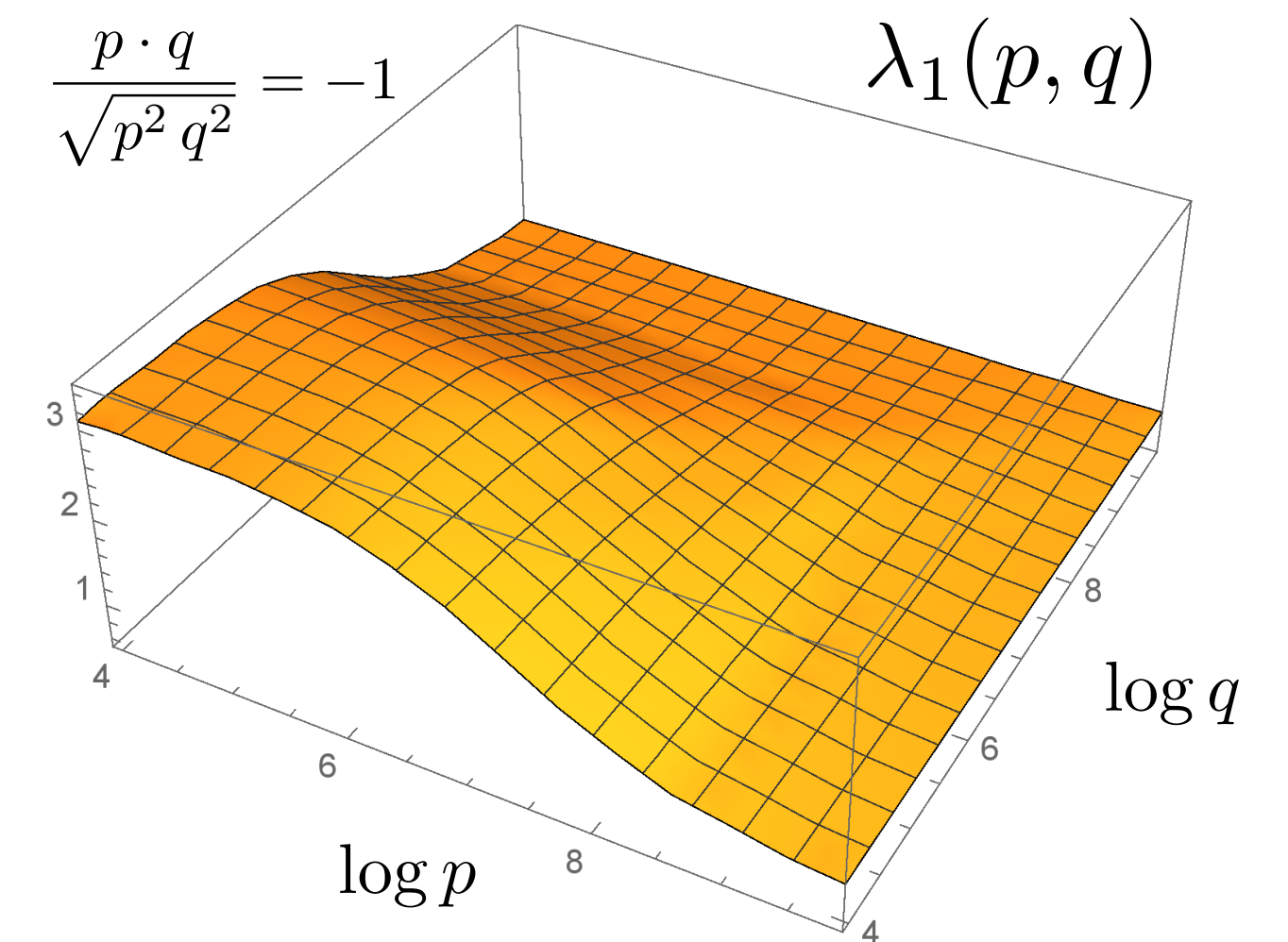
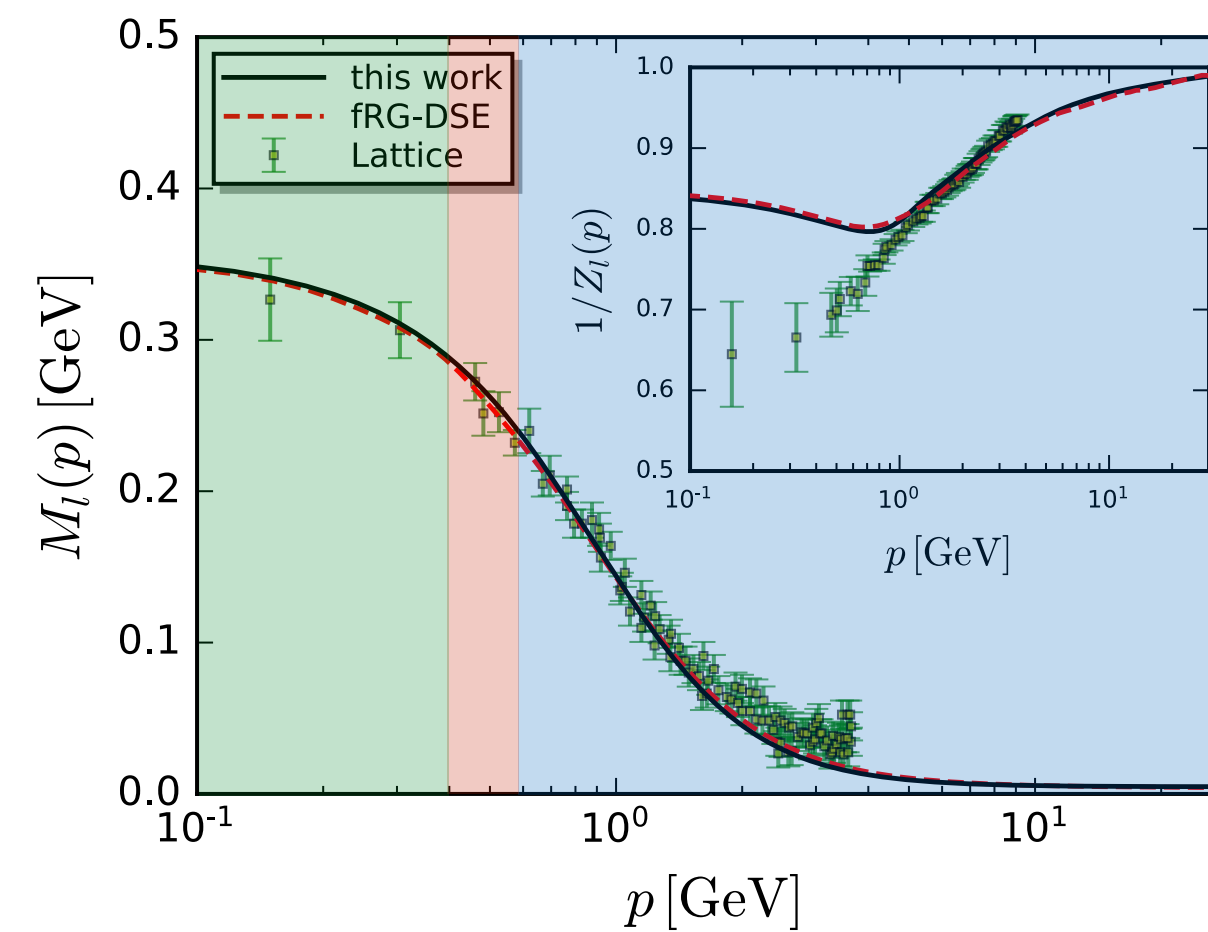


gluon two-point correlator

quark-gluon scattering



quark two-point correlator



$\lambda_1(p, q), \dots, \lambda_8(p, q)$

Relevant: $\lambda_1, \lambda_4, \lambda_7$

VertEXpand

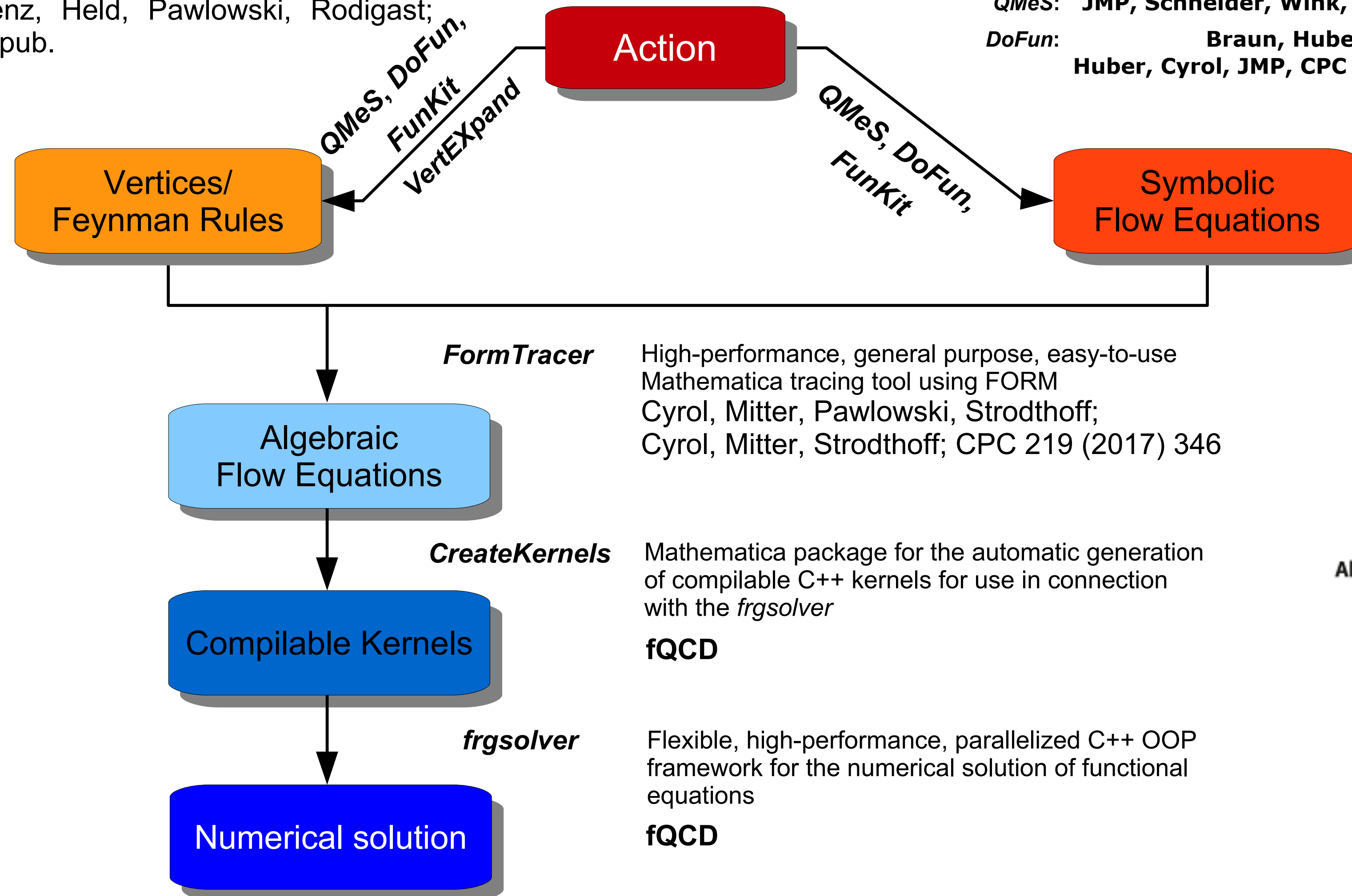
Mathematica package for the derivation of vertices from a given action using FORM
Denz, Held, Pawlowski, Rodigast; unpub.

QMeS, DoFun

Mathematica packages for the derivation of functional equations

QMeS: JMP, Schneider, Wink, CPC287 (2023) 108711

DoFun: Braun, Huber, CPC 183 (2012) 1290
Huber, Cyrol, JMP, CPC 183 248 (2020) 107058



FormTracer

High-performance, general purpose, easy-to-use Mathematica tracing tool using FORM
Cyrol, Mitter, Pawlowski, Strodthoff;
Cyrol, Mitter, Strodthoff; CPC 219 (2017) 346

CreateKernels

Mathematica package for the automatic generation of compilable C++ kernels for use in connection with the *frgsolver*

fQCD

Flexible, high-performance, parallelized C++ OOP framework for the numerical solution of functional equations

fQCD

GEFÖRDERT VOM



Bundesministerium
für Bildung
und Forschung



Alexander von Humboldt
Stiftung/Foundation

FWF

Der Wissenschaftsfonds.

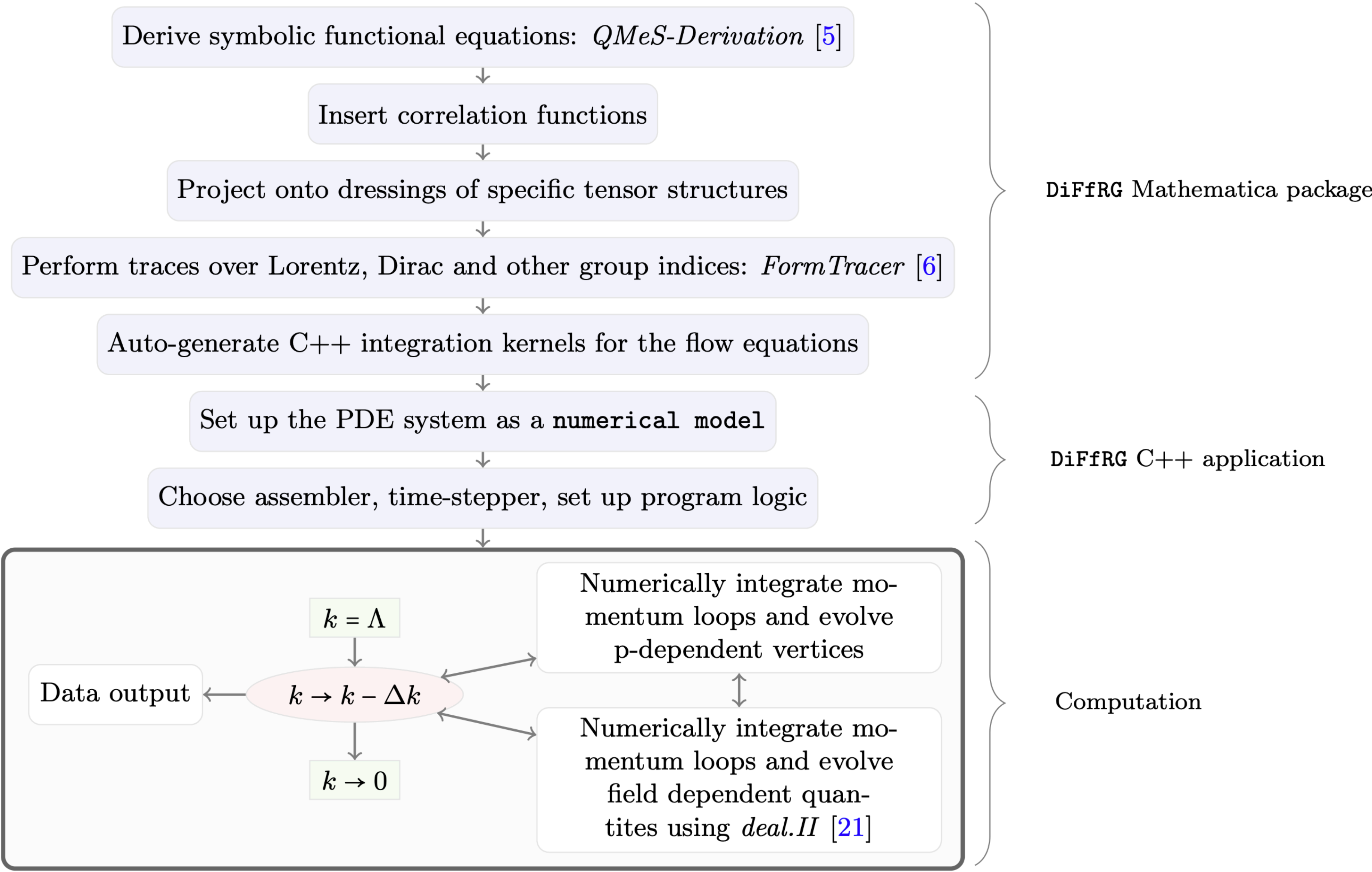


European Research Council

Established by the European Commission

fQCD: workflow

DiFfRG
github



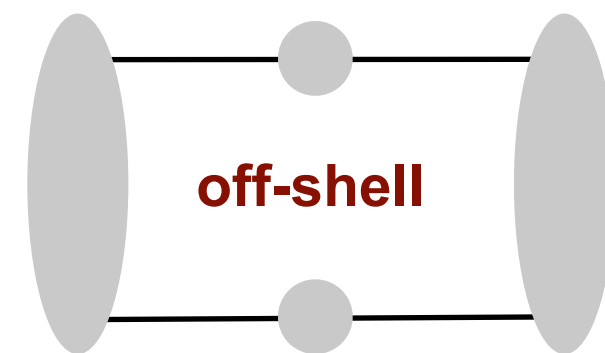


How to: three remarks on functional methods

'... and now for something
completely different...'

- off-shell representation of thermodynamic observables

e.g. $\text{Tr} \langle q(x) \bar{q}(x) \rangle$



pressure, trace anomaly,
fluctuations, volume flucs., ...

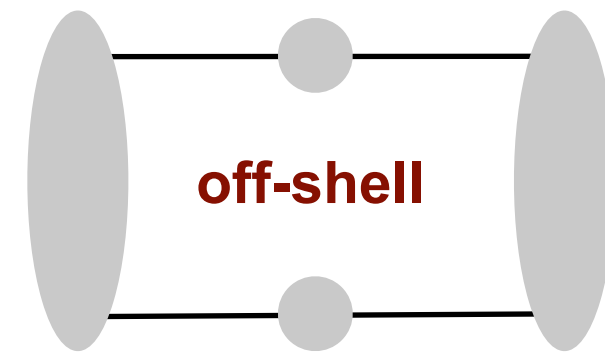


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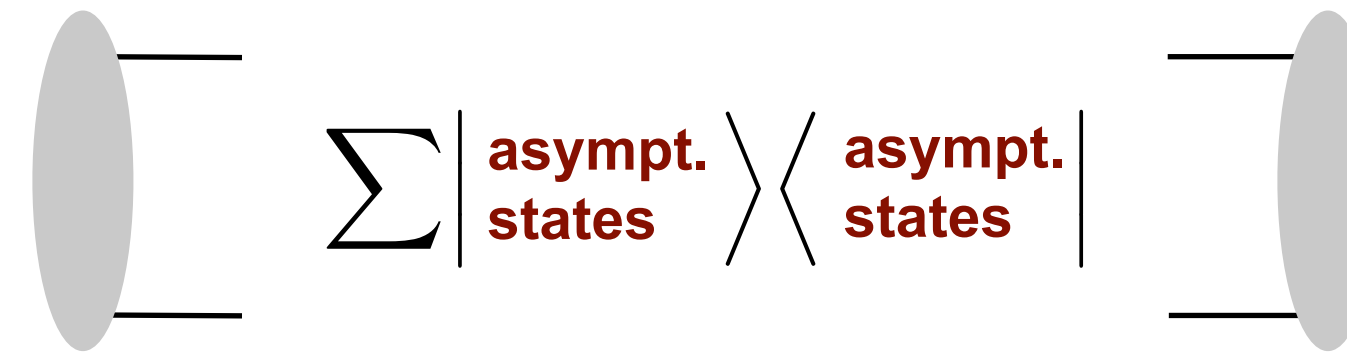
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on-shell



e.g. hadron resonances

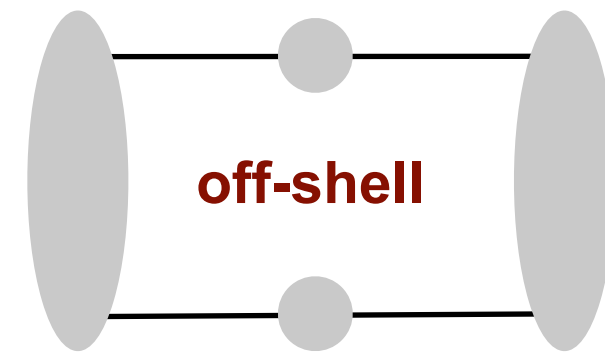


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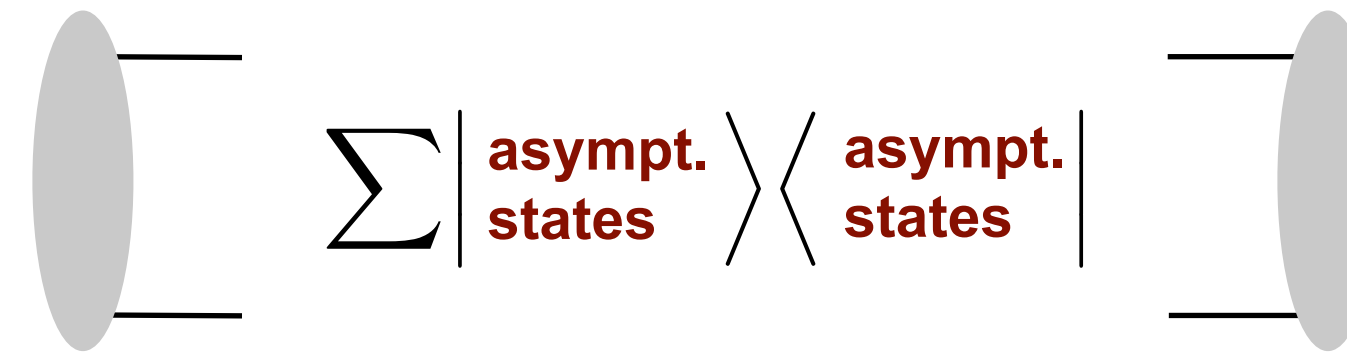
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on-shell



e.g. hadron resonances

- gauge fixing = parameterisation

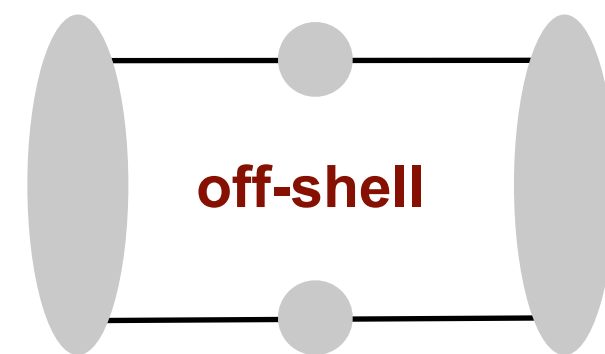


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e.g. $\text{Tr} \langle q(x) \bar{q}(x) \rangle$



pressure, trace anomaly,
fluctuations, volume flucs., ...



on-shell



e.g. hadron resonances

▪ gauge fixing = parameterisation

$$\langle q(x_1) \cdots \bar{q}(x_{2n}) A_\mu(y_1) \cdots A_\mu(y_m) h(z_1) \cdots h(z_l) \rangle$$

Consequences

I: simple correlations

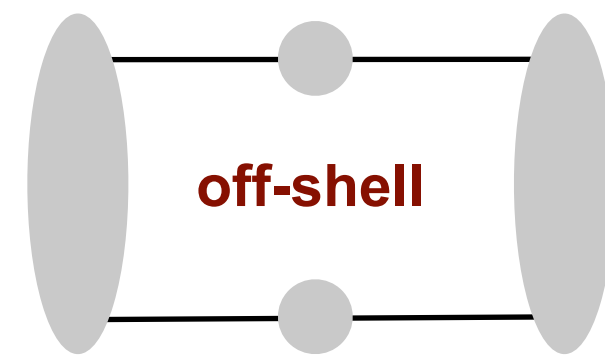


How to: three remarks on functional methods

'... and now for something completely different...'

▪ off-shell representation of thermodynamic observables

e.g. $\text{Tr} \langle q(x) \bar{q}(x) \rangle$



pressure, trace anomaly,
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Consequences

I: simple correlations

II: Difficult access to some observables
'No free lunch theorem'

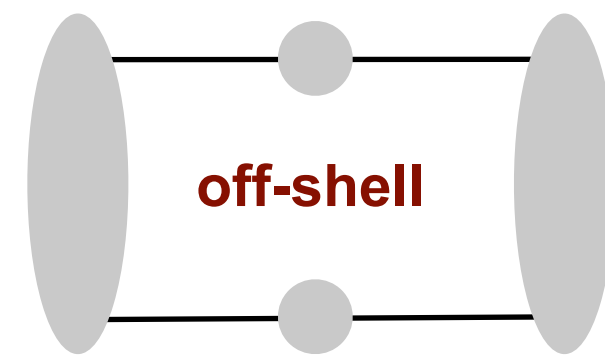


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Consequences

I: simple correlations

II: Difficult access to some observables
'No free lunch theorem'

▪ 'Your mean field is not my mean field'

$$\left. \frac{\delta S_{\text{cl}}[\phi]}{\delta \phi} \right|_{\phi=\bar{\phi}} = 0$$

$$\left. \frac{\delta \Gamma[\phi]}{\delta \phi} \right|_{\phi=\bar{\phi}_{\text{quant}}} = 0$$



How to: direct computations vs extrapolations

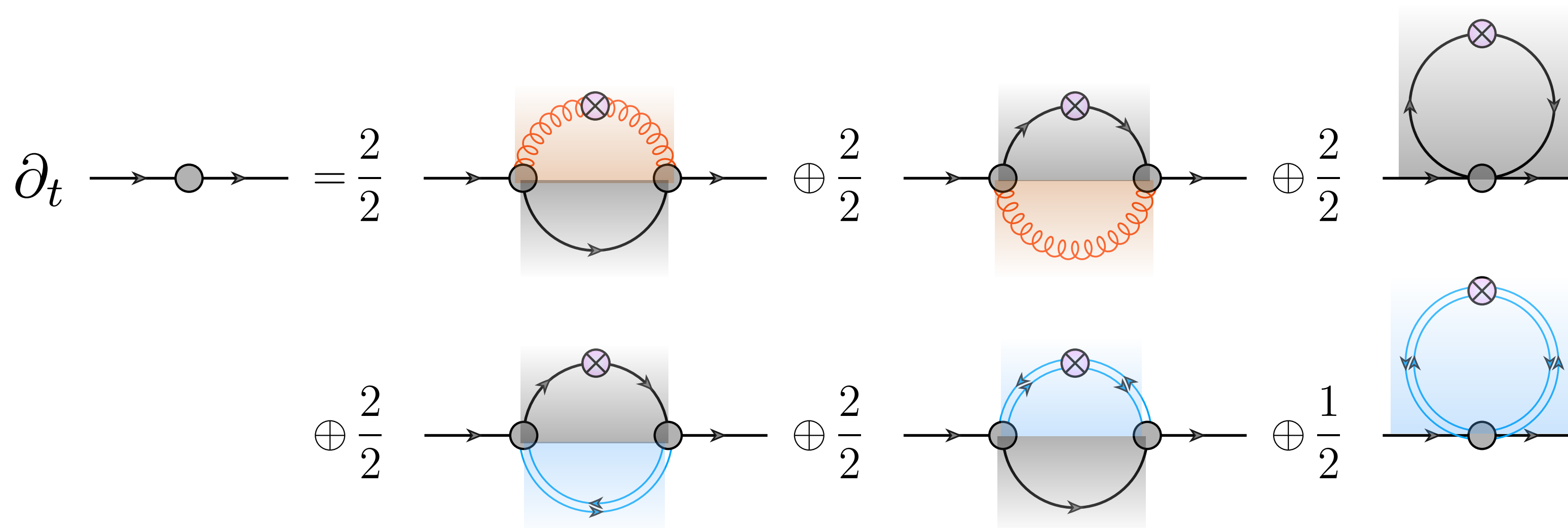
'... and now for something completely different...'

To QCD or not to QCD....a minimal point of view

Those are my interpretations,
and if you don't like them....
well, I have others

- **Self-consistent truncations to functional relations define analytic functions in μ_B , eg:**

$$\partial_t \left\langle q(x) \bar{q}(y) \right\rangle^{-1}(\mu_B) = \text{Loop} \left[\left\langle q(x) \bar{q}(y) \right\rangle(\mu_B), \left\langle q(x) A_\mu(y) \bar{q}(z) \right\rangle(\mu_B), \dots; \mu_B \right]$$





'... and now for something
completely different...'

Extrapolations & von Neumann's Elephant

Extrapolations are ill-conditioned problems

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'There are two ways of doing calculations in theoretical physics.

One way is to have a clear physical picture of the process that you are calculating.

The other way is to have a precise and self-consistent mathematical formalism.

...You have neither'

Fermi to Dyson



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*'With four parameters I can fit an elephant,
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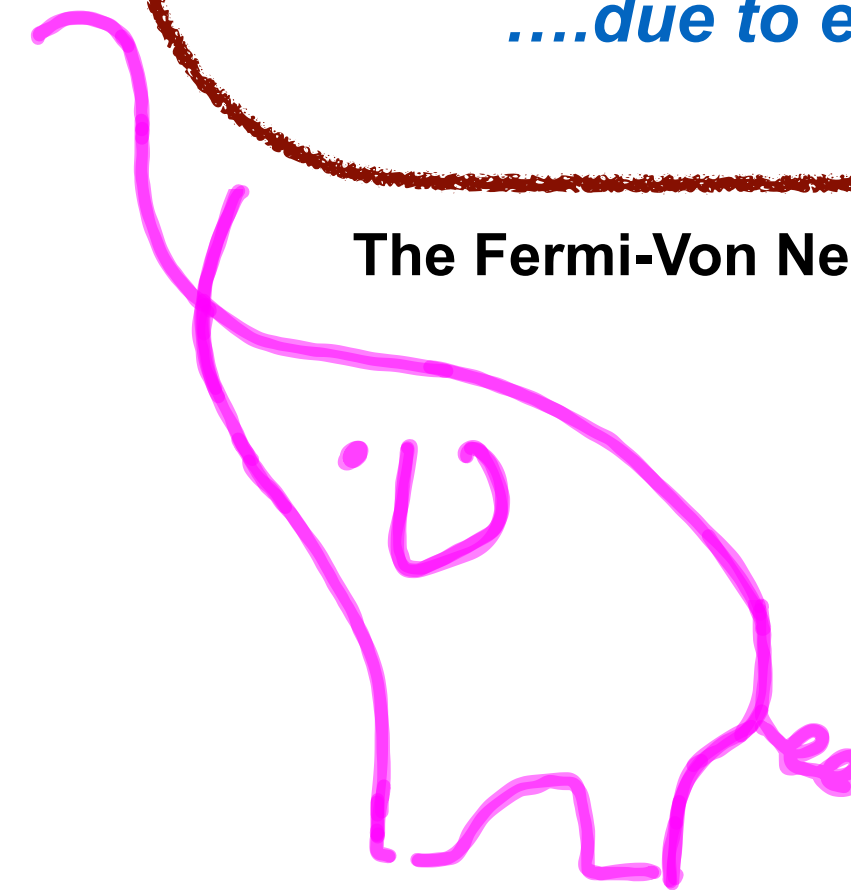
Von Neumann

*With four parameter models we can fit many data
but we cannot extrapolate without external information*

**In particular, extrapolations cannot predict the existence
and location of the CEP without external information**

*The other way is to have a precise and self-consistent
effective theory whose systematic error is under control
....due to external QCD information*

The Fermi-Von Neumann QCD phase structure elephant





Extrapolations & von Neumann's Elephant

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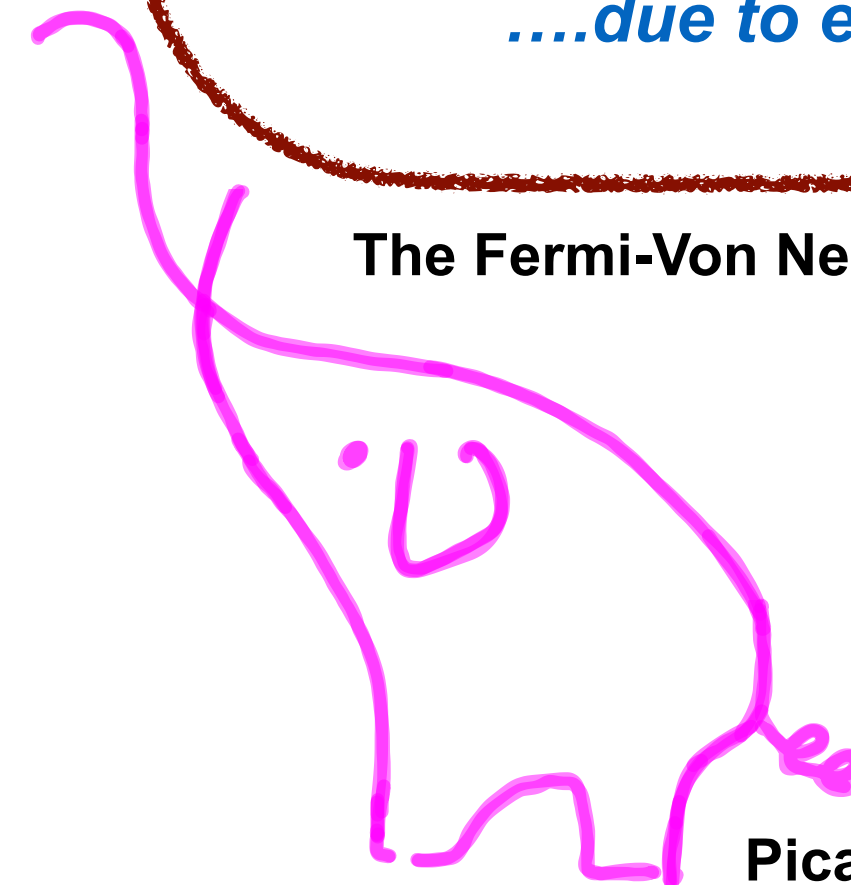
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Picasso-assisted Pawlowski elephant



How to: direct computations and the minimal point of view

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Unique: QCD-based analytic continuations
that satisfy the lattice benchmarks
at small chemical potential.



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Unique: QCD-based analytic continuations
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- ? Corroborated?

By definition, LEFTs (including extrapolations) cannot assess
the existence or absence of emergent phenomena!



How to: direct computations and the minimal point of view

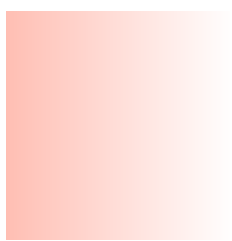
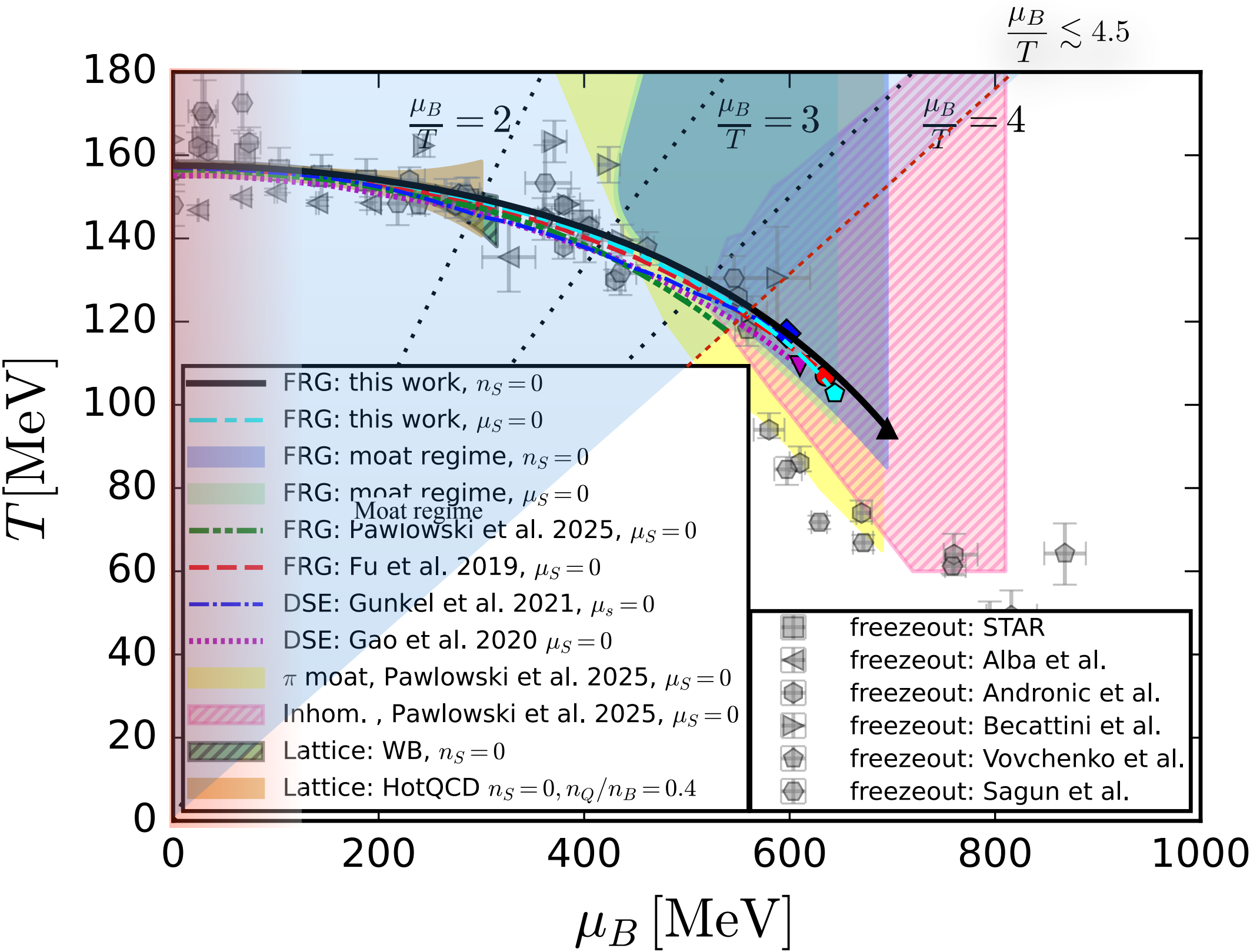
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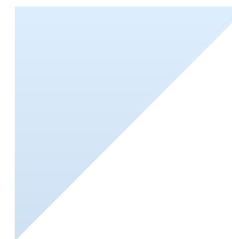
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- Consequences for functional QCD predictions at finite density



By now the best truncations to functional QCD pass lattice benchmark tests at vanishing and small chemical potential



Regime of quantitative reliability of current best truncation

$$\frac{\mu_B}{T} \lesssim 4.5$$

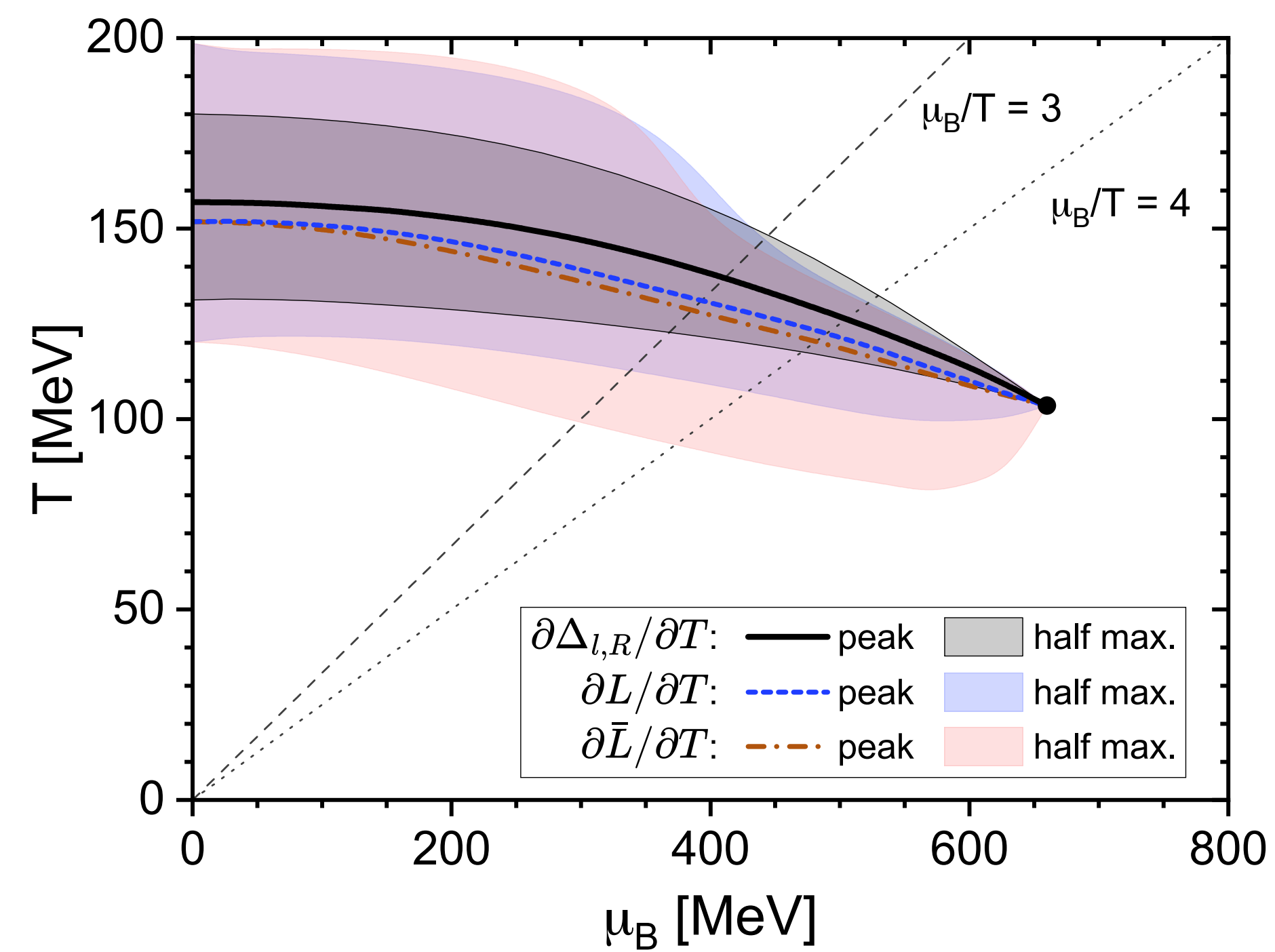
Unique: QCD-based analytic continuations that satisfy the lattice benchmarks at small chemical potential.



Great opportunity for a combined analysis of high density QCD (Exp. data + lattice QCD + functional QCD + LEFTs)

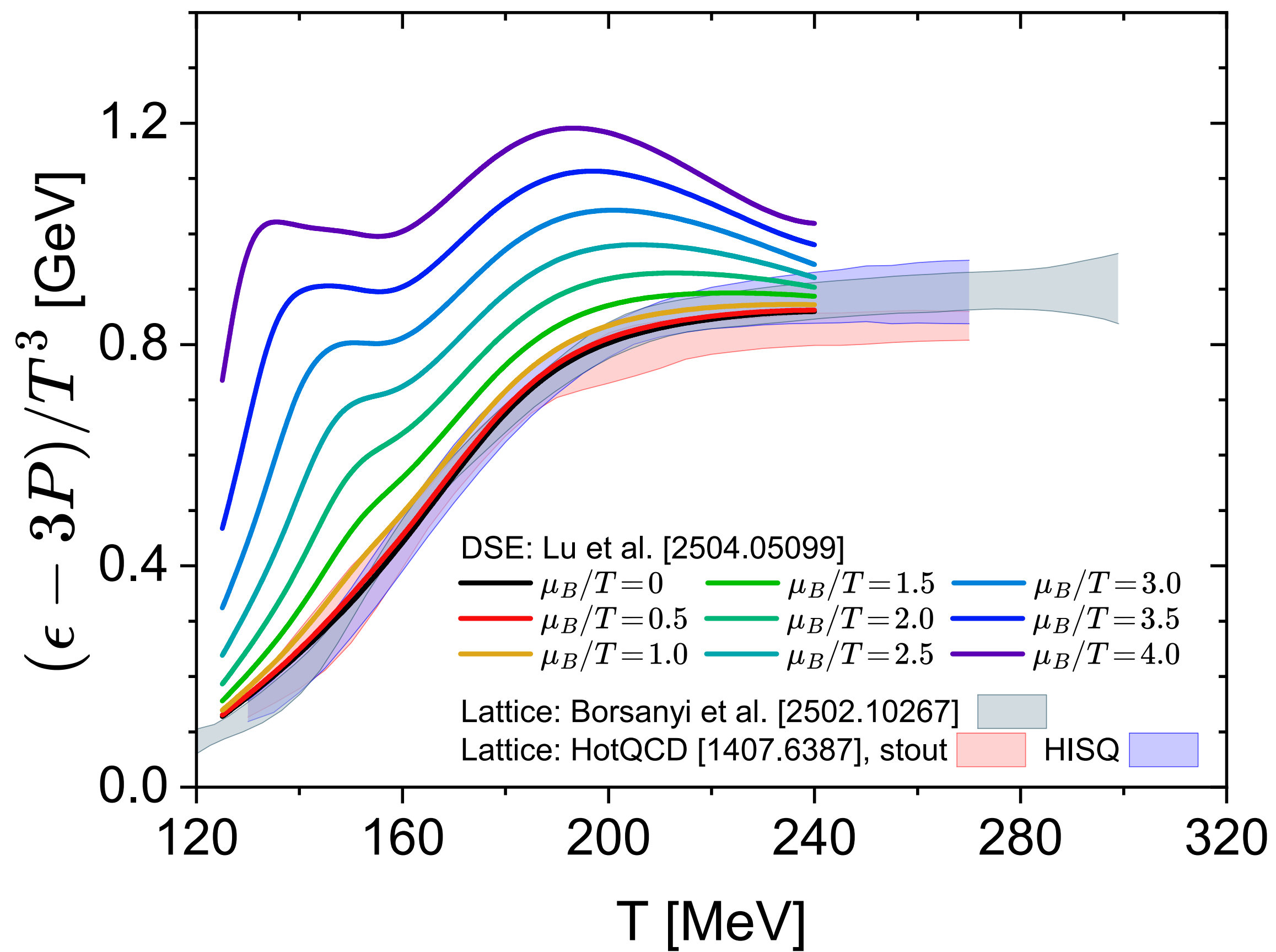
scales, scales, scales

Strong correlations above the chiral crossover



Strongly correlated physics

Strongly correlated phase for temperatures $T_\chi \lesssim T \lesssim 2T_\chi$

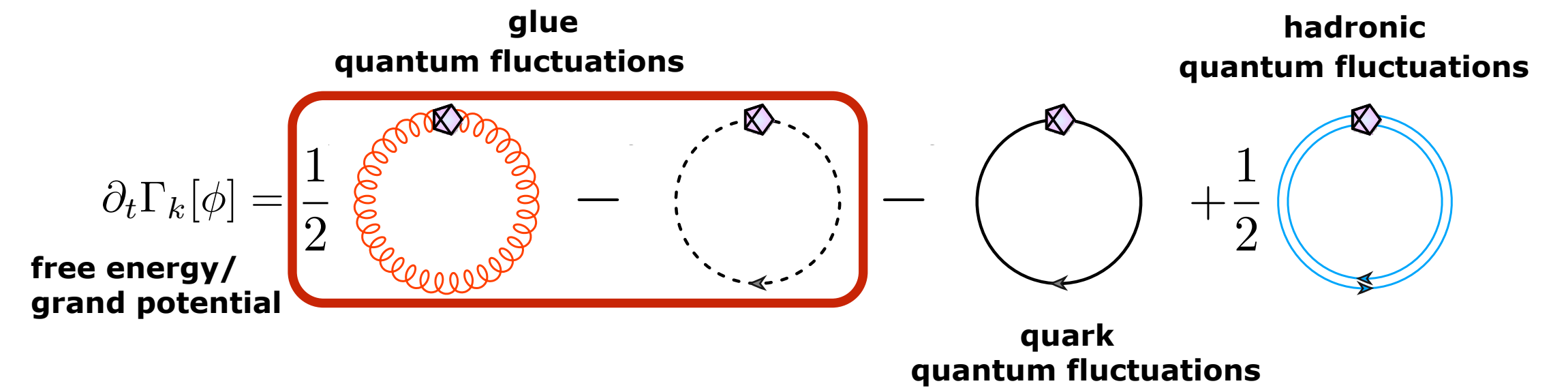


Functional confinement & the dynamical gluonic background

FRG: Braun, Gies, JMP, PLB 684 (2010) 262

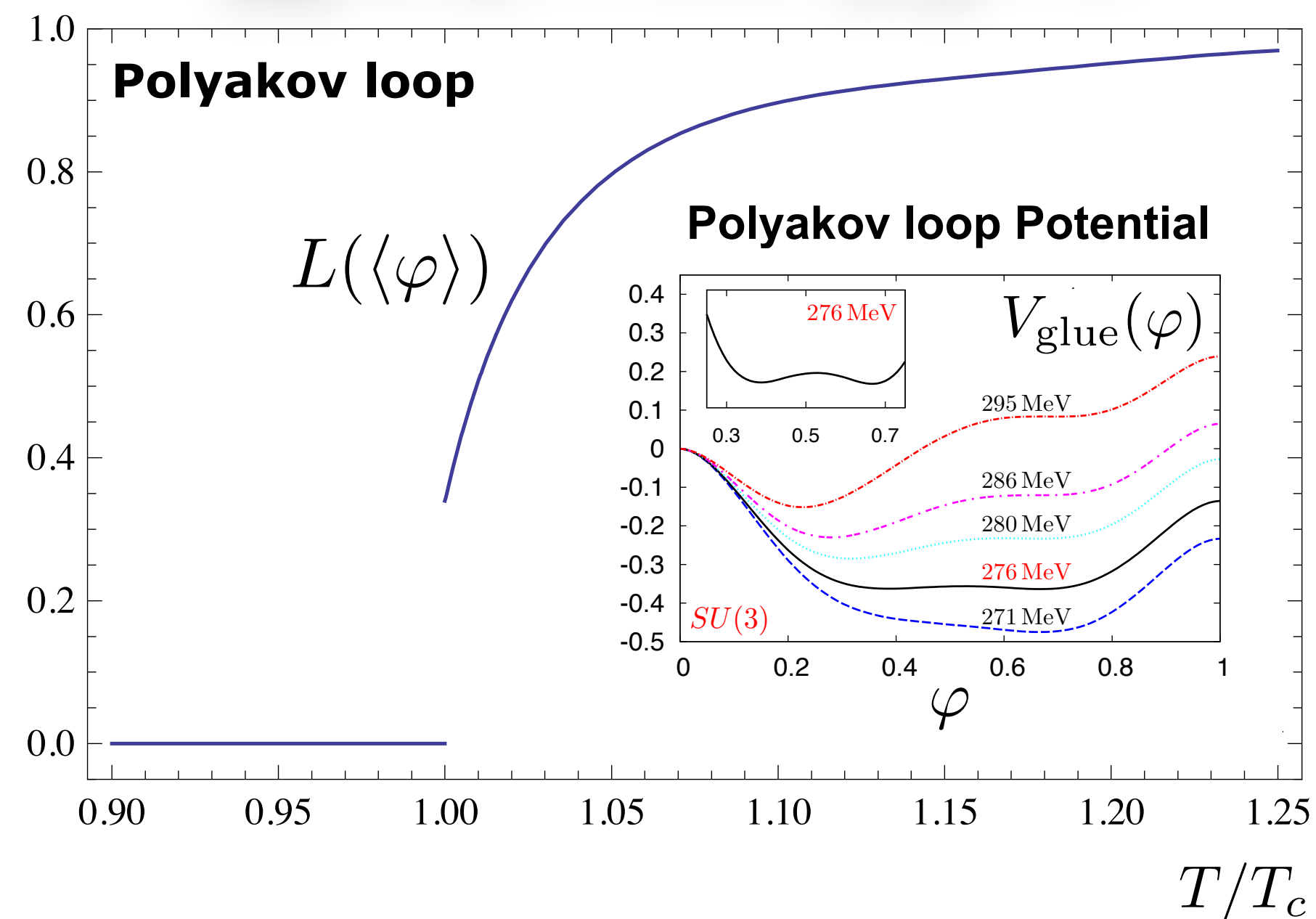
FRG, DSE, 2PI: Fister, JMP, PRD 88 (2013) 045010

$$L(A_0) = \frac{1}{N_c} \text{tr} \mathcal{P} e^{i g_s \int_0^\beta dt A_0(x)}$$



Polyakov loop Potential: $V_{\text{glue}}(\varphi)$

$$\mathcal{P} e^{i g_s \int_0^\beta dt A_0(x)} = e^{2\pi i \varphi}$$



$$T_c/\sqrt{\sigma} = 0.658 \pm 0.023$$

$$\text{lattice : } T_c/\sqrt{\sigma} = 0.646$$

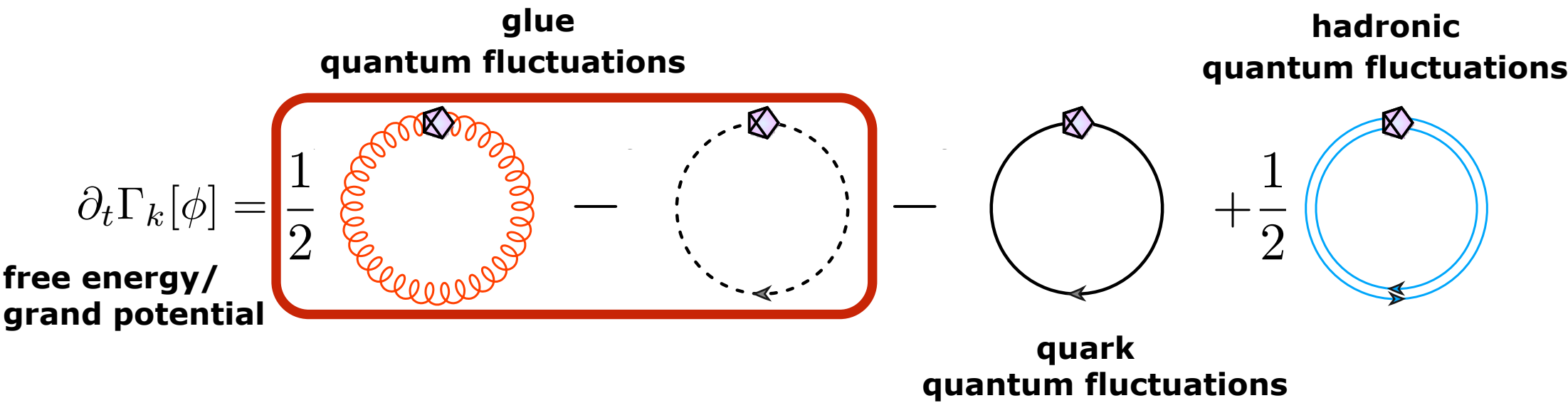
scales, scales, scales

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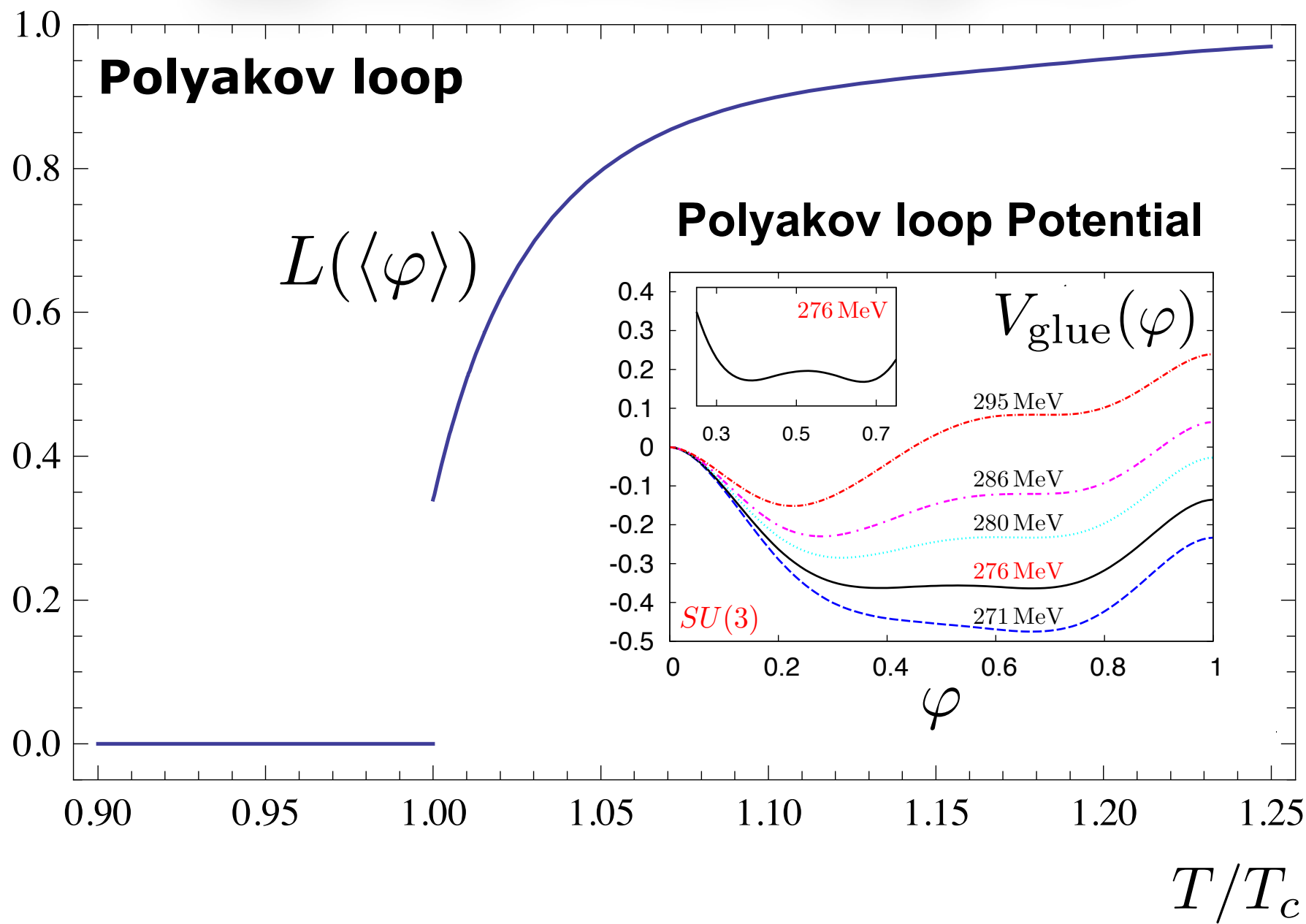
FRG, DSE, 2PI: Fister, JMP, PRD 88 (2013) 045010

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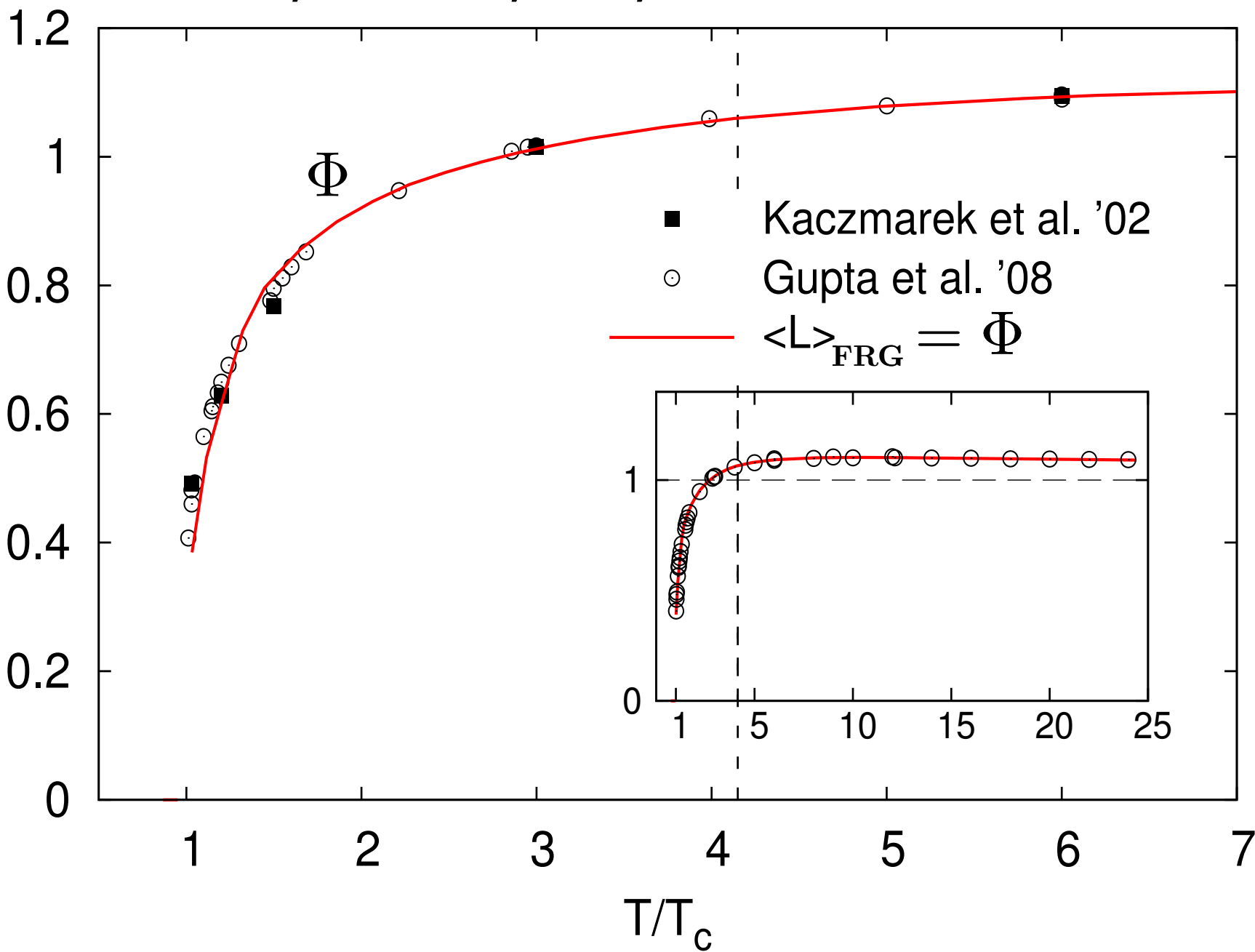


Polyakov loop Potential: $V_{\text{glue}}(\varphi)$

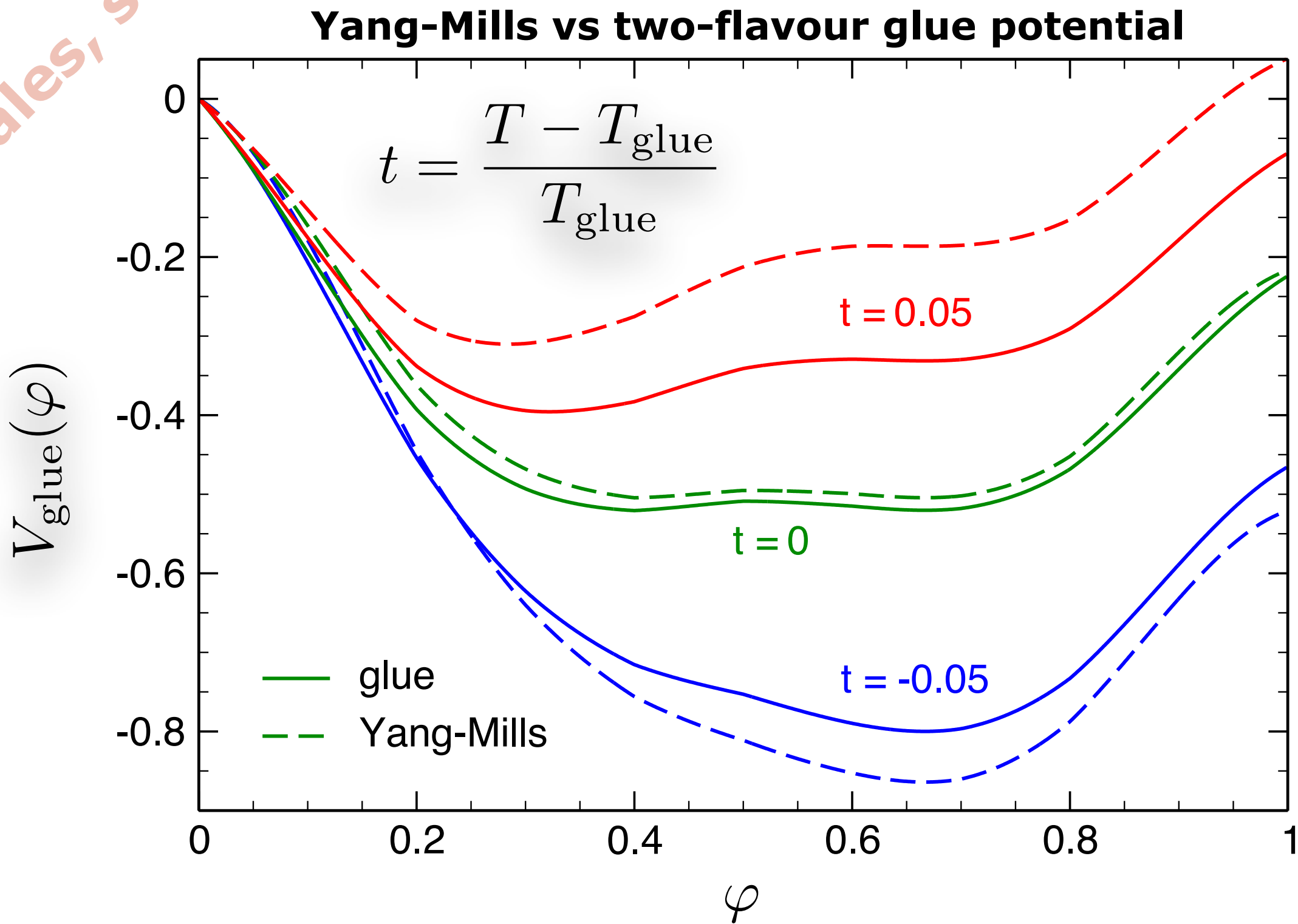
$$\mathcal{P} e^{i g_s \int_0^\beta dt A_0(x)} = e^{2\pi i \varphi}$$



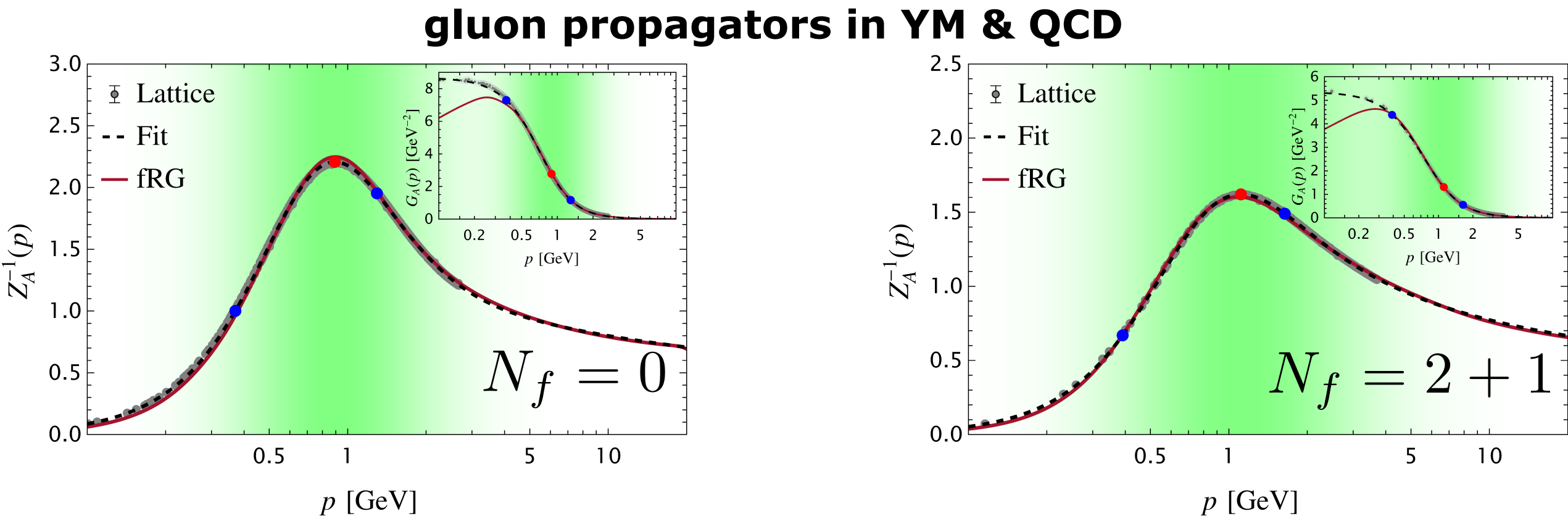
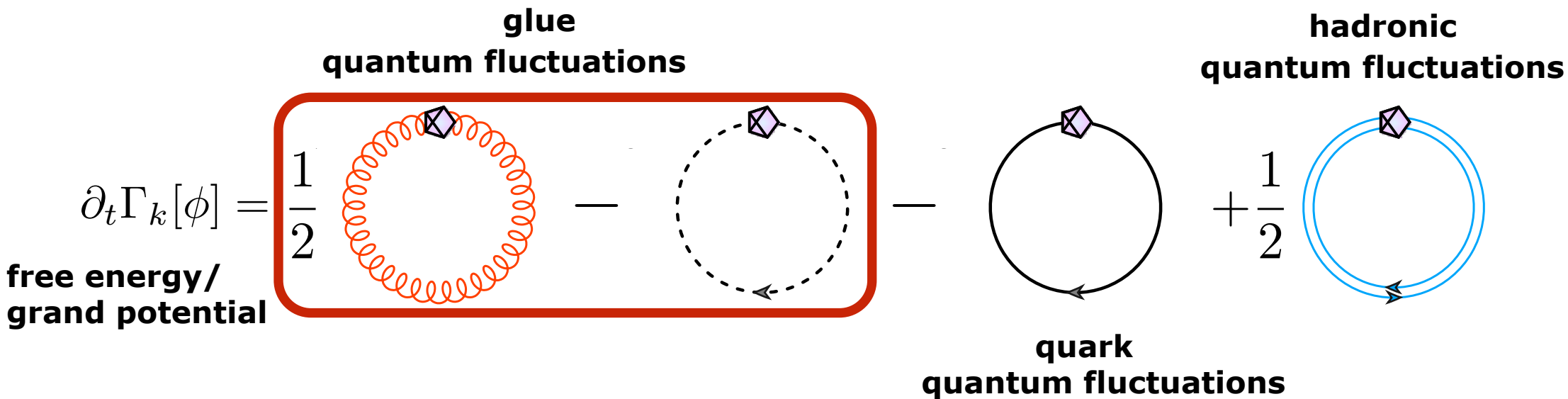
Herbst, Luecker, JMP, arXiv:1510.03830



Functional confinement & the dynamical gluonic background



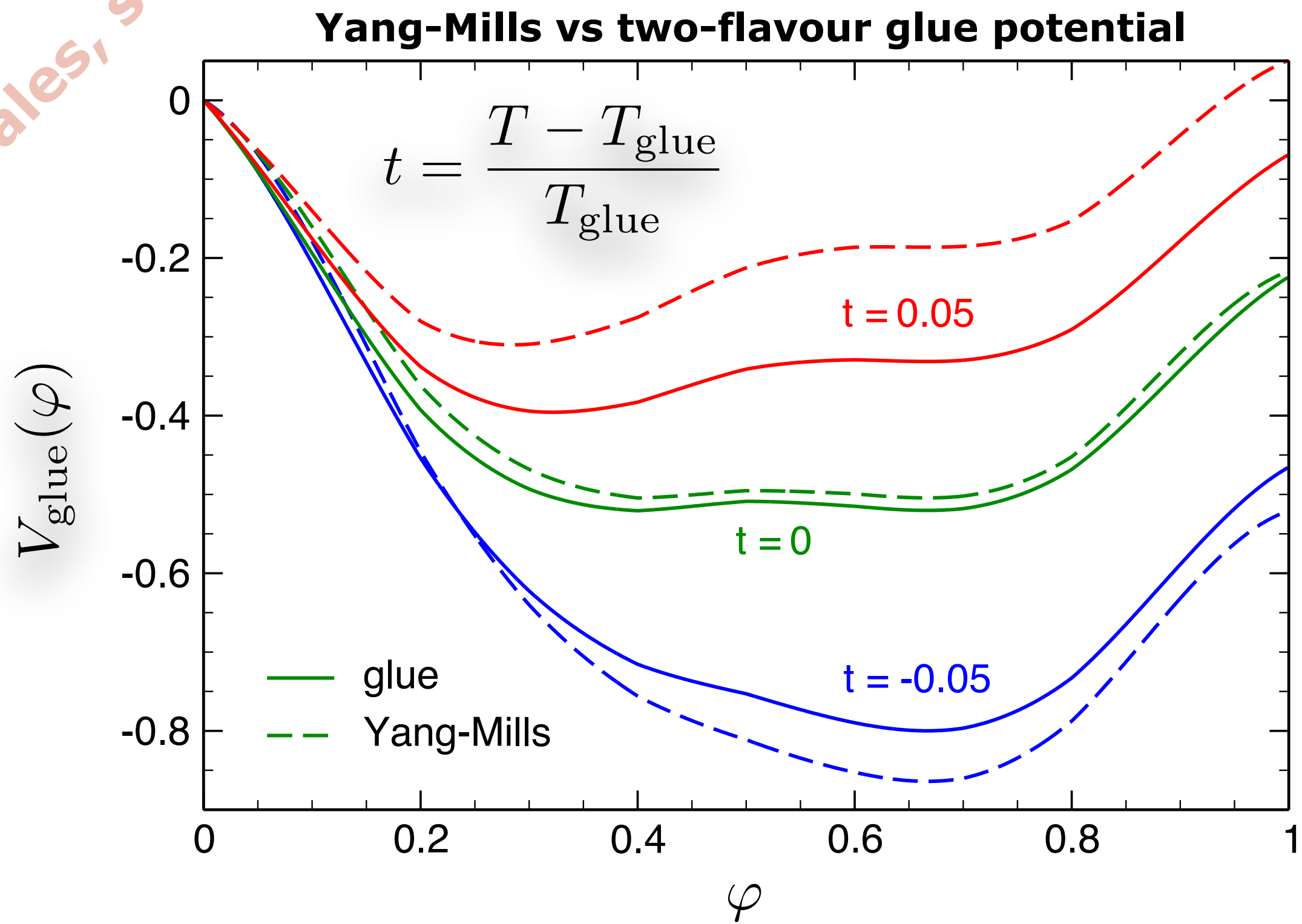
Haas, Stiele, Braun, JMP, Schaffner-Bielich, PRD 87 (2013) 076004



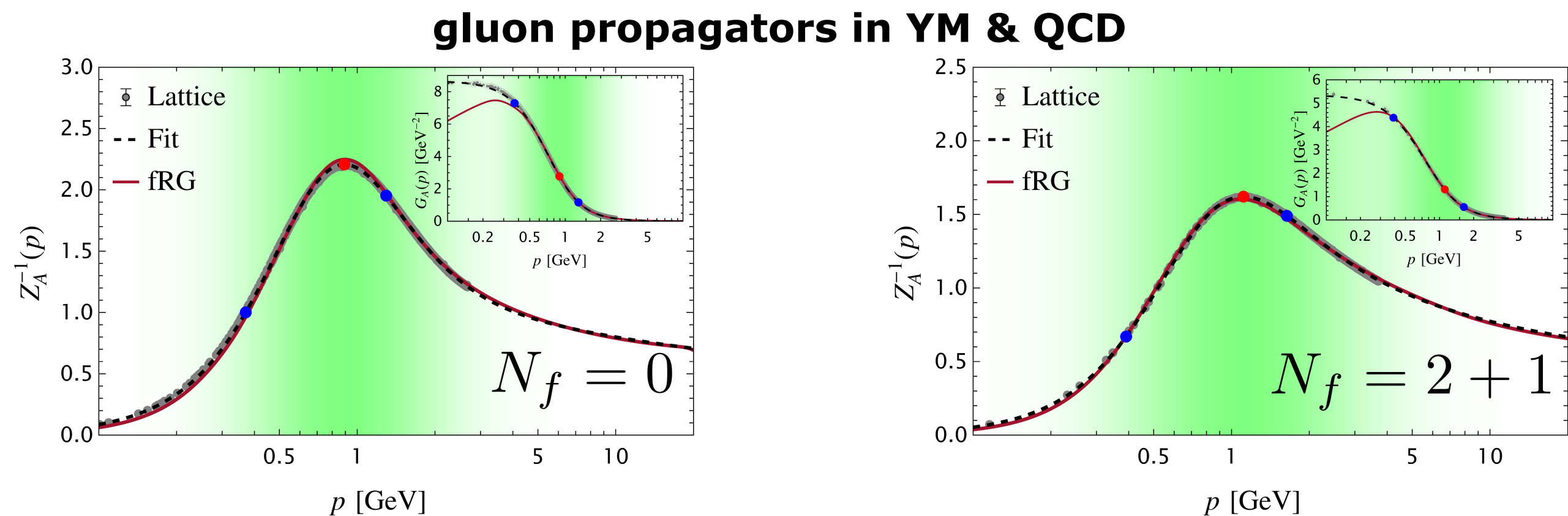
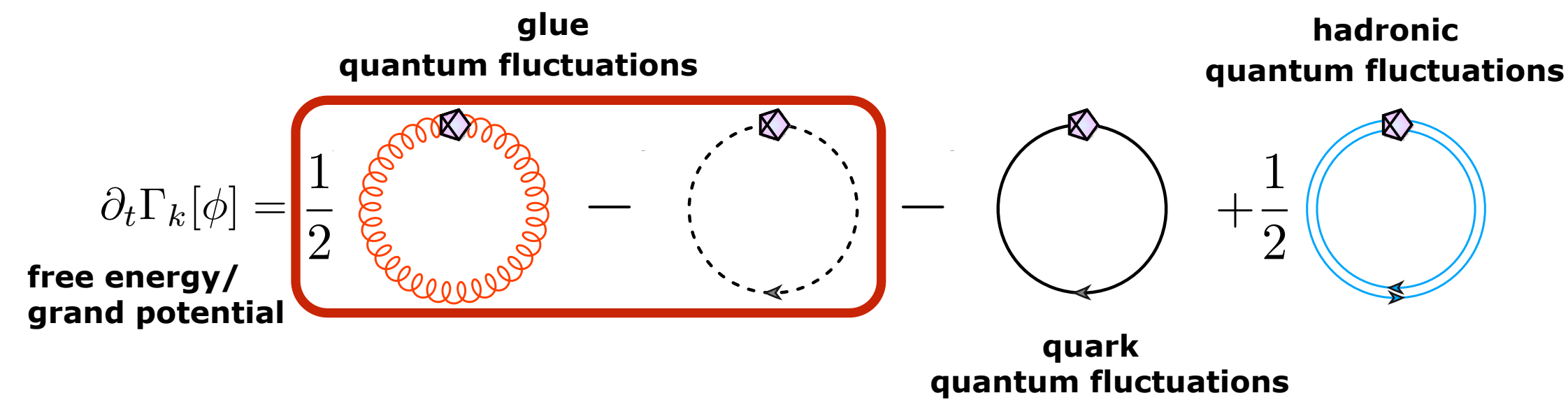
Ferreira, Papavassiliou, JMP, EPJC 85 (2025) 11, 1339

$$T_{\text{glue}}^{(\text{QCD})} \approx 200 \text{ MeV}$$

Functional confinement & the dynamical gluonic background



Haas, Stiele, Braun, JMP, Schaffner-Bielich, PRD 87 (2013) 076004



Ferreira, Papavassiliou, JMP, EPJC 85 (2025) 11, 1339

Direct applications to the QCD phase structure

Braun, Haas, Marhauser, JMP, PRL 106 (2011) 022002

Fischer, Fister, Lücker, JMP, PLB 732 (2014) 273

Fischer, Lücker, Welzbacher, PRD 90 (2014) 034022

Fischer, Lücker, JMP, PRD 91 (2015) 014024

Lu, Gao, Liu, JMP, 113 (2026) 054019

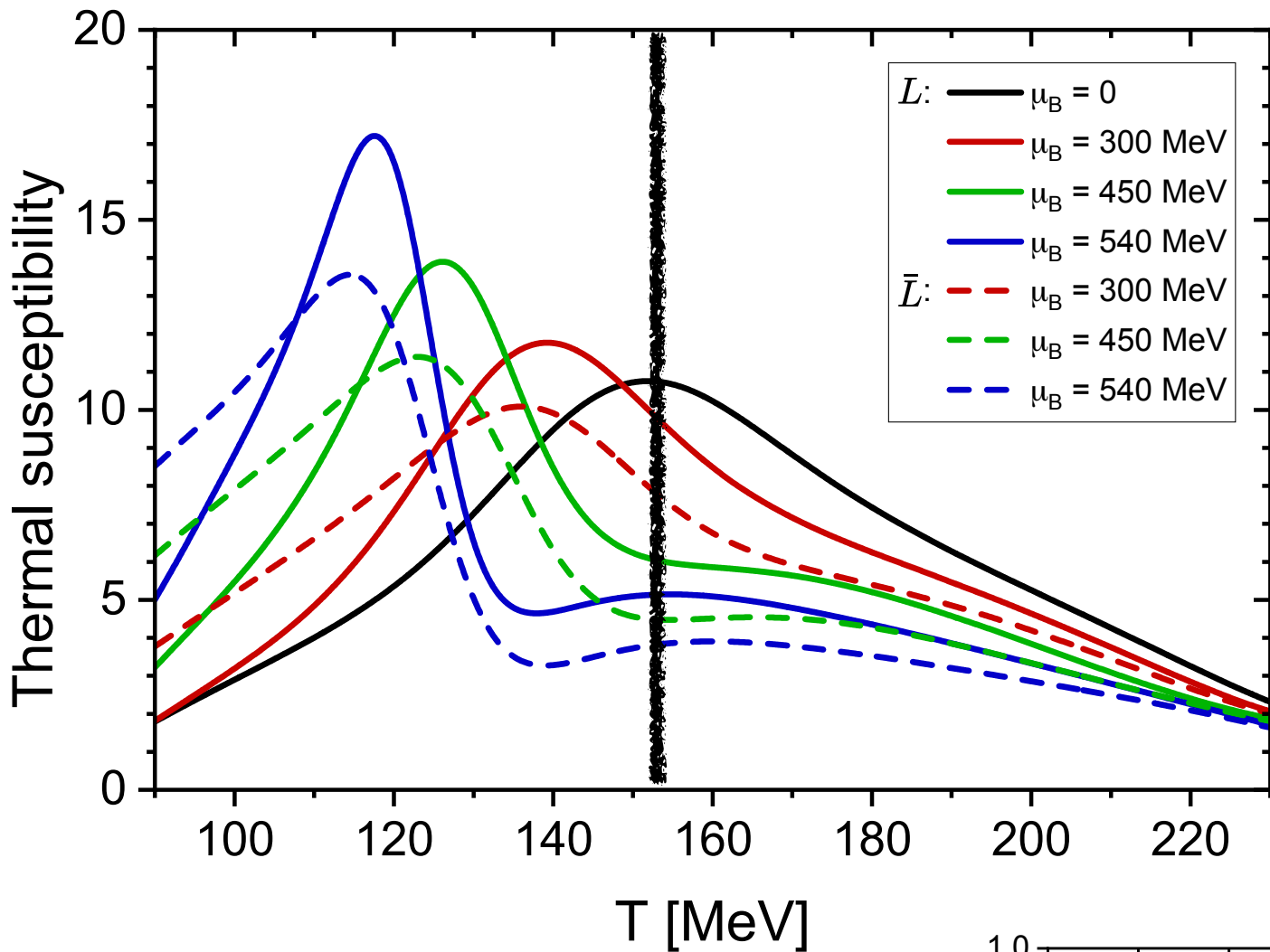
Lu, Fischer, Gao, Liu, JMP, arXiv:2603.09336

fRG & DSE

Strongly correlated physics

‘Confinement’ related ‘crossovers’ at $T_\chi \lesssim T_{\text{conf}} \lesssim 2 T_\chi$

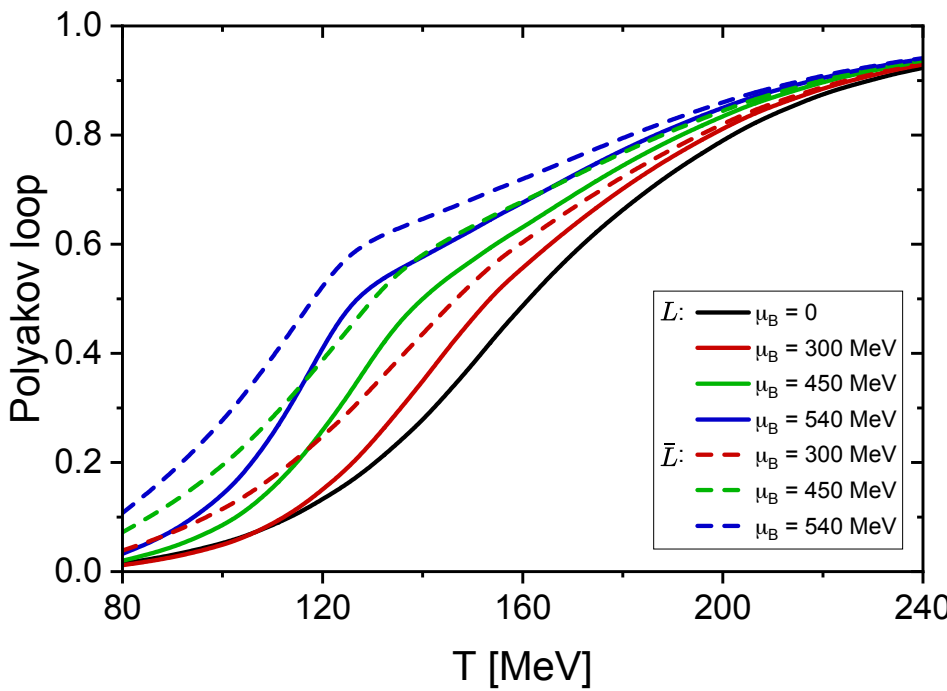
Functional QCD



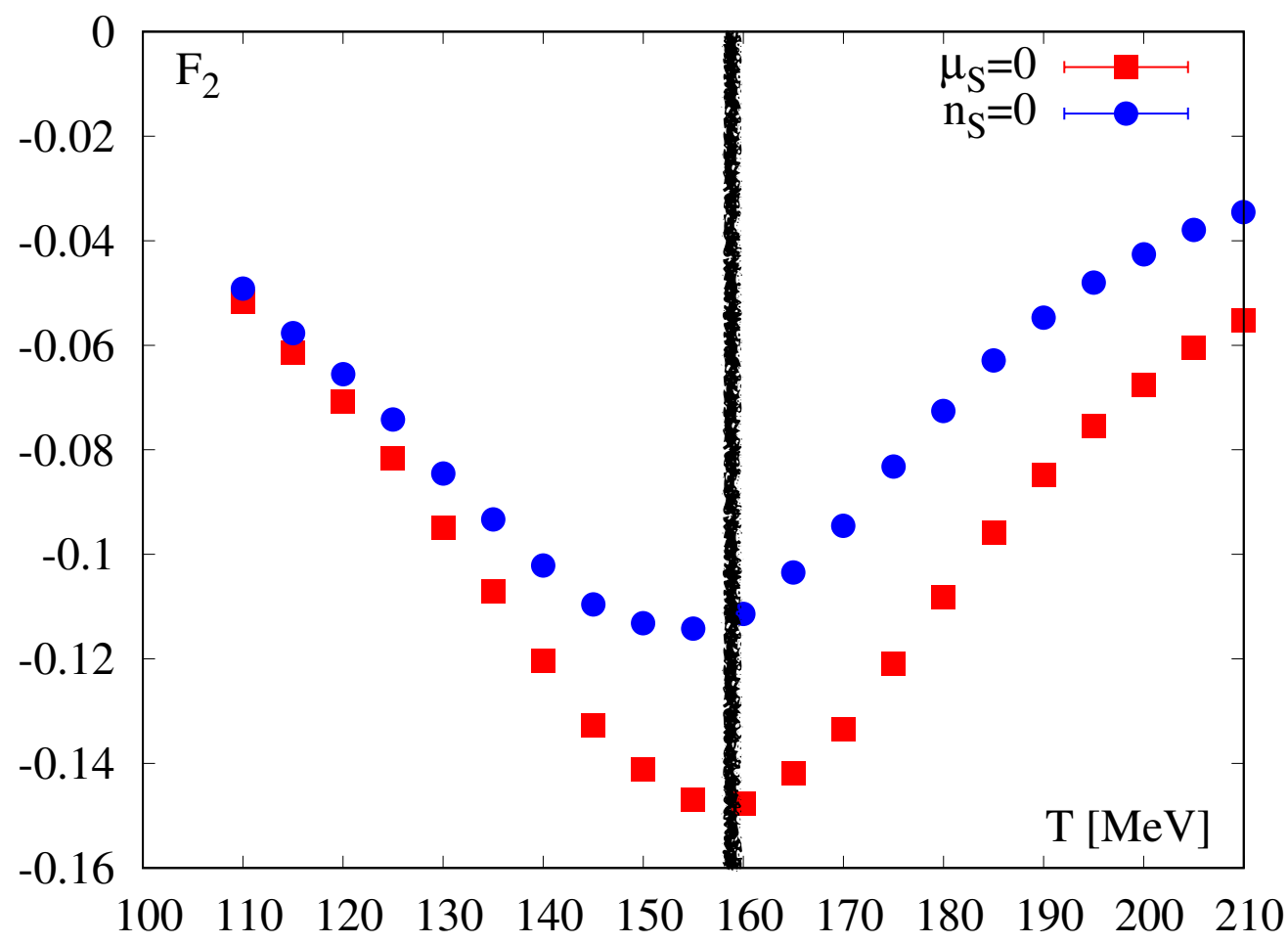
Variants of the traced Polyakov loop

$$P = \frac{1}{N_c} \text{tr} e^{2\pi i \varphi}$$

$$\propto P(\langle \text{ev}(\varphi) \rangle)$$

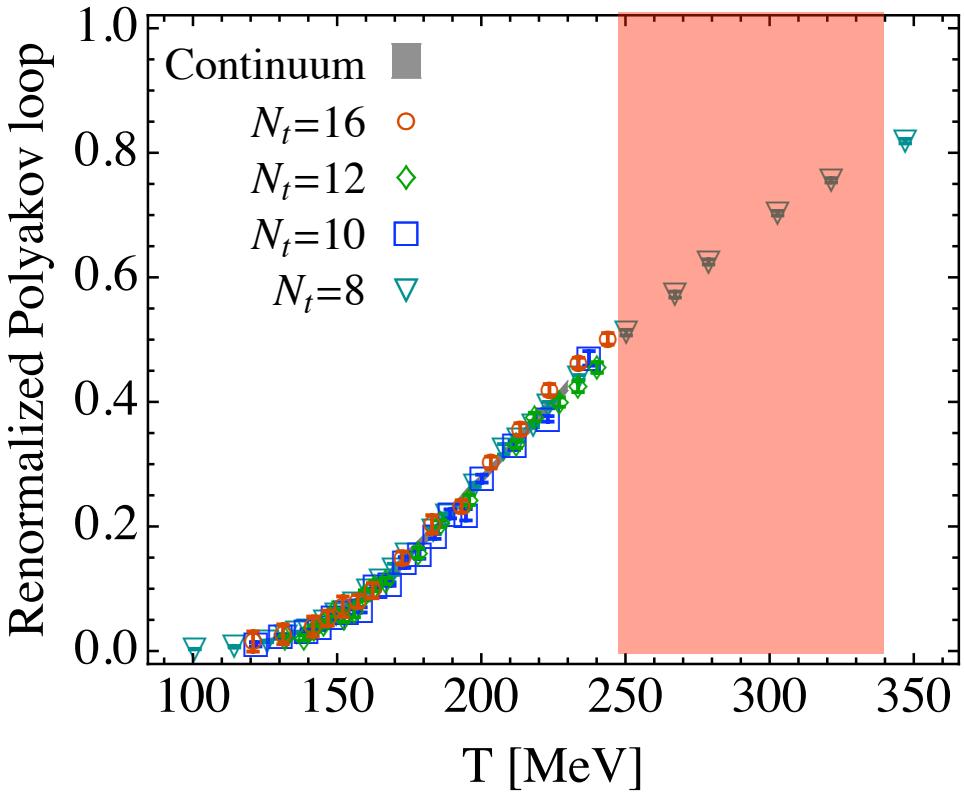


Lattice QCD



$$\propto T \frac{\partial^2 \log |\langle P \rangle|}{(\partial \mu_B / T)^2}$$

$$\propto \langle P \rangle$$



⋮

⋮

Lu, Gao, Liu, JMP, 113 (2026) 054019

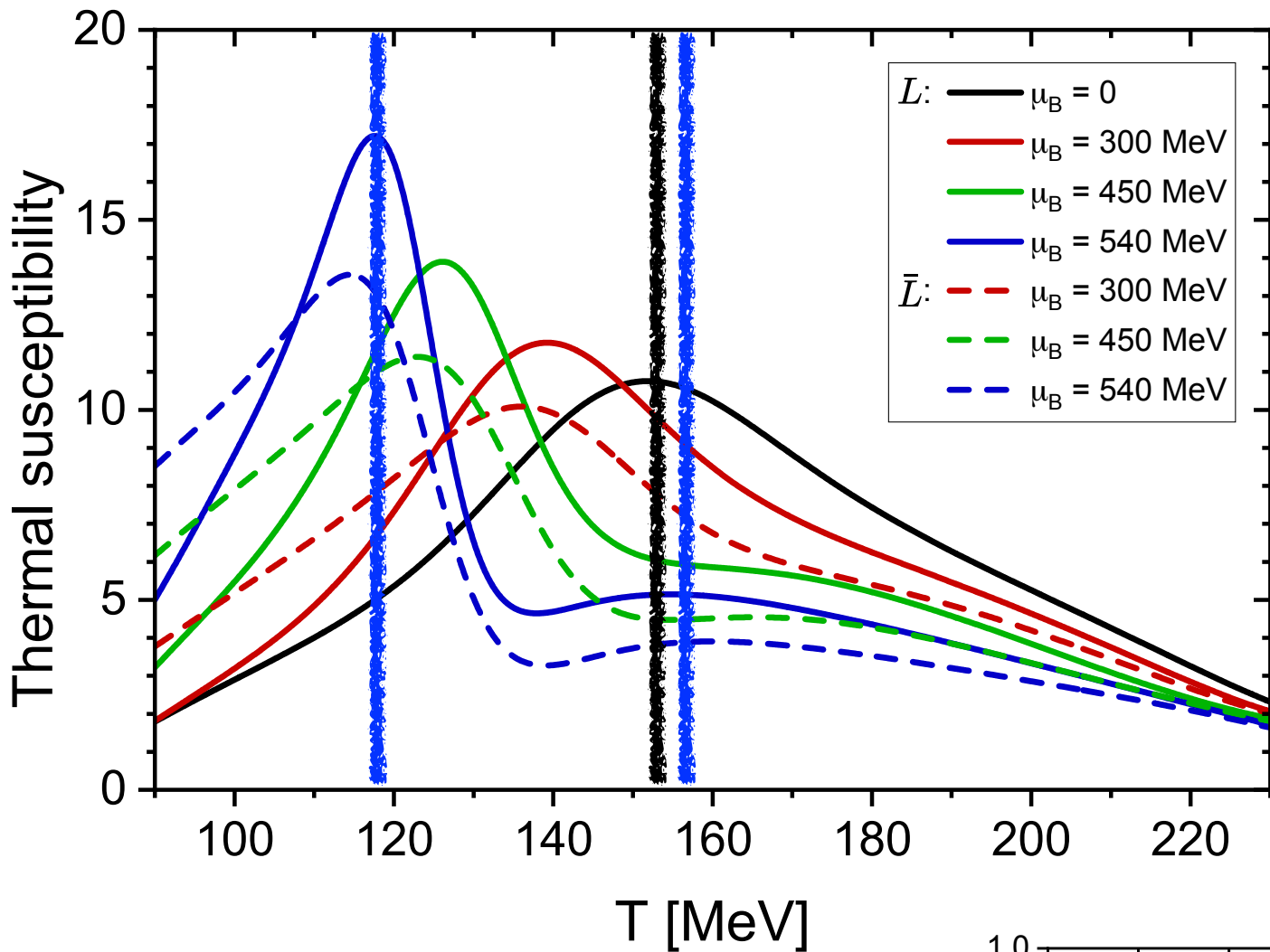
Bazavov, Brambilla, Ding, Petreczky, Schadler, Vairo, Weber, PRD 93 (2016) 11, 114502

Borsányi, Fodor, Guenther, Kara, Paolo Parotto, Pásztor, Pirelli, Wong, PRD 110 (2024) 114507

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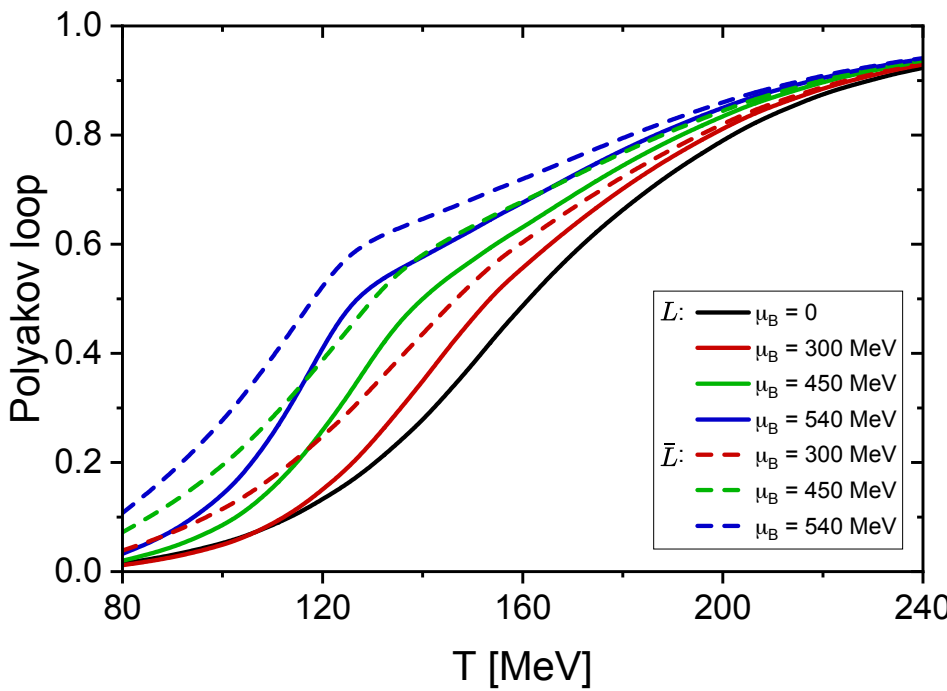
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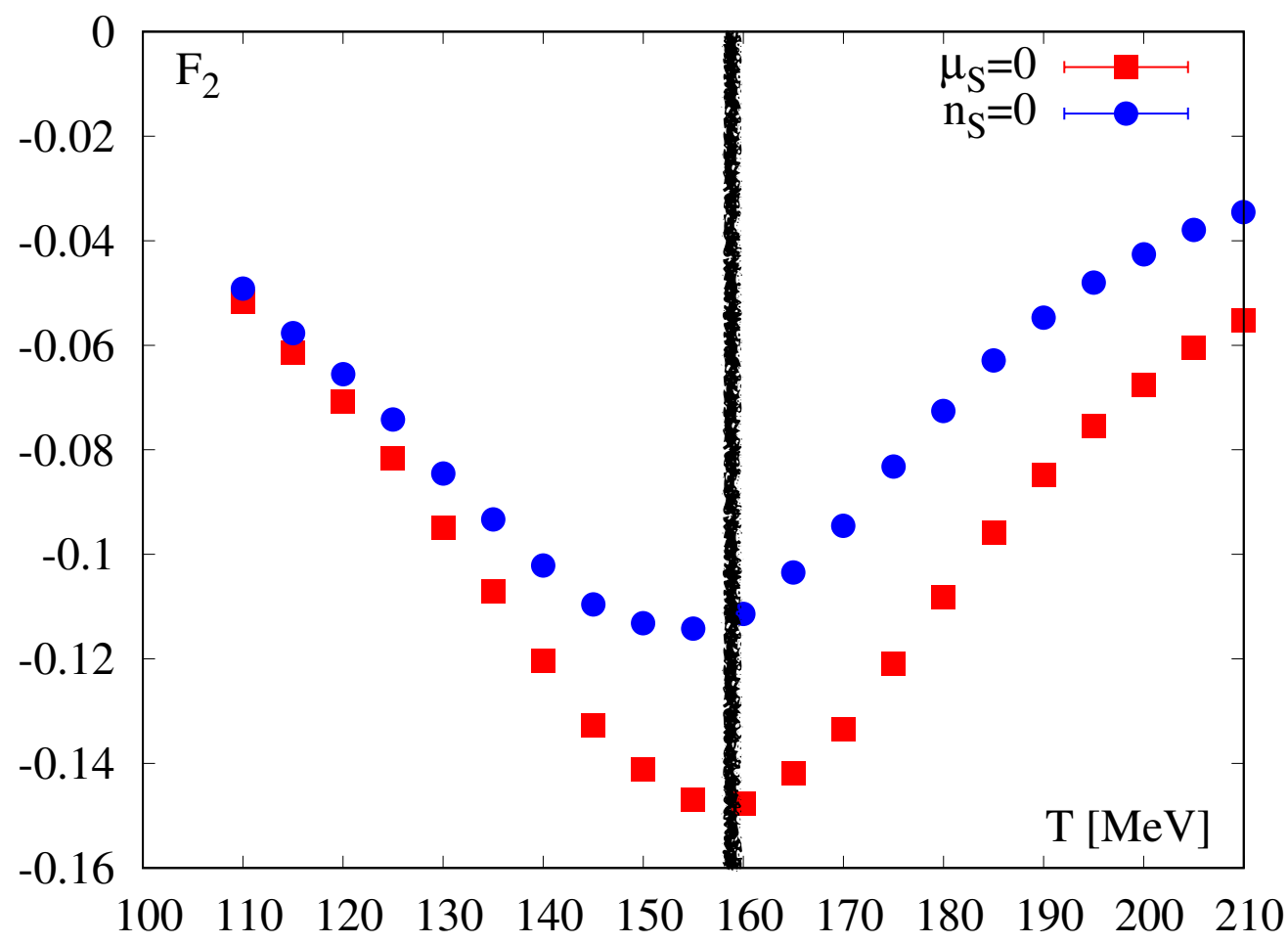
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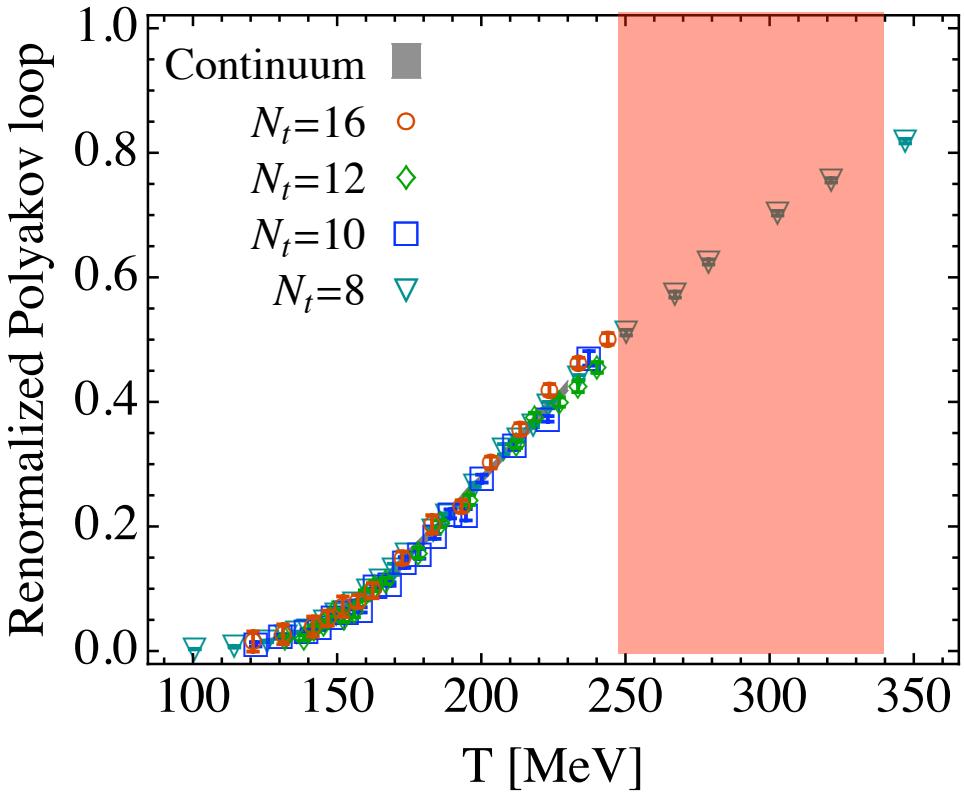


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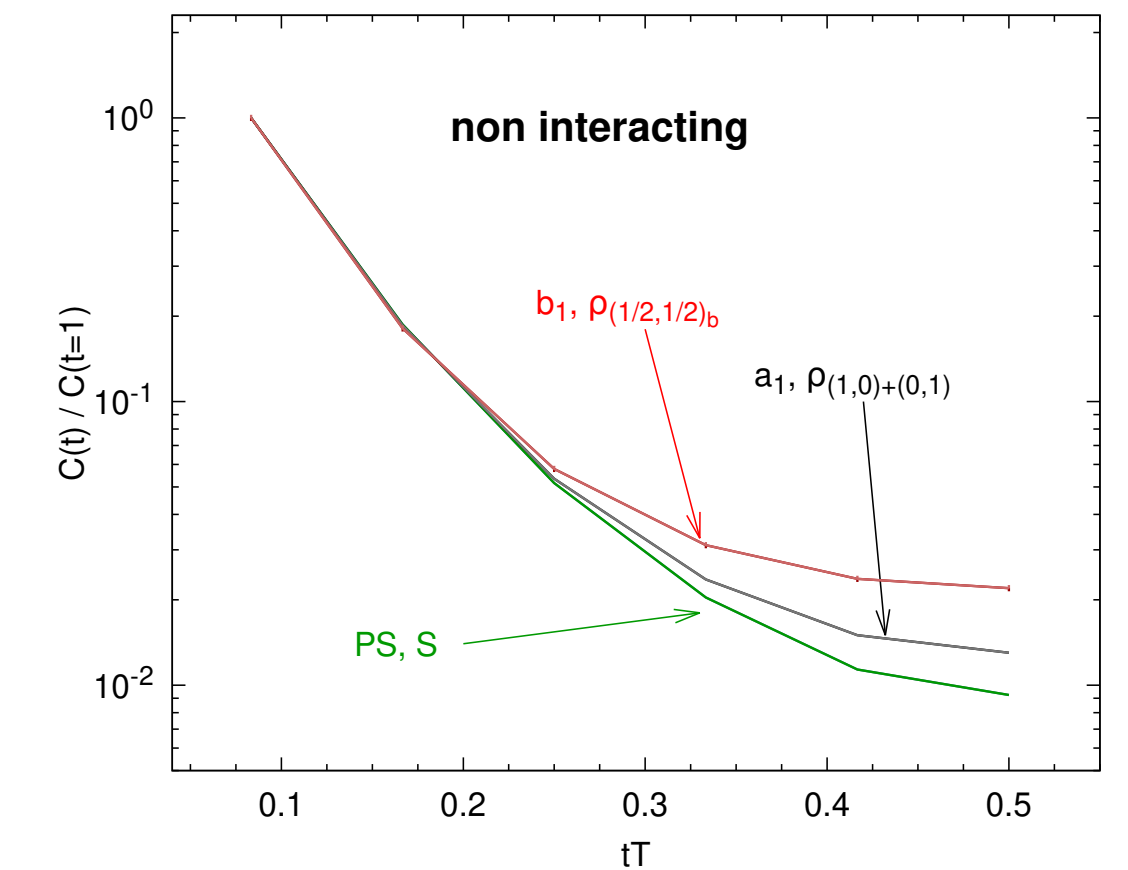
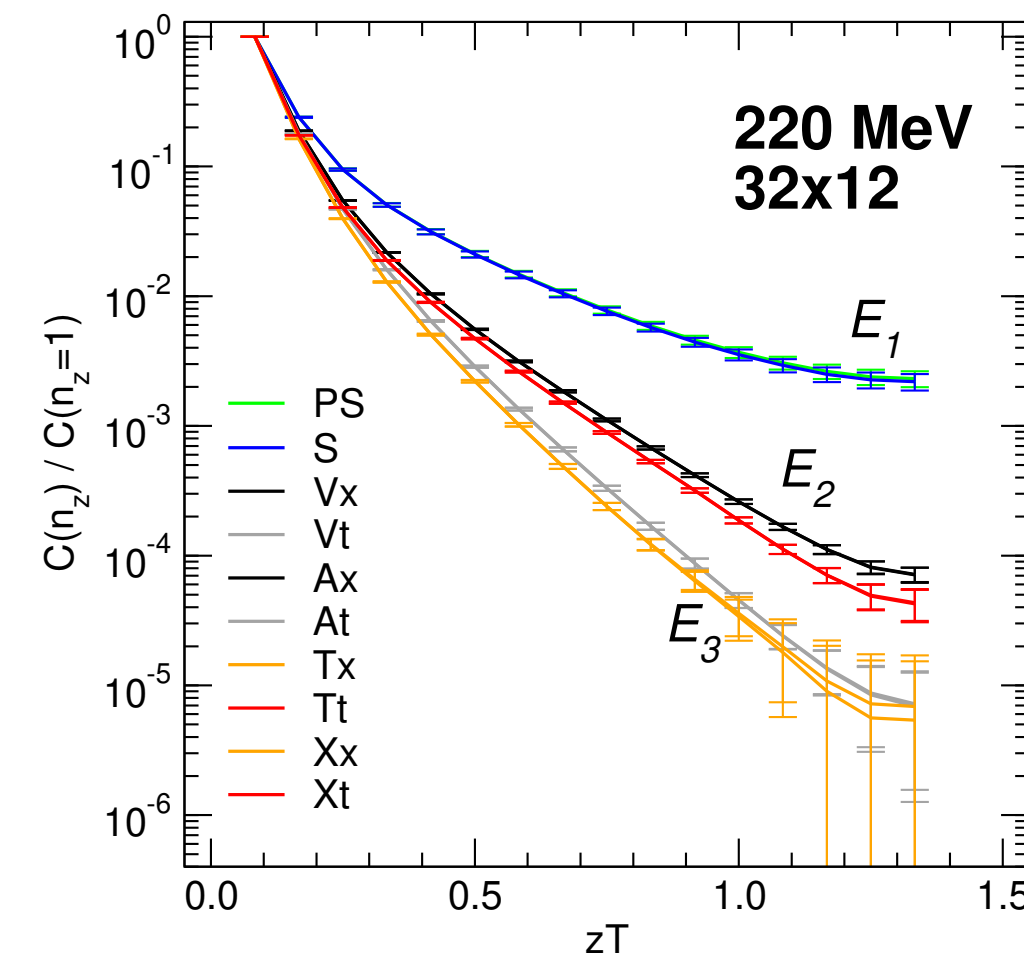
Bazavov, Brambilla, Ding, Petreczky, Schadler, Vairo, Weber, PRD 93 (2016) 11, 114502

Borsányi, Fodor, Guenther, Kara, Paolo Parotto, Pásztor, Pirelli, Wong, PRD 110 (2024) 114507

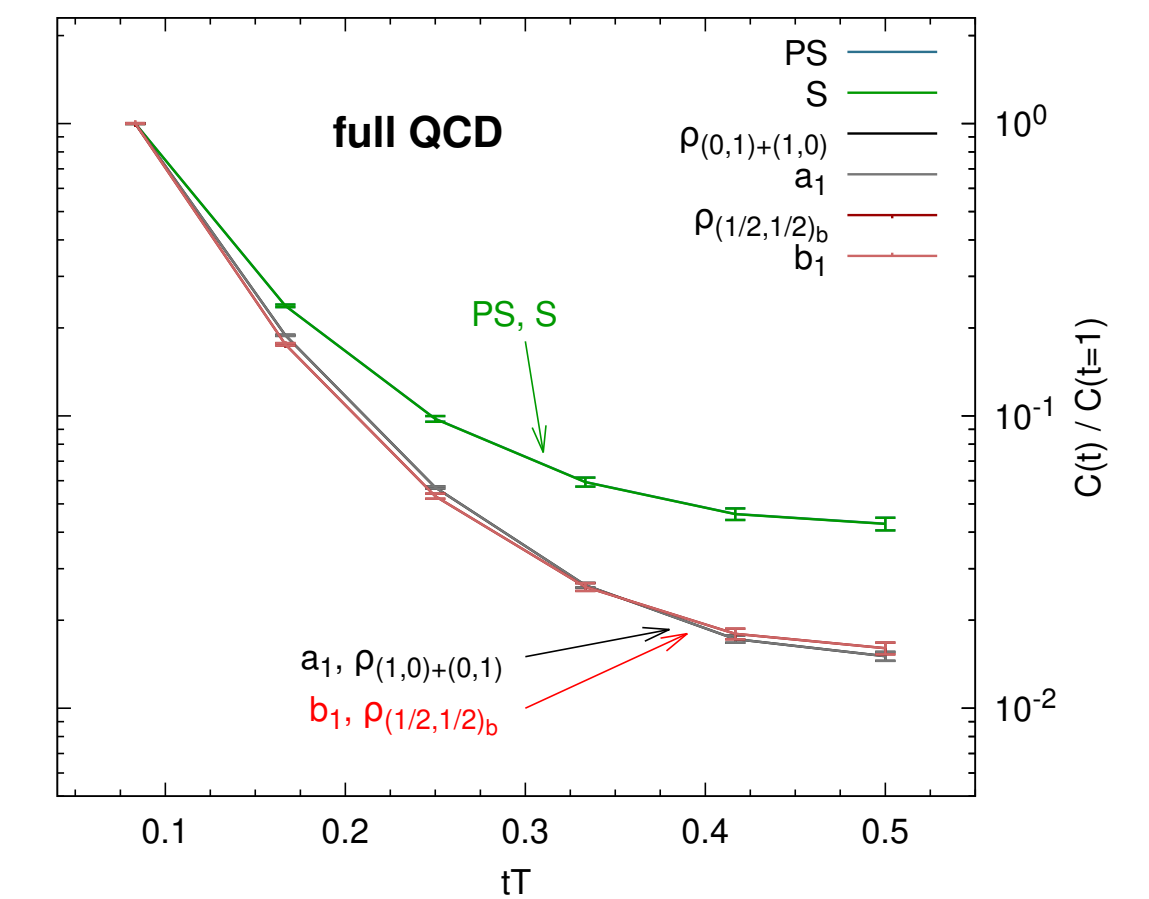
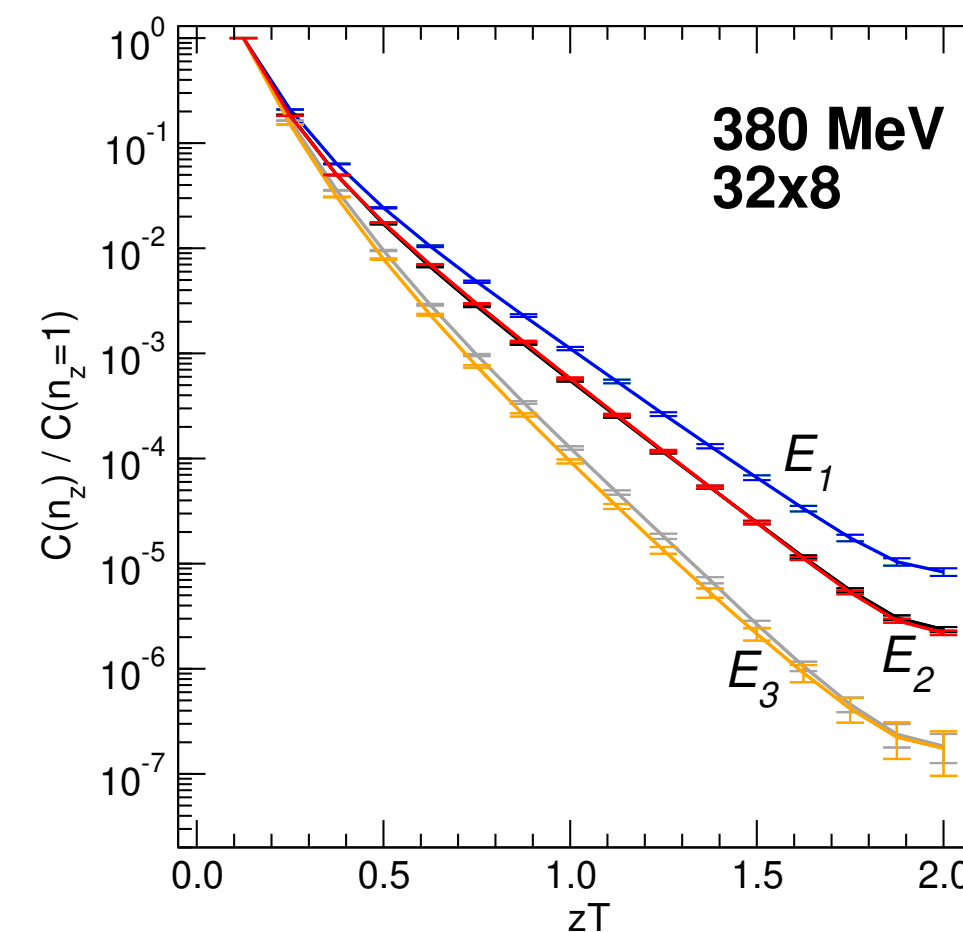
Strongly correlated physics

Spin symmetry for $T_\chi \lesssim T \lesssim 2T_\chi$

Rohrhofer, Aoki, Cossu, Fukaya, Glozman, Hashimoto,
Lang, Prelovsek, PRD 96 (2017) 094501



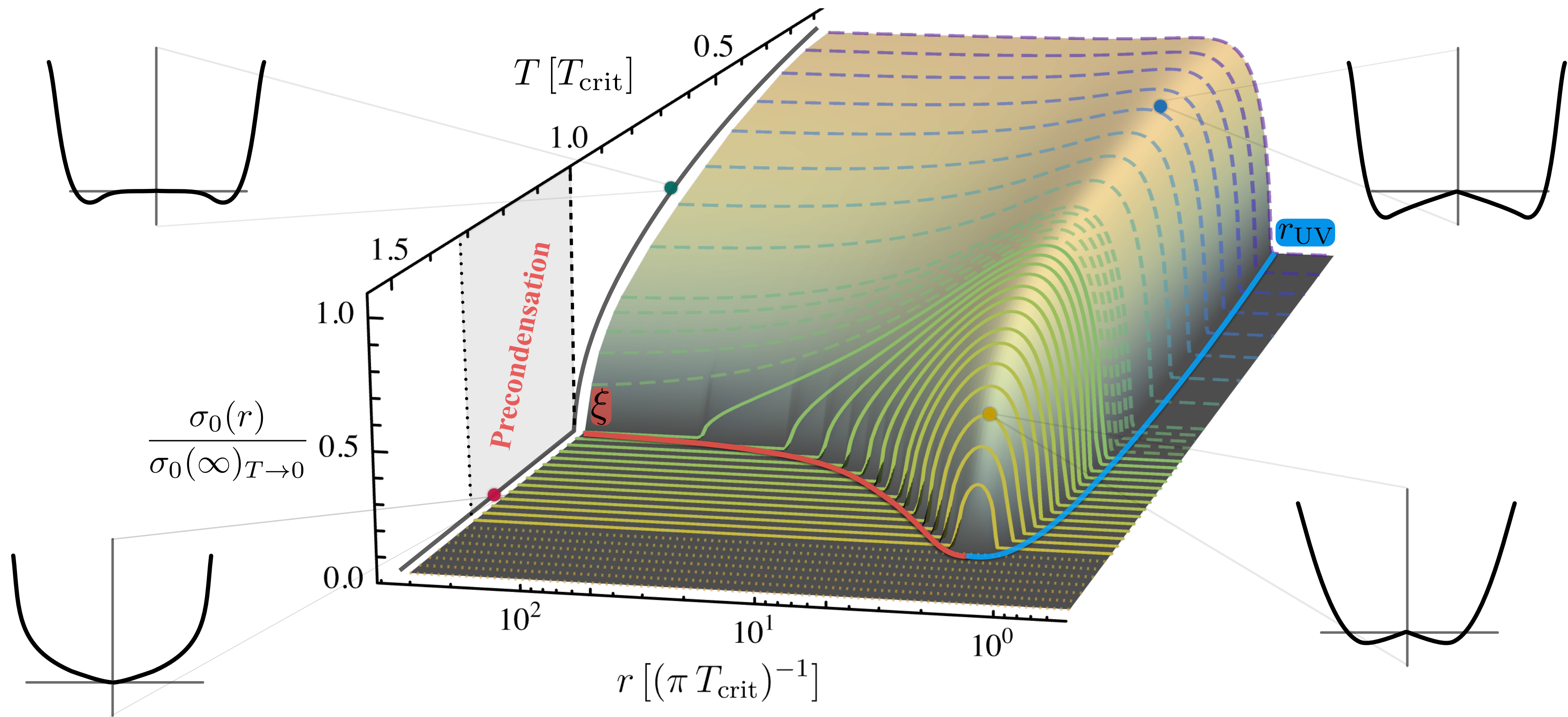
Rohrhofer, Aoki, Glozman, Hashimoto, PLB 802 (2020) 135245



Relation?

Strongly correlated physics

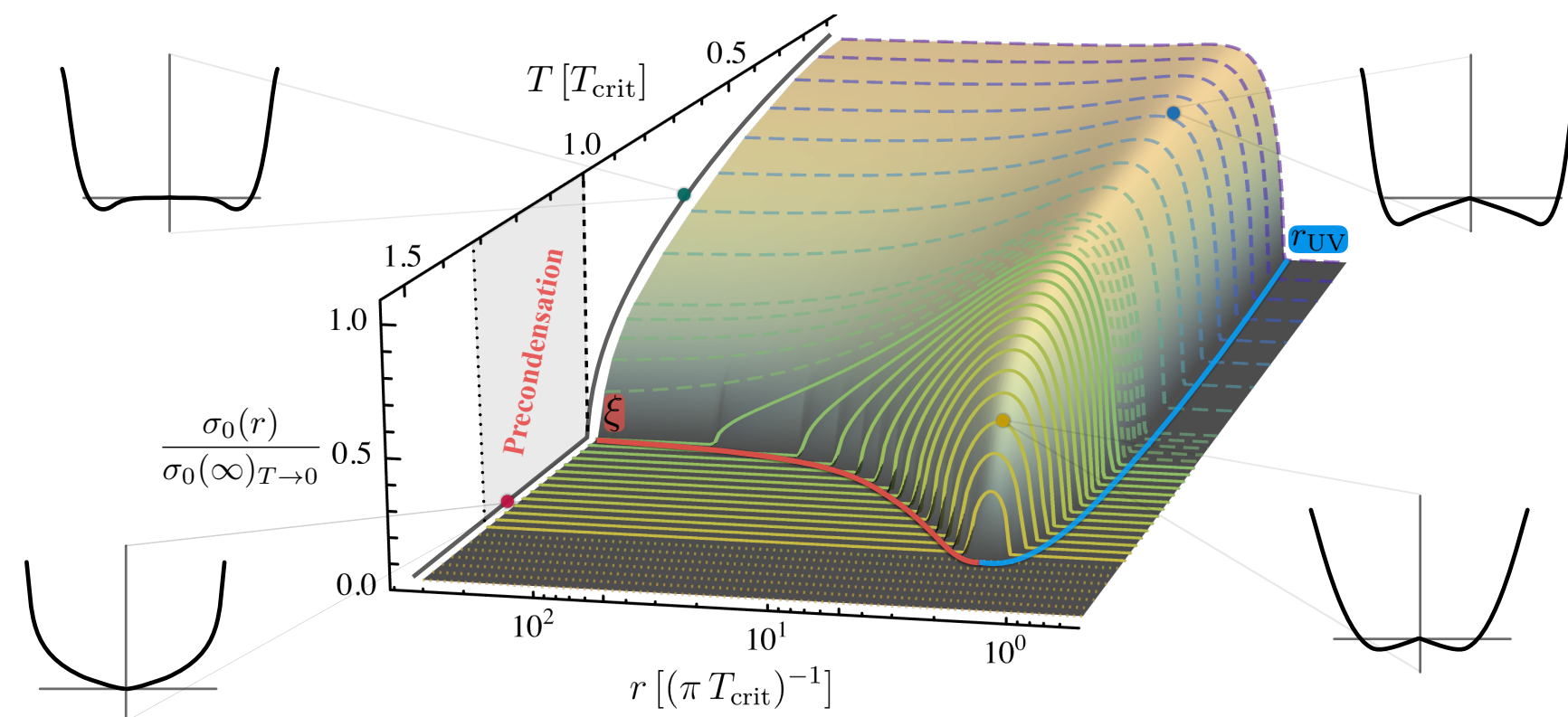
Precondensation in chiral QCD



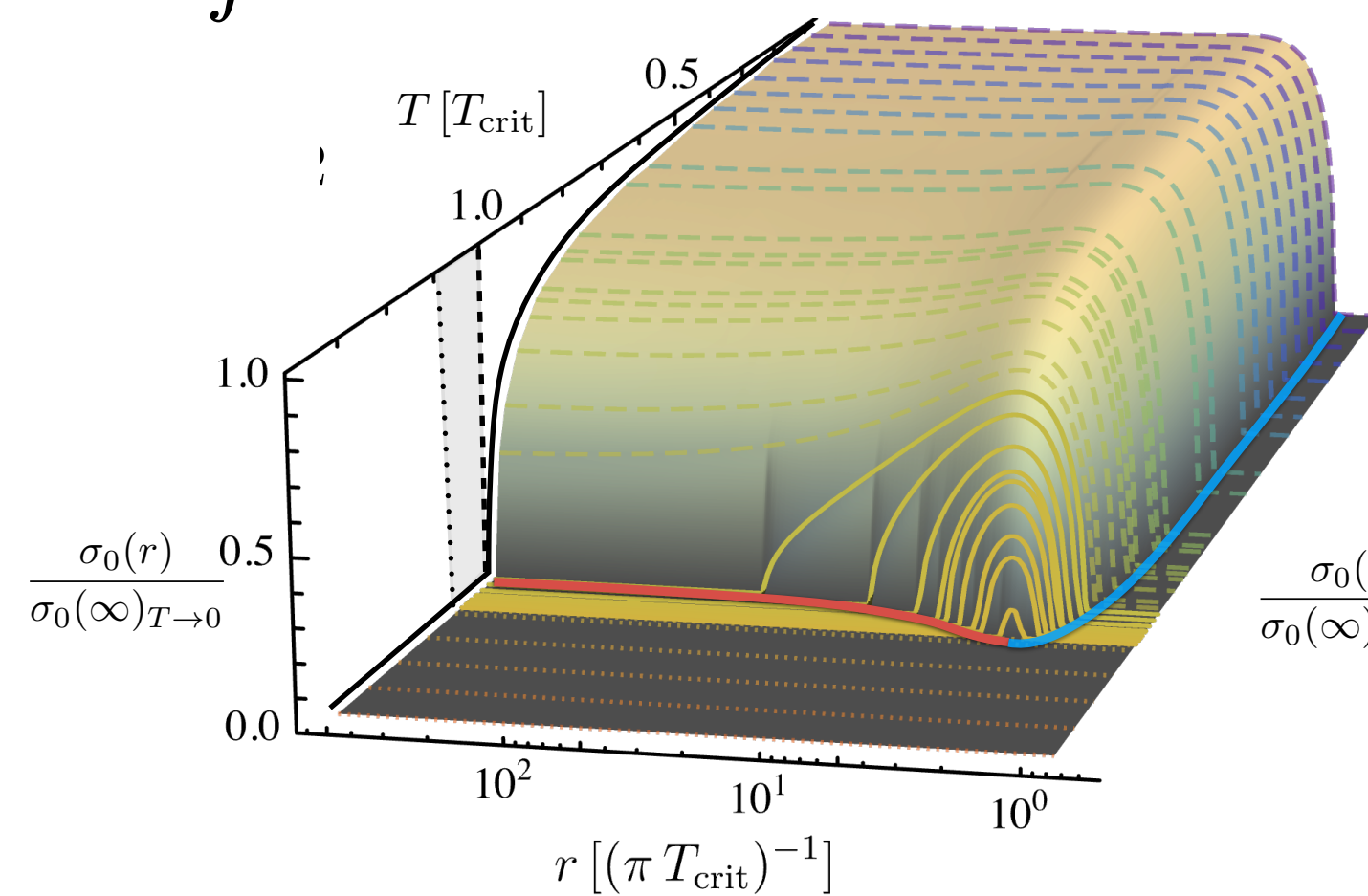
Strongly correlated physics

Precondensation for different flavours

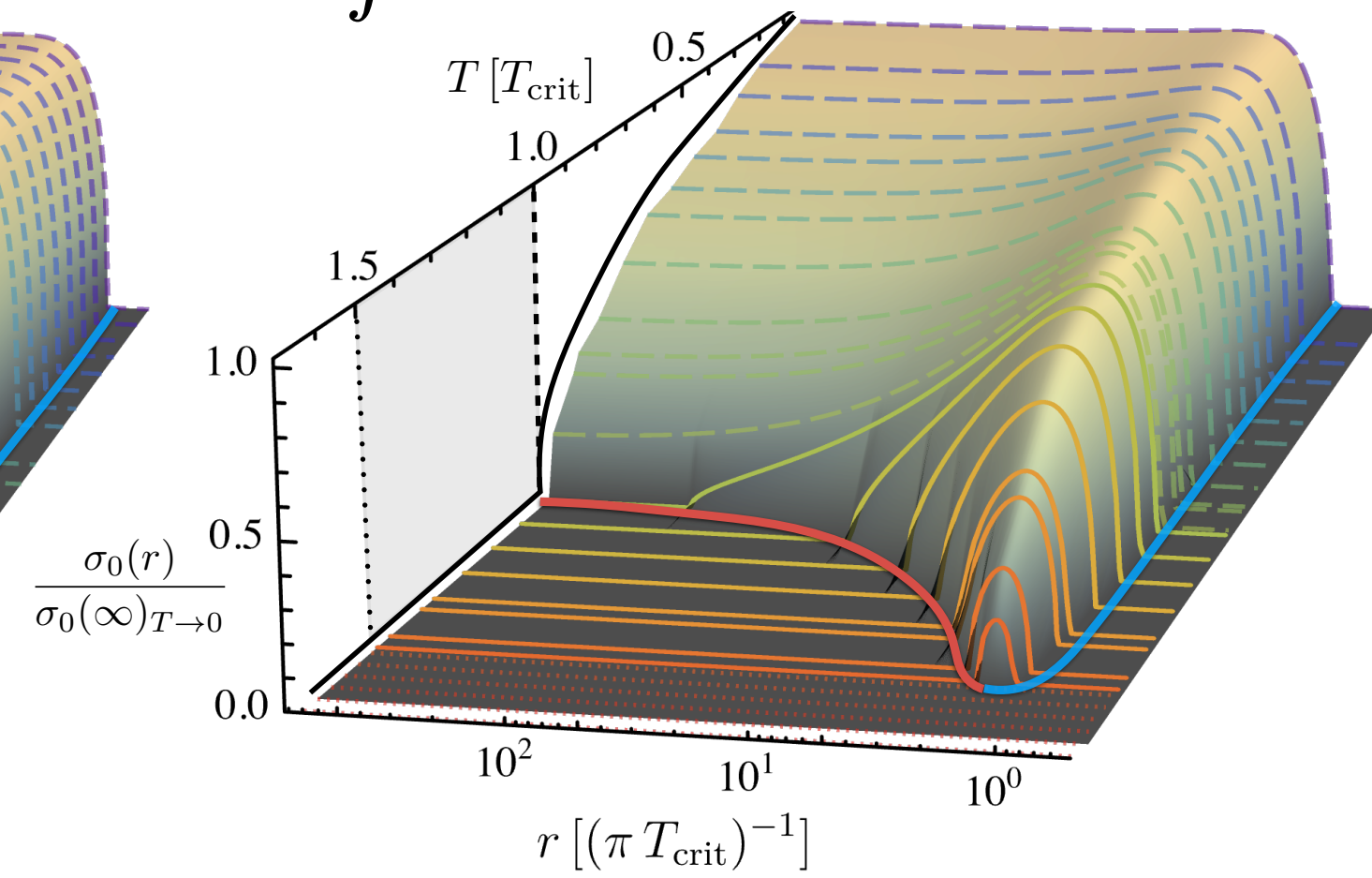
$$N_f = 3$$



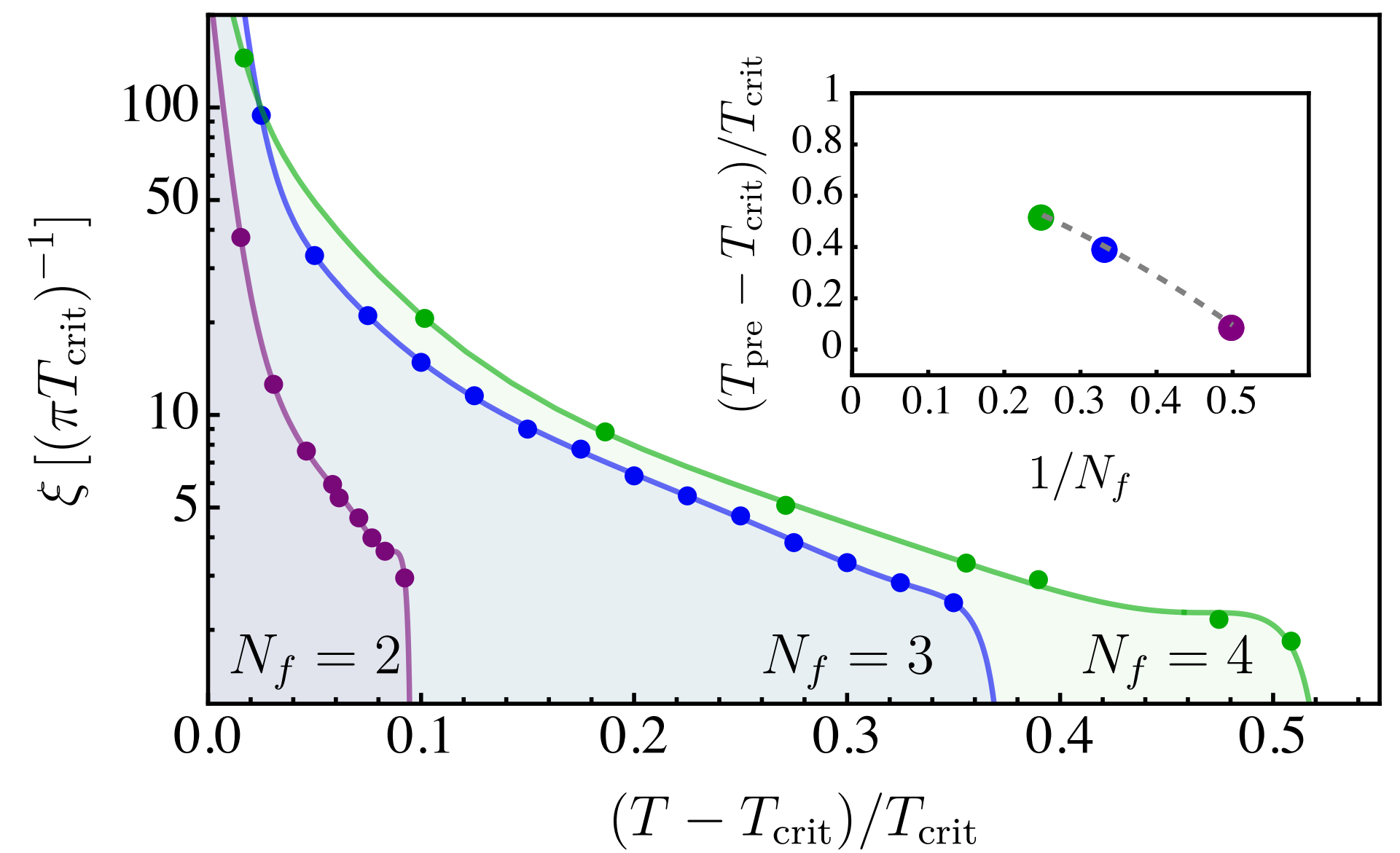
$$N_f = 2$$

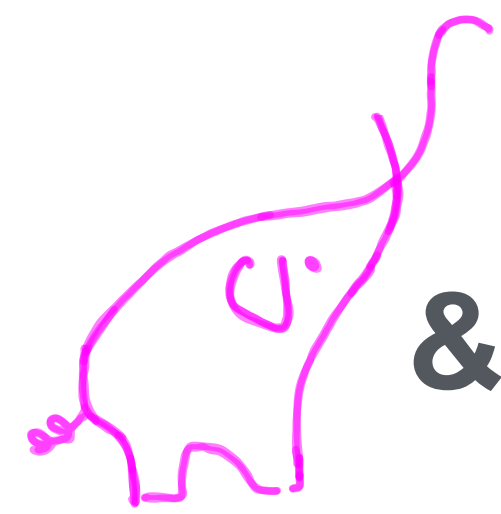


$$N_f = 4$$



Correlation length of 'domains'



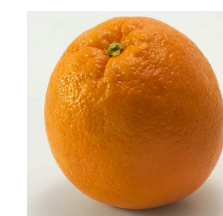


& functional QCD

Ripples of the CEP

aka

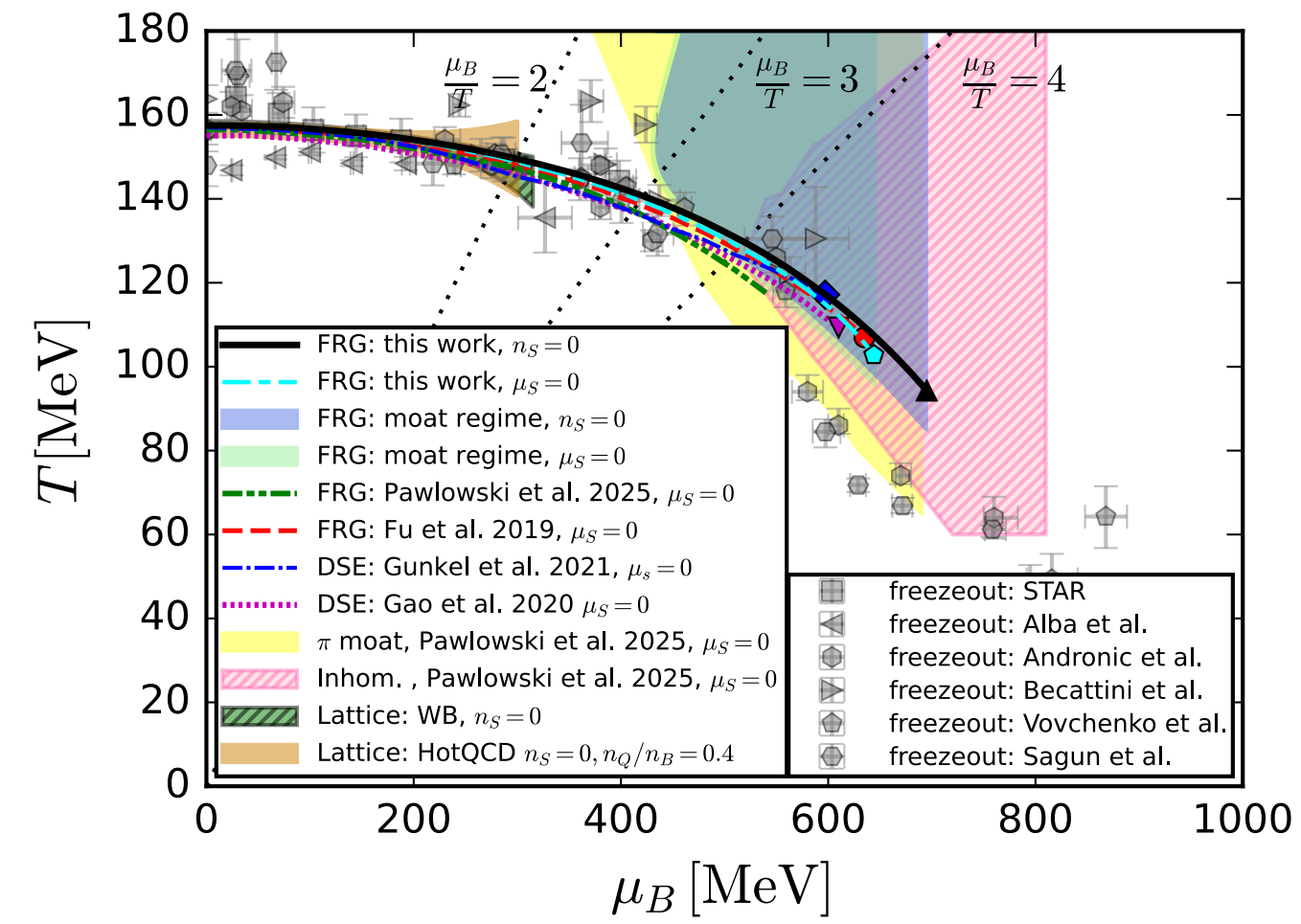
(Strong) signals of the ONP



QCD-assisted low energy effective theory

Direct QCD input

chiral phase structure

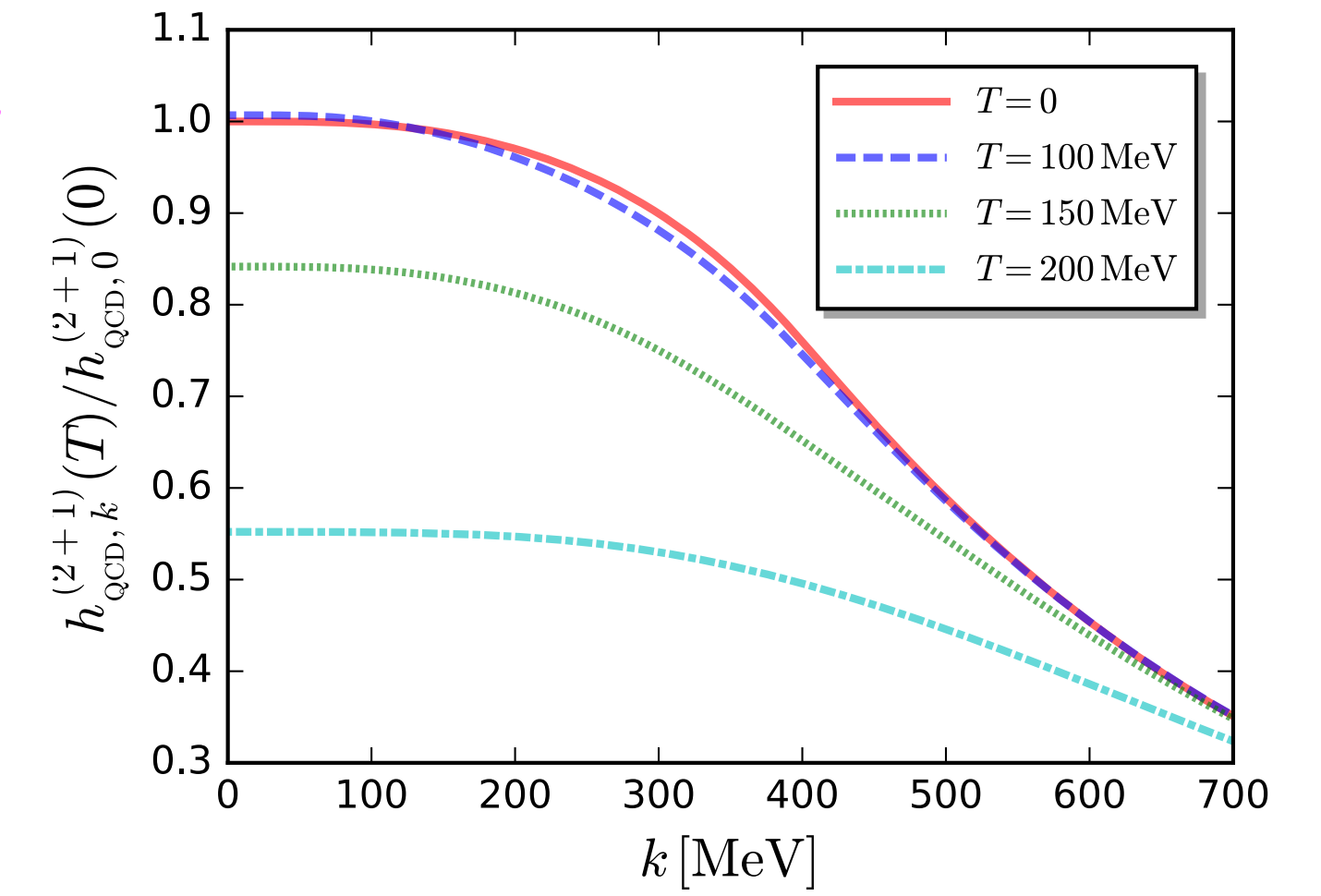


+



Fu, JMP, Rennecke, PRD 101 (2020) 054032

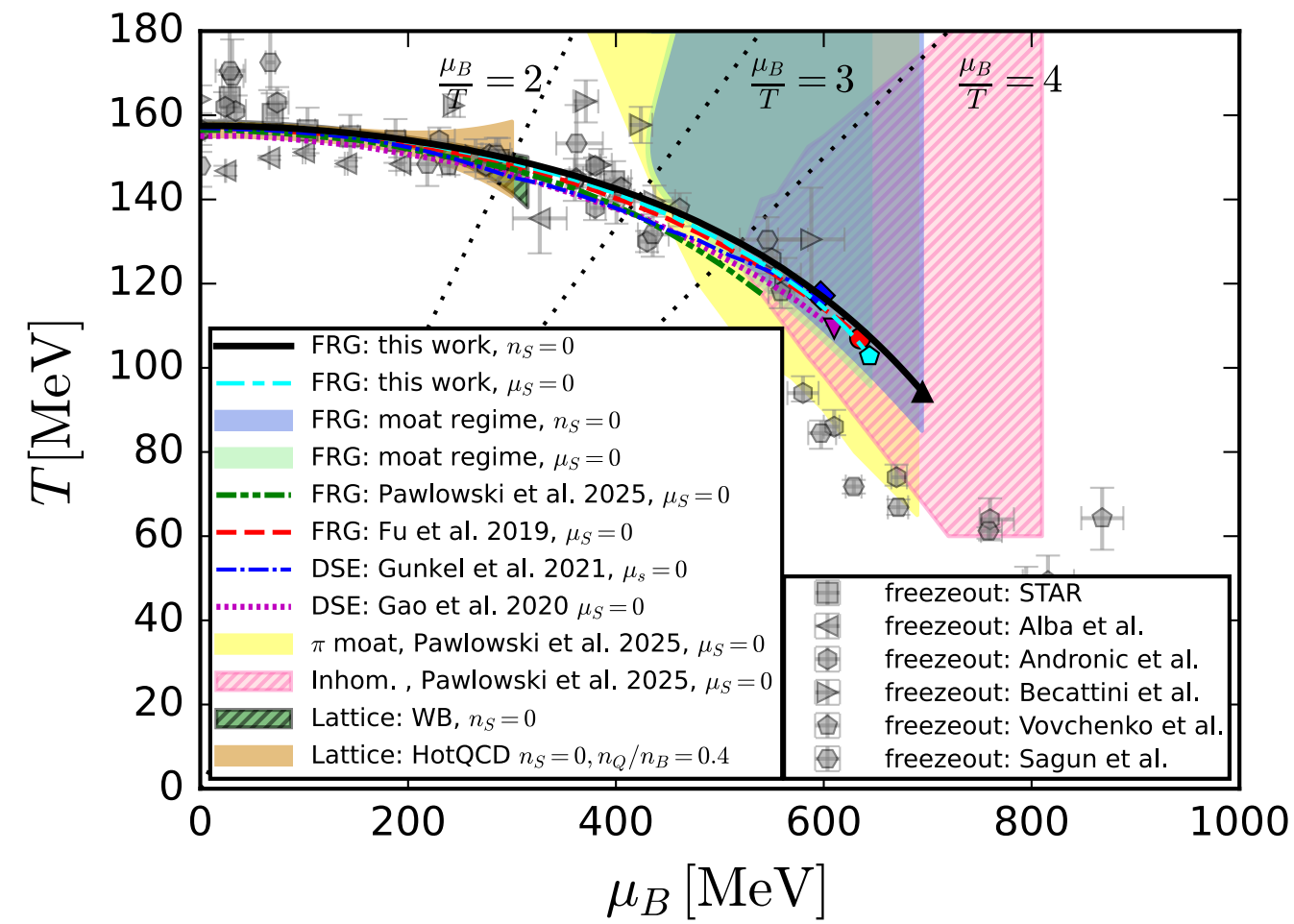
quark-antiquark — meson scattering



QCD-assisted low energy effective theory

Direct QCD input

chiral phase structure

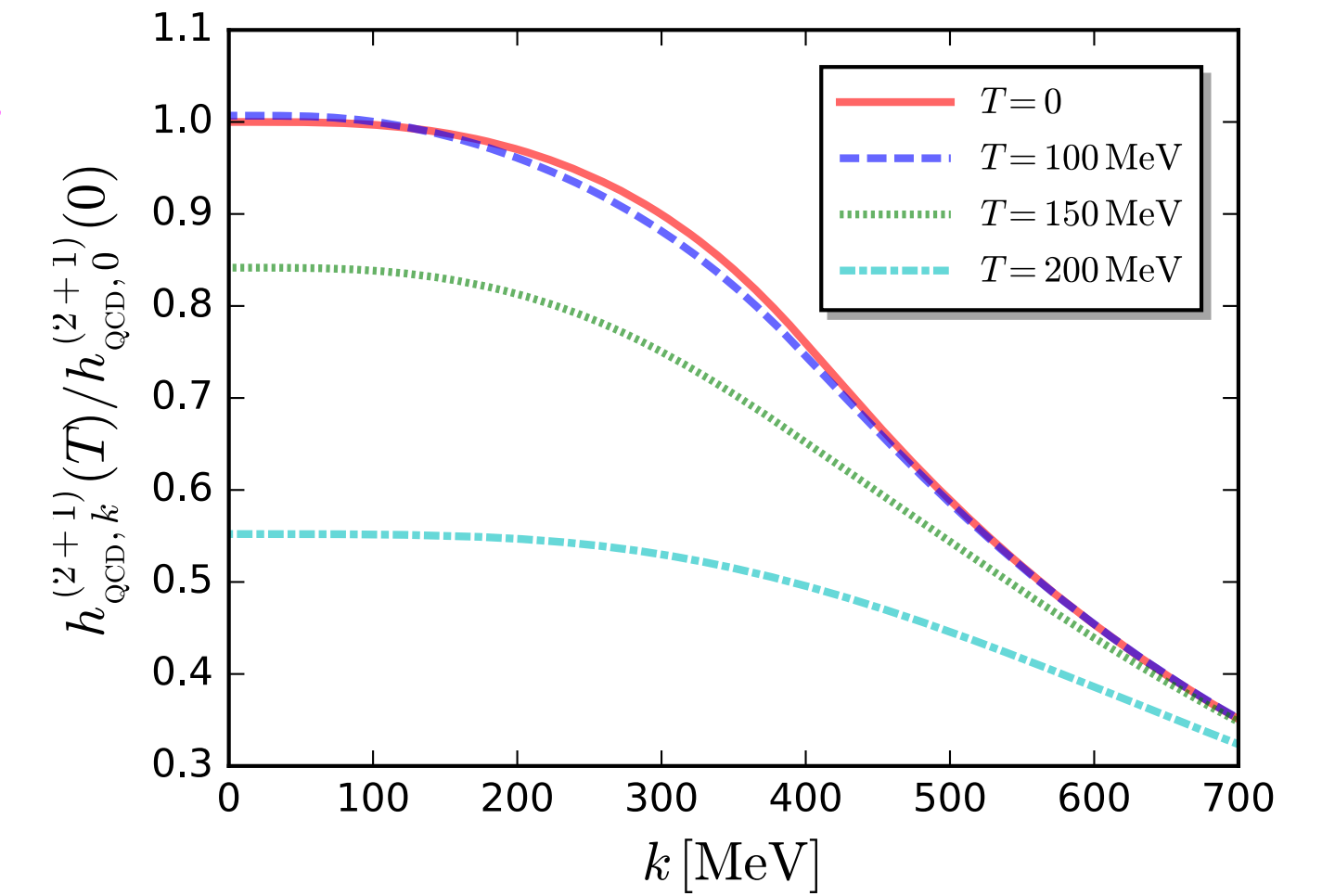


+



Fu, JMP, Rennecke, PRD 101 (2020) 054032

quark-antiquark — meson scattering

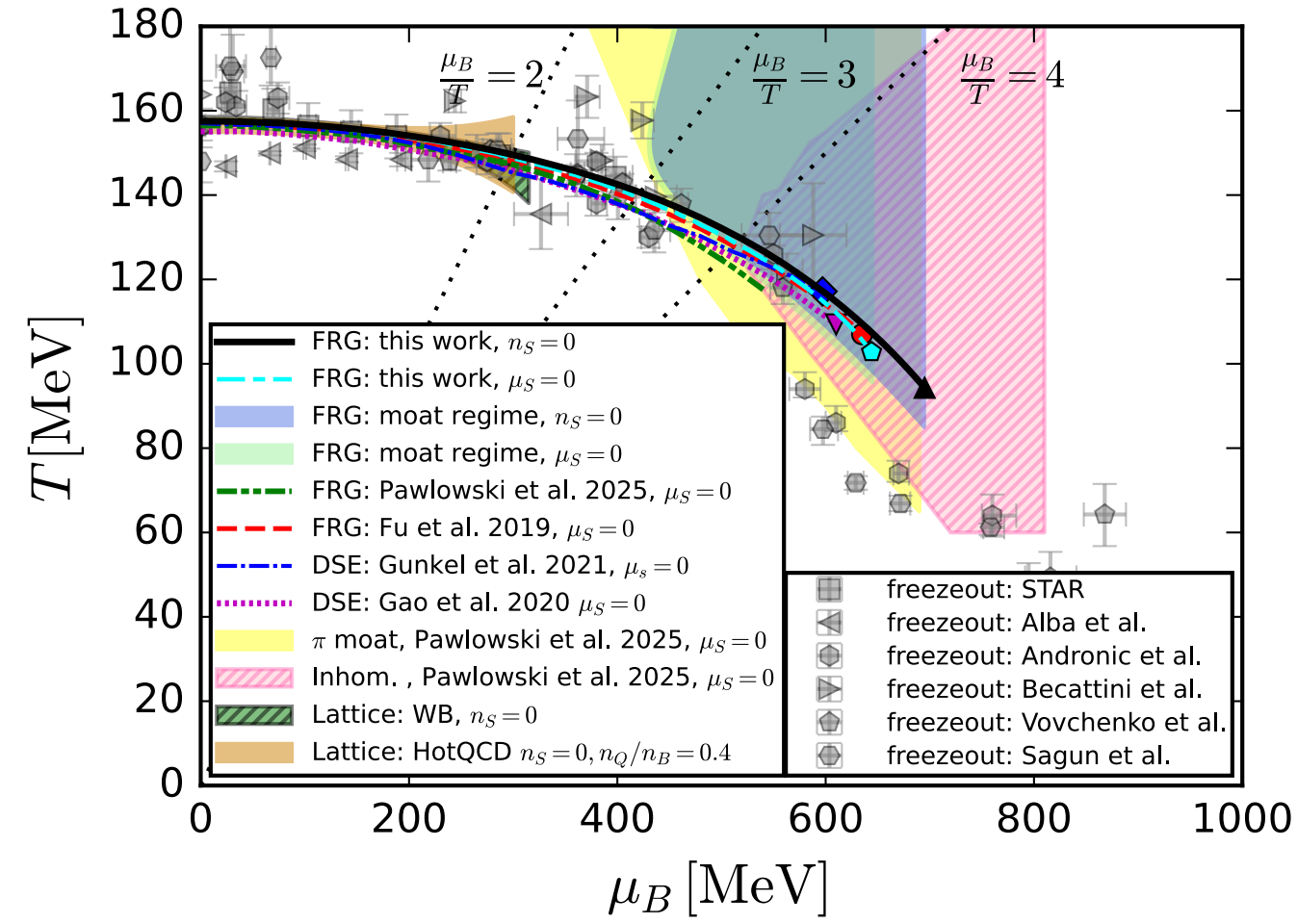


low energy quantum, thermal & density fluctuations via fRG (QCD-assisted PQM model)

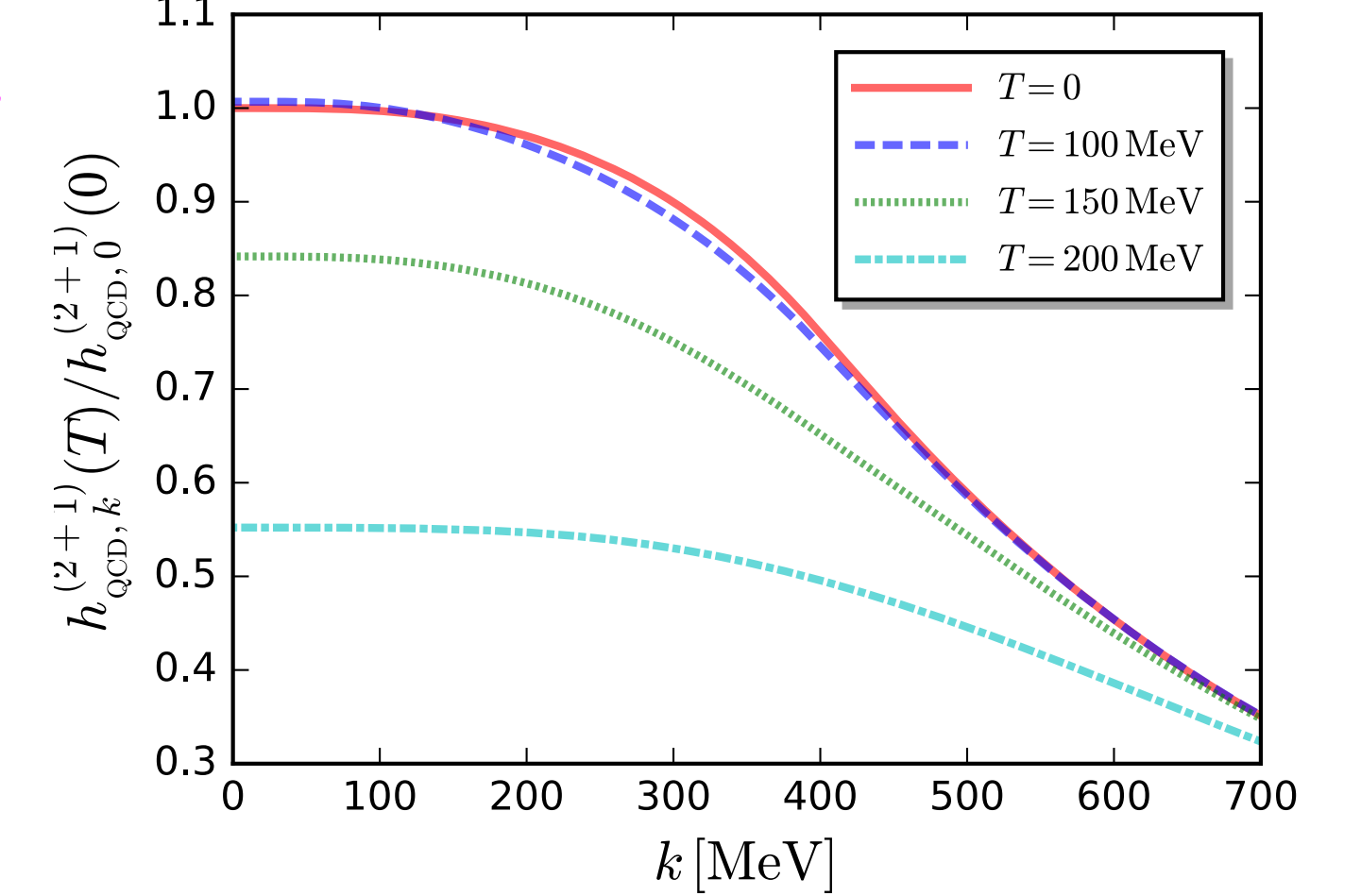
QCD-assisted low energy effective theory

Direct QCD input

chiral phase structure



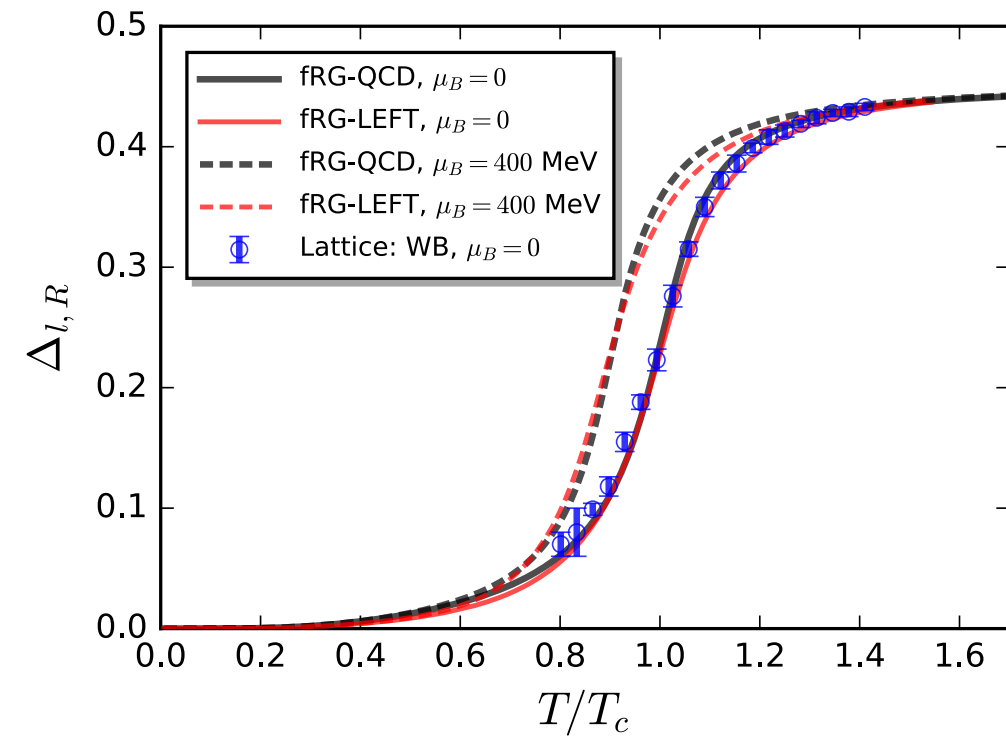
quark-antiquark — meson scattering



Fu, JMP, Rennecke, PRD 101 (2020) 054032

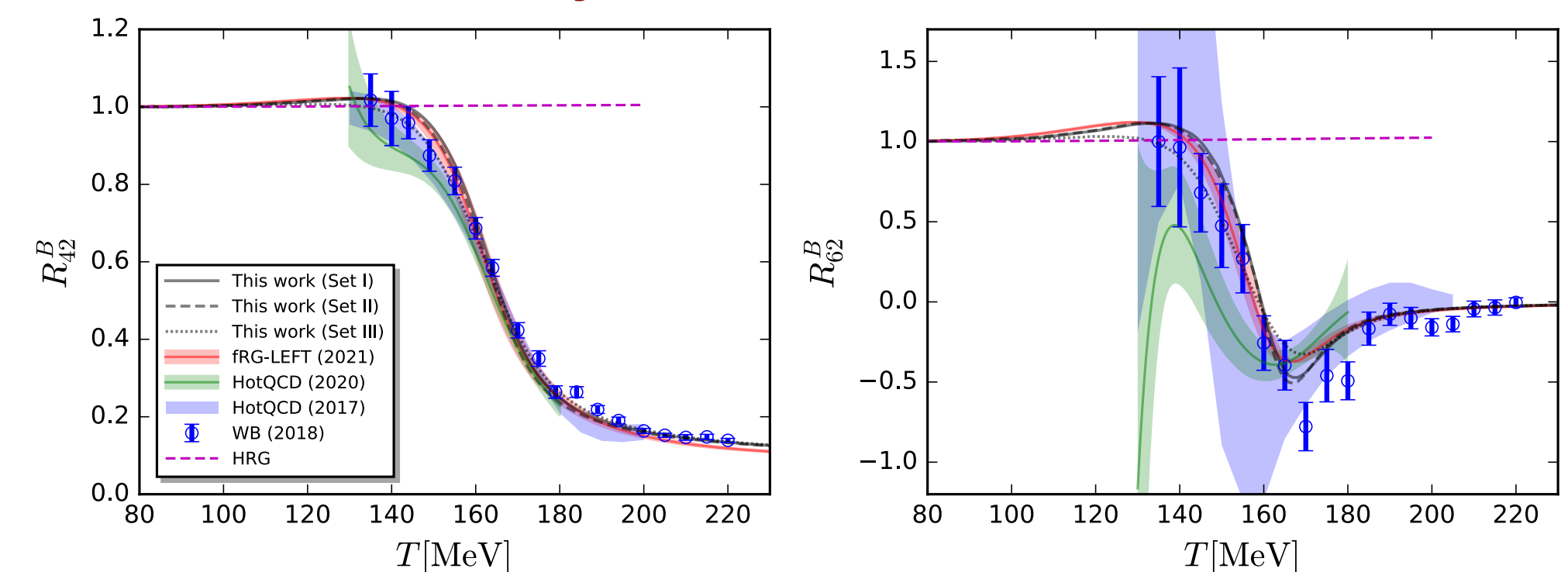
low energy quantum, thermal & density fluctuations via fRG (QCD-assisted PQM model)

renormalised chiral condensate



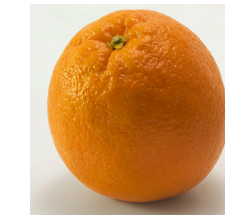
**Benchmarks with lattice and fQCD
at
vanishing density and fQCD at finite density**

baryon number fluctuations



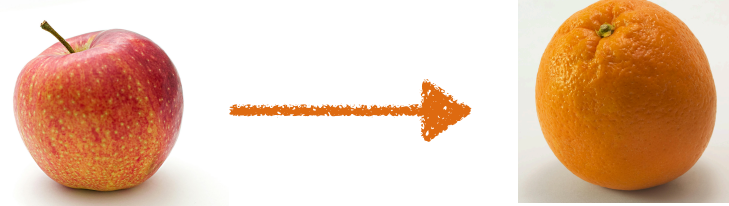


Ripples of the critical point at chemical freeze-out



Canonical corrections via subensemble acceptance

Vovchenko, Savchuk, Poberezhnyuk, Gorenstein, Koch, PLB 811, 135868 (2020)



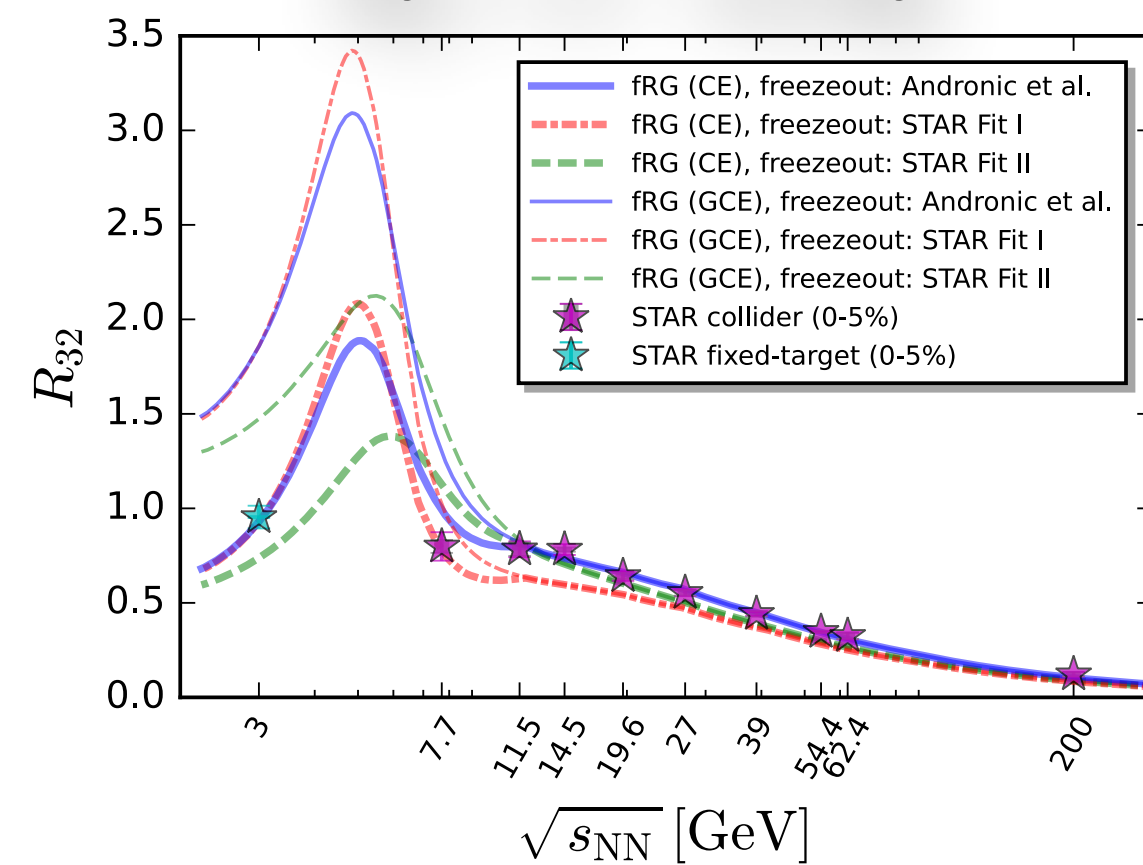
fixing the subensemble volume

subensemble volume

system volume

$$V_1 = \alpha V$$

$$\bar{R}_{32}^B = (1 - 2\alpha) R_{32}^B$$



qualitative adjustment

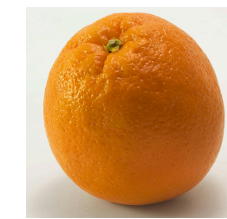
$$\alpha(\bar{s}) = a \left(1 - \sqrt{\bar{s}}\right) \theta(1 - \bar{s})$$

$$a = 0.33$$

$$\sqrt{\bar{s}} = \frac{\sqrt{s_{\text{NN}}}}{11.9 \text{ GeV}}$$

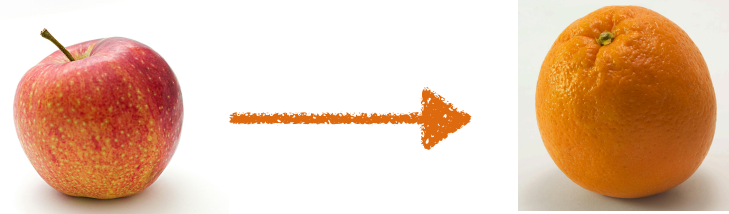


Ripples of the critical point at chemical freeze-out



Canonical corrections via subensemble acceptance

Vovchenko, Savchuk, Poberezhnyuk, Gorenstein, Koch, PLB 811, 135868 (2020)



fixing the subensemble volume

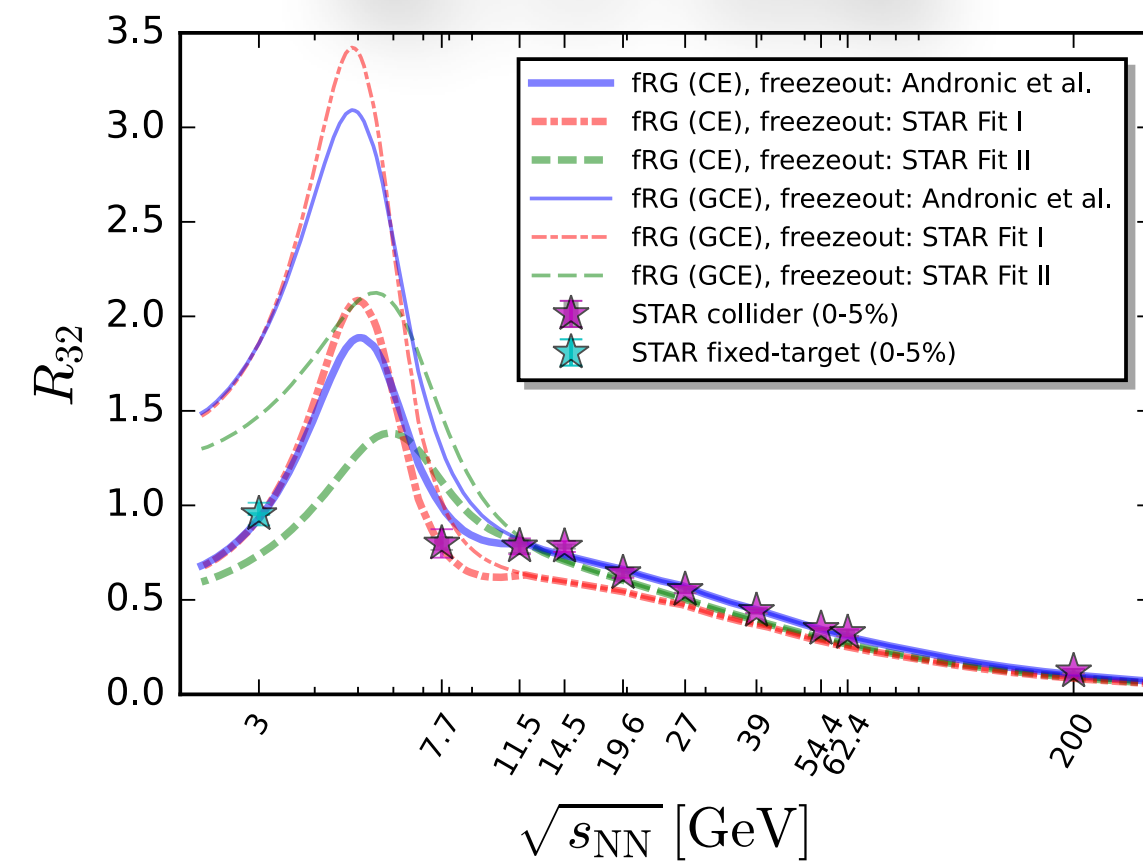
baryon & proton number fluctuations

baryon fluctuations with functional QCD

subensemble volume system volume

$$V_1 = \alpha V$$

$$\bar{R}_{32}^B = (1 - 2\alpha) R_{32}^B$$

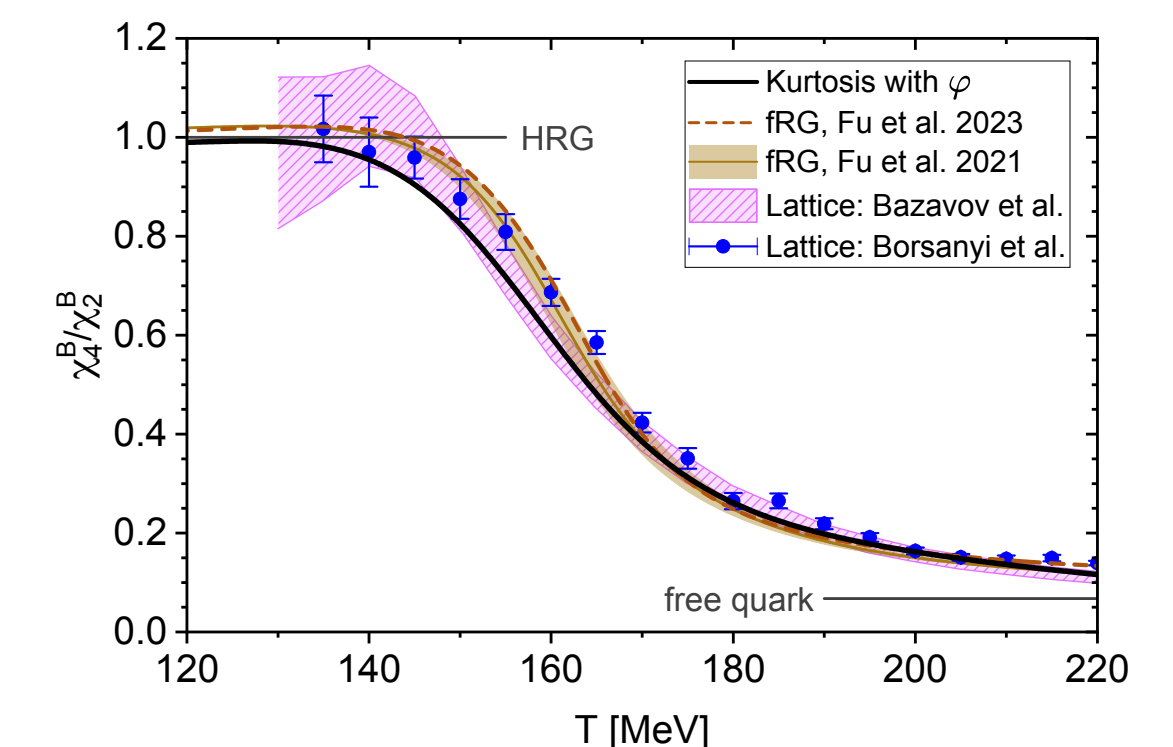
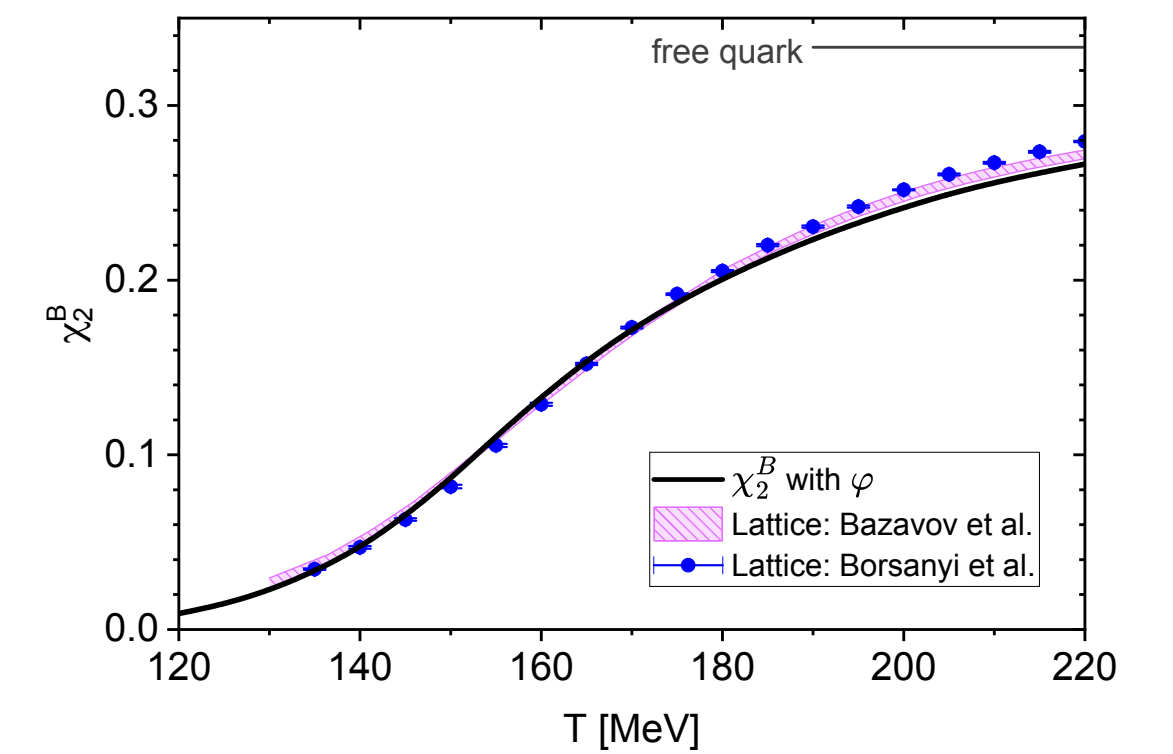
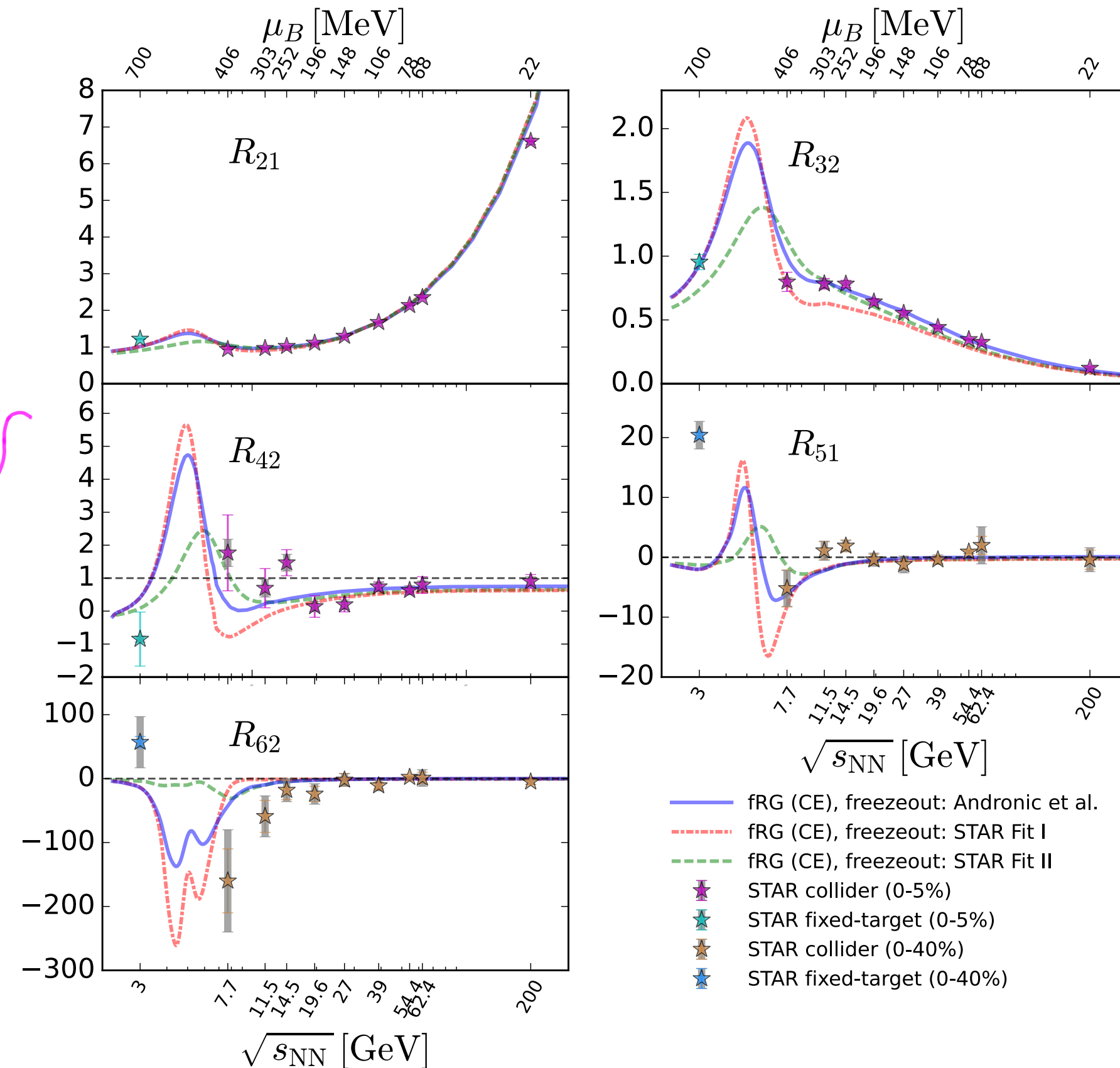


qualitative adjustment

$$\alpha(\bar{s}) = a \left(1 - \sqrt{\bar{s}}\right) \theta(1 - \bar{s})$$

$$a = 0.33$$

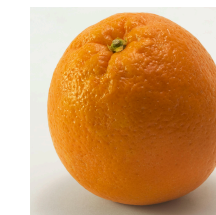
$$\sqrt{\bar{s}} = \frac{\sqrt{s_{NN}}}{11.9 \text{ GeV}}$$



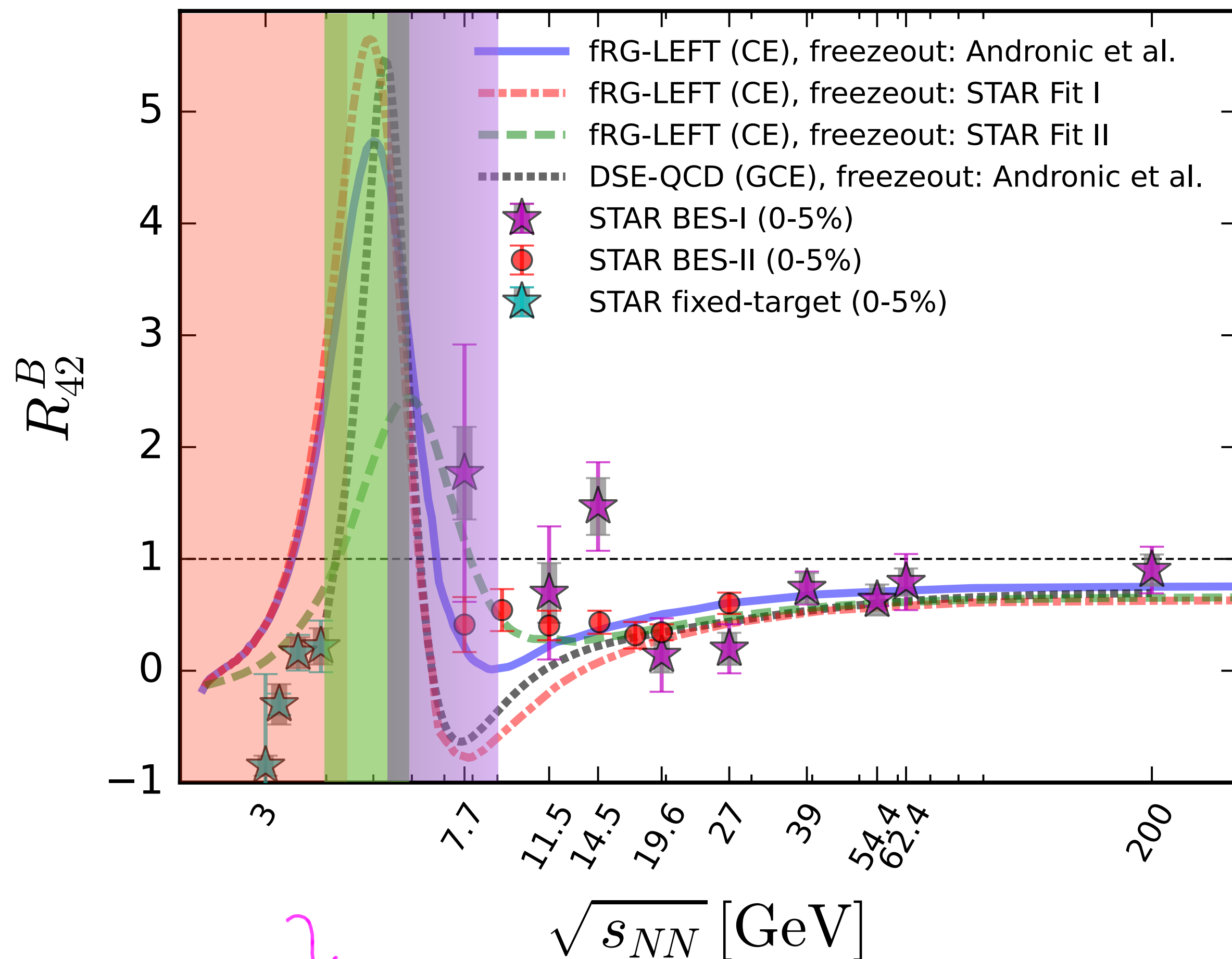
Gao, Liu, Lu, JMP, PRD 113 (2026) 054019



Ripples of the critical point at chemical freeze-out



net-baryon fluctuations in QCD vs net-proton fluctuations at STAR



Dominated by non-critical soft modes

Results:

1st principles functional QCD computations
& low energy effective theories/extrapolations



& functional QCD



Fu, Luo, JMP, Rennecke, Wen, Yin, PRD 104 (2021) 9

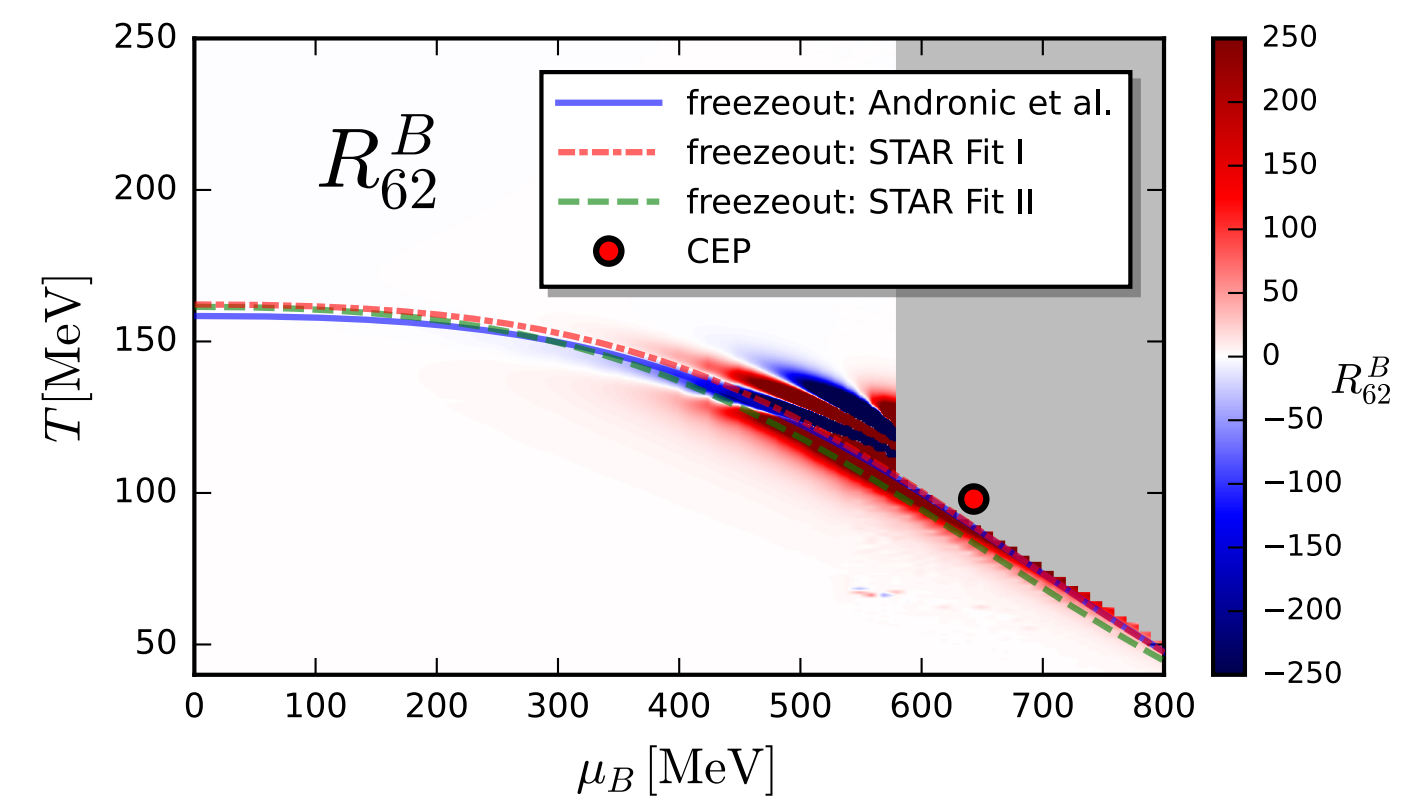
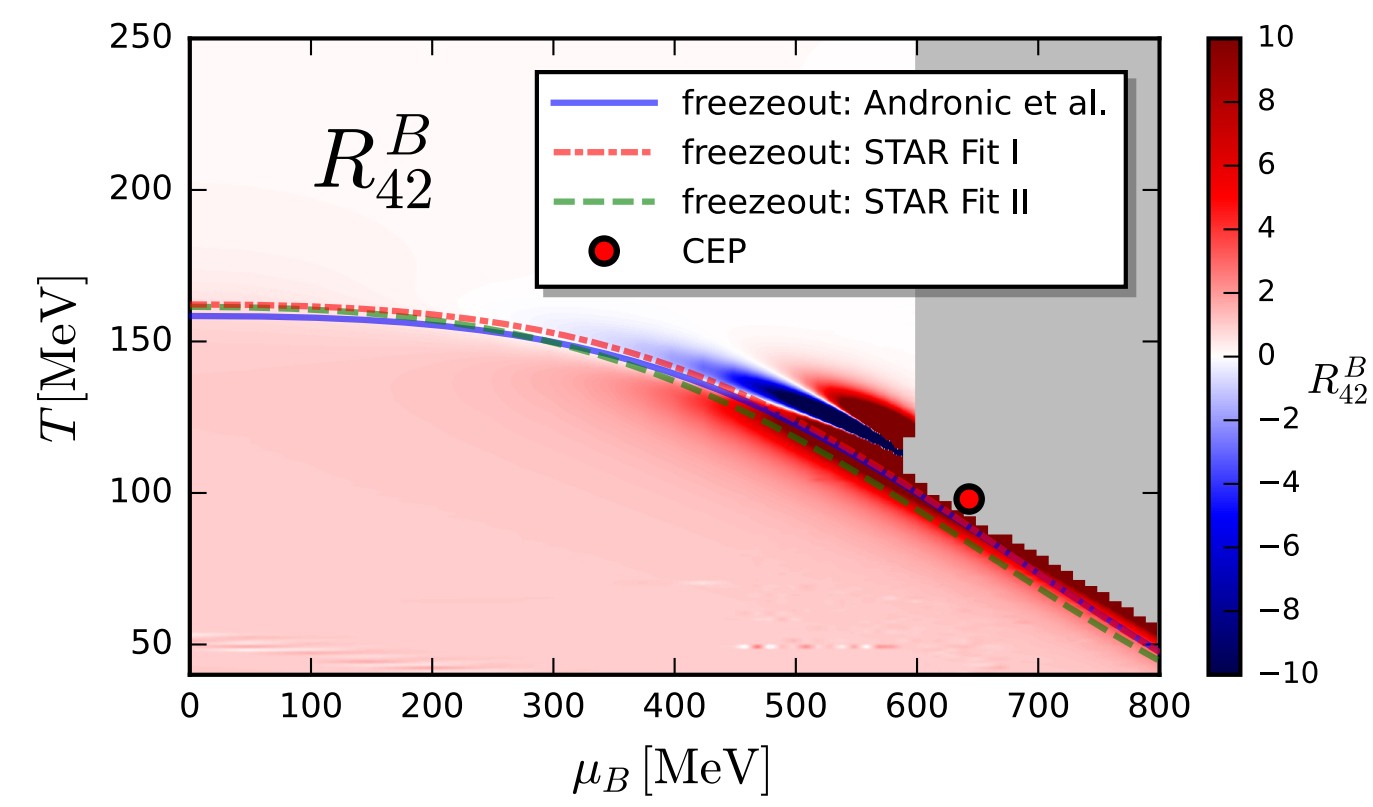
Fu, Luo, JMP, Rennecke, Yin, PRD 111, L031502

Functional QCD: Lu, Gao, Liu, JMP, 113 (2026) 054019

Ripples of the critical point

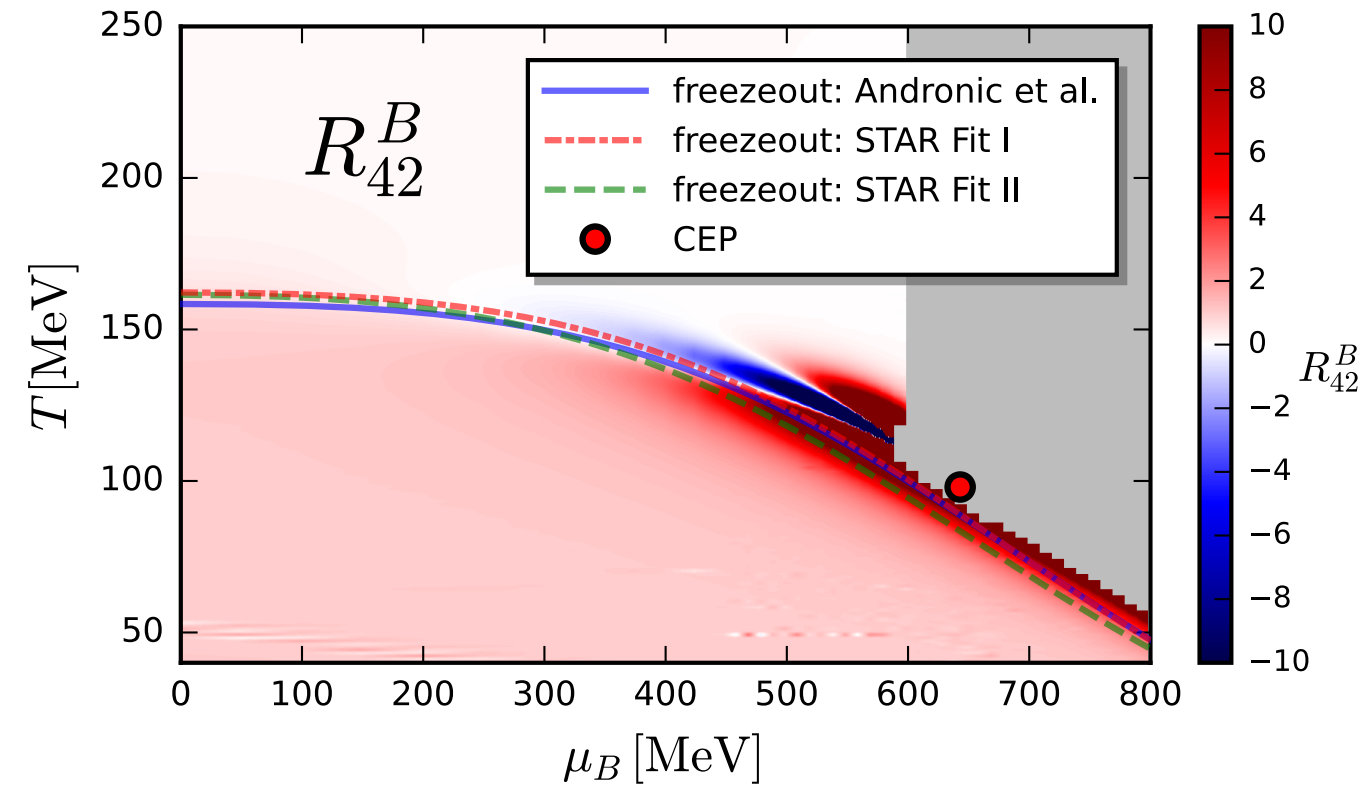
baryon number fluctuations in the phase structure

$$(T_{\text{CEP}}, \mu_{B_{\text{CEP}}}) = (98, 643) \text{ MeV}$$



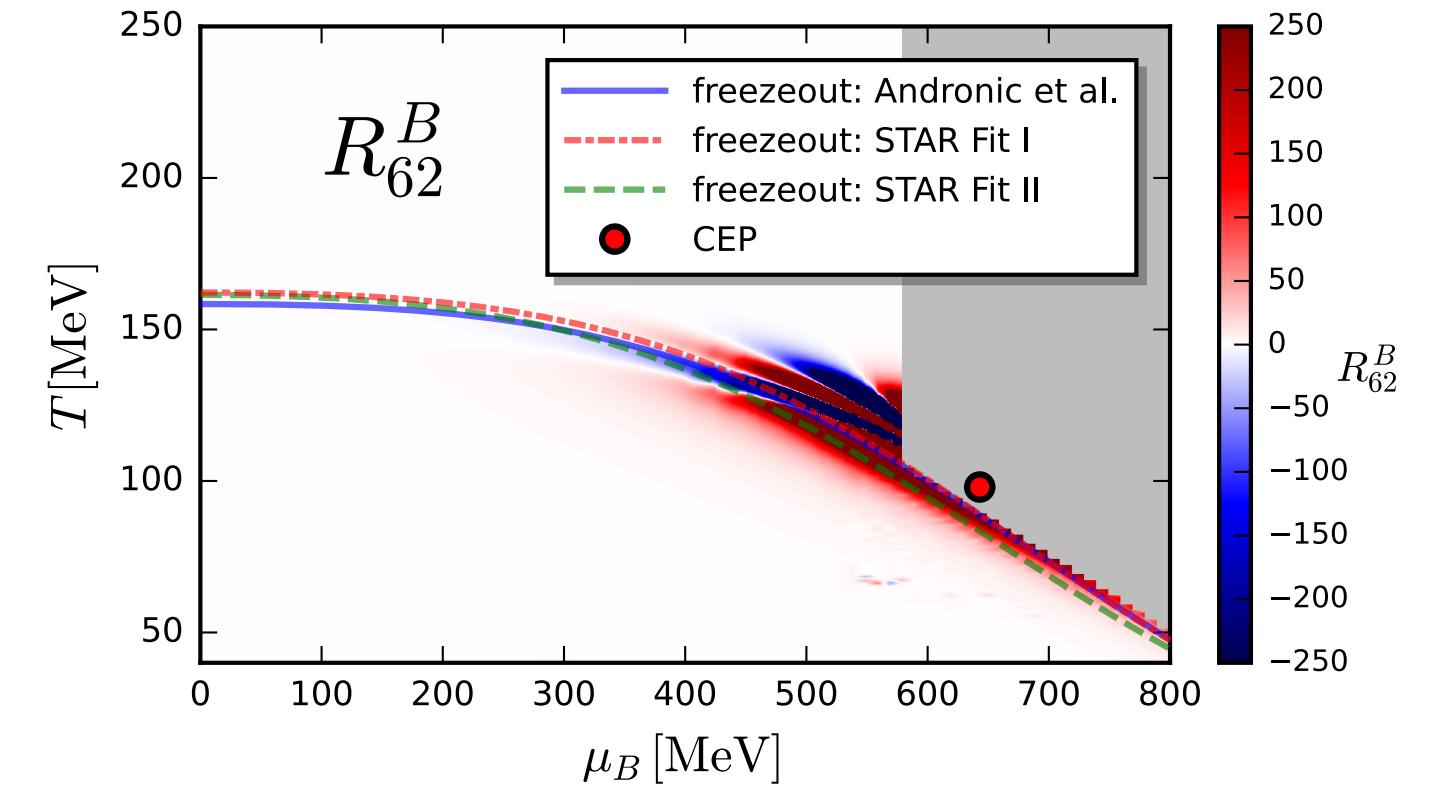
Ripples of the critical point

baryon number fluctuations in the phase structure

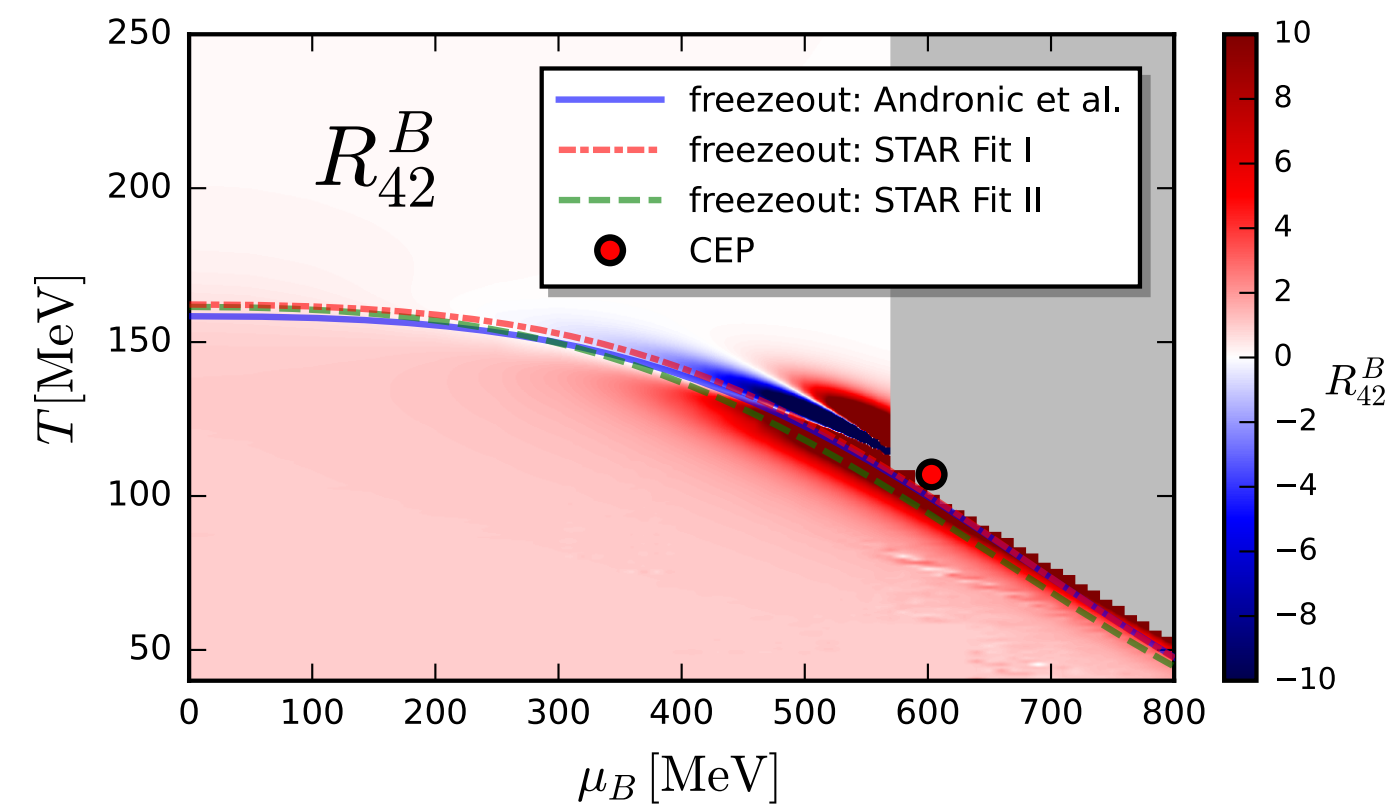


$$(T_{\text{CEP}}, \mu_{B_{\text{CEP}}}) = (98, 643) \text{ MeV}$$

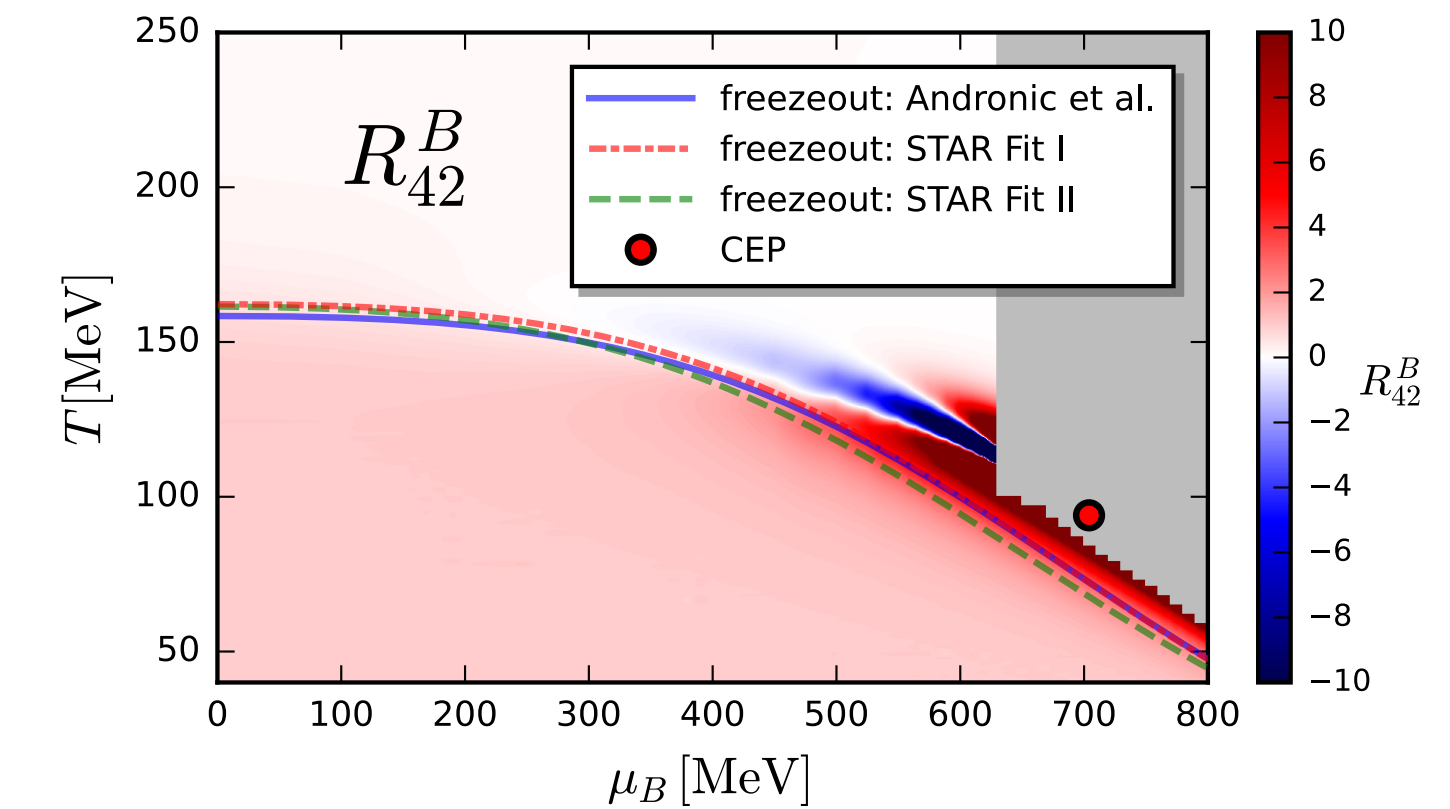
Ride the



Variations of the CEP in the allowed estimate regime



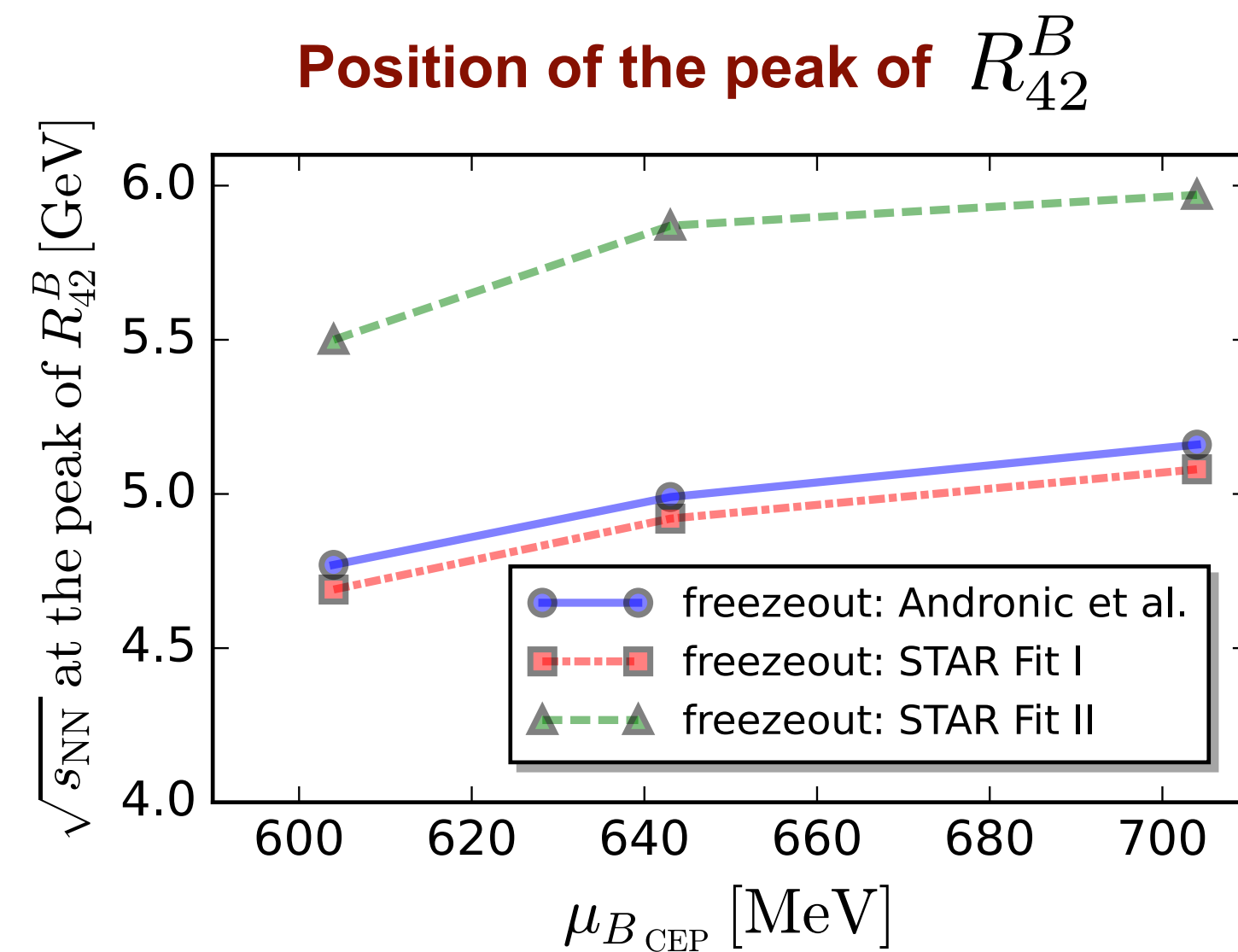
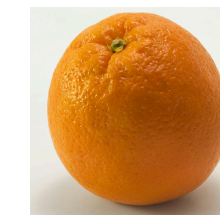
$$(T_{\text{CEP}}, \mu_{B_{\text{CEP}}}) = (108, 604) \text{ MeV}$$



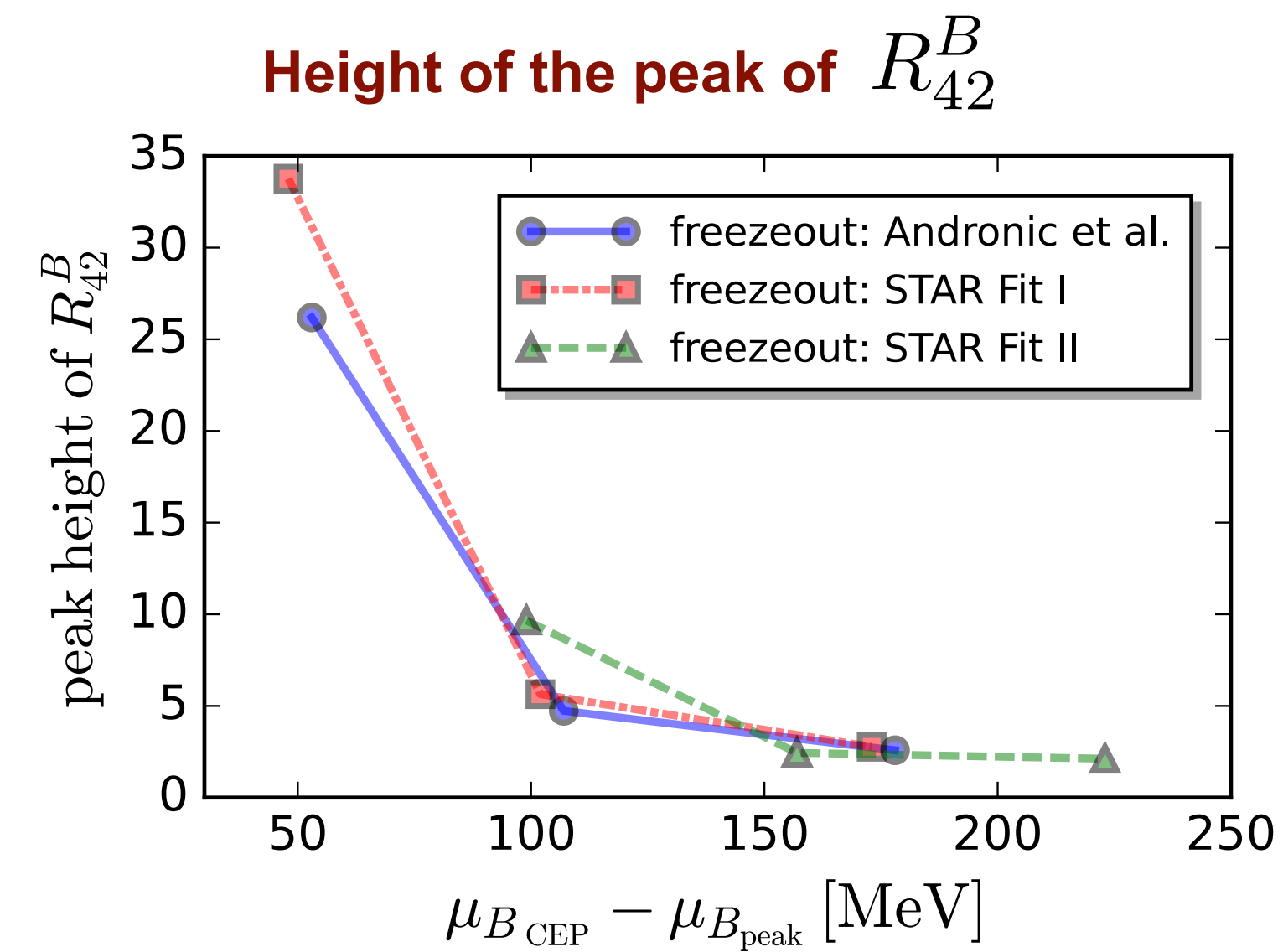
$$(T_{\text{CEP}}, \mu_{B_{\text{CEP}}}) = (94, 704) \text{ MeV}$$



Ripples of the critical point at chemical freeze-out

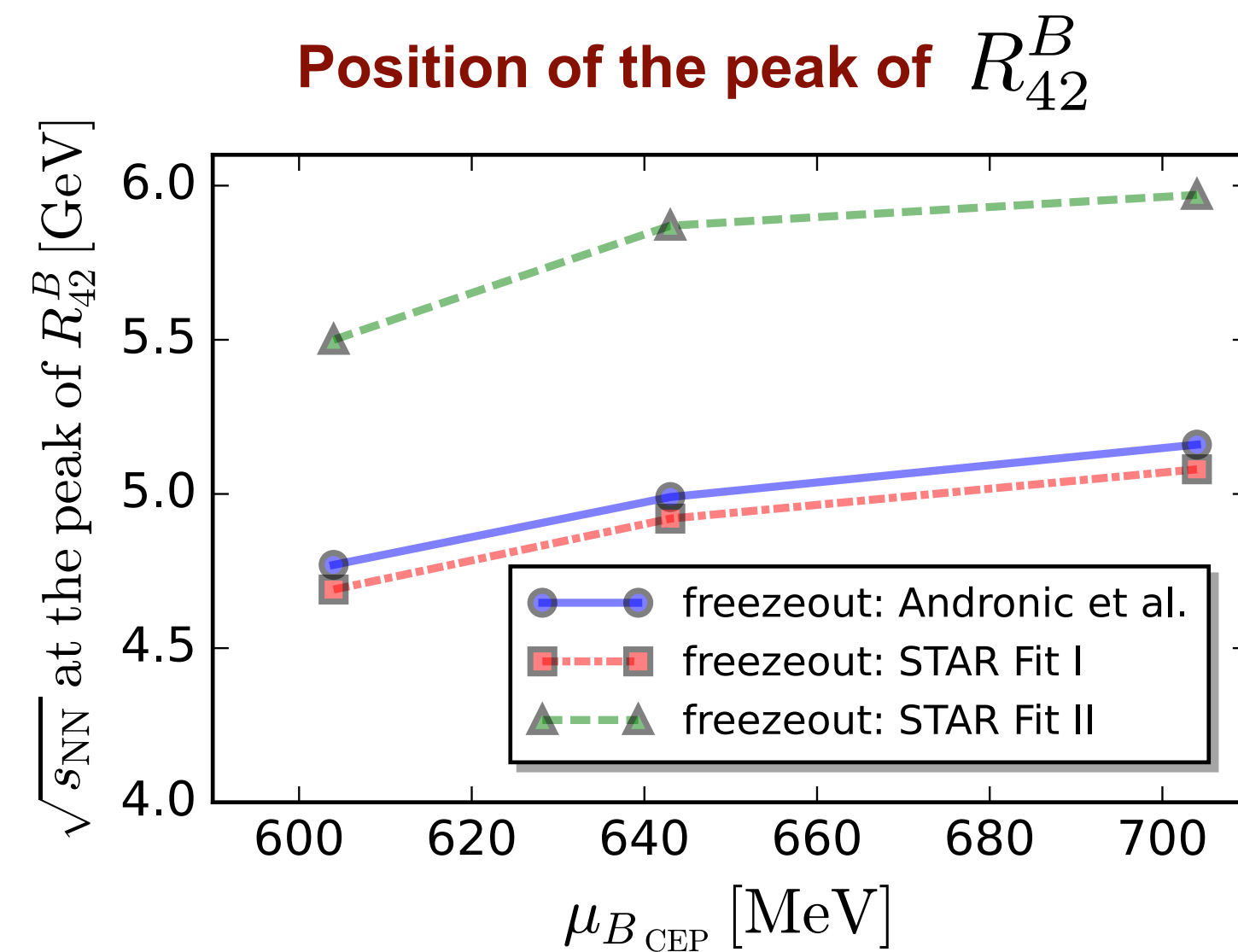
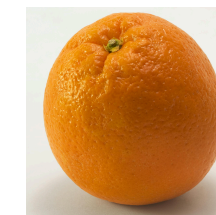


Reconstructing the CEP

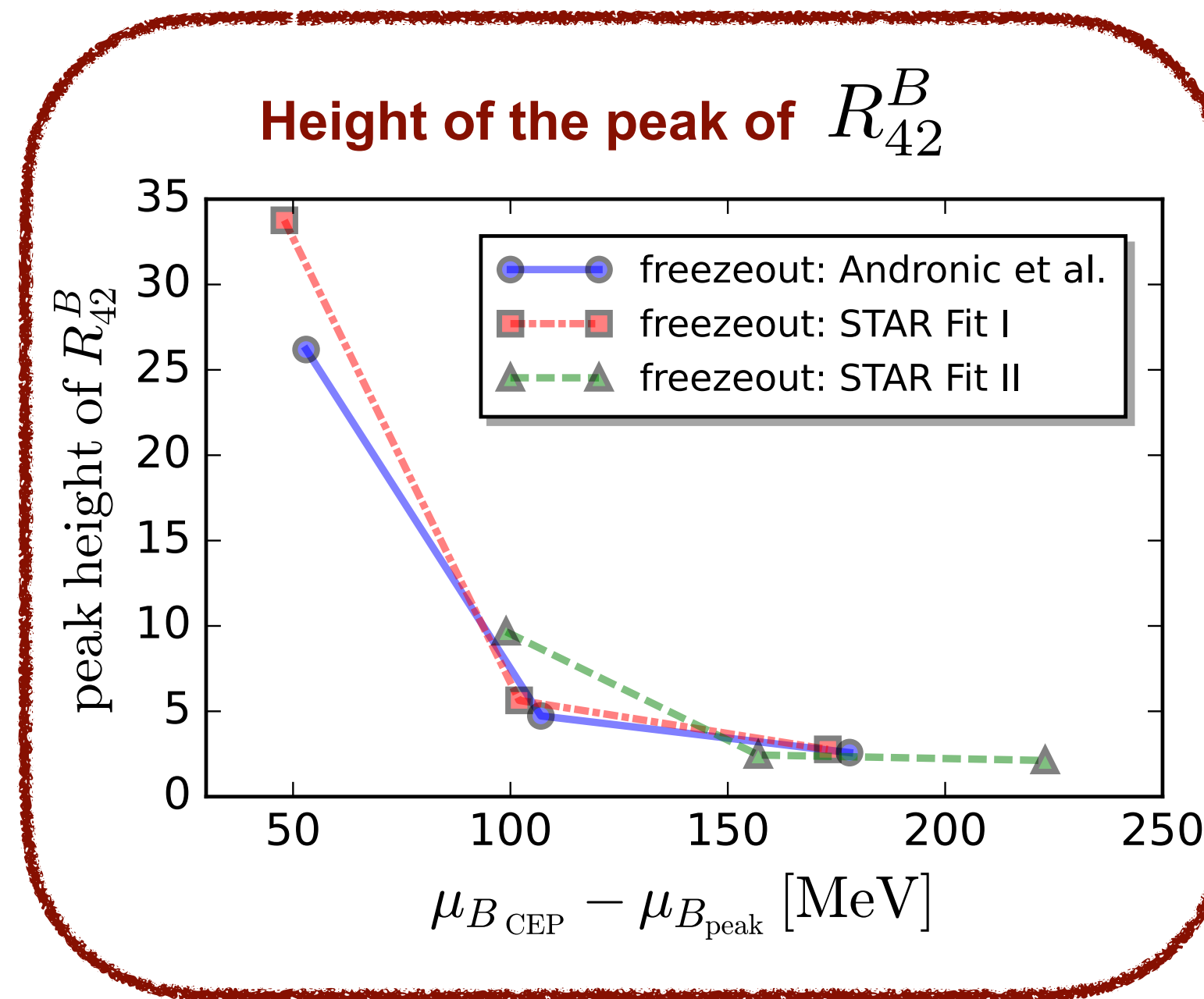




Ripples of the critical point at chemical freeze-out

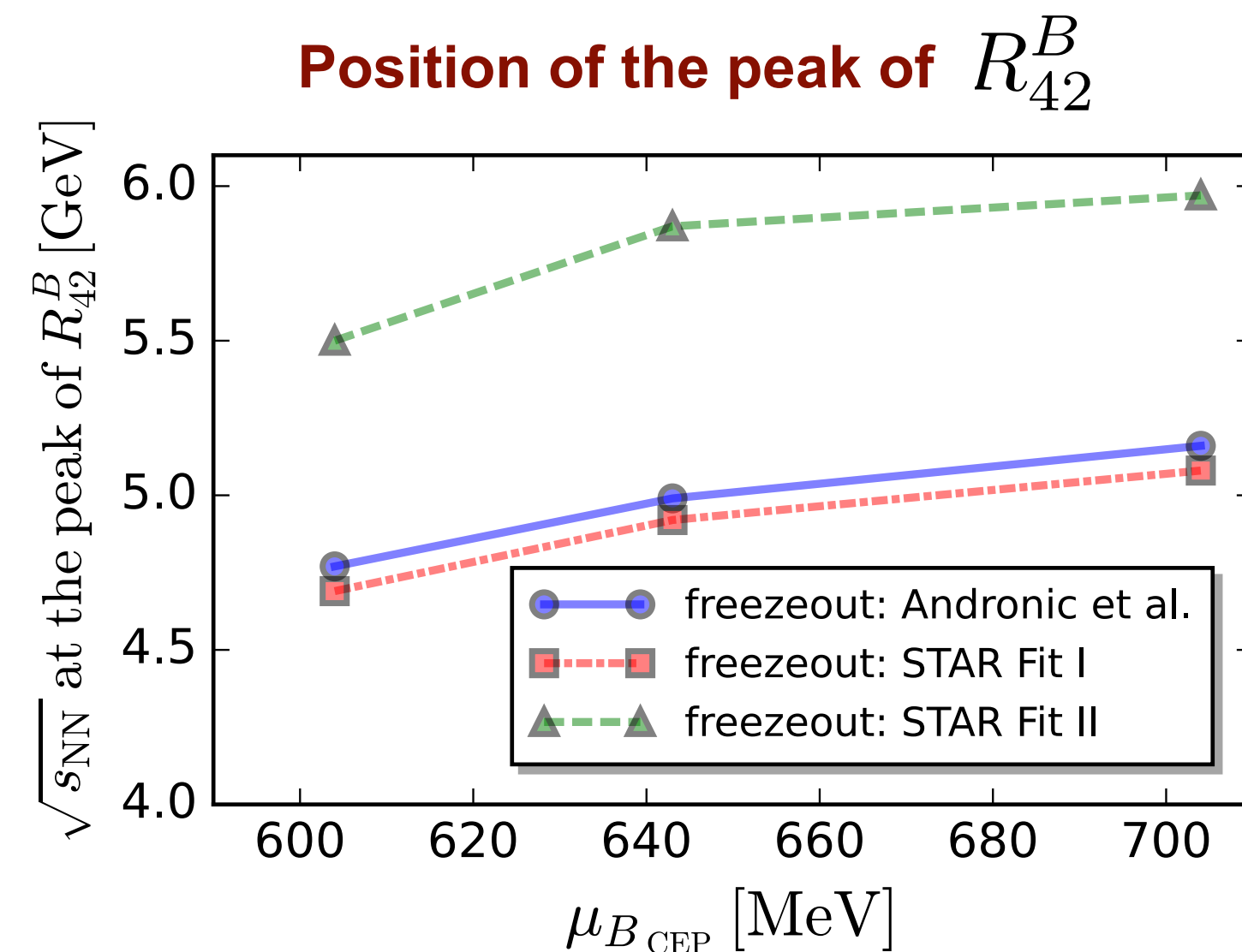
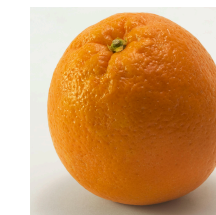


Reconstructing the CEP

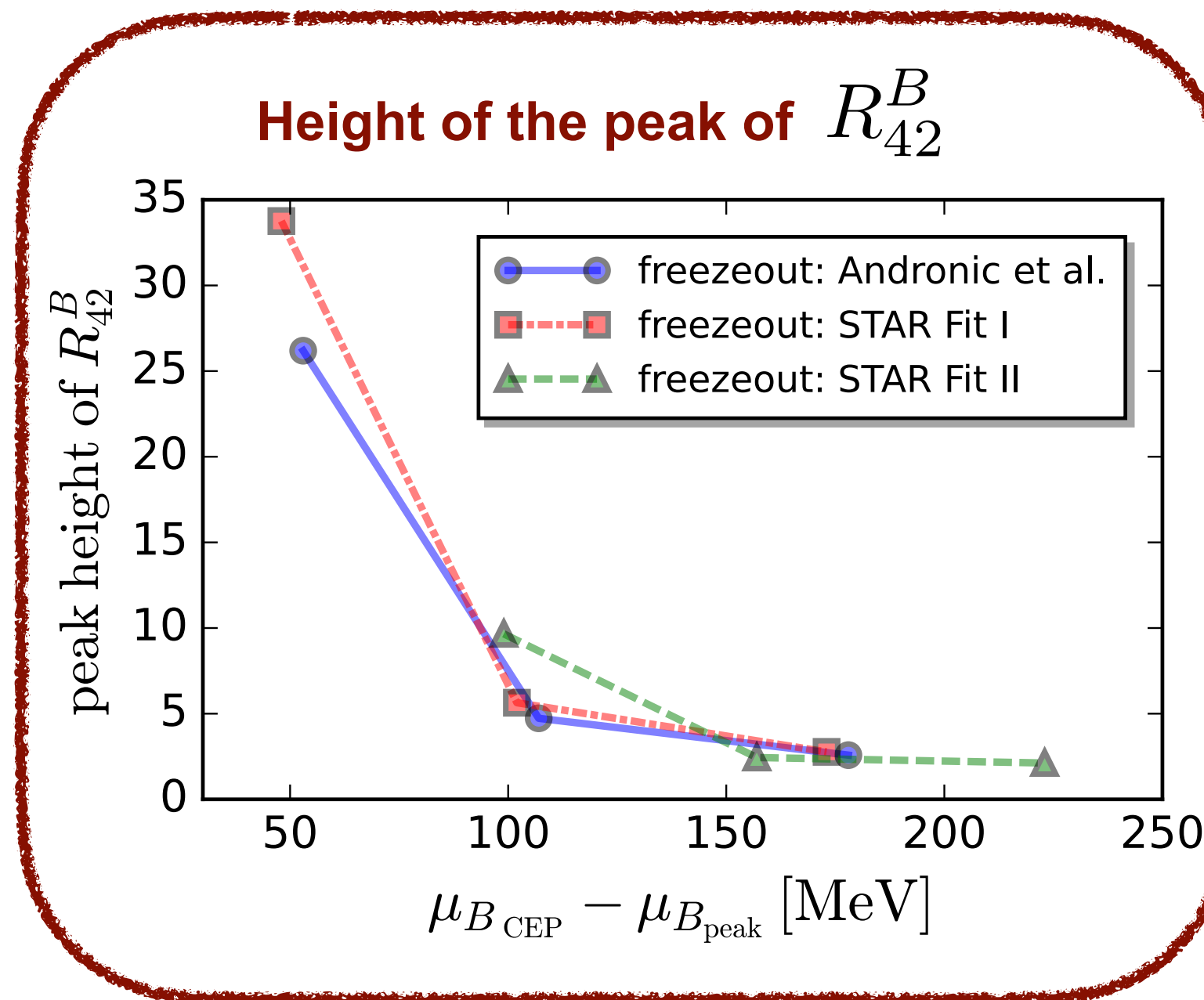




Ripples of the critical point at chemical freeze-out



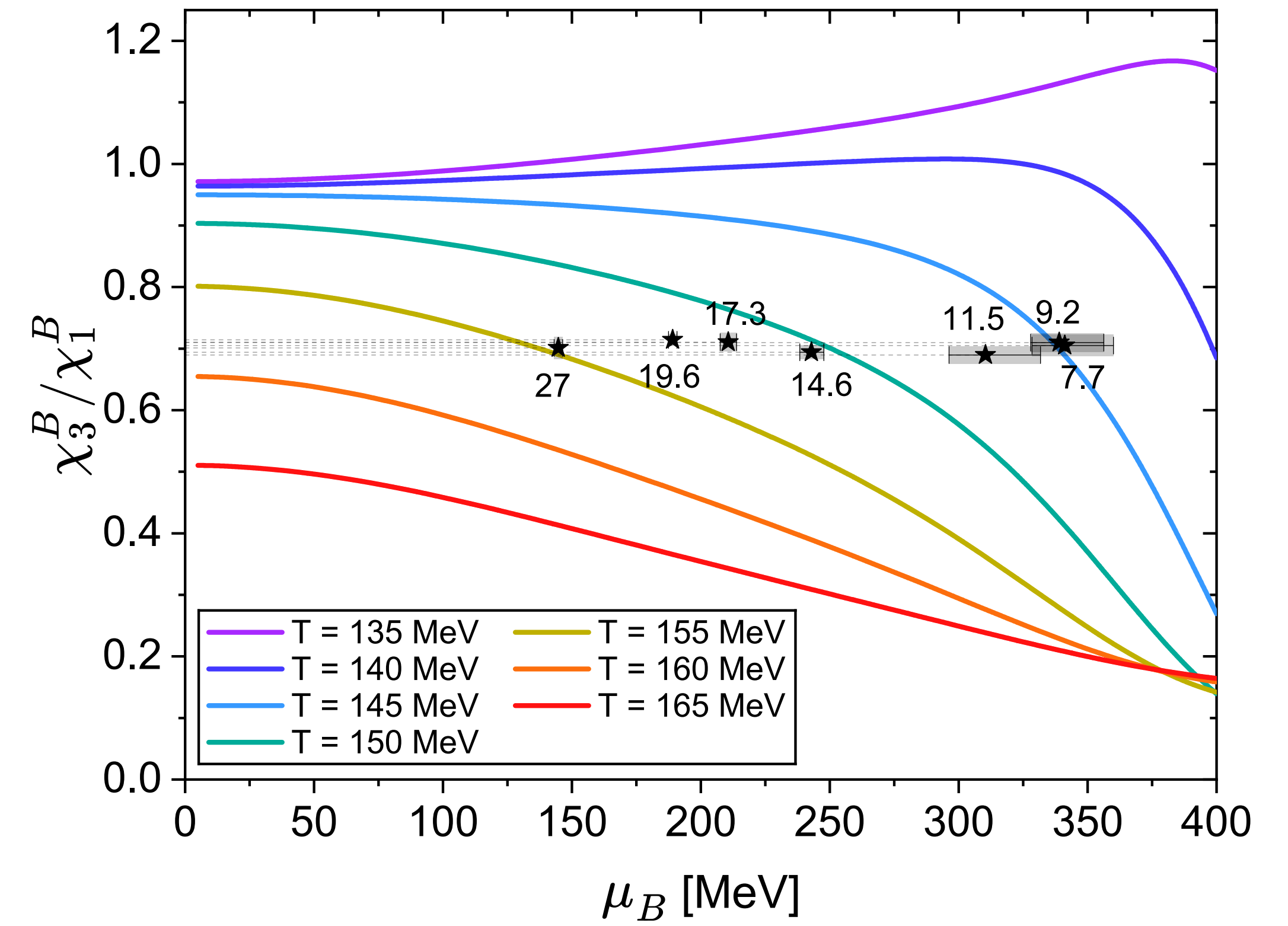
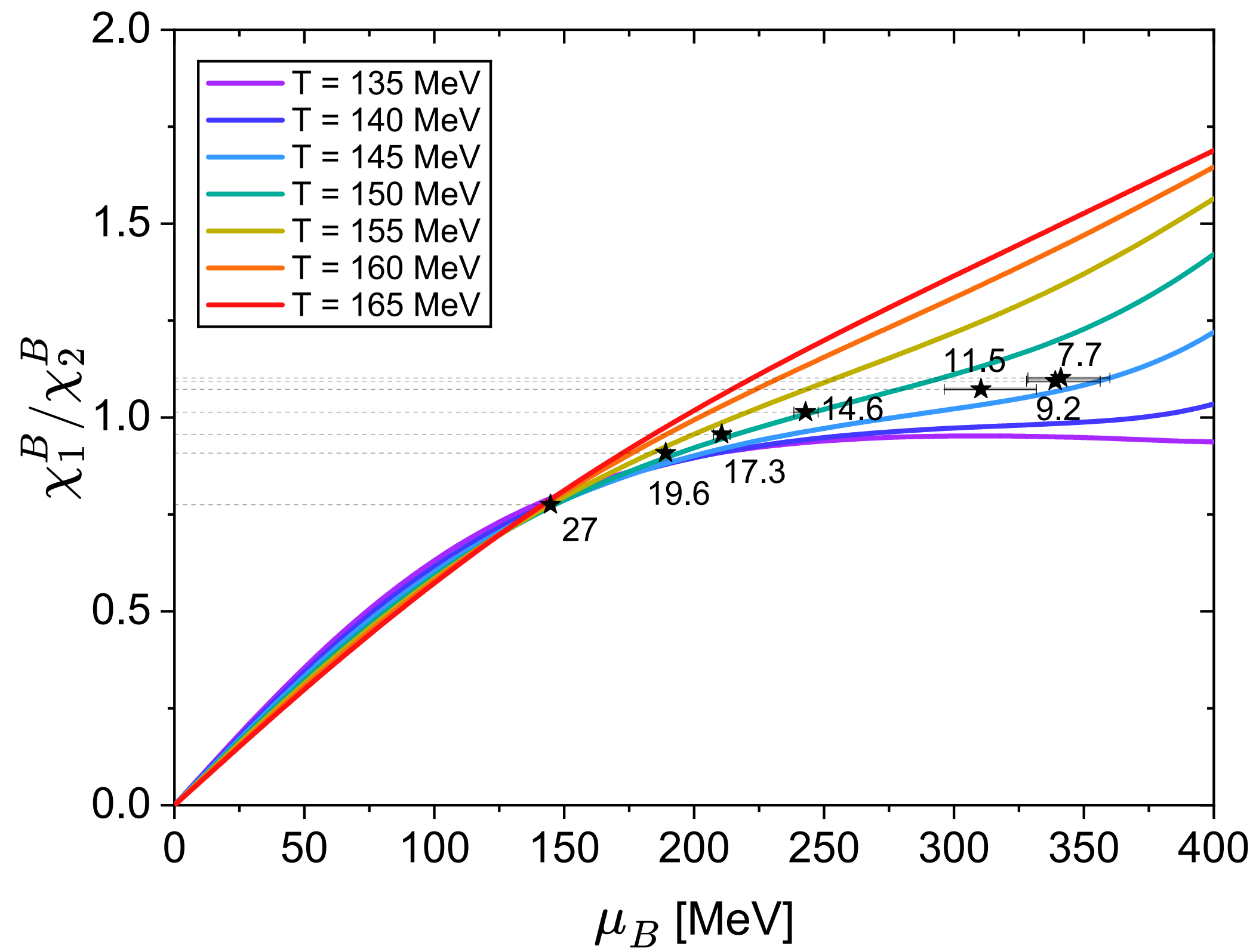
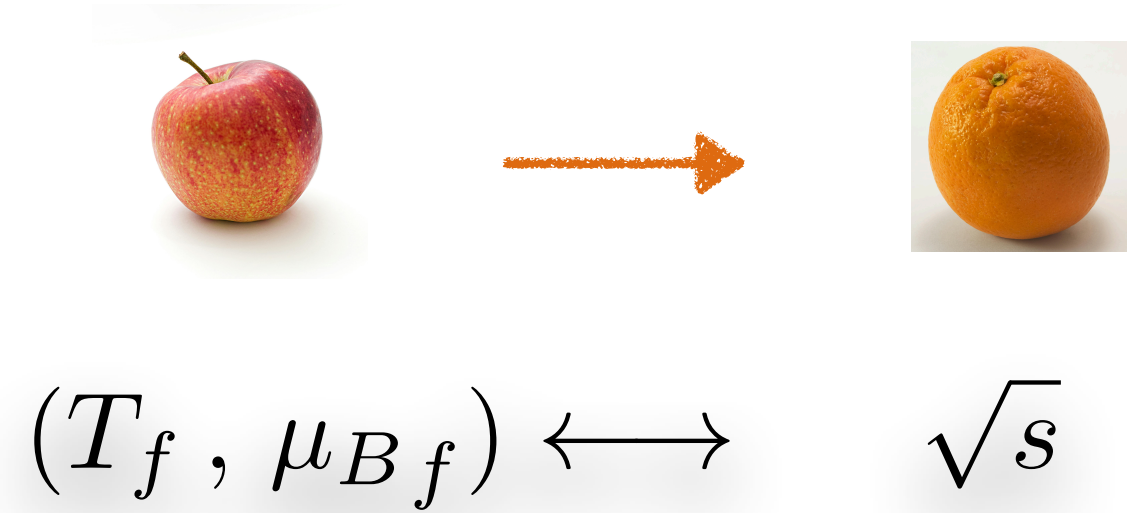
Reconstructing the CEP



Unfolding the high density regime with new phases & physics

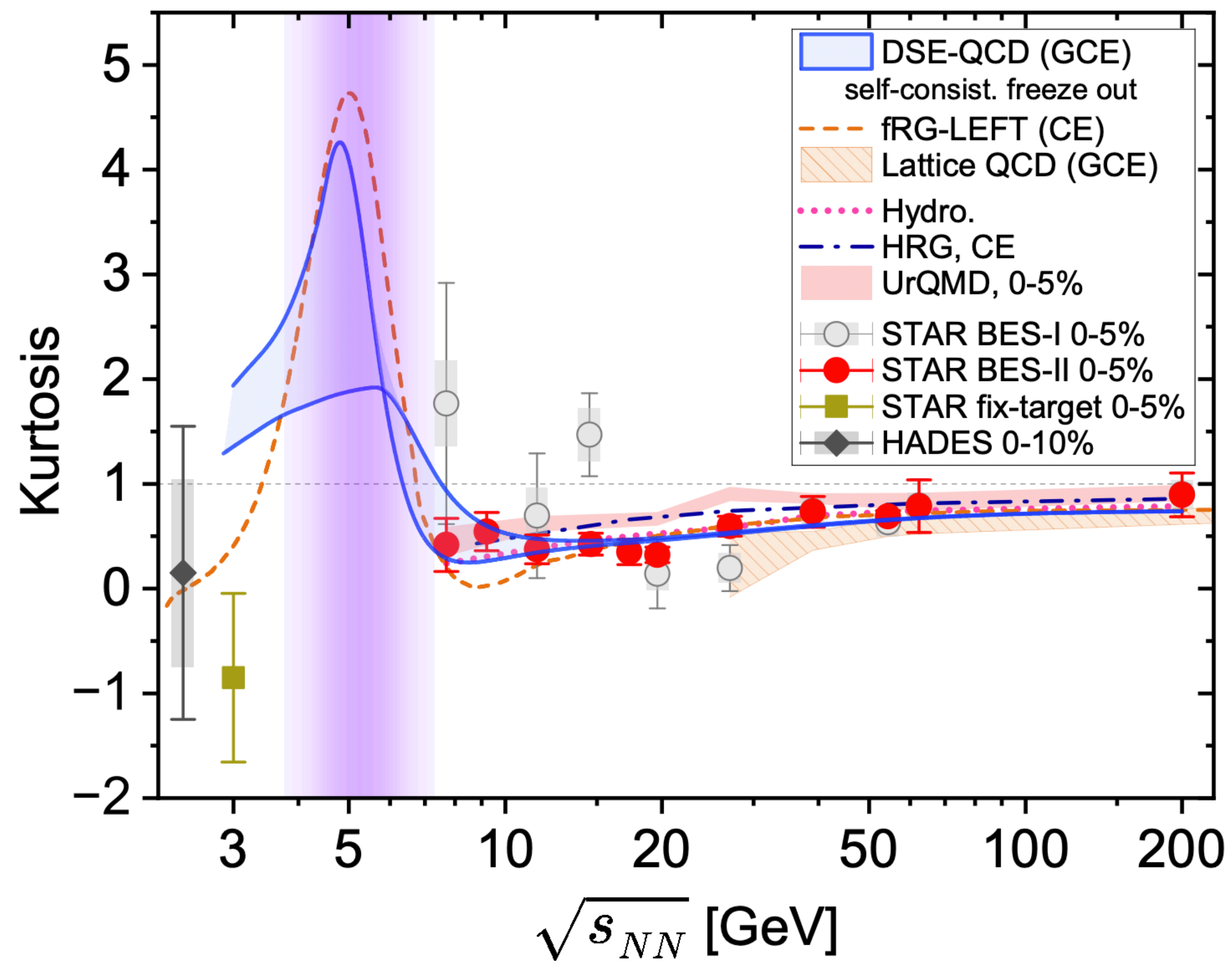
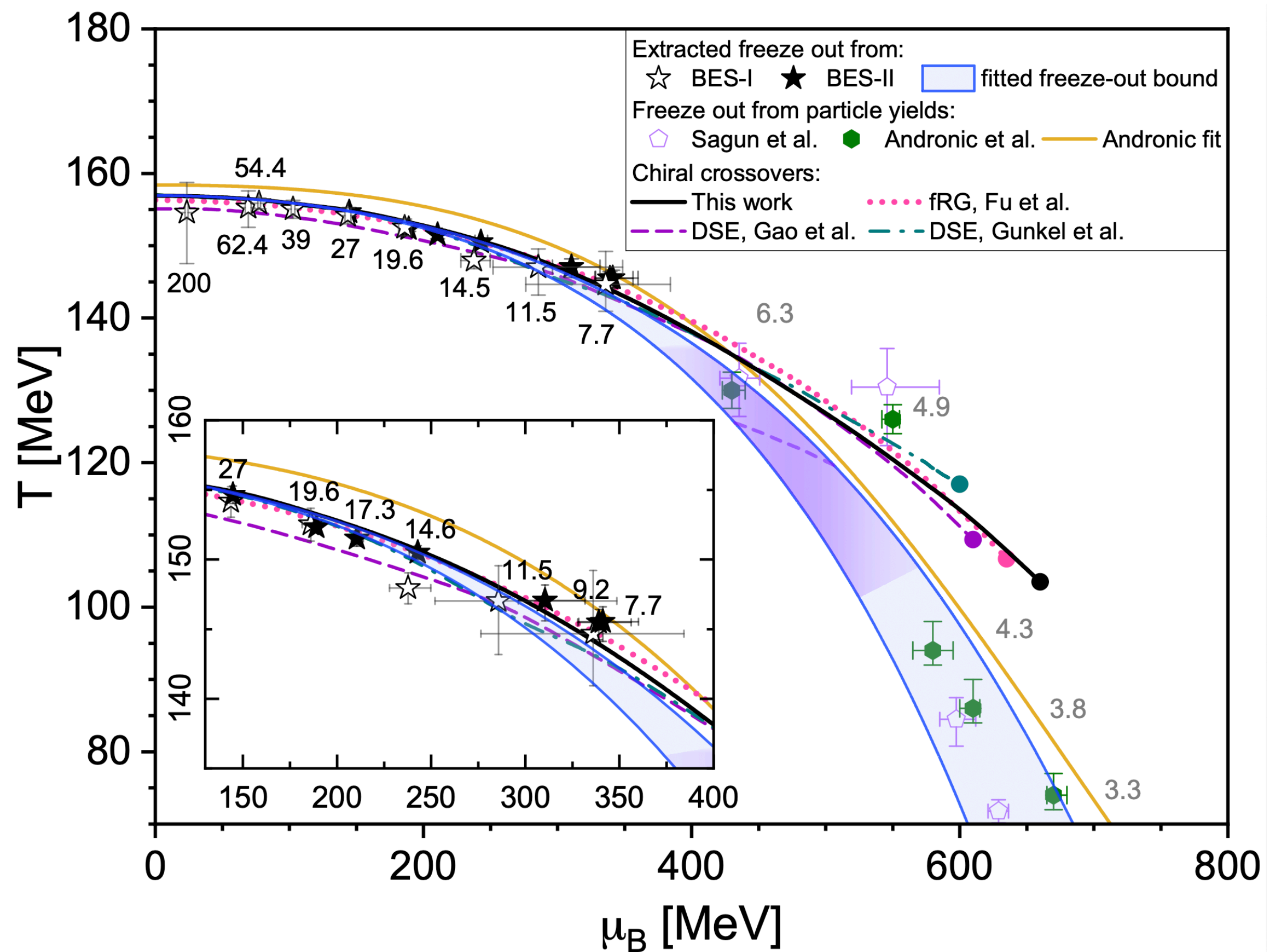
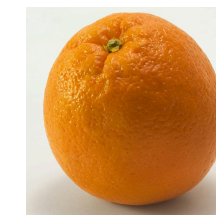
Great opportunity for a combined high precision analysis of high density QCD (Exp. data + lattice QCD + functional QCD)

Functional Freeze-out



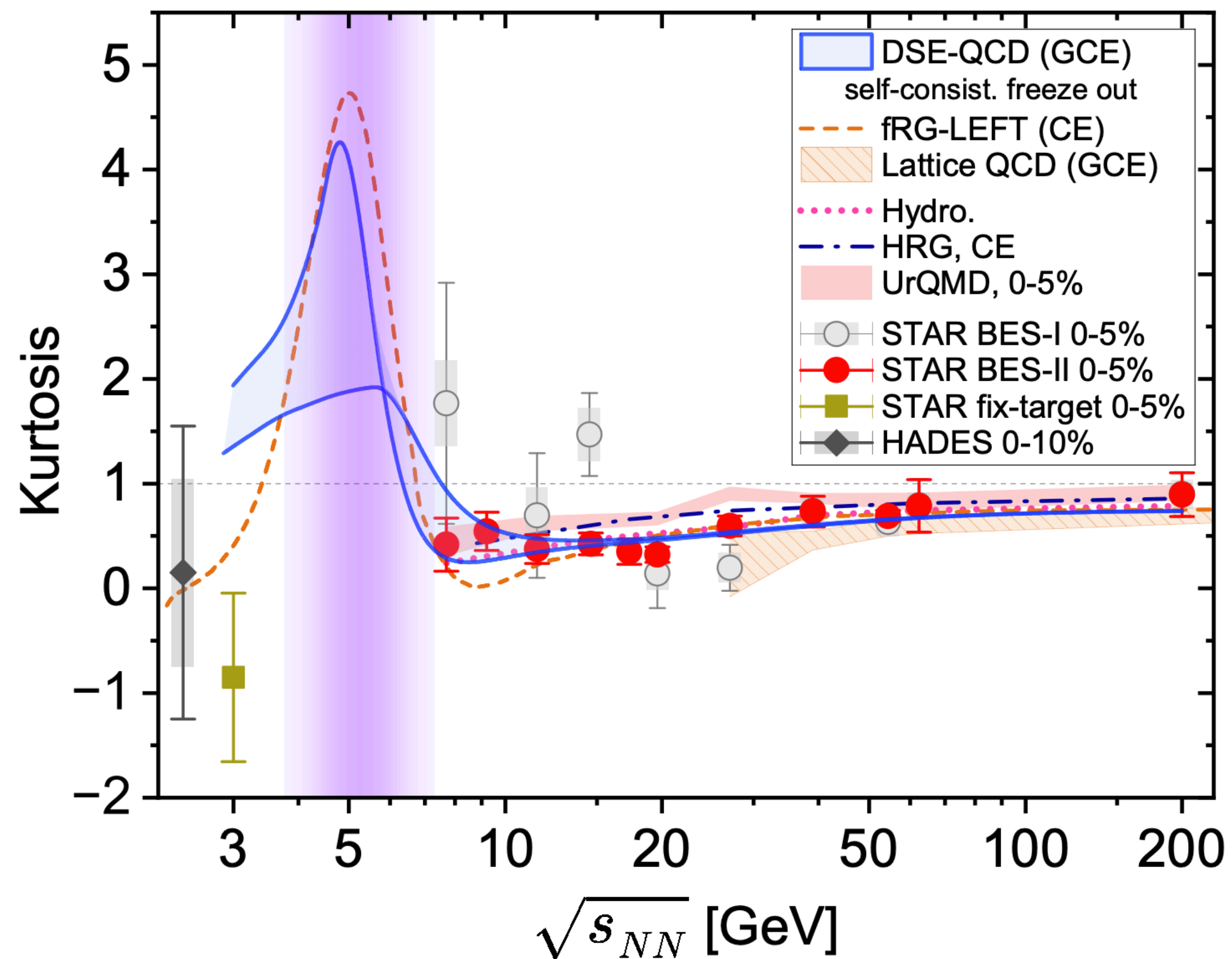
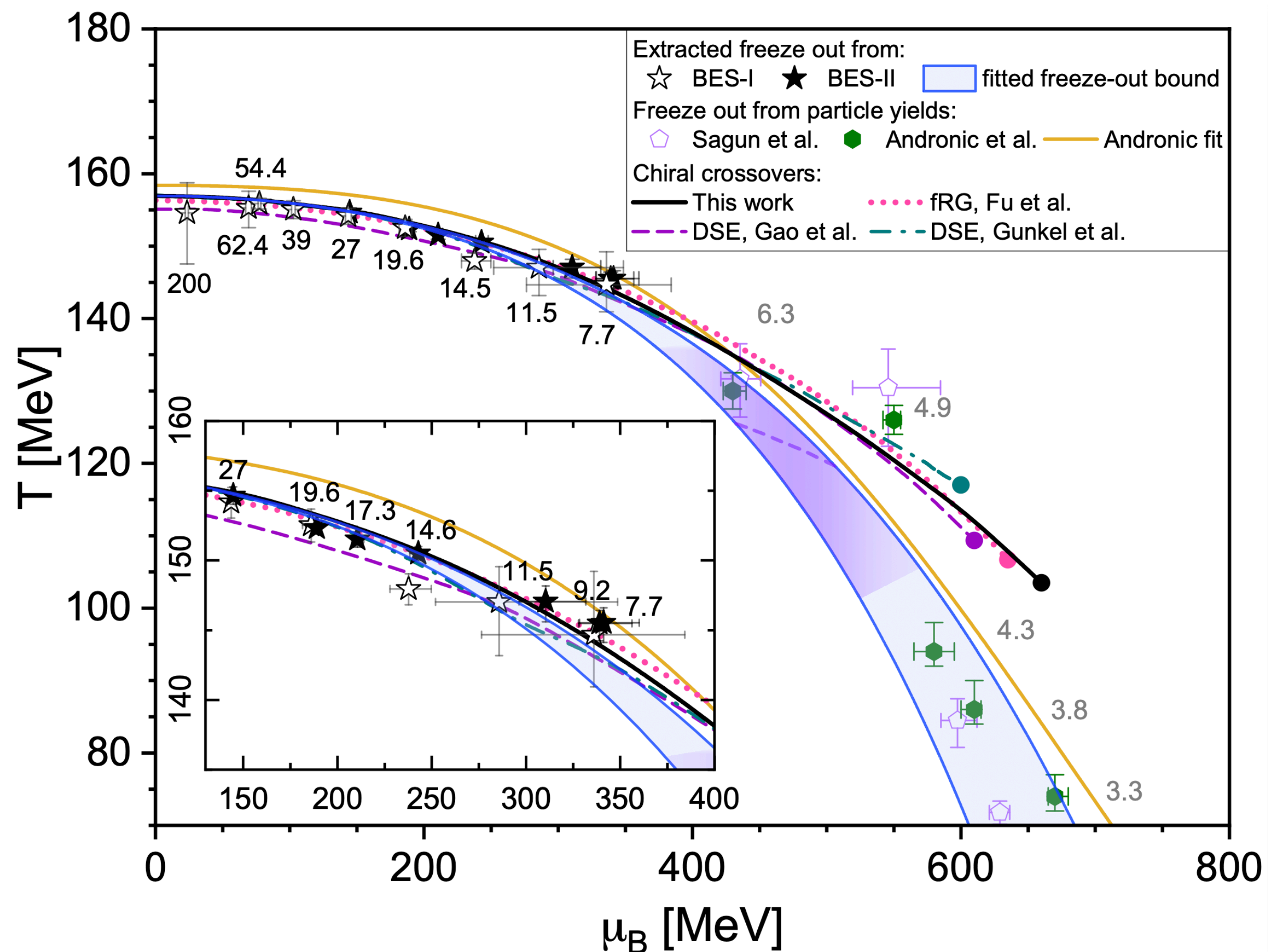
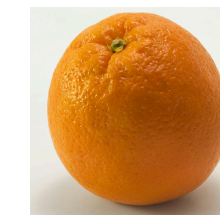


Ripples of the critical point at chemical freeze-out





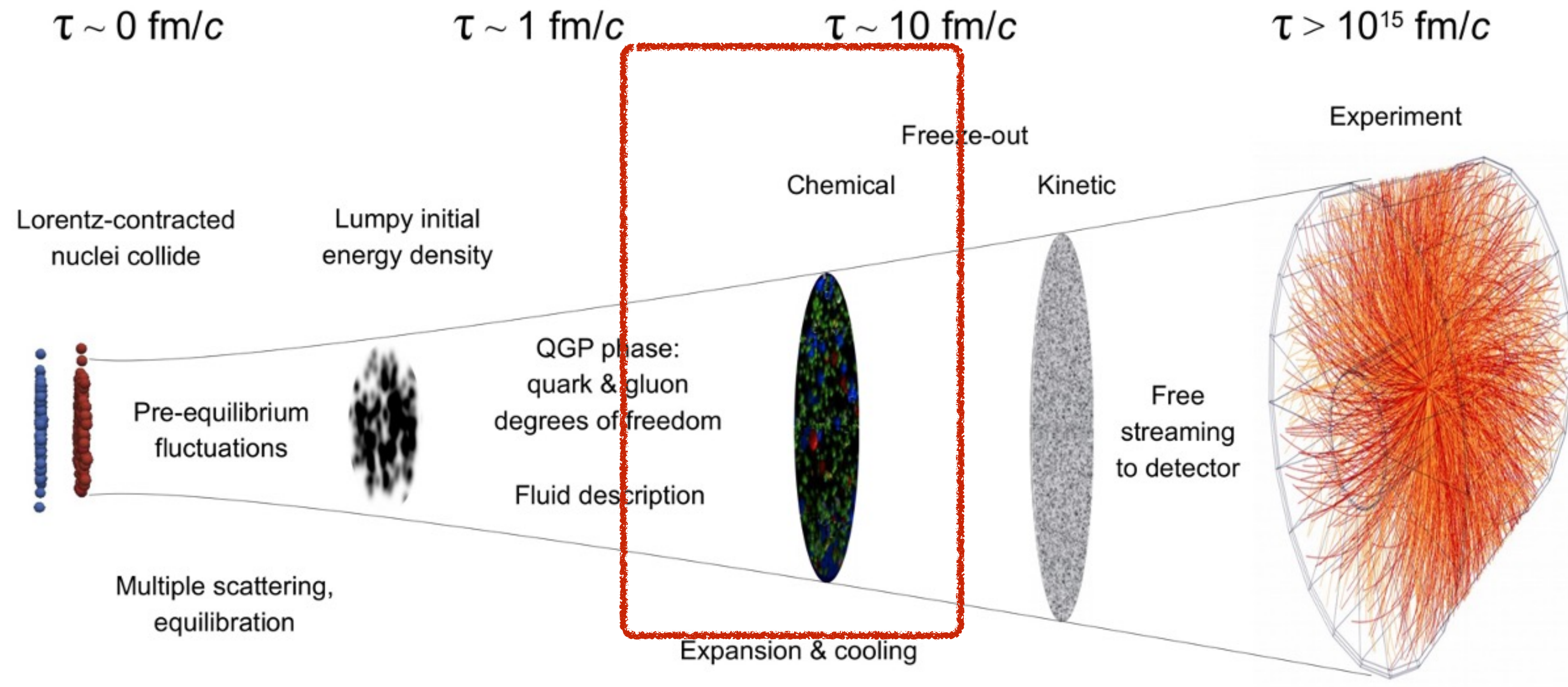
Ripples of the critical point at chemical freeze-out



Demand: build in dynamics

EMMI workshop: Collective phenomena and the EoS of dense baryonic matter

Unfolding strongly correlated QCD with heavy ion collisions (extreme QCD)



How to unfold: inverse problem!

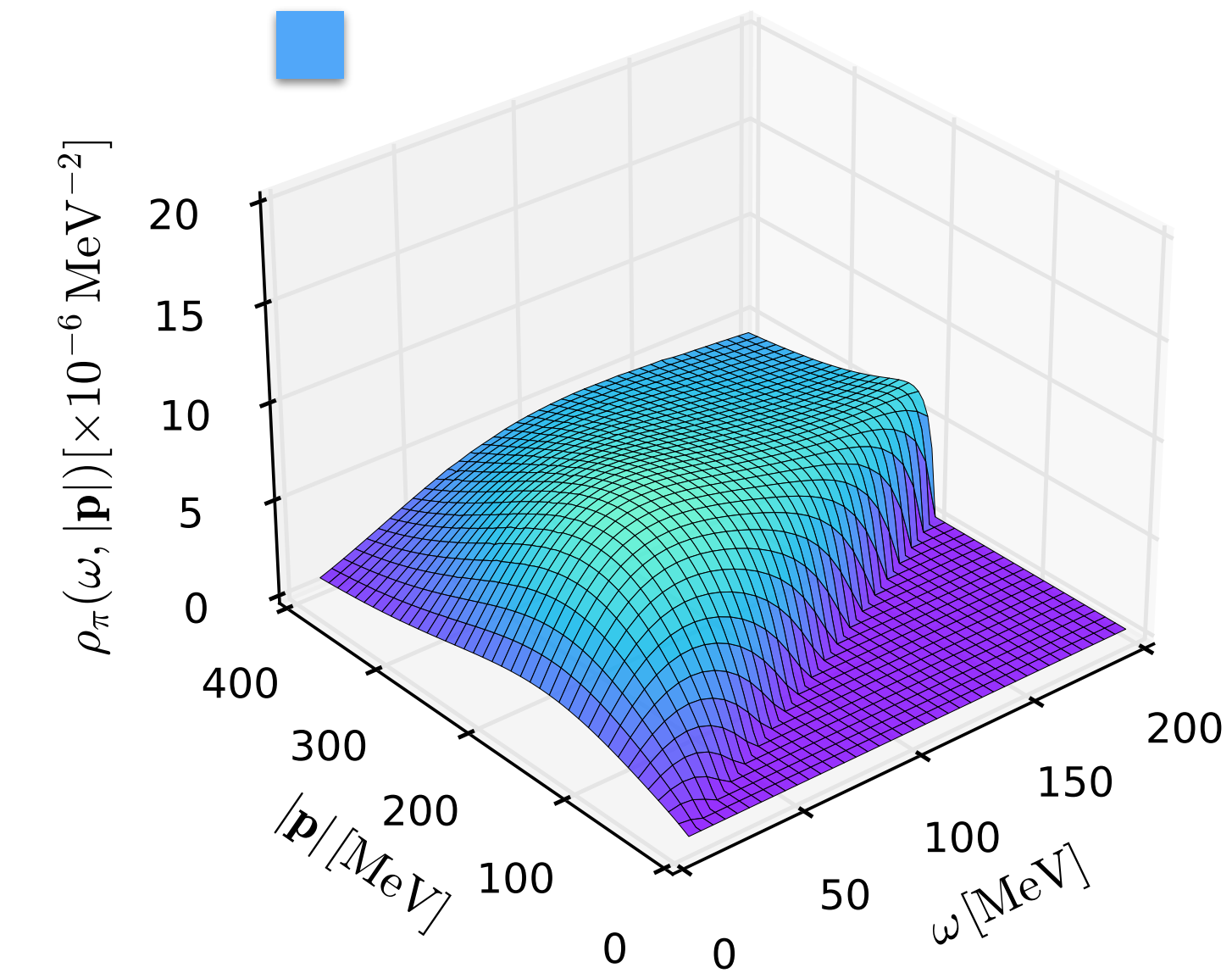
Demand: unfolding HICs

The QCD moat and inhomogeneous phases

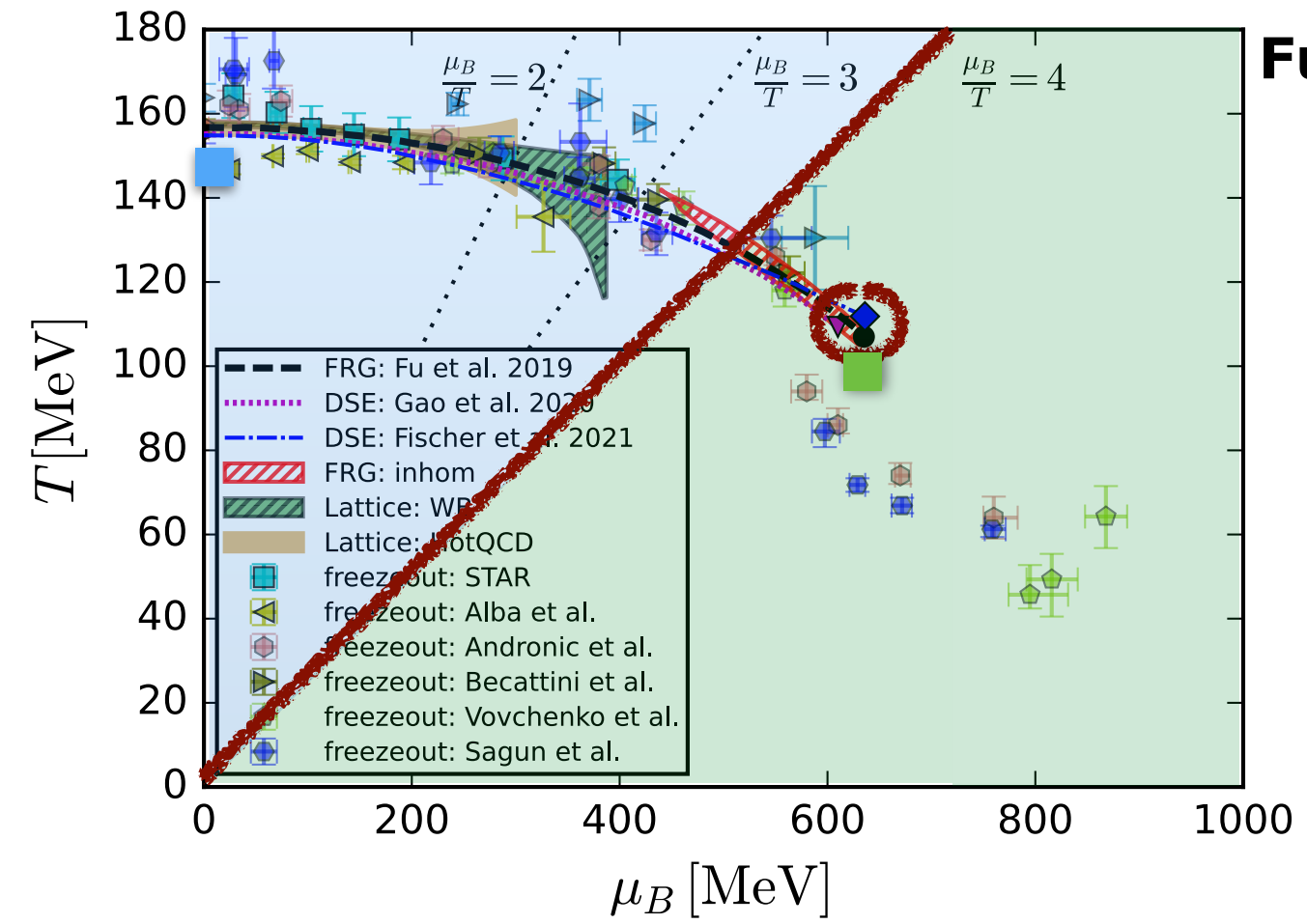
Moat regime

Pisarski, Rennecke, PRL 127 (2021) 152302

$T=160 \text{ MeV}$ & $\mu_B=0 \text{ MeV}$



Prediction



Fu, JMP, Rennecke, PRD 101 (2020) 054032

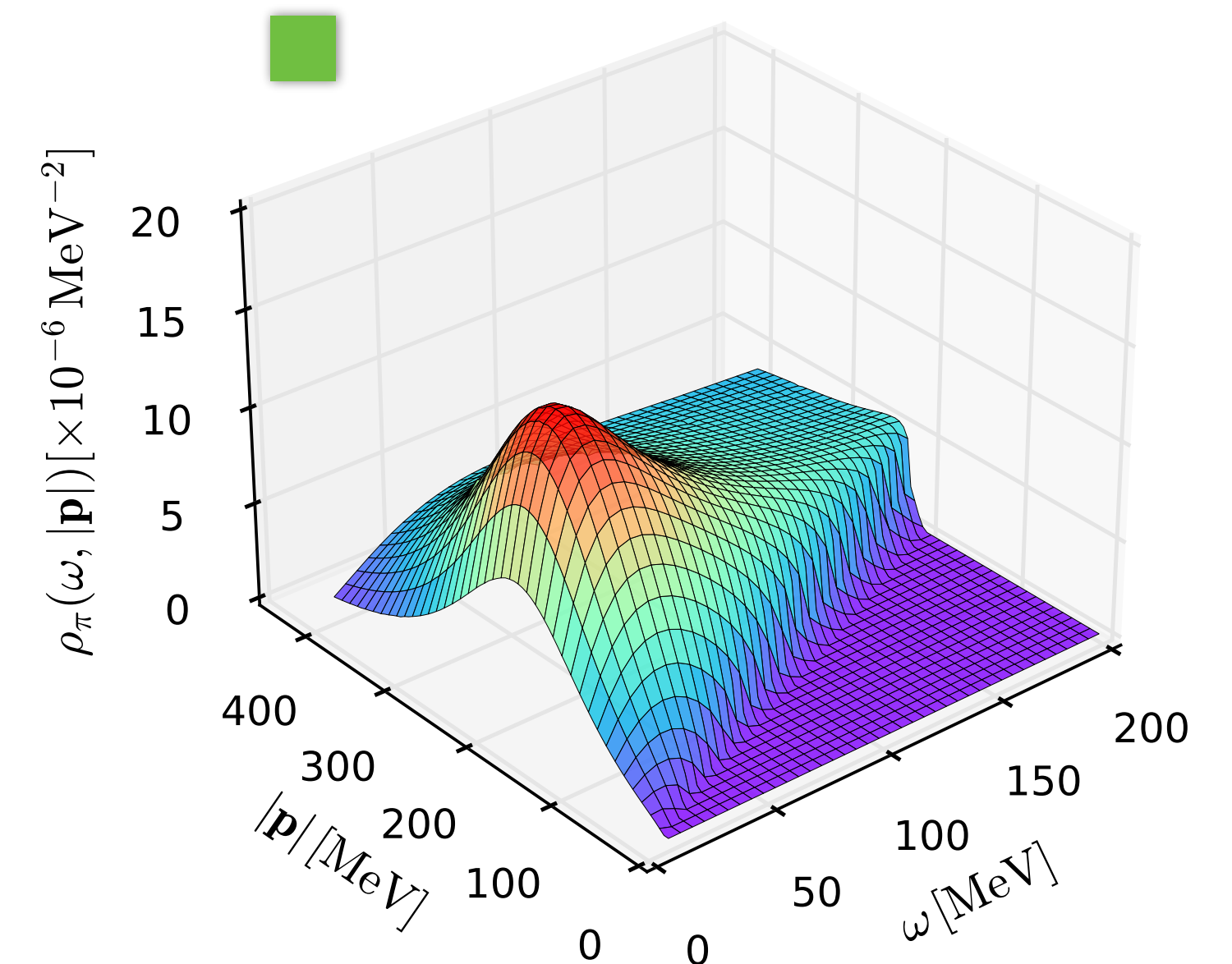
Regime of quantitative reliability
of
current best truncation

Estimate

Moat regime was not captured quantitatively

Pion spectral functions

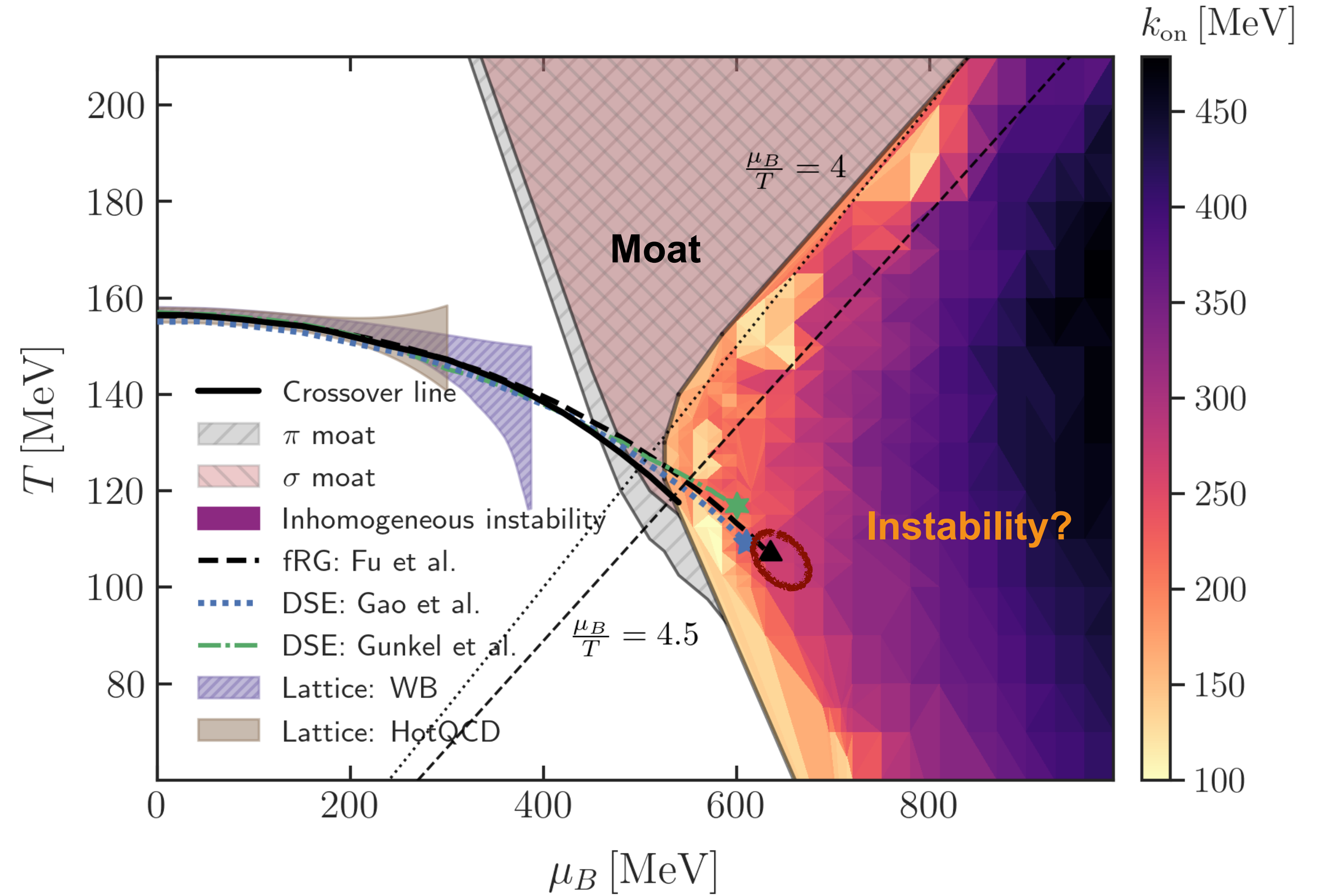
$T=114 \text{ MeV}$ & $\mu_B=630 \text{ MeV}$



Fu, JMP, Pisarski, Rennecke, Wen, Yin, PRD 111 (2025) 094026

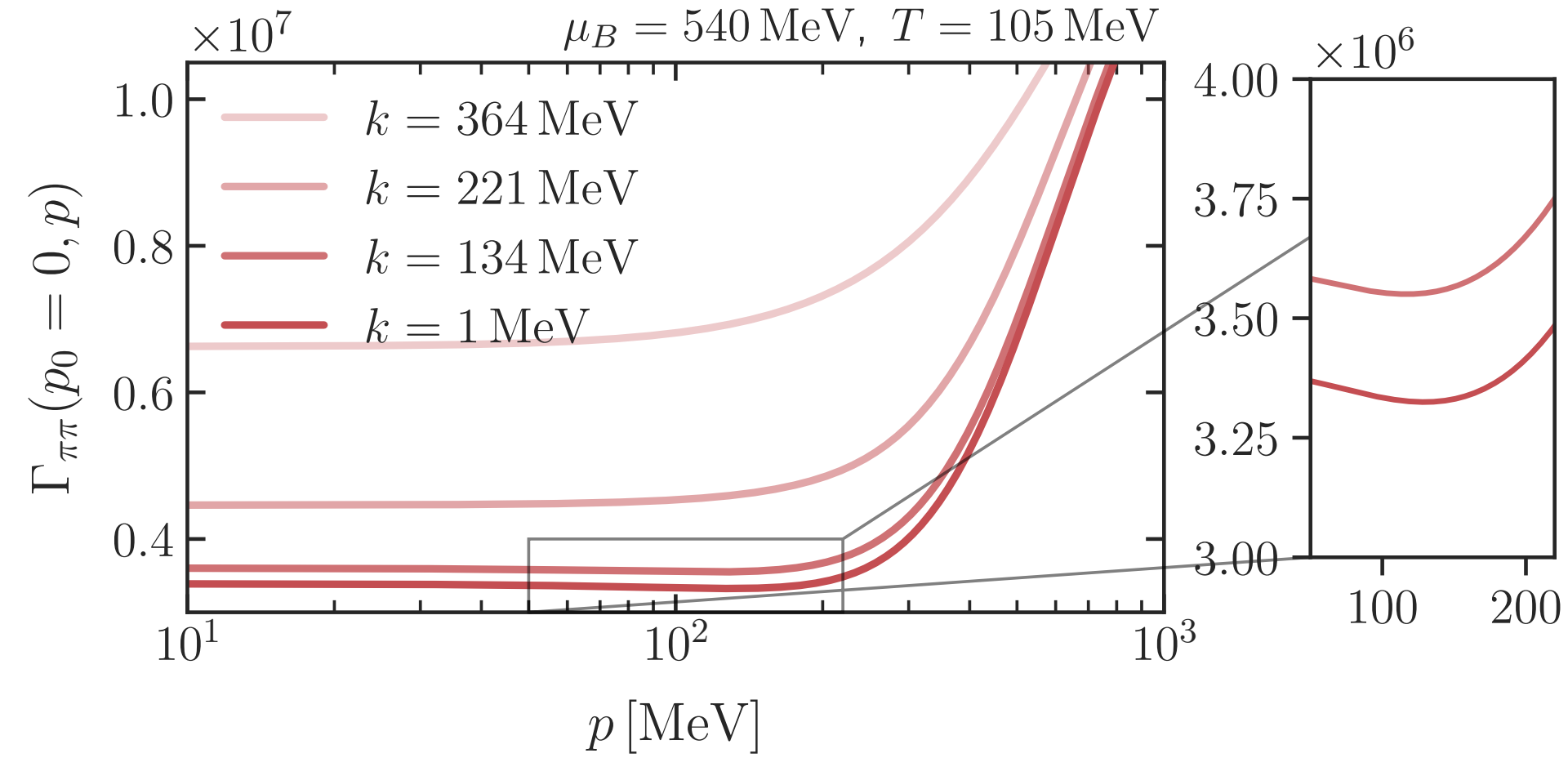
The QCD Moat reloaded

 Estimated location of CEP

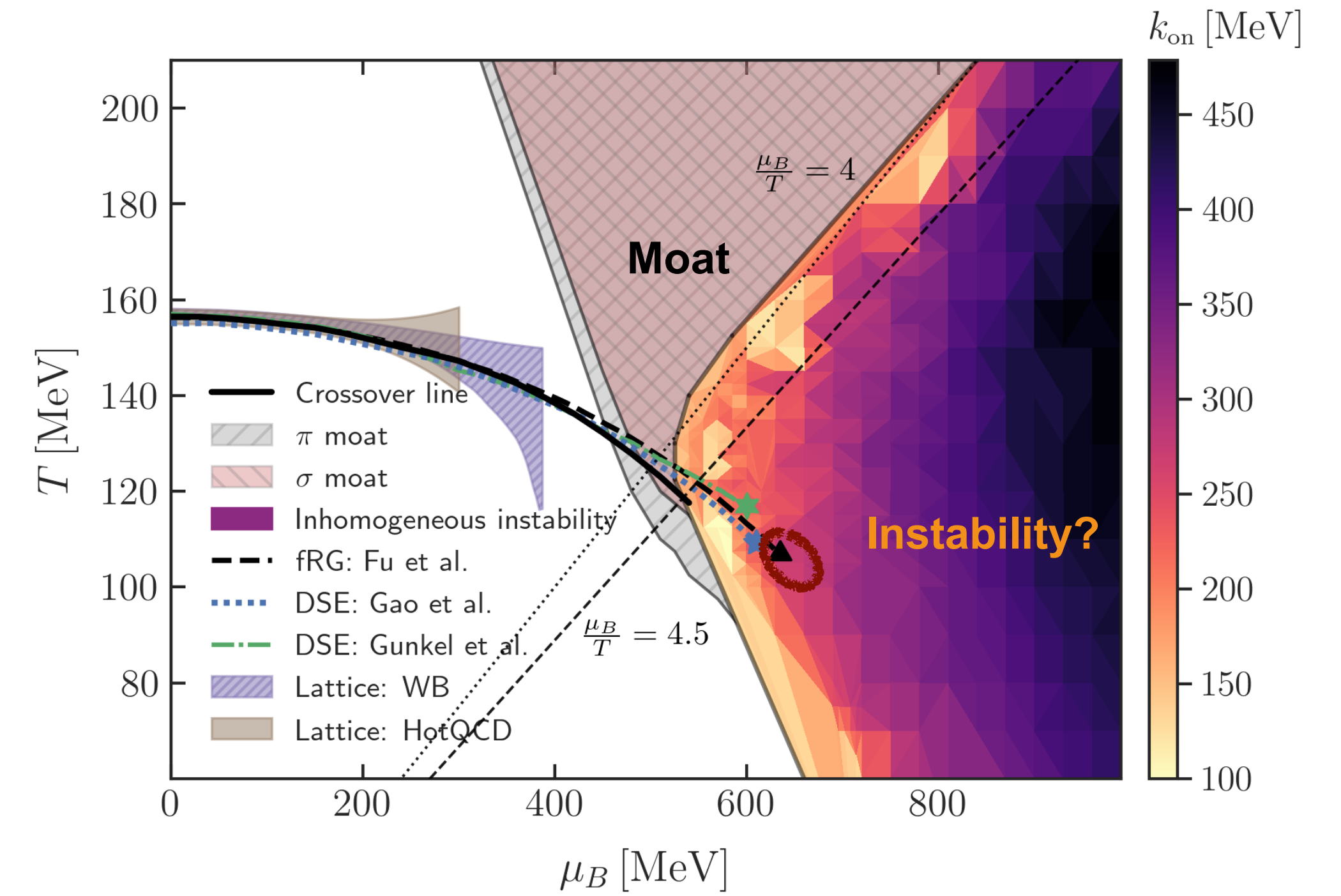
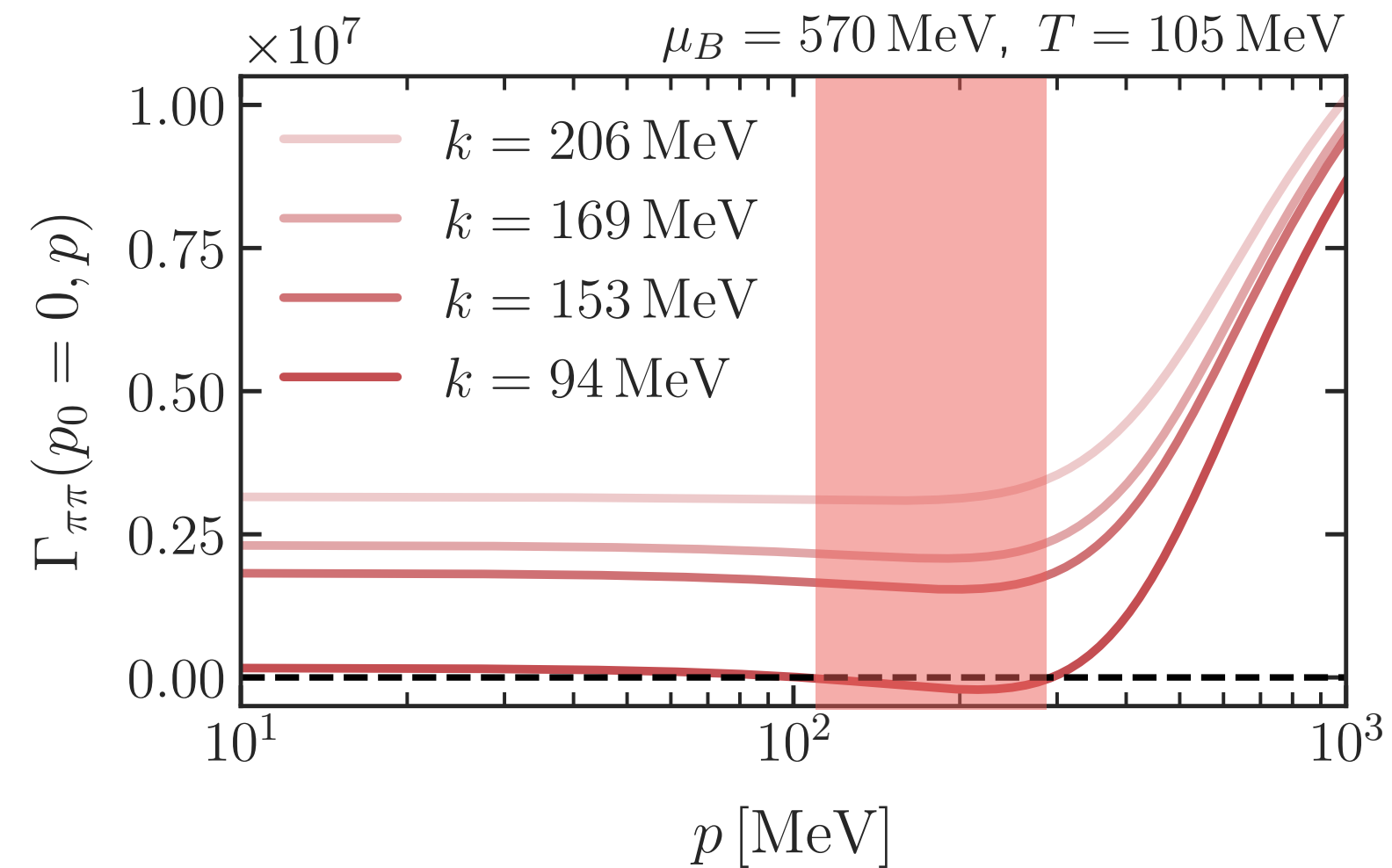


The QCD Moat reloaded

Moat



Instability



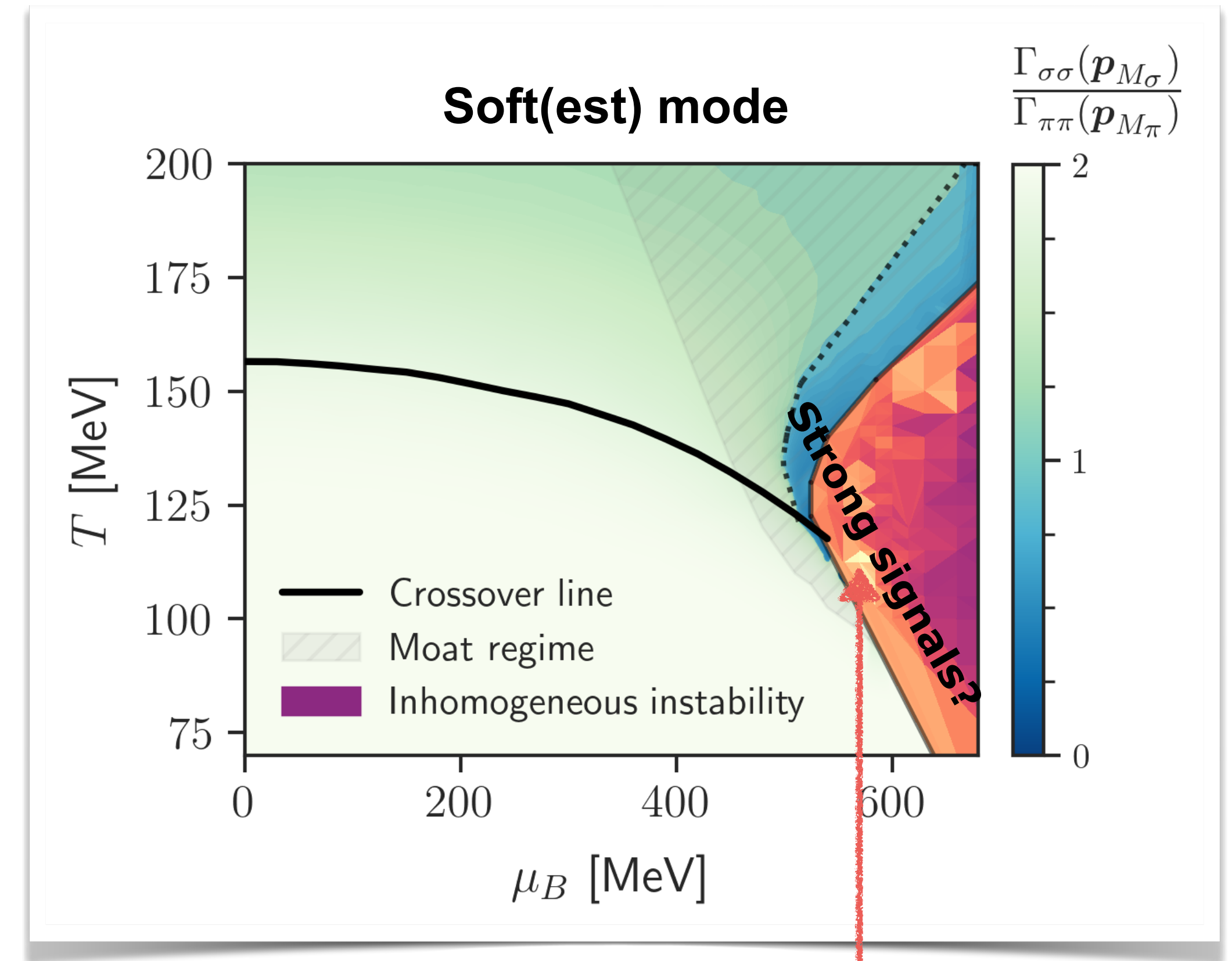
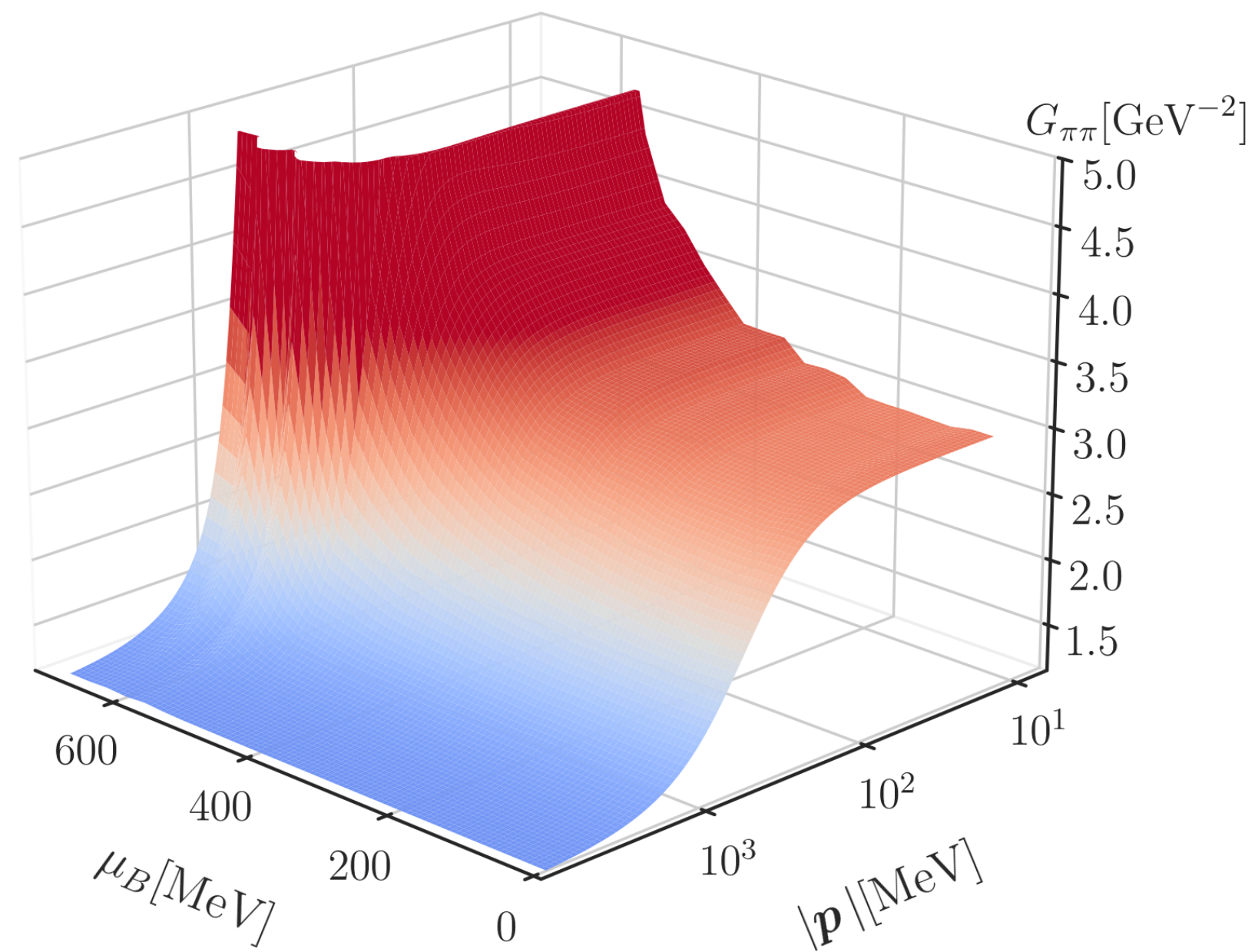
No full computation deep inside the instability regime

Estimated location of CEP

The QCD Moat reloaded

Soft mode

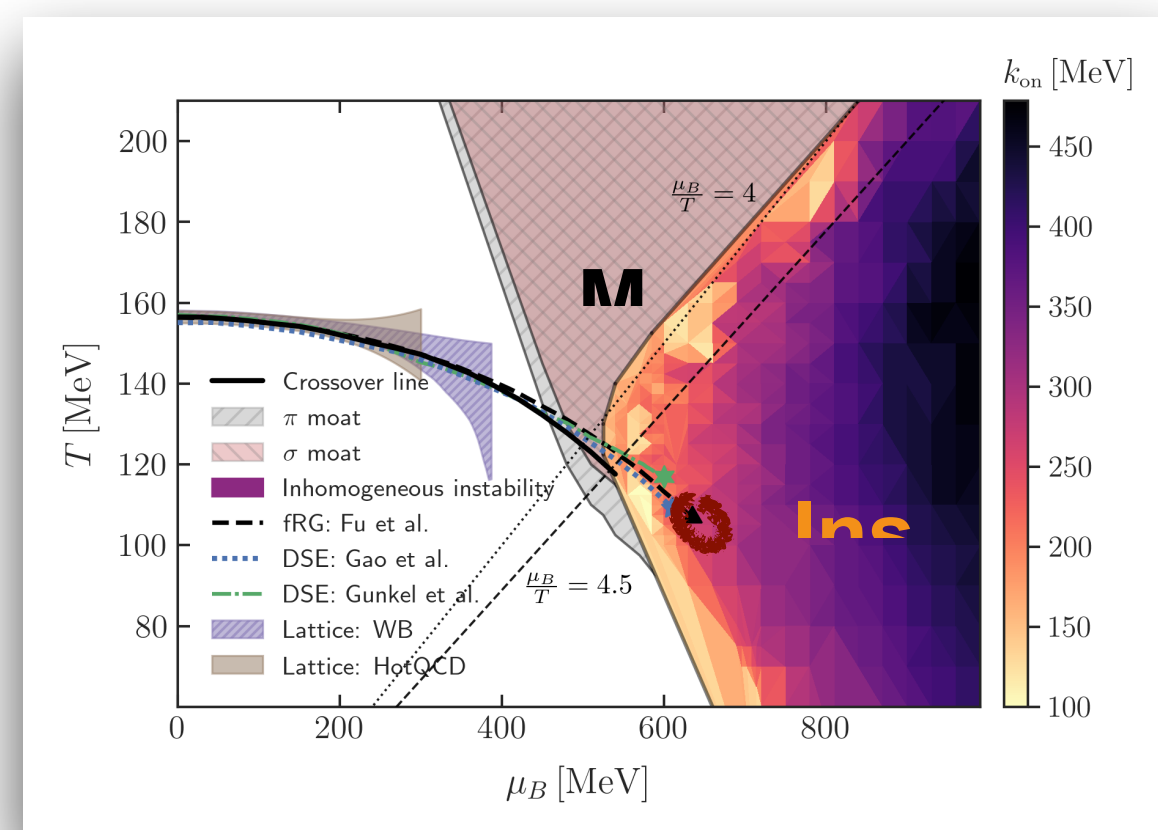
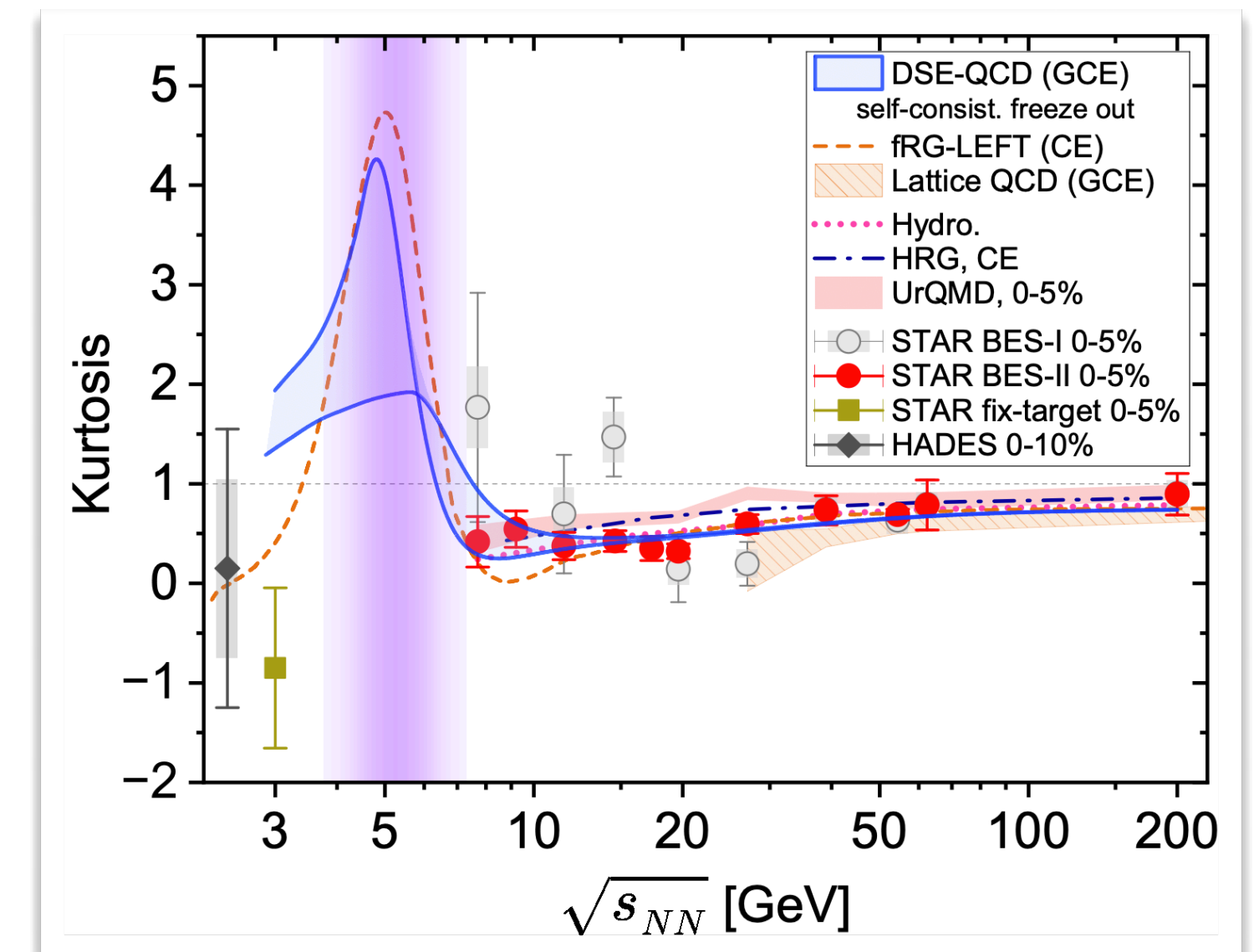
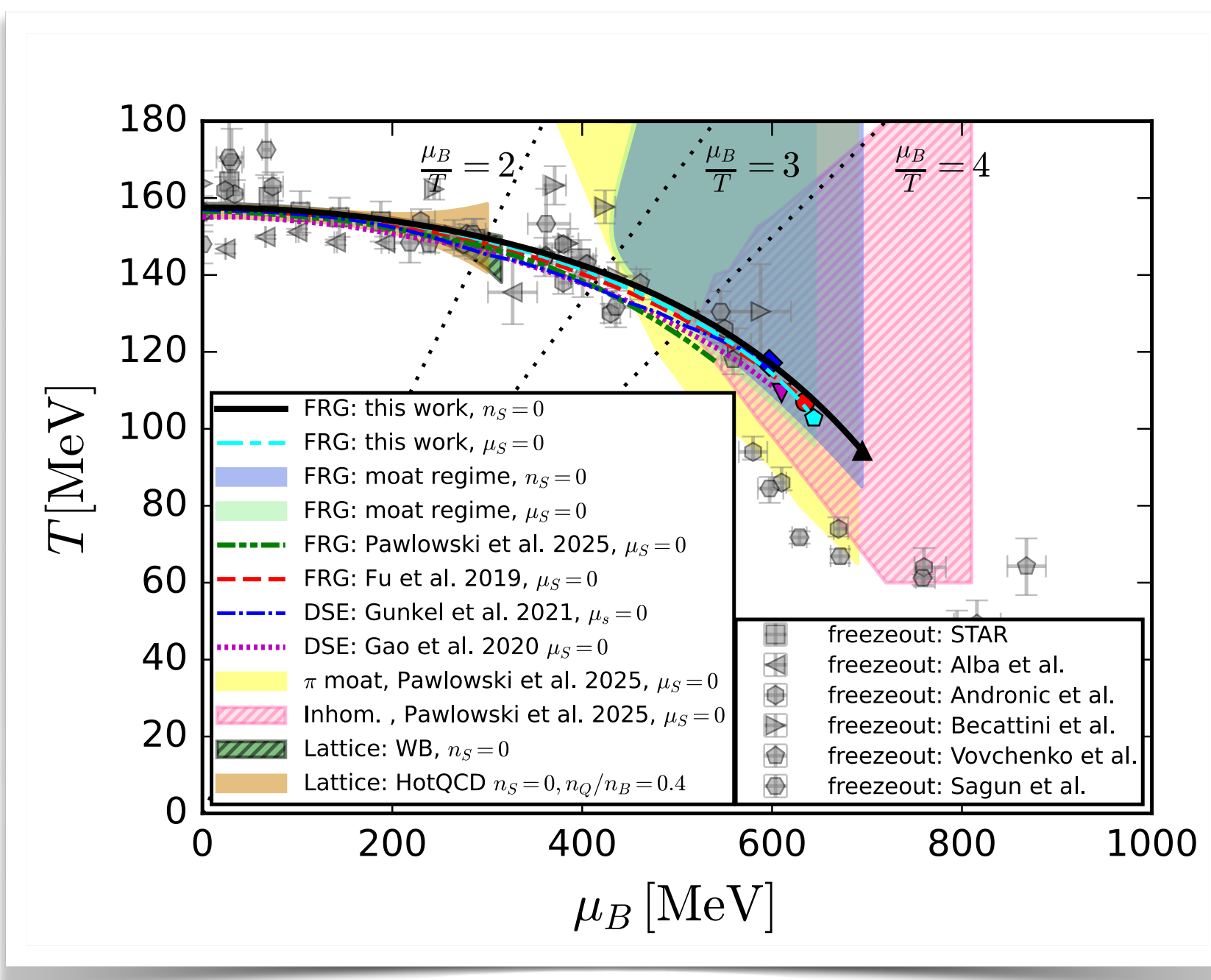
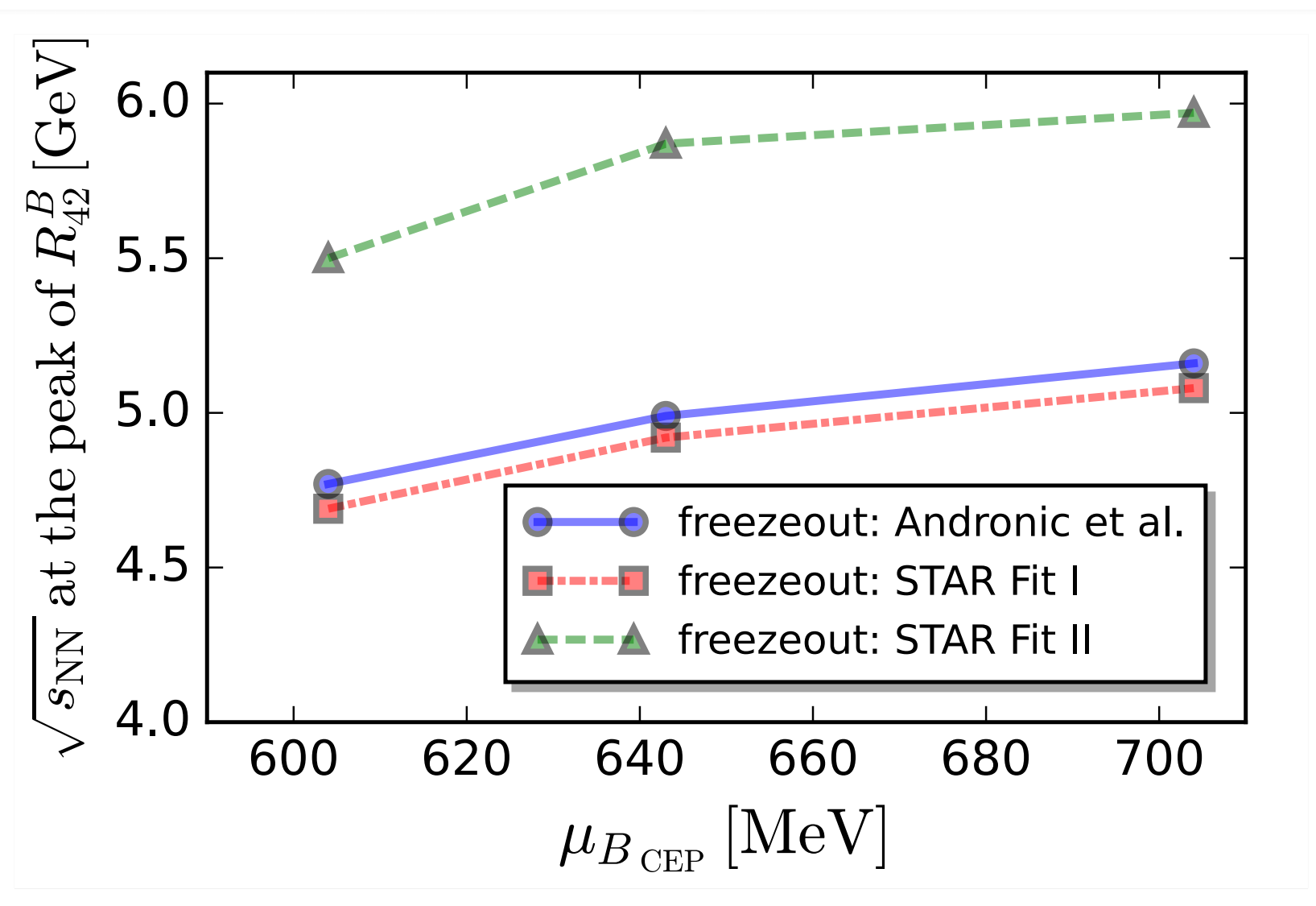
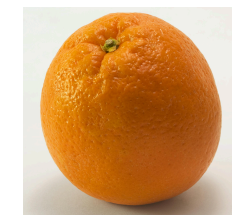
Pion propagator



$$k_{\text{on}} \lesssim 100 \text{ MeV}$$

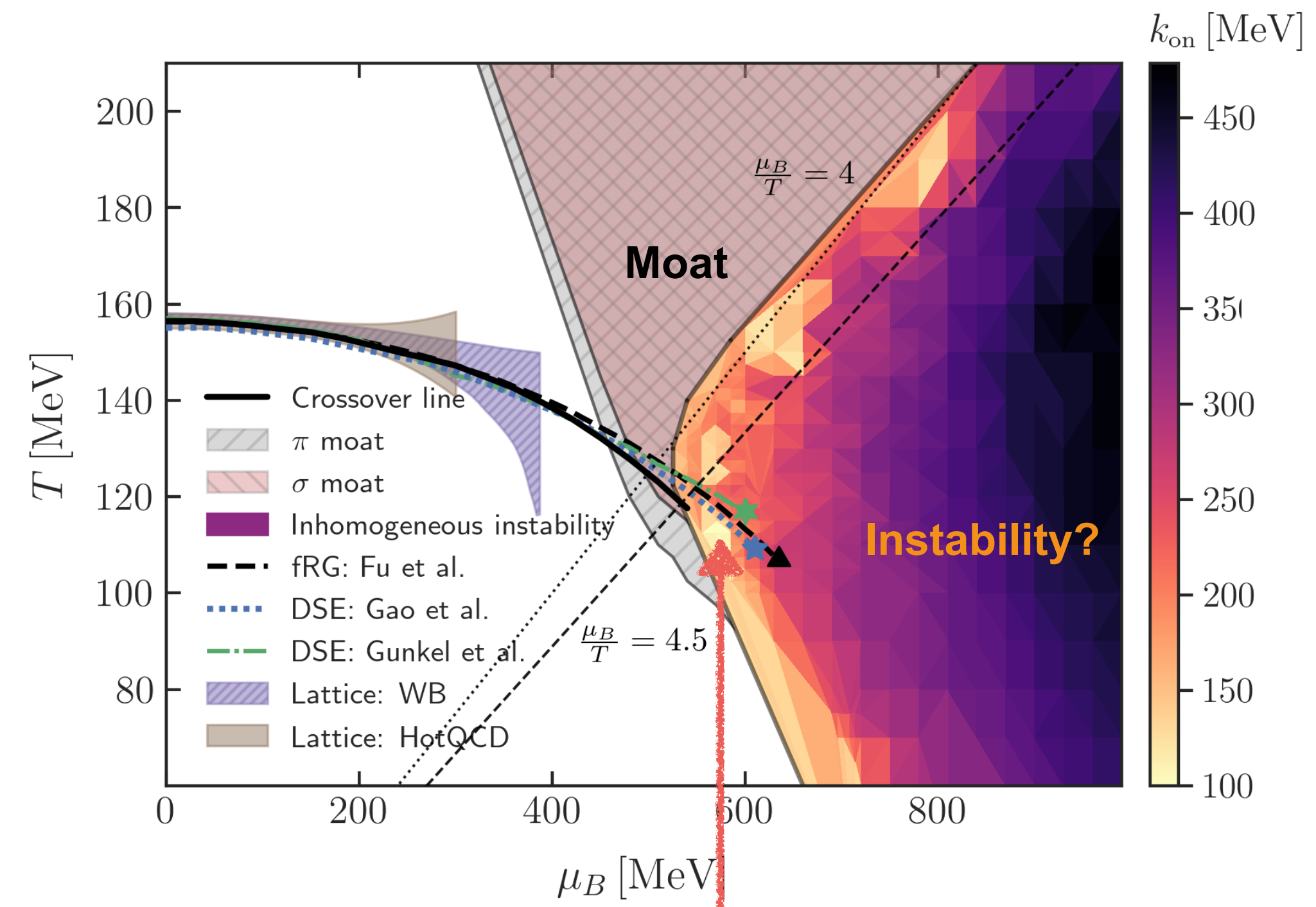


First functional shot on the experimental location of the peak



$$\sqrt{s_{\text{peak}}} \approx 5 \text{ GeV}$$

Experimental signatures for the onset of new phases

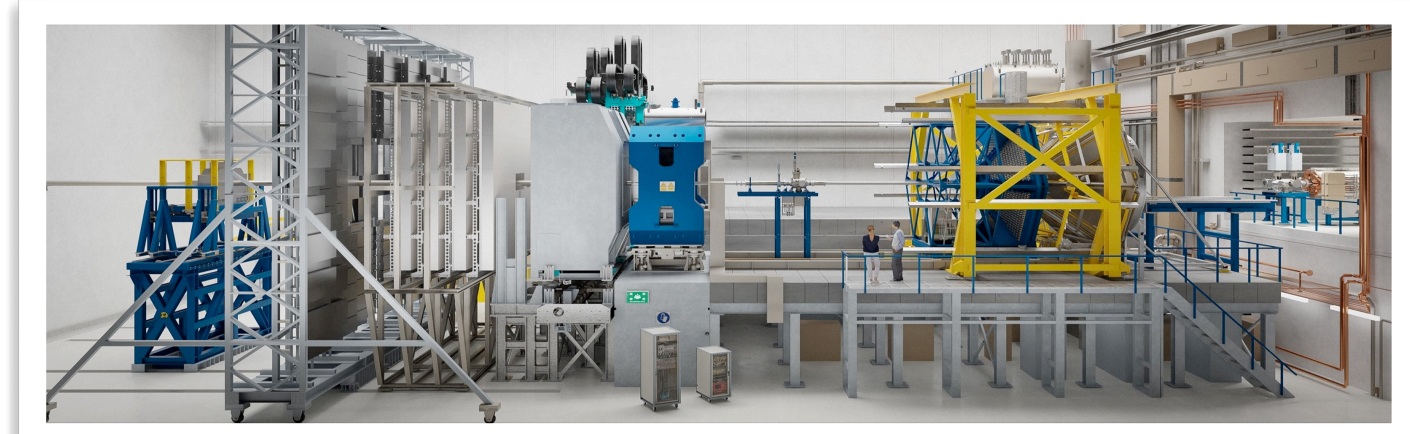


The moat
&
the regime with new phases

are in the CBM regime
and
require high statistics

$k_{\text{on}} \lesssim 100 \text{ MeV}$

$\sqrt{s_{\text{peak}}} \approx 5$



!Go for it!

Summary

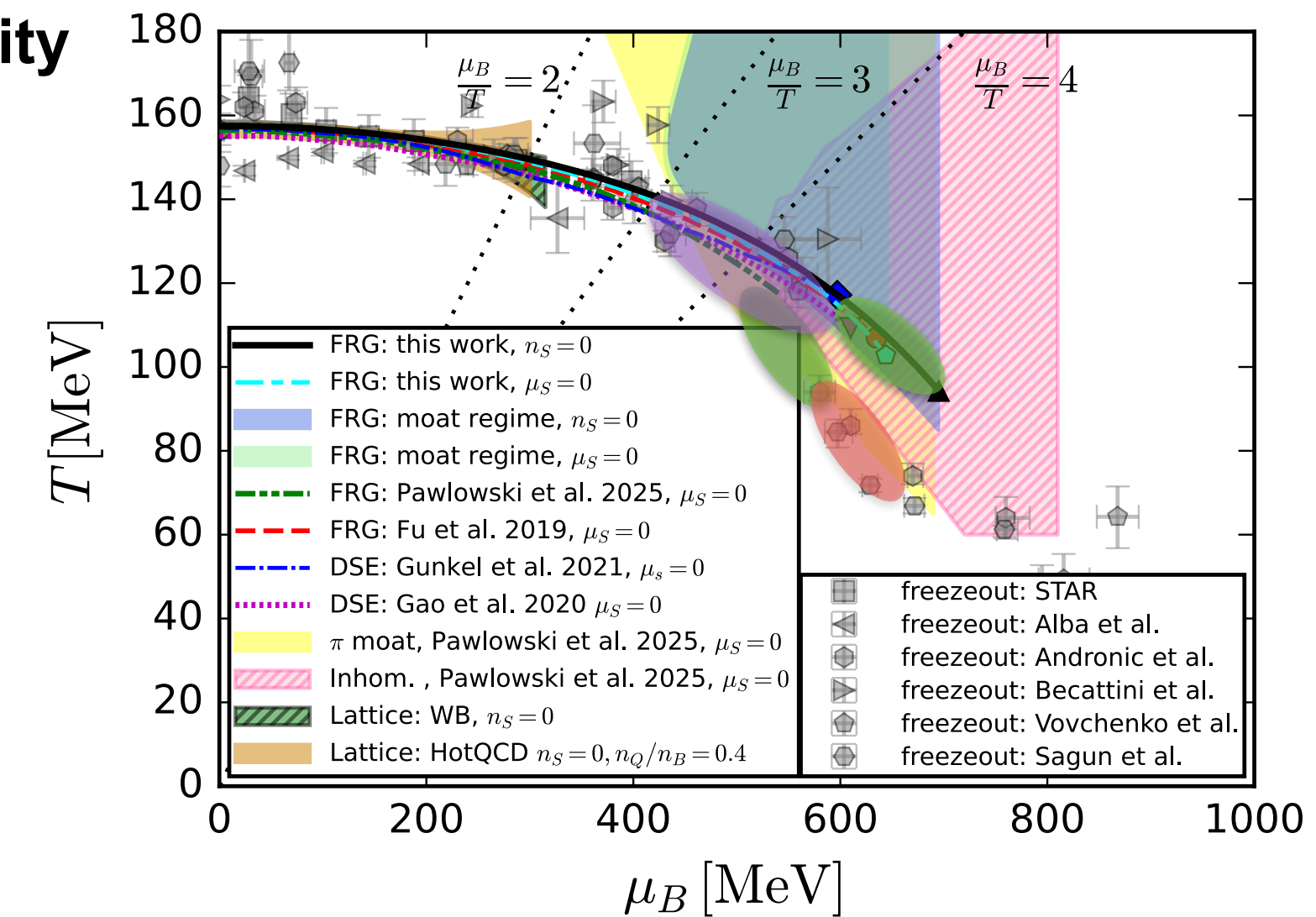
- Functional QCD provides direct 1st principle results for the phase structure at finite density

▪ Predictions: $\frac{\mu_B}{T} \lesssim 4.5$

▪ Estimates: $4.5 \lesssim \frac{\mu_B}{T} \lesssim 8$

- Soft modes are commonplace, critical regimes are small

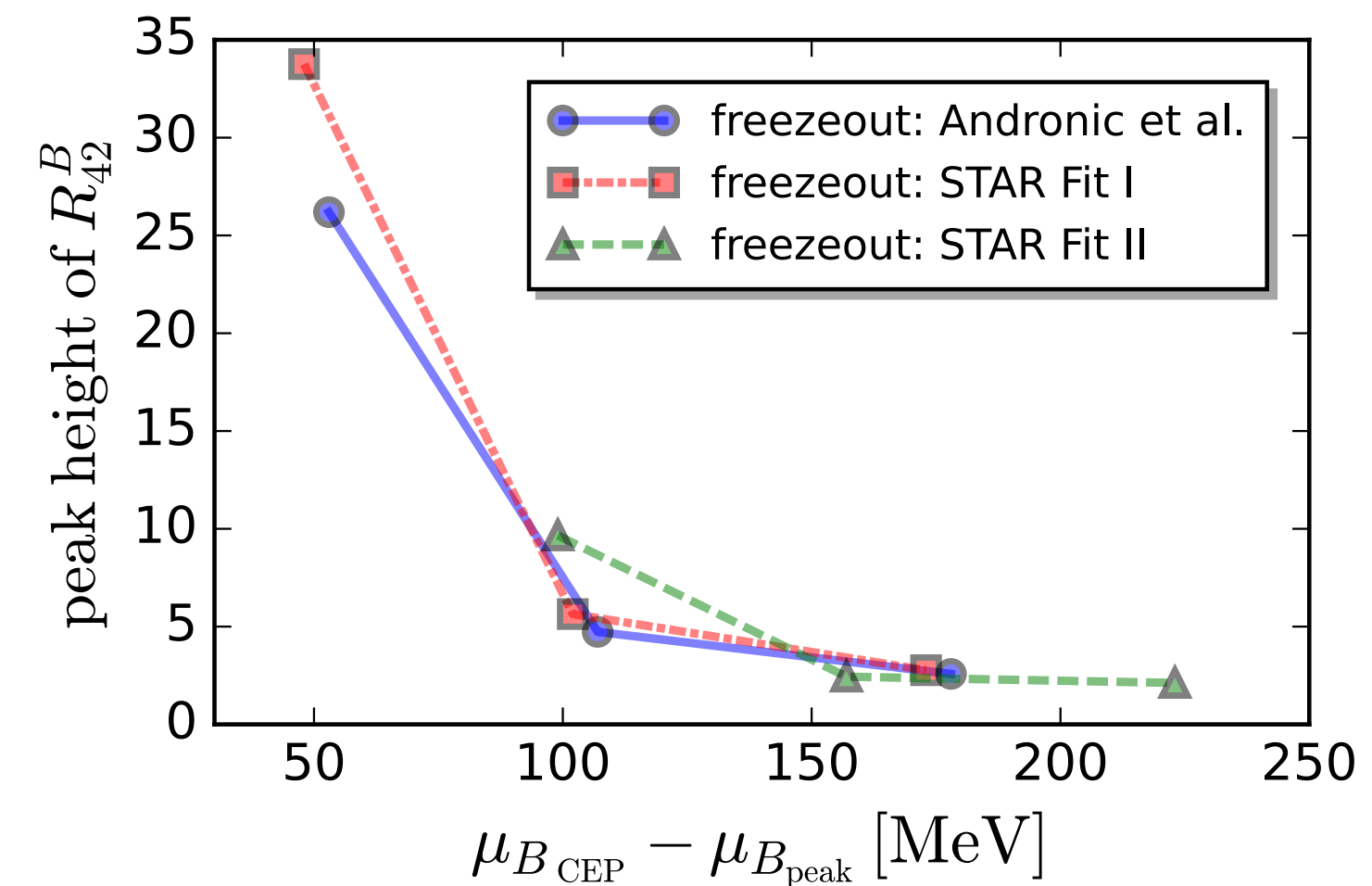
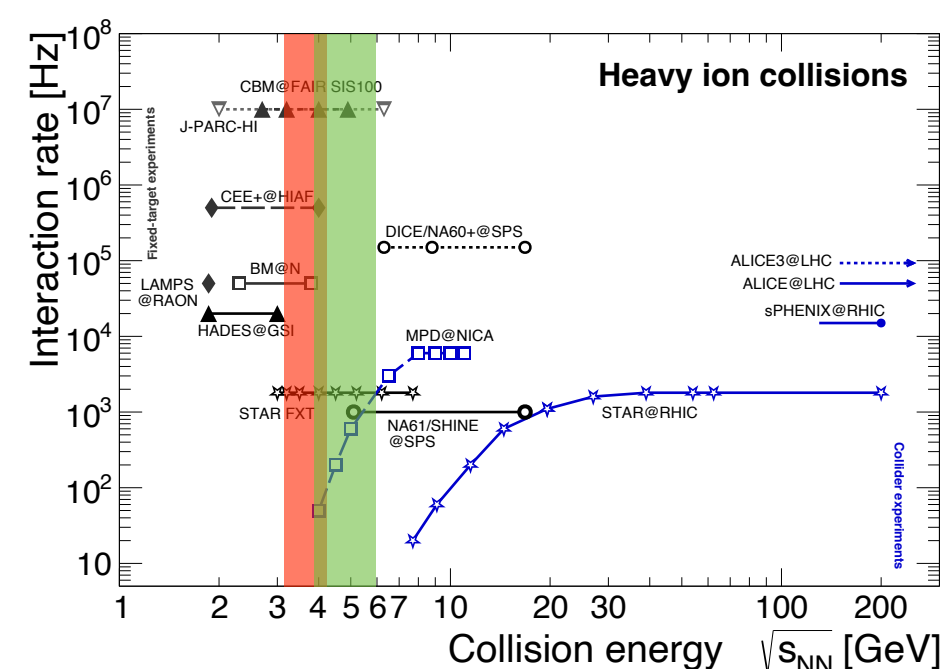
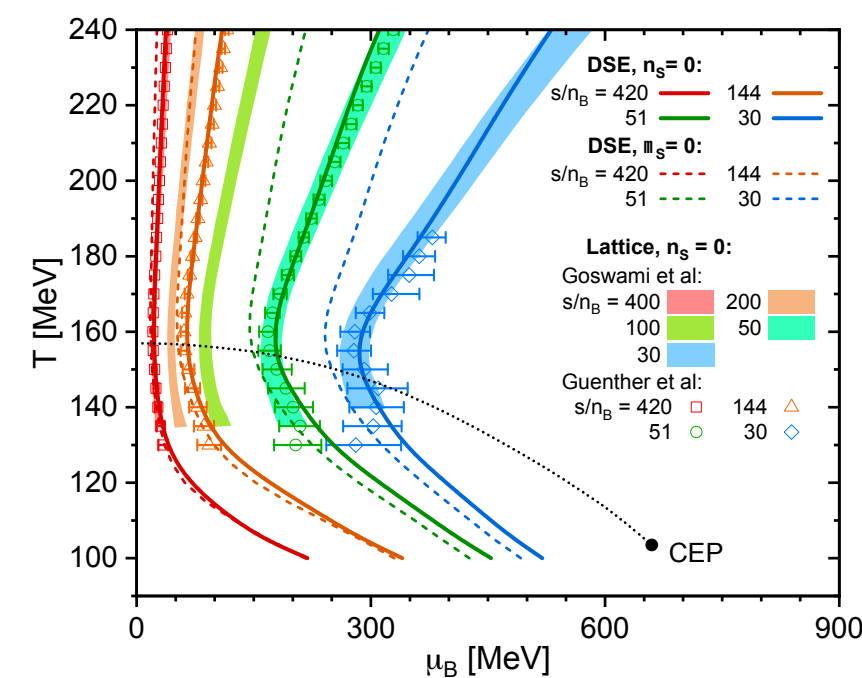
- Ripples of new phases aka of the critical end point



- Explanation for the convergence of CEP locations of extrapolation approaches

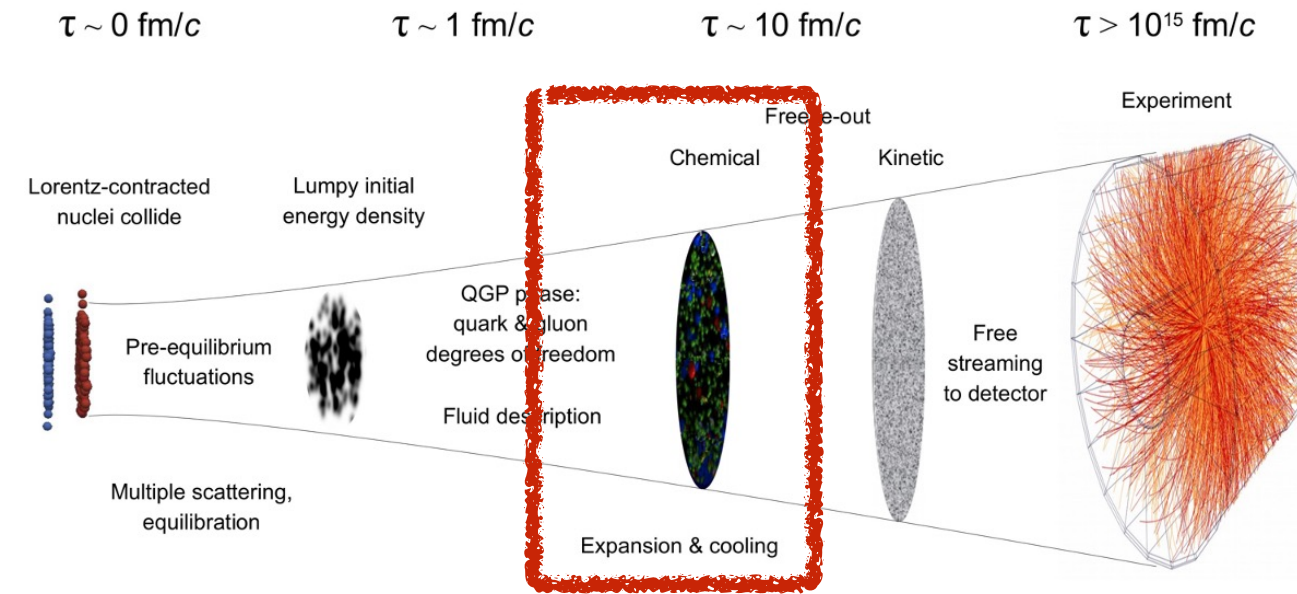
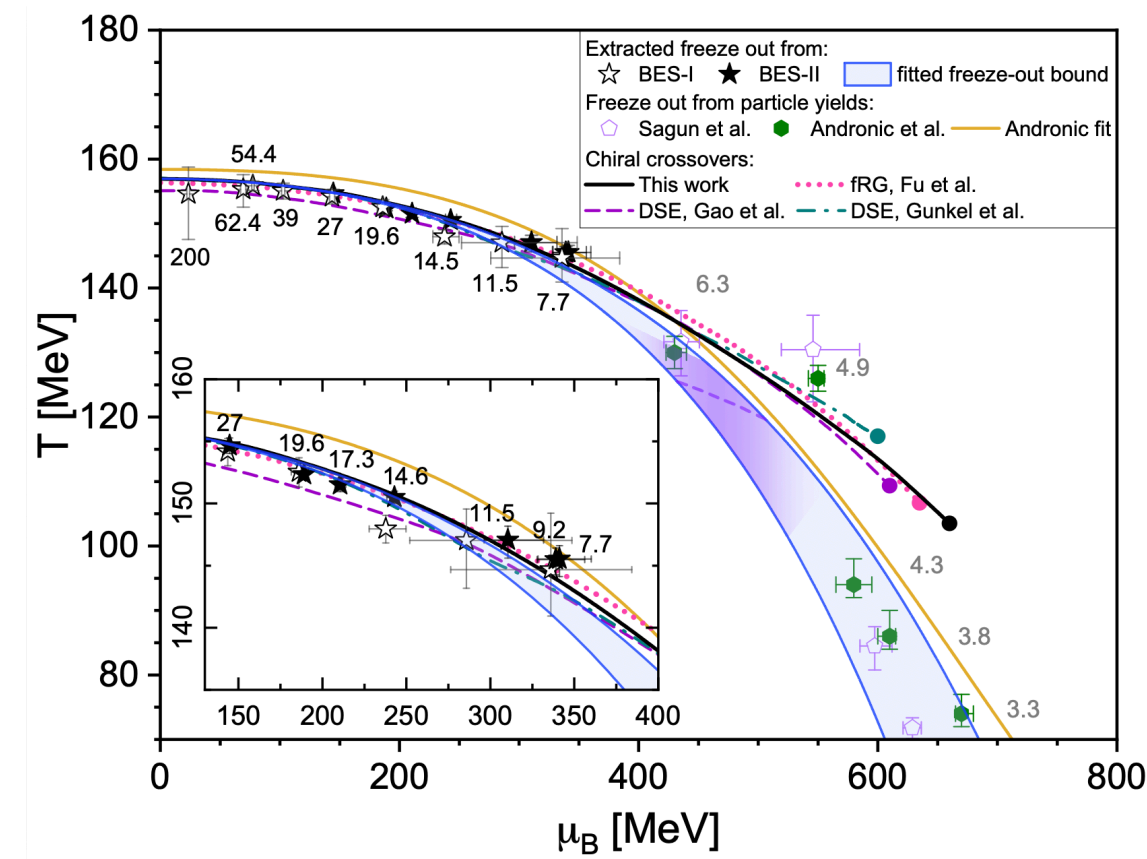
- Results & observables: EoS, fluctuations of conserved charges, tiny critical regime & onset of new phases; moat regime,

- Peak height of kurtosis carries the location of the CEP



Summary

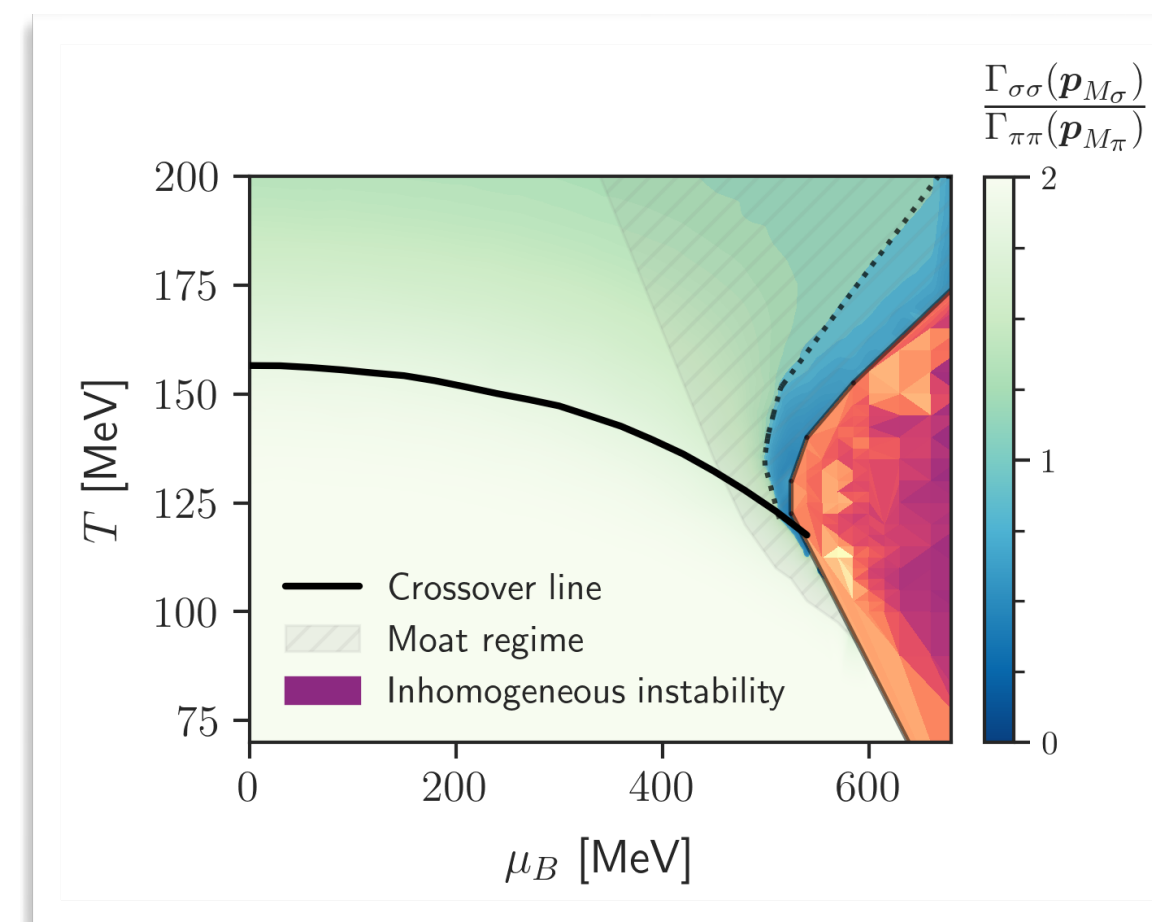
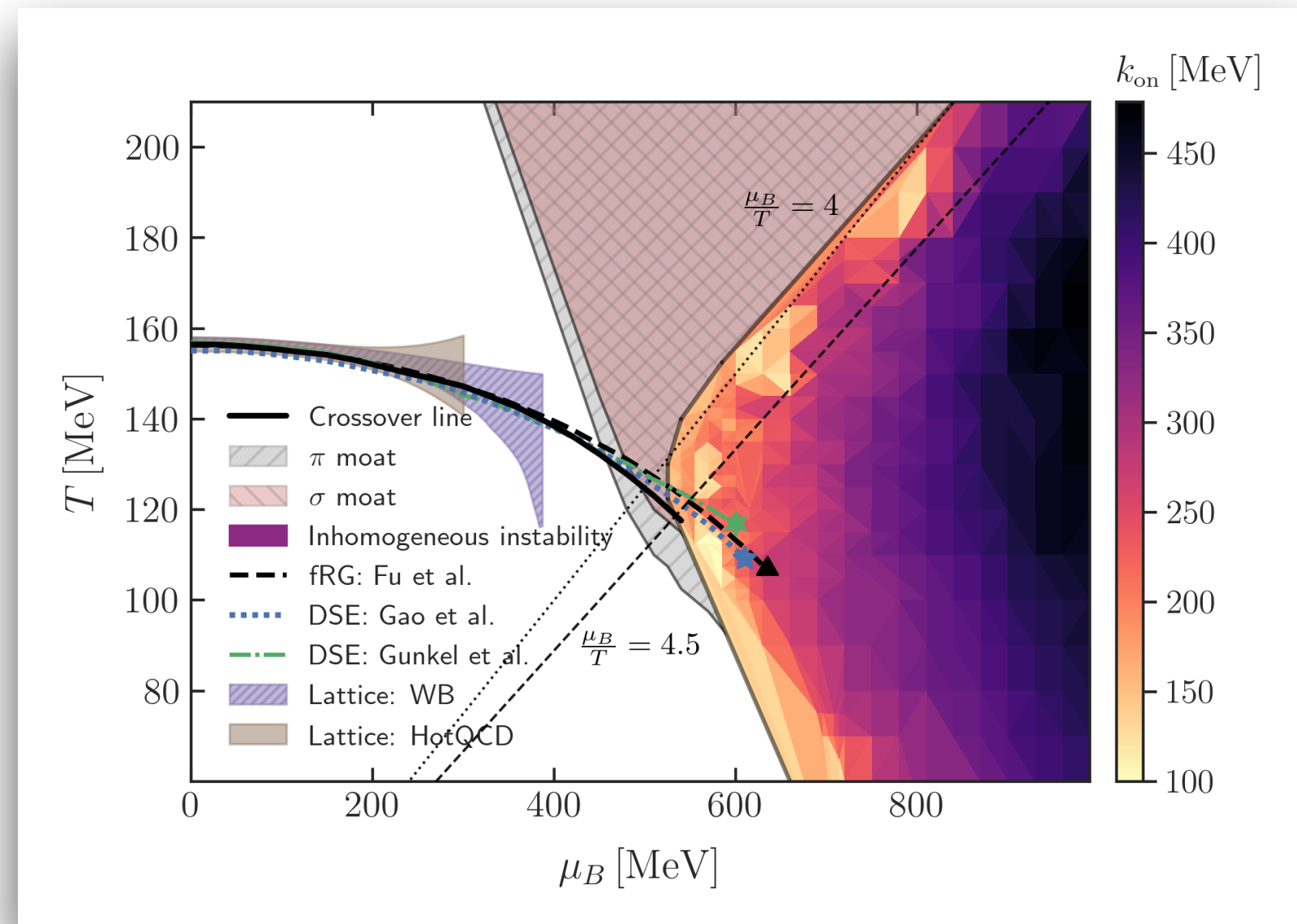
Functional freeze out & towards real time dynamics



‘Combined unfolding’

$$\sqrt{s_{\text{peak}}} \approx 5 \text{ GeV}$$

First steps in the QCD moat and inhomogeneous phases



Material

Collection of talks, colloquia & slides from schools

Topical reviews

Collection of reviews & lecture notes on the FRG & DSE

The nonperturbative functional renormalization group and its applications

Dupuis et al, Phys.Rep. 910 (2021) 1-114

QCD at finite temperature and density within the fRG approach: An overview

Fu, Commun.Theor.Phys. 74 (2022) 9, 097304

QCD phase structure & equation of state: A functional perspective

Rennecke, QM plenary talk, arxiv:2510.11270

Phase structure and observables at high densities from first principles QCD

Fischer, JMP, arxiv: 2603.11135, to be published in EPJST

Phase structure of strong interaction matter from Functional QCD

Fischer, JMP, arxiv: 2606.03703, contribution to Encyclopedia of Nuclear Physics

Just a reminder: It was your choice!

The phase structure of QCD

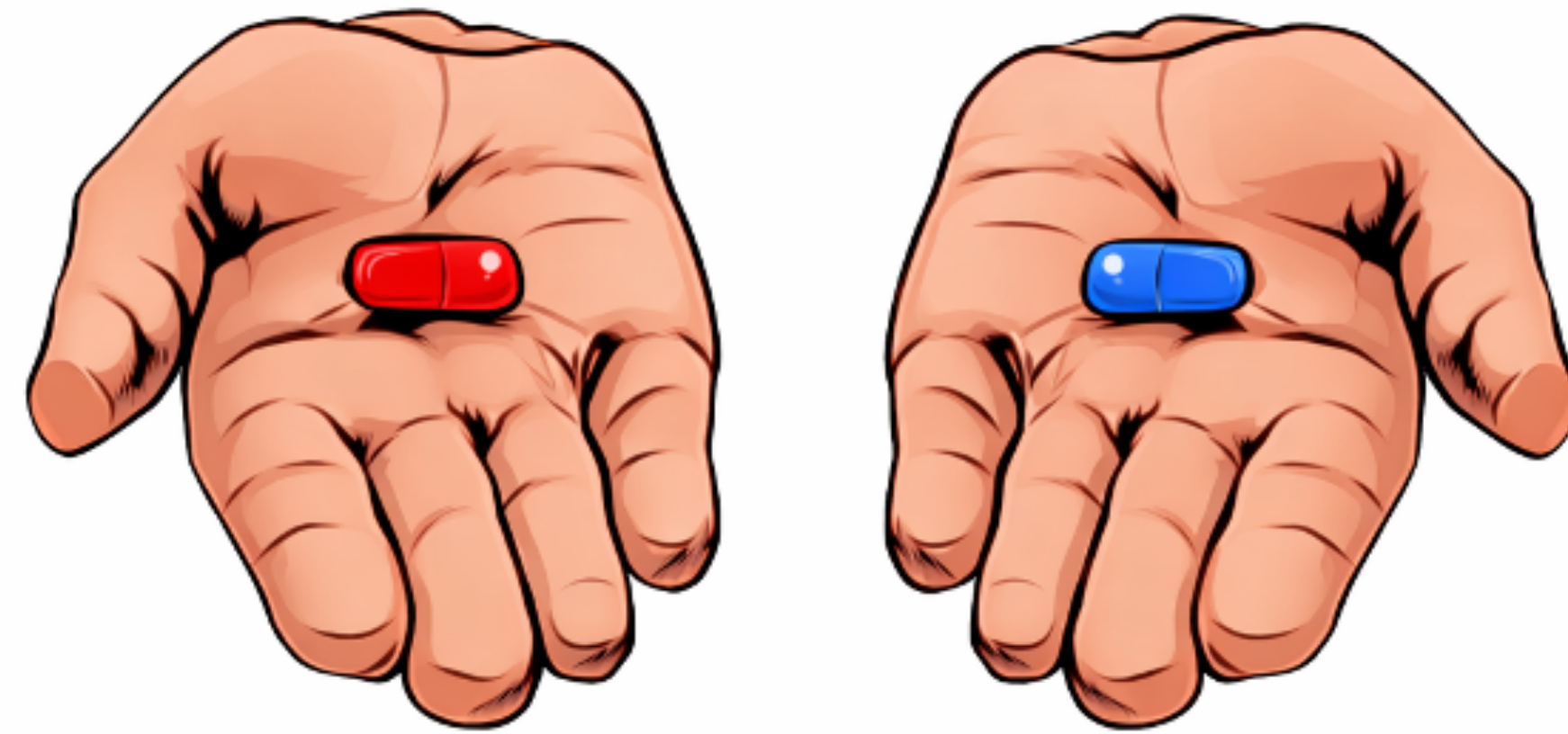
from functional the RG with all its g(l)ory details



How to compute

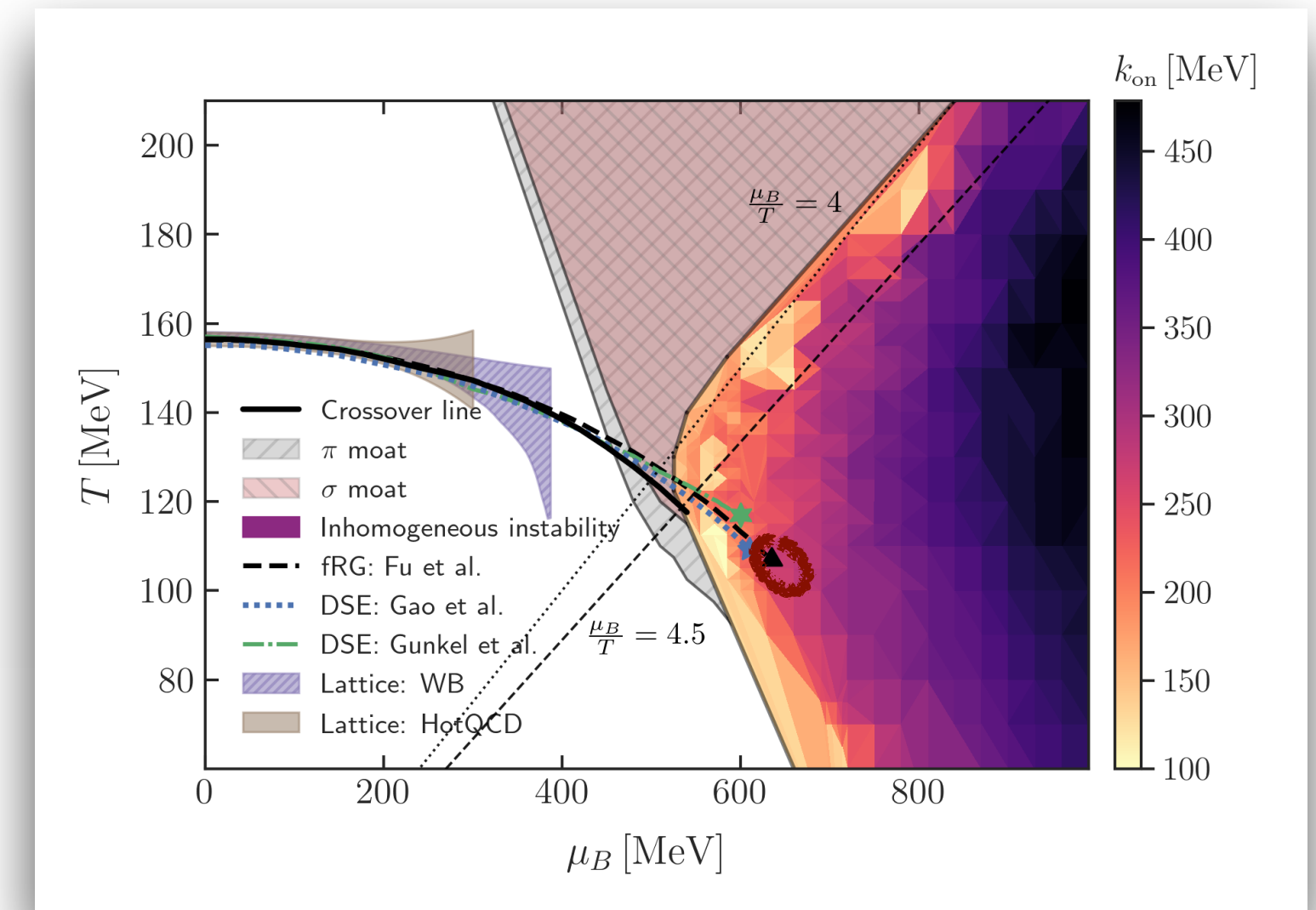
&

How to judge



Colloquium on the phase structure of QCD

from functional methods



Outline

- **(I) Functional renormalisation group for QCD**
- **(II) Functional QCD and the QCD phase structure**

(I) Functional renormalisation group for QCD

- **Introduction to the functional renormalisation group**

- Derivation of the flow equation ✓
- Spontaneous symmetry breaking ✓
- Systematic error control & optimisation ✗

- **Functional flows for QCD**

- Flows for correlation functions & chiral symmetry breaking ✓
- Getting dynamical: emergent hadrons & diquarks ✗
- Dynamical hadronisation at work ✗

(II) Functional QCD and the QCD phase structure

- **QCD at finite temperature and density**

- Benchmarks in the vacuum X
- Correlation functions at finite temperature X
- Polyakov loop from functional approaches X

- Phase structure from functional QCD: how to ✓

- **QCD phase structure**

- Locating the QCD phase boundary and the critical end point ✓
- Fluctuations of conserved charges: Ripples of the critical end point X

(I) Functional renormalisation group for QCD

- **Introduction to the functional renormalisation group**

- Derivation of the flow equation ✓
- Spontaneous symmetry breaking ✓
- Systematic error control & optimisation ✗

- **Functional flows for QCD**

- Flows for correlation functions & chiral symmetry breaking ✓
- Getting dynamical: emergent hadrons & diquarks ✗
- Dynamical hadronisation at work ✗

Introduction to the functional renormalisation group

Derivation of the flow equation

Functional renormalisation group

Generating functional Z

$$Z[J] = \frac{1}{\mathcal{N}} \int d\varphi e^{-S[\varphi] + \int_x J\varphi}$$

partition function

$$\langle \varphi \rangle_J = \phi$$

$$S[\varphi] = \frac{1}{2} \int_x \left[\partial_\mu \varphi \partial_\mu \varphi + m^2 \varphi^2 + \frac{\lambda}{4} \varphi^4 \right]$$

classical action

zero-dimensional example: 'Functional' flows for integrals

Functional renormalisation group

Generating functional Z

$$Z[J] = \frac{1}{\mathcal{N}} \int d\varphi e^{-S[\varphi] + \int_x J\varphi}$$

partition function

$$\langle \varphi \rangle_J = \phi$$

Effective action Γ

$$\Gamma[\phi] = -\log \int d\hat{\varphi} e^{-S[\hat{\varphi} + \phi] + \int_x \hat{\varphi} \frac{\delta \Gamma[\phi]}{\delta \phi}}$$

free energy

$$\varphi = \hat{\varphi} + \phi$$
$$\langle \hat{\varphi} \rangle_{\frac{\delta \Gamma}{\delta \phi}} = 0$$

$$J = \frac{\delta \Gamma}{\delta \phi}$$

Functional renormalisation group

Generating functional Z

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partition function

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Effective action Γ

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free energy

$$\varphi = \hat{\varphi} + \phi$$
$$\langle \hat{\varphi} \rangle_{\frac{\delta \Gamma}{\delta \phi}} = 0$$

$$J = \frac{\delta \Gamma}{\delta \phi}$$

$$\Gamma[\phi] = \sup_J \left(\int_x J \cdot \phi - \log Z[J] \right)$$

Legendre transform

Functional renormalisation group

Generating functional Z

$$Z[J] = \frac{1}{\mathcal{N}} \int d\varphi e^{-S[\varphi] + \int_x J\varphi}$$

partition function

$$\langle \varphi \rangle_J = \phi$$

Effective action Γ

$$\Gamma[\phi] = -\log \int d\hat{\varphi} e^{-S[\hat{\varphi} + \phi] + \int_x \hat{\varphi} \frac{\delta \Gamma[\phi]}{\delta \phi}}$$

free energy

$$\varphi = \hat{\varphi} + \phi$$
$$\langle \hat{\varphi} \rangle_{\frac{\delta \Gamma}{\delta \phi}} = 0$$

$$J = \frac{\delta \Gamma}{\delta \phi}$$

Dyson-Schwinger equation

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = \left\langle \frac{\delta S[\hat{\varphi} + \phi]}{\delta \phi(x)} \right\rangle$$

quantum equation of motion

Functional renormalisation group

Dyson-Schwinger equation

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = \left\langle \frac{\delta S[\hat{\varphi} + \phi]}{\delta\phi(x)} \right\rangle$$

Diagrammatics

$$S[\phi] = \frac{1}{2} \int_x \left[\partial_\mu \phi \partial_\mu \phi + m^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right]$$

Functional renormalisation group

Dyson-Schwinger equation

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = \left\langle \frac{\delta S[\hat{\varphi} + \phi]}{\delta\phi(x)} \right\rangle$$

Diagrammatics

$$S[\phi] = \frac{1}{2} \int_x \left[\partial_\mu \phi \partial_\mu \phi + m^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right]$$

$$\frac{\lambda}{2} \langle [\hat{\varphi}(x) + \phi(x)]^3 \rangle = \frac{\lambda}{2} \phi^3(x) + \frac{3\lambda}{2} \phi(x) \text{ (loop with one orange vertex)} - \frac{\lambda}{2} \text{ (loop with three orange vertices and one grey vertex)}$$

$$S^{(n)} = \text{ (V vertex with } n \text{ external lines)}$$

$$\Gamma^{(n)} = \text{ (Feynman vertex with } n \text{ external lines)}$$

$$G = \text{ (propagator with orange vertex)} = \langle \hat{\varphi}(x) \hat{\varphi}(y) \rangle_c$$

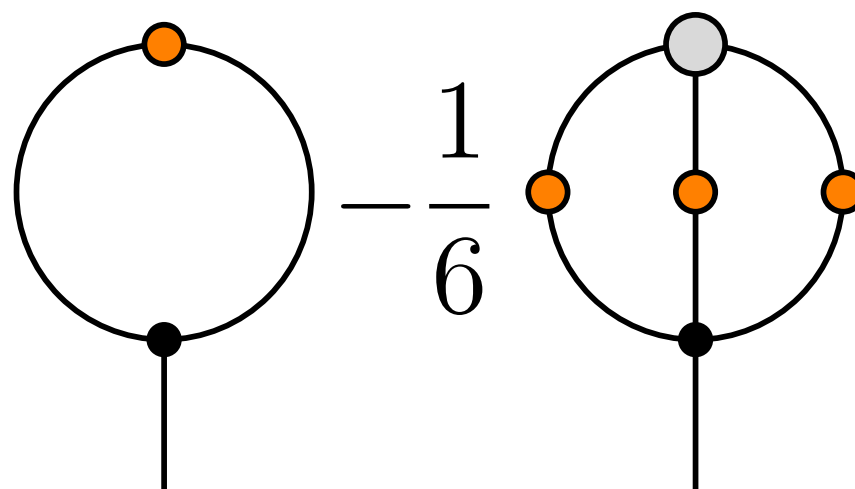
Functional renormalisation group

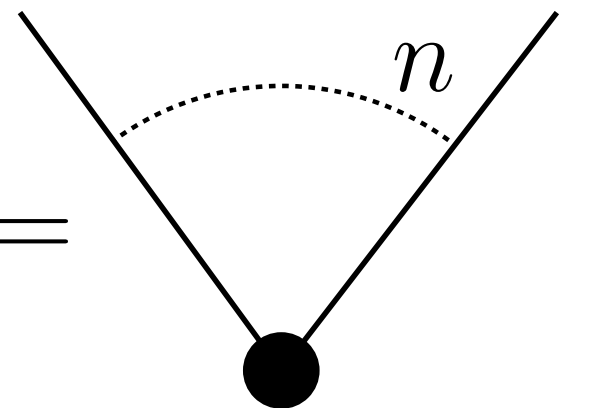
Dyson-Schwinger equation

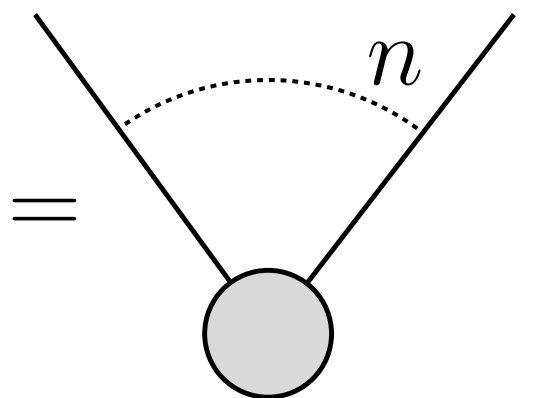
$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = \left\langle \frac{\delta S[\hat{\varphi} + \phi]}{\delta\phi(x)} \right\rangle$$

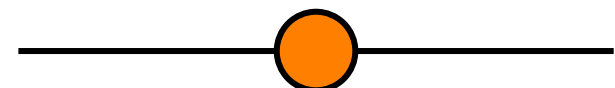
Diagrammatics

$$S[\phi] = \frac{1}{2} \int_x \left[\partial_\mu \phi \partial_\mu \phi + m^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right]$$

$$\frac{\delta\Gamma}{\delta\phi} = \frac{\delta S}{\delta\phi} + \frac{1}{2} \text{ (loop with orange dots) } - \frac{1}{6} \text{ (loop with grey and orange dots) }$$


$$S^{(n)} = \text{ (V vertex with orange dot) }$$


$$\Gamma^{(n)} = \text{ (V vertex with grey dot) }$$


$$G = \text{ (propagator with orange dot) } = \langle \hat{\varphi}(x) \hat{\varphi}(y) \rangle_c$$


Functional renormalisation group

Effective action Γ

$$\Gamma[\phi] = -\log \int d\hat{\varphi} e^{-S[\hat{\varphi}+\phi] + \int_x \hat{\varphi} \frac{\delta \Gamma[\phi]}{\delta \phi}}$$

No quantum fluctuations

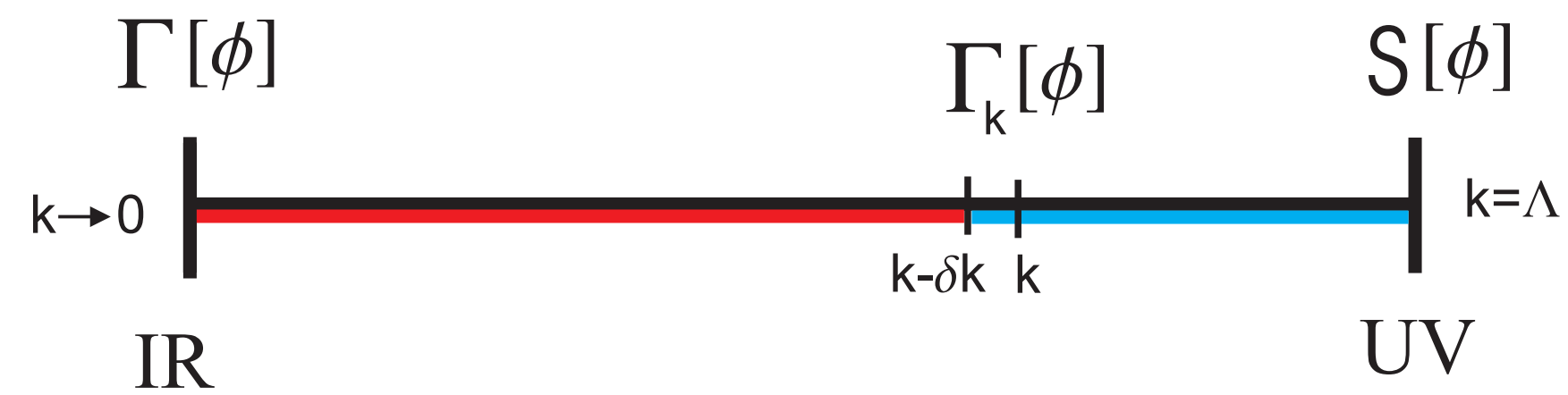
$$\Gamma[\phi] = -\log e^{-S[\phi]} = S[\phi]$$

Functional renormalisation group

Effective action Γ

$$\Gamma[\phi] = -\log \int d\hat{\varphi} e^{-S[\hat{\varphi}+\phi] + \int_x \hat{\varphi} \frac{\delta \Gamma[\phi]}{\delta \phi}}$$

UV quantum fluctuations up to $p^2 \approx k^2$



Functional renormalisation group

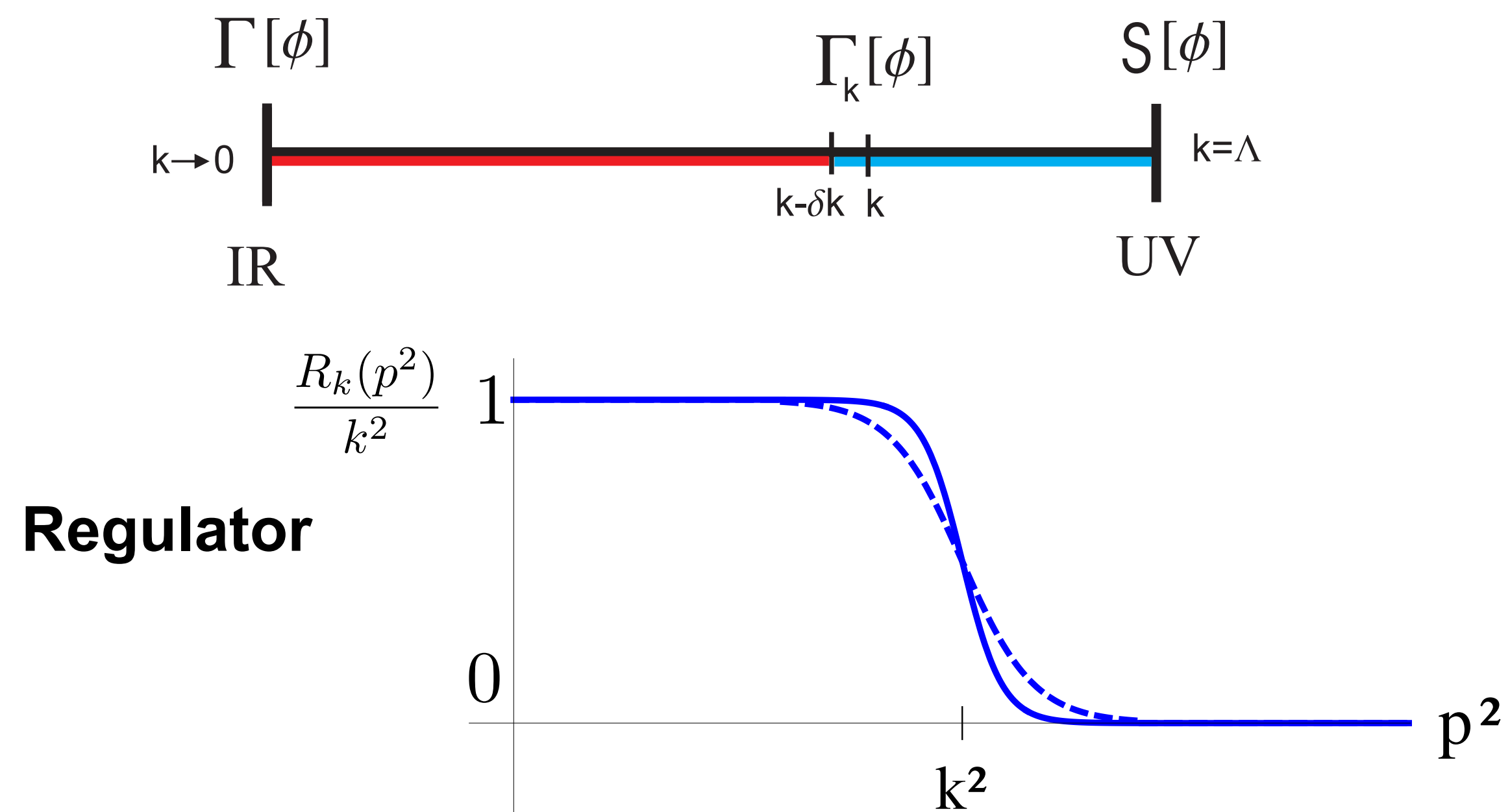
Effective action Γ

$$\Gamma_k[\phi] = -\log \int d\hat{\varphi} e^{-S[\phi+\hat{\varphi}] + \frac{1}{2} \int_p \hat{\varphi}(p) R_k(p^2) \hat{\varphi}(-p) + \int_x \hat{\varphi}(x) \frac{\delta \Gamma_k[\phi]}{\delta \phi(x)}}$$

DSE

$$\frac{\delta \Gamma_k[\phi]}{\delta \phi(x)} = \left\langle \frac{\delta S[\hat{\varphi} + \phi]}{\delta \phi(x)} \right\rangle$$

UV quantum fluctuations up to $p^2 \approx k^2$

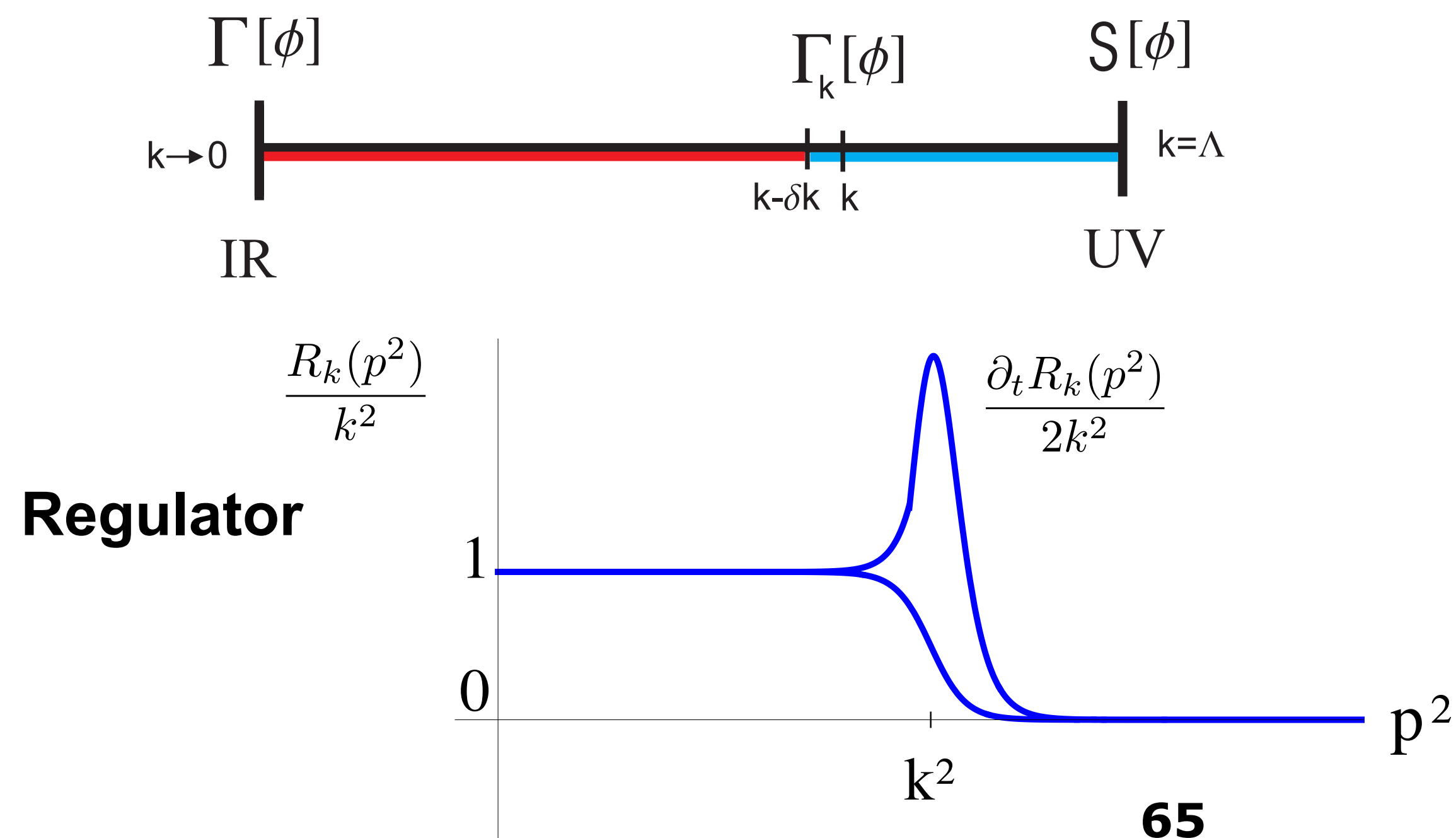


Functional renormalisation group

Effective action Γ

$$\Gamma_k[\phi] = -\log \int d\hat{\varphi} e^{-S[\phi+\hat{\varphi}] + \frac{1}{2} \int_p \hat{\varphi}(p) R_k(p^2) \hat{\varphi}(-p) + \int_x \hat{\varphi}(x) \frac{\delta \Gamma_k[\phi]}{\delta \phi(x)}}$$

UV quantum fluctuations up to $p^2 \approx k^2$



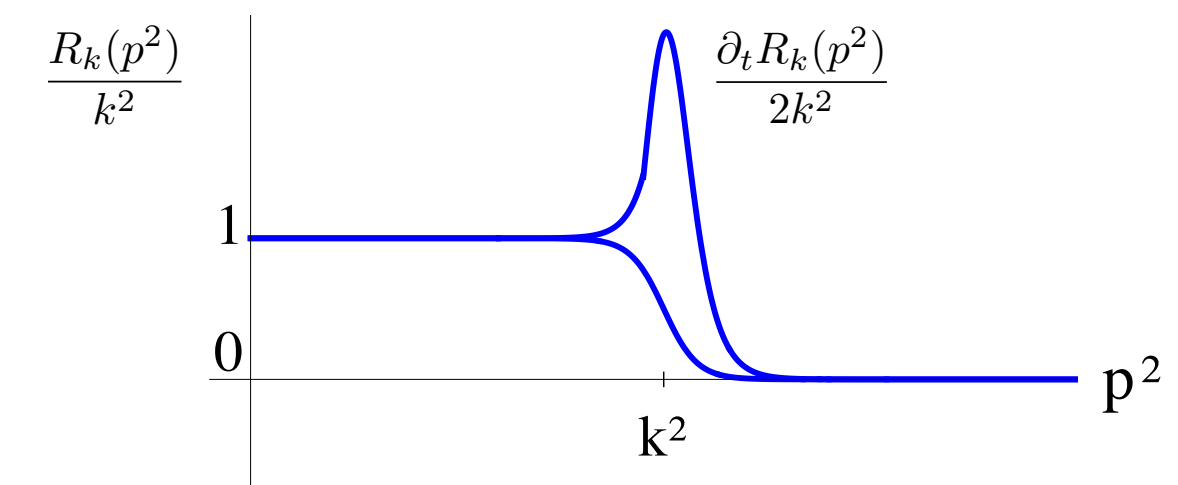
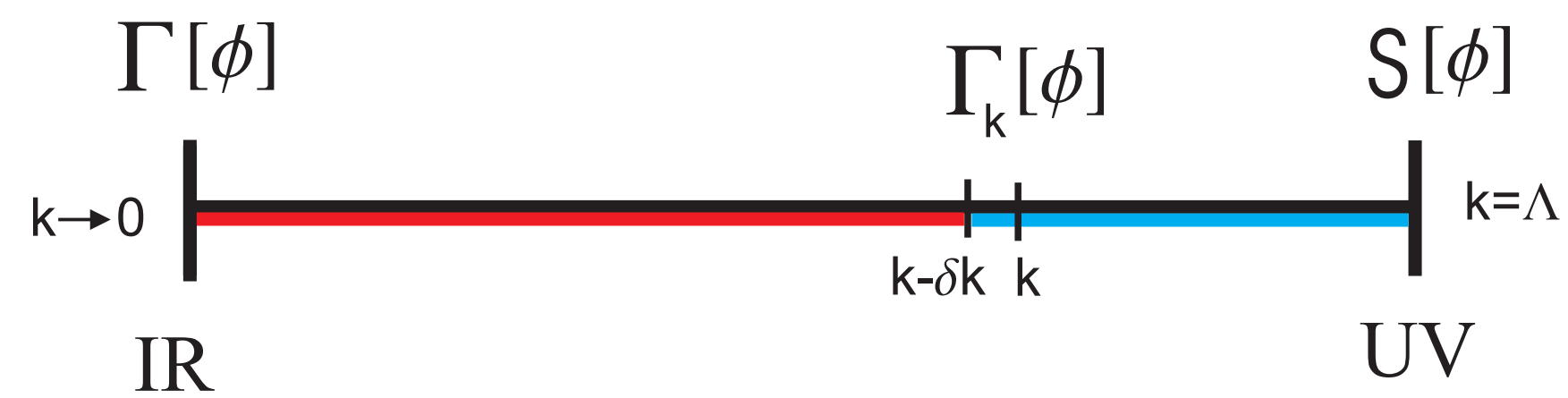
$$t = \log \frac{k}{\Lambda}$$

Functional renormalisation group

Effective action Γ

$$\Gamma_k[\phi] = -\log \int d\hat{\varphi} e^{-S[\phi+\hat{\varphi}] + \frac{1}{2} \int_p \hat{\varphi}(p) R_k(p^2) \hat{\varphi}(-p) + \int_x \hat{\varphi}(x) \frac{\delta \Gamma_k[\phi]}{\delta \phi(x)}}$$

UV quantum fluctuations up to $p^2 \approx k^2$



Flow

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \langle \hat{\varphi}(p) \hat{\varphi}(-p) \rangle \partial_t R_k(p^2)$$

$$t = \log \frac{k}{\Lambda}$$

Functional renormalisation group

Flow

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \langle \hat{\varphi}(p) \hat{\varphi}(-p) \rangle \partial_t R_k(p^2)$$

$$t = \log \frac{k}{\Lambda}$$

Propagator

$$G = \text{---} \bigcirc \text{---} = \langle \hat{\varphi}(x) \hat{\varphi}(y) \rangle_c$$

$$G^{-1}[\phi] = \Gamma_k^{(2)}[\phi] + R_k$$

Functional renormalisation group

Flow

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \langle \hat{\varphi}(p) \hat{\varphi}(-p) \rangle \partial_t R_k(p^2)$$

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Propagator

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$$G^{-1}[\phi] = \Gamma_k^{(2)}[\phi] + R_k$$

DSE

$$\frac{\delta \Gamma_k[\phi]}{\delta \phi(x)} = \left\langle \frac{\delta S[\hat{\varphi} + \phi]}{\delta \phi(x)} \right\rangle$$

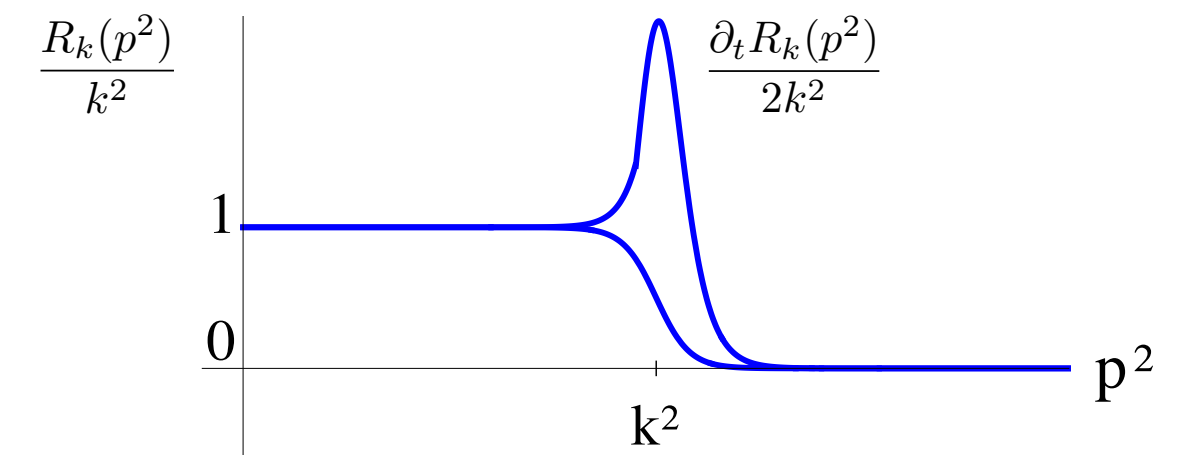
$$\Gamma_k[\phi] = -\log \int d\hat{\varphi} e^{-S[\hat{\varphi}+\phi] + \frac{1}{2} \int_p \hat{\varphi}(p) R_k(p^2) \hat{\varphi}(-p) + \int_x \hat{\varphi} \frac{\delta \Gamma_k[\phi]}{\delta \phi}}$$

$$\Gamma_k^{(2)}[\phi] = S^{(2)}[\phi] + \frac{1}{2} \text{ (loop with 1 orange vertex) } - \frac{1}{2} \text{ (loop with 2 orange vertices) } - \frac{1}{6} \text{ (loop with 3 orange vertices) } + \frac{1}{2} \text{ (loop with 4 orange vertices) }$$

Functional renormalisation group

Flow

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$



regulator

Diagrammatics

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left(\text{Diagram} \right)$$

The diagram is a circle with a cross (X) at the top and an orange circle at the bottom, representing the regulator. The label $[\phi]$ is to the right of the circle.

(Inverse) propagator

$$\partial_t \Gamma_k^{(2)}[\phi] = -\frac{1}{2} \frac{\delta}{\delta \phi} \left(\text{Diagram 1} \right) = -\frac{1}{2} \left(\text{Diagram 2} \right) + \left(\text{Diagram 3} \right)$$

Diagram 1: A circle with a cross (X) at the top, an orange circle at the bottom, and a grey circle at the bottom. A vertical line extends from the grey circle.

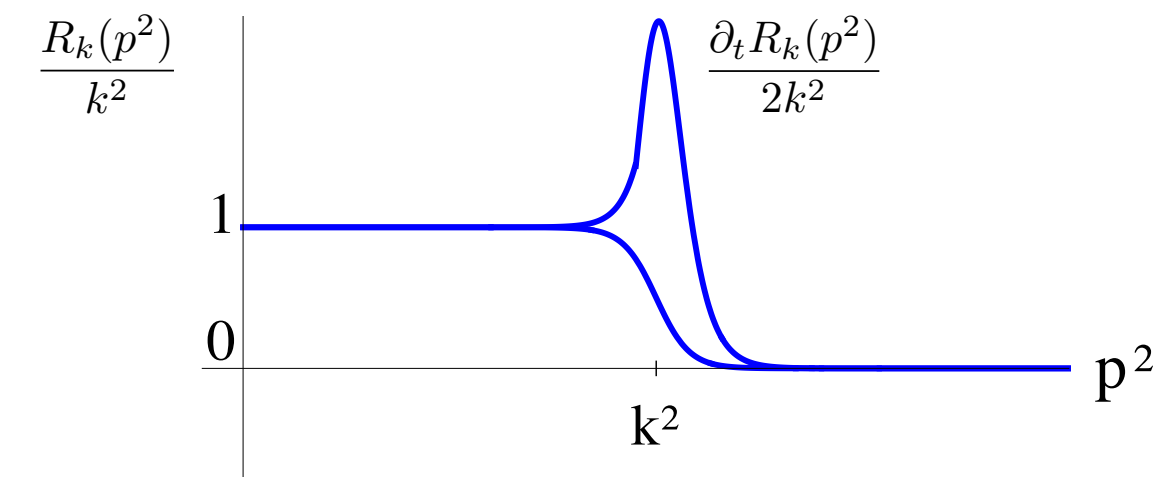
Diagram 2: A circle with a cross (X) at the top, an orange circle at the bottom, and a grey circle at the bottom. Two lines extend from the grey circle.

Diagram 3: A circle with a cross (X) at the top, an orange circle at the bottom, and a grey circle at the bottom. Two lines extend from the grey circle.

Functional renormalisation group

Flow

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$



(Inverse) propagator

fRG

$$\partial_t \Gamma_k^{(2)}[\phi] = -\frac{1}{2} \text{ (diagram with 2 orange and 1 grey nodes) } + \text{ (diagram with 3 orange and 2 grey nodes) }$$

The first diagram is a loop with two orange nodes and one grey node. The second diagram is a loop with three orange nodes and two grey nodes.

DSE

$$\Gamma_k^{(2)}[\phi] - S^{(2)}[\phi] = \frac{1}{2} \text{ (diagram 1) } - \frac{1}{2} \text{ (diagram 2) } - \frac{1}{6} \text{ (diagram 3) } + \frac{1}{2} \text{ (diagram 4) }$$

The diagrams represent various loop corrections: a tadpole, a self-energy loop, a sunset diagram, and a more complex multi-loop diagram.

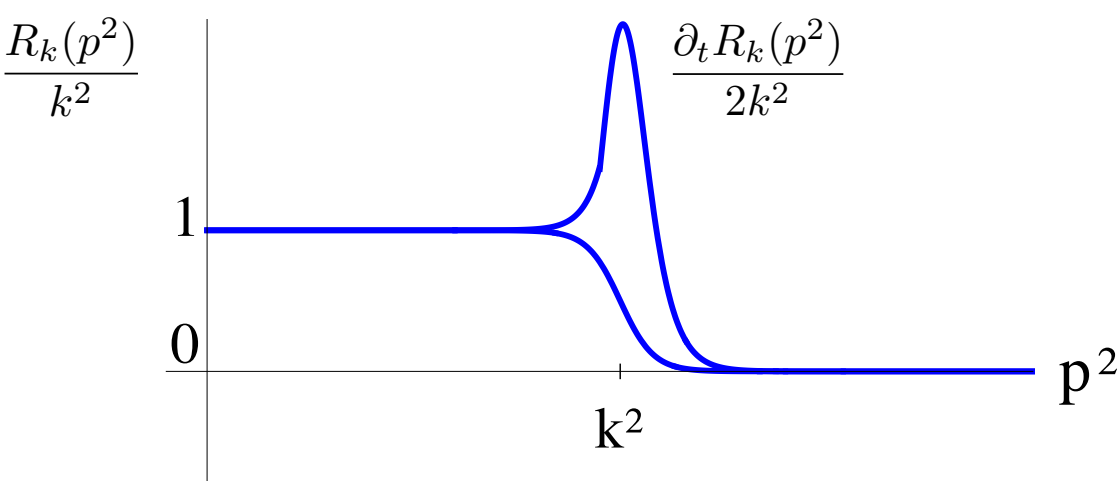
$$\partial_t \Gamma^{(n)} = \text{Flow}_n[\Gamma^{(m)}; m = 2, \dots, n + 2]$$

$$\Gamma^{(n)} = \text{DSE}_n[S^{(m)}, \Gamma^{(m)}; m = 2, \dots, n + 2]$$

Functional renormalisation group

Flow

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$



Properties

	fRG	DSE	2PI	3PI	4PI
● 1-loop exact	✓	—			
● closed	✓	✓			
● RG-scaling	✓	—	—	—	✓
● energy/particle-number conservation	—	—	✓	✓	✓



automatic

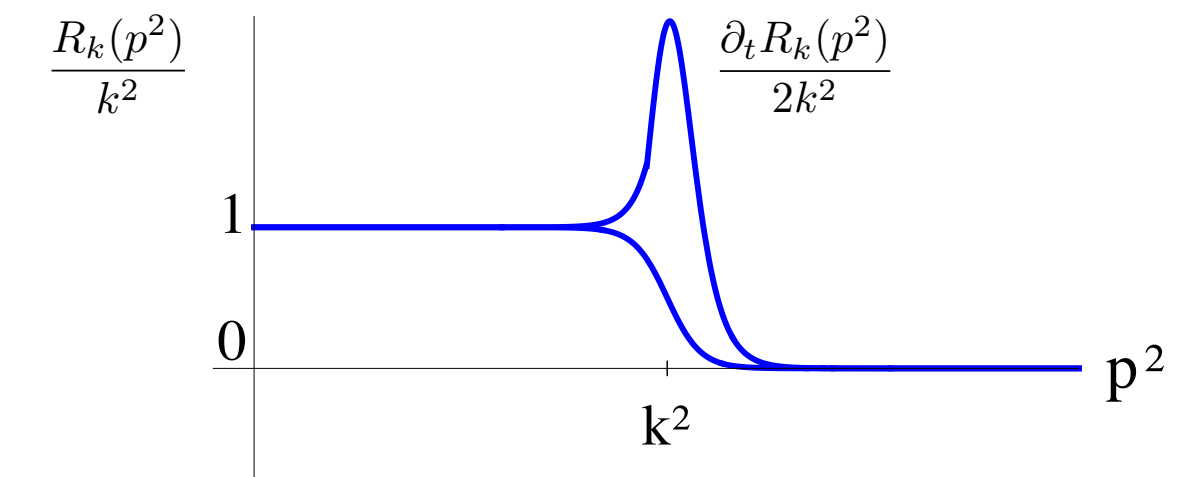


only in specific approximation schemes

Functional renormalisation group

Flow

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$



Properties

FunApproaches

- 1-loop exact
- closed
- RG-scaling
- energy/particle-number conservation



automatic



only in specific approximation schemes

Approximation schemes

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$

Derivative expansion

- Expansion in powers of momenta
- Controlled in the presence of a mass gap m_{gap}
- Expansion parameter $\frac{p^2}{\max(k^2, m_{\text{gap}}^2)}$

Vertex expansion

- Expansion in number n of external fields
- Controlled in perturbation theory/presence of symmetries
- Expansion parameter n

Mixtures, exact resummation schemes,

Approximation schemes

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$

$$\partial_t \Gamma^{(n)} = \text{Flow}_n[\Gamma^{(m)}; m = 2, \dots, n + 2]$$

Derivative expansion

- Expansion in powers of momenta
- Controlled in the presence of a mass gap

m_{gap}

- Expansion parameter

$$\frac{p^2}{\max(k^2, m_{\text{gap}}^2)}$$

Lowest order: 0th order

$$\Gamma_k[\phi] = \frac{1}{2} \int_p \phi p^2 \phi + \int_x V_k(\phi) + O(p^2)$$

$$\Gamma_k^{(2)}[\phi](p, q) = (p^2 + V_k''(\phi)) (2\pi)^d \delta(p - q)$$

Approximation schemes

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$

Derivative expansion

Lowest order: 0th order

$$\Gamma_k[\phi] = \frac{1}{2} \int_p \phi p^2 \phi + \int_x V_k(\phi) + O(p^2)$$

$$\Gamma_k^{(2)}[\phi](p, q) = (p^2 + V_k''(\phi)) (2\pi)^d \delta(p - q)$$

$$\partial_t V_k(\phi) = \frac{1}{d} \frac{\Omega_d}{(2\pi)^d} \frac{k^{2+d}}{k^2 + V_k''(\phi)}$$

$$R_{k,\text{opt}}(p^2) = (k^2 - p^2) \theta(k^2 - p^2)$$

$$\partial_t R_{k,\text{opt}}(p^2) = 2k^2 \theta(k^2 - p^2)$$

$$\Gamma^{(2)}[\phi](p) + R_{k,\text{opt}}(p^2) = [k^2 + V''(\phi)] \theta(k^2 - p^2) + (p^2 + V''(\phi)) \theta(p^2 - k^2)$$

Approximation schemes

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$

Derivative expansion

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Approximation schemes

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$

Derivative expansion

Lowest order: 0th order

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Flow

$$\partial_t V_k(\phi) = \frac{1}{d} \frac{\Omega_d}{(2\pi)^d} \frac{k^{2+d}}{k^2 + V_k''(\phi)}$$

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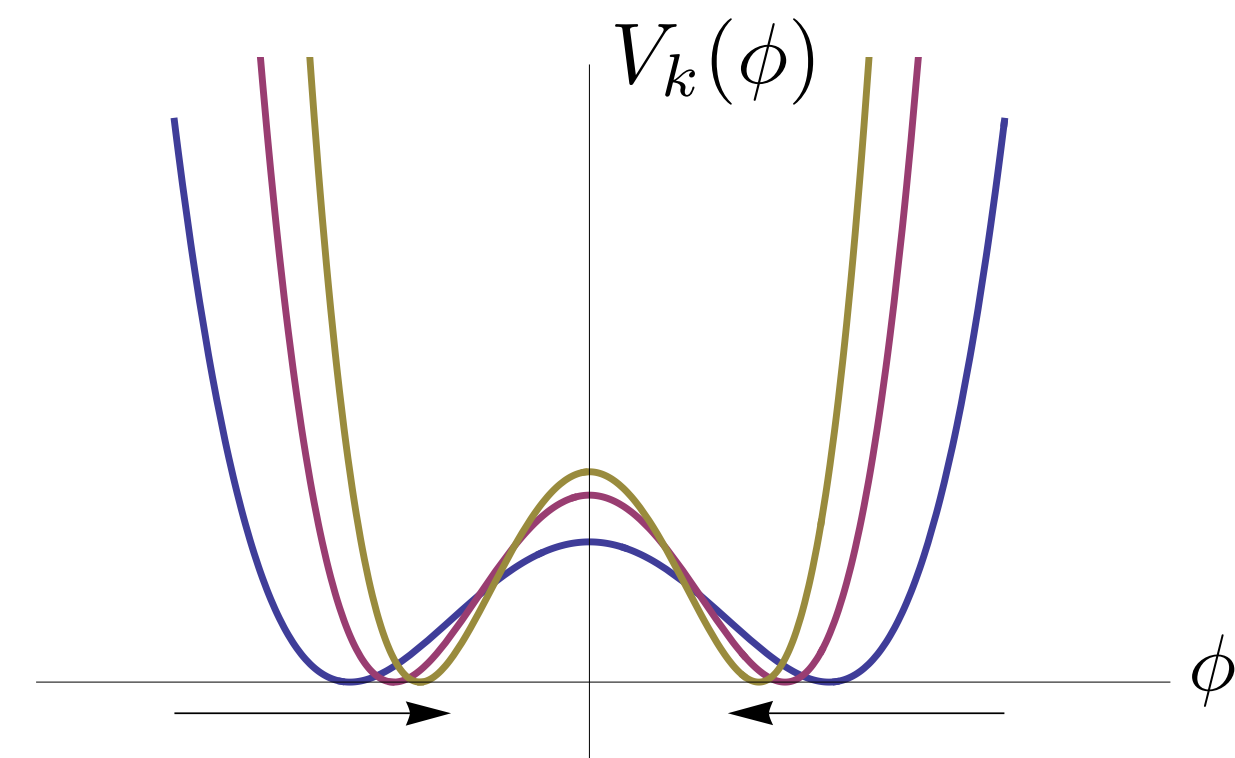
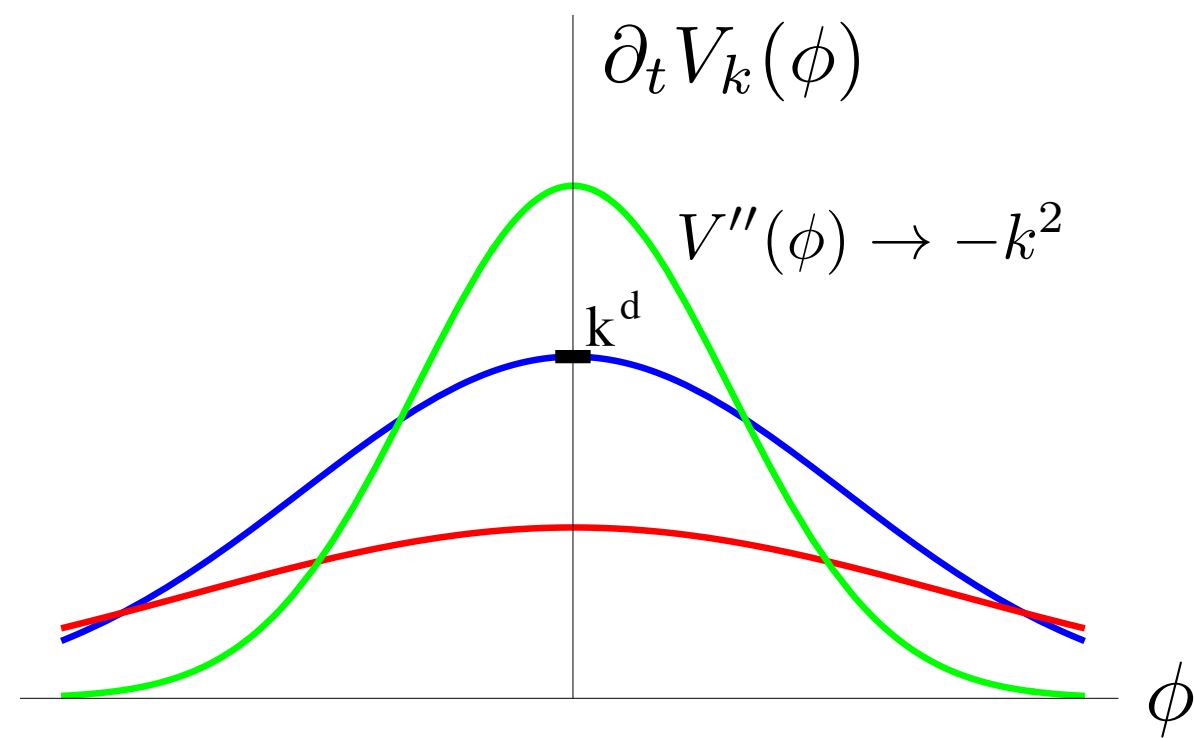
$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$\Gamma^{(2)}[\phi](p) + R_{k,\text{opt}}(p^2) = [k^2 + V''(\phi)] \theta(k^2 - p^2) + (p^2 + V''(\phi)) \theta(p^2 - k^2)$$

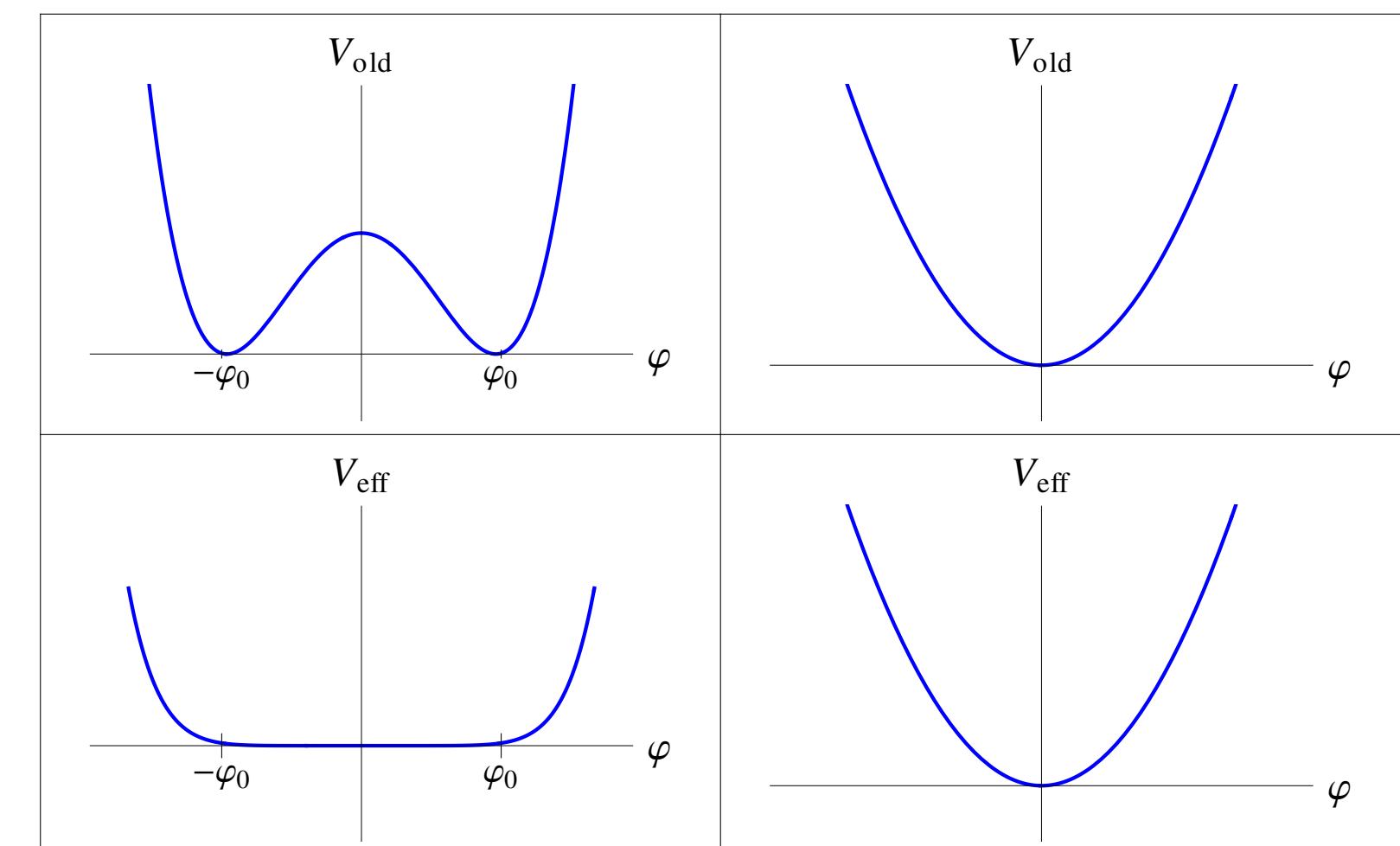
Spontaneous symmetry breaking

Approximation schemes & phase structure

$$\partial_t V_k(\phi) = \frac{1}{d} \frac{\Omega_d}{(2\pi)^d} \frac{k^{2+d}}{k^2 + V_k''(\phi)}$$

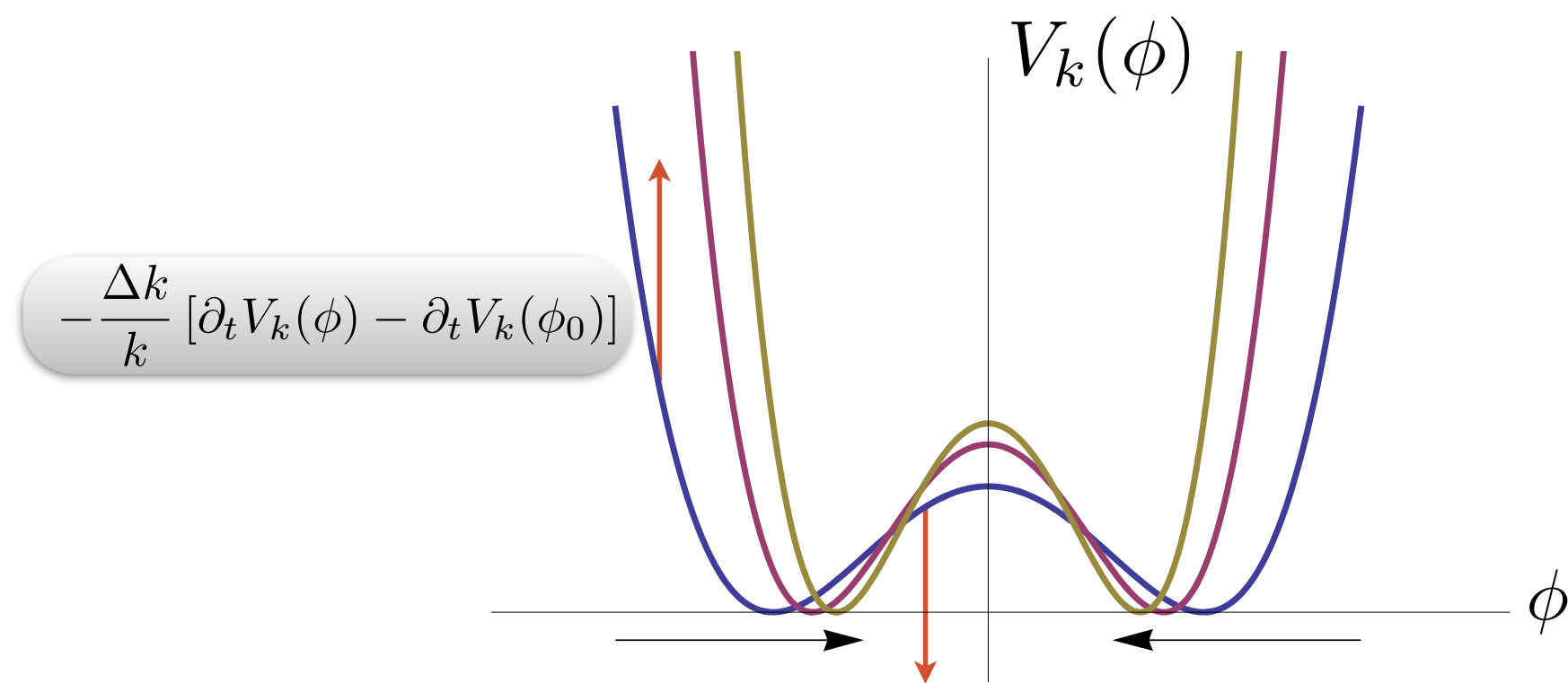
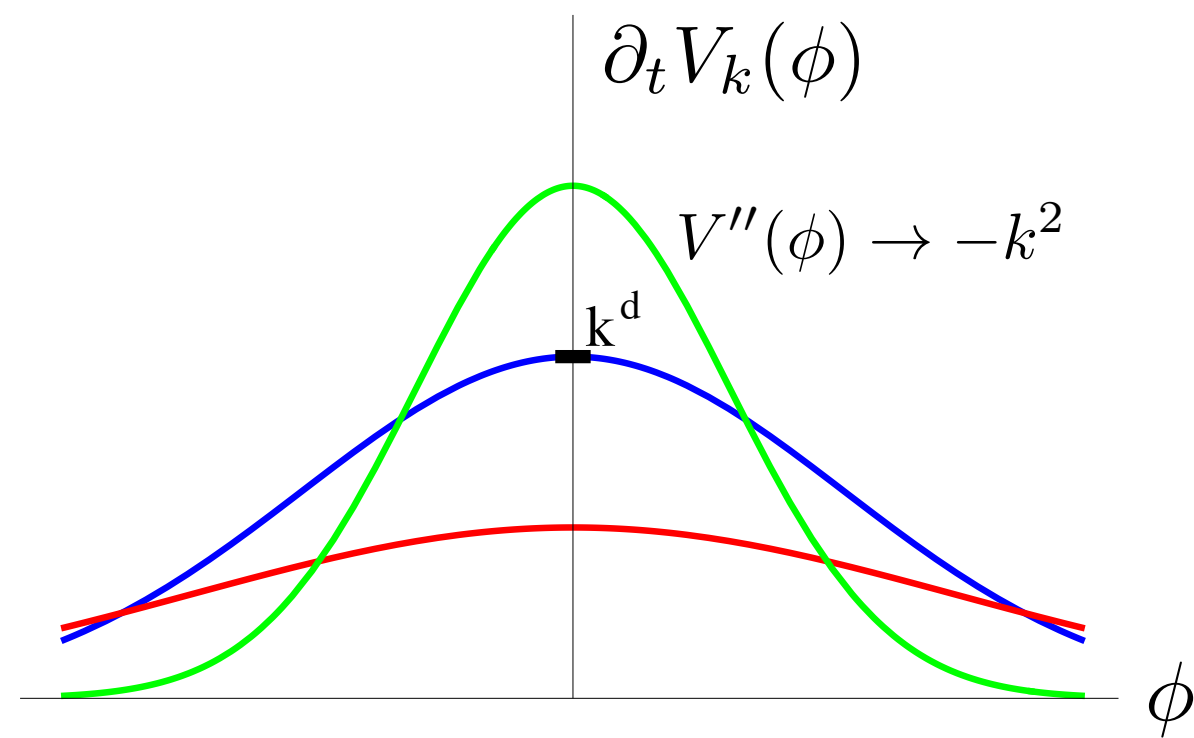


- bosonic flow is symmetry-restoring
- flow guarantees convexity of effective action

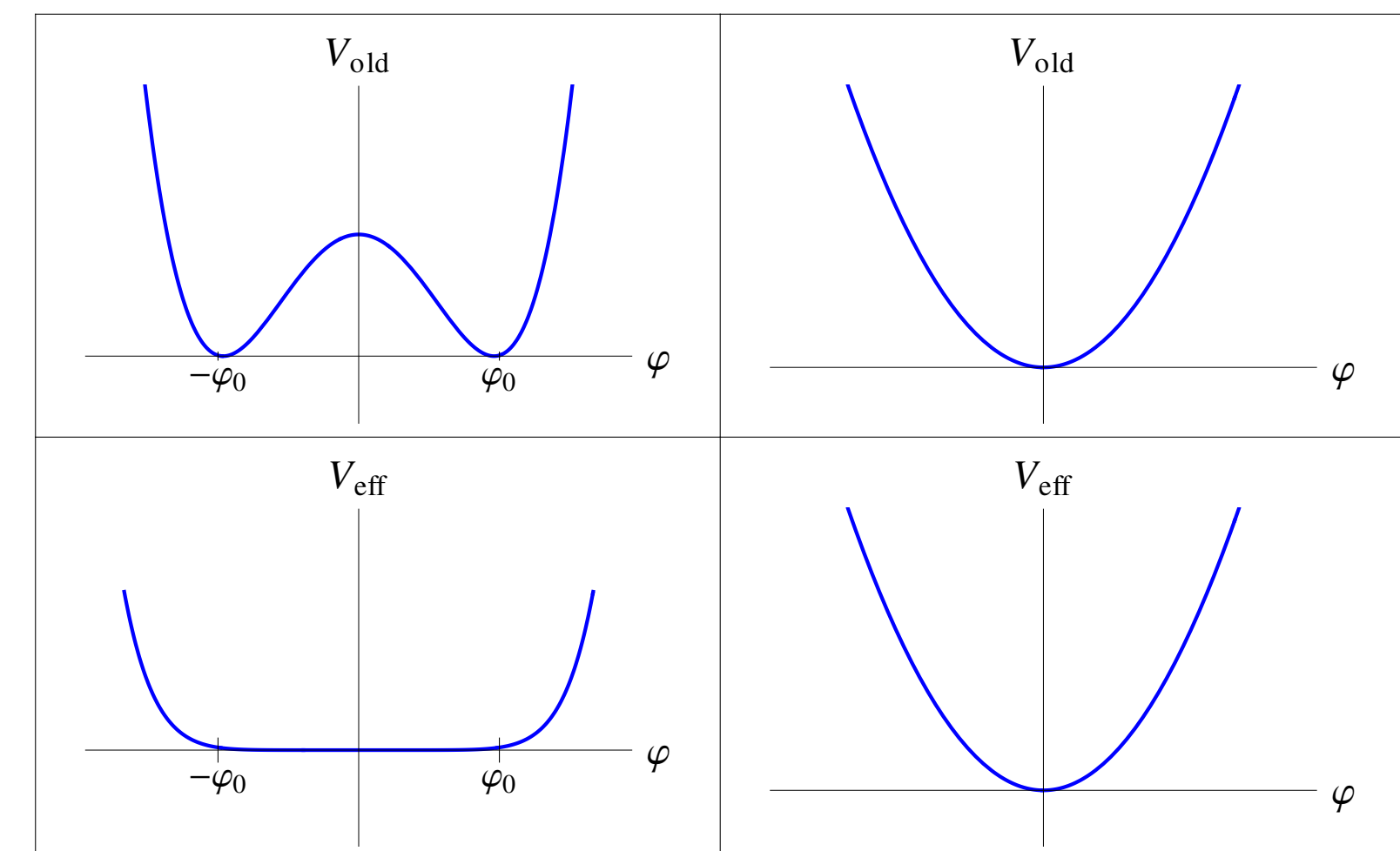


Approximation schemes & phase structure

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- **bosonic flow is symmetry-restoring**
- **flow guarantees convexity of effective action**

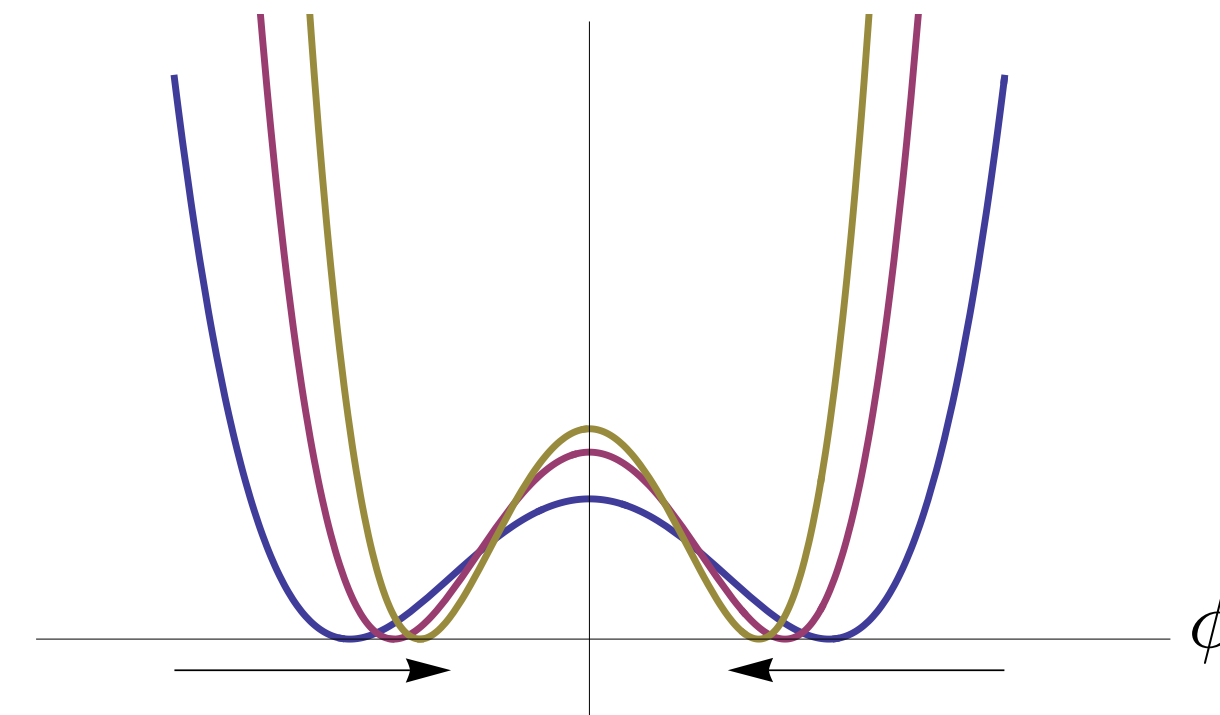
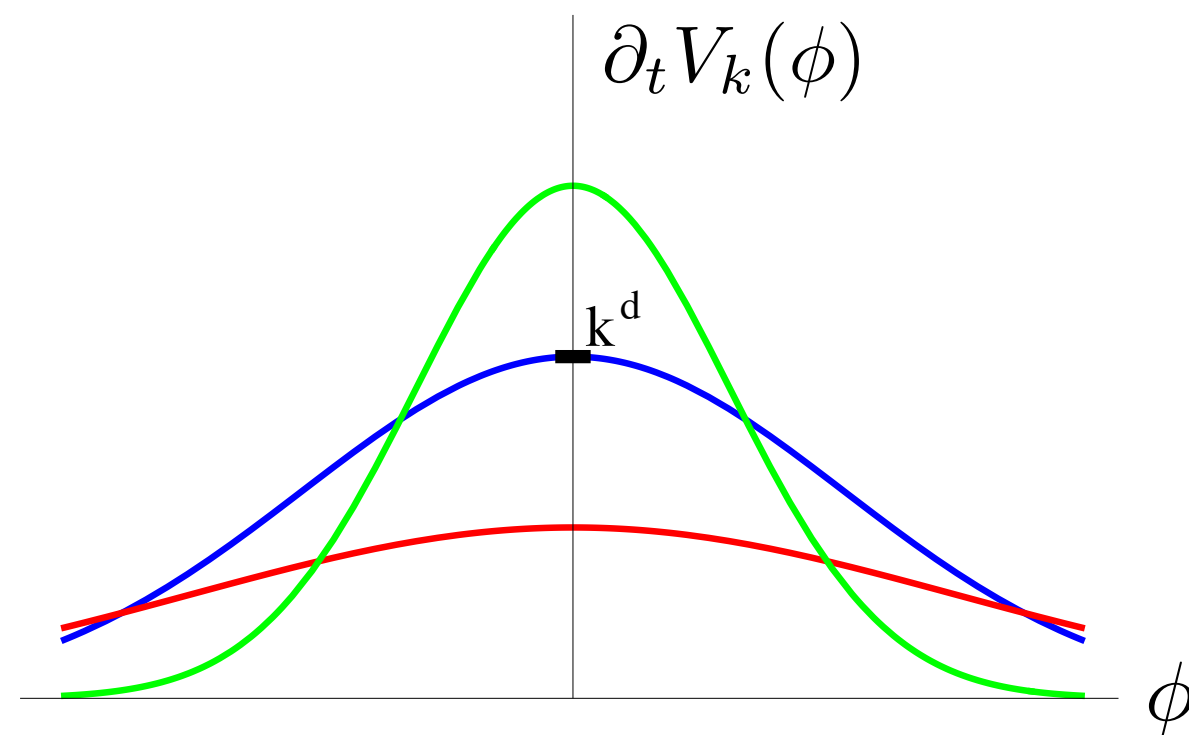


Approximation schemes & phase structure

$$\partial_t V_k(\phi) = \frac{k^d}{d} \frac{\Omega_d}{(2\pi)^d} \frac{1 - \frac{\eta_\phi}{d+2}}{1 + \frac{V_k''(\phi)}{k^2}}$$

Anomalous dimension

$$\eta_\phi = -\frac{\partial_t Z_\phi}{Z_\phi}$$



- bosonic flow is symmetry-restoring
- flow guarantees convexity of effective action

Litim, JMP, Vergara, hep-th/0602140

Example: 3d critical exponents with fRG

Simple approximation: LPA'

$$\Gamma_k[\phi] = \frac{1}{2} \int_p Z_\phi \phi(p) p^2 \phi(-p) + \int_x V_k(\phi)$$

Taylor expansion

$$V_k(\phi) = \sum_{n=1}^{N_{\max}} \frac{\lambda_n}{n!} (\phi^2 - \phi_{0,k}^2)^n$$

Ising universality

$$N = 1 : \nu_{\text{Ising}} = 0.630\dots$$

fRG: LPA'

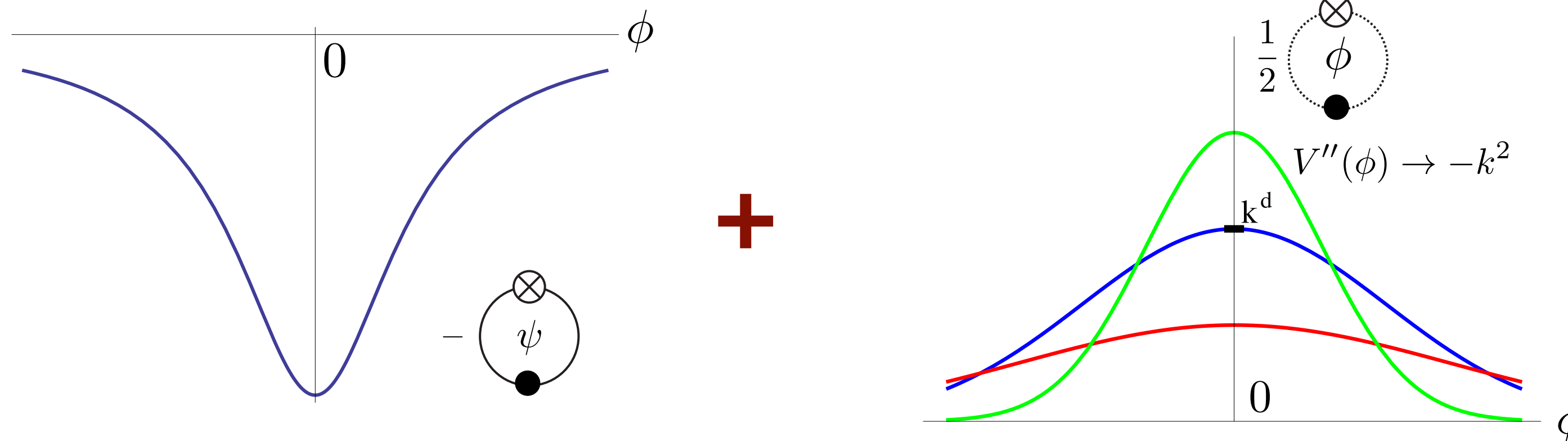
$$N = 1 : \nu_{\text{Ising}} = 0.637\dots$$

A simple program to compute critical exponents in O(N)-models with the Wetterich equation

Michael Scherer

Spontaneous symmetry breaking

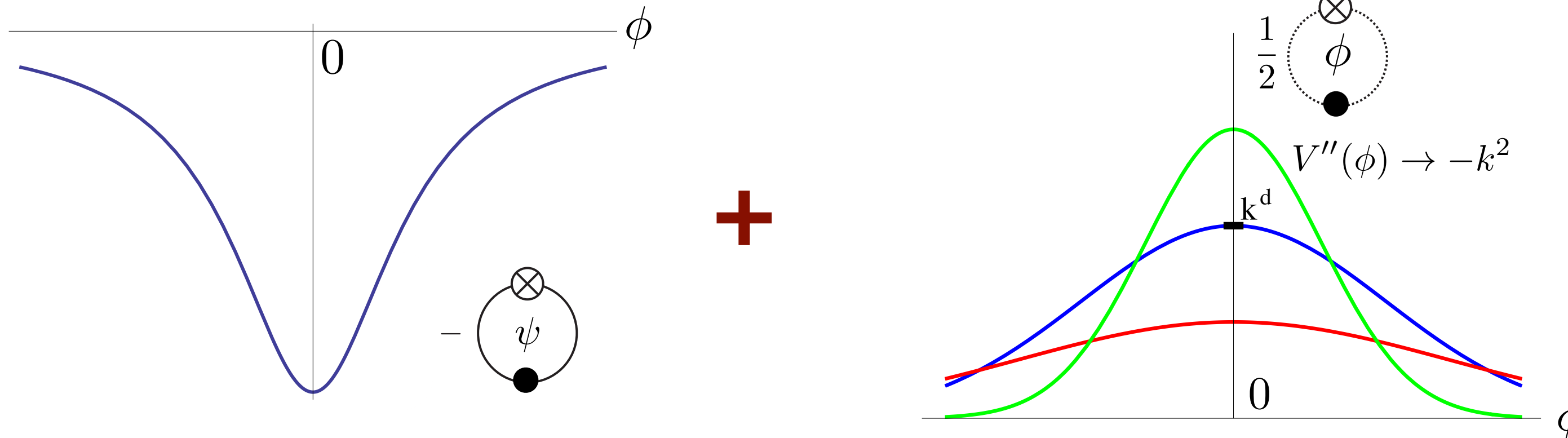
$$\partial_t V_k(\phi) = - \text{[Feynman diagram: circle with } \psi \text{ and a crossed circle]} + \frac{1}{2} \text{[Feynman diagram: circle with } \phi \text{ and a crossed circle]}$$



- bosonic flow is symmetry-restoring
- fermionic flow is symmetry-breaking
- competing dynamics decides about fate of symmetries
- flow guarantees convexity

Spontaneous symmetry breaking

$$\partial_t V_k(\phi) = - \text{[Feynman diagram: circle with } \psi \text{ and a cross]} + \frac{1}{2} \text{[Feynman diagram: dashed circle with } \phi \text{ and a cross]}$$

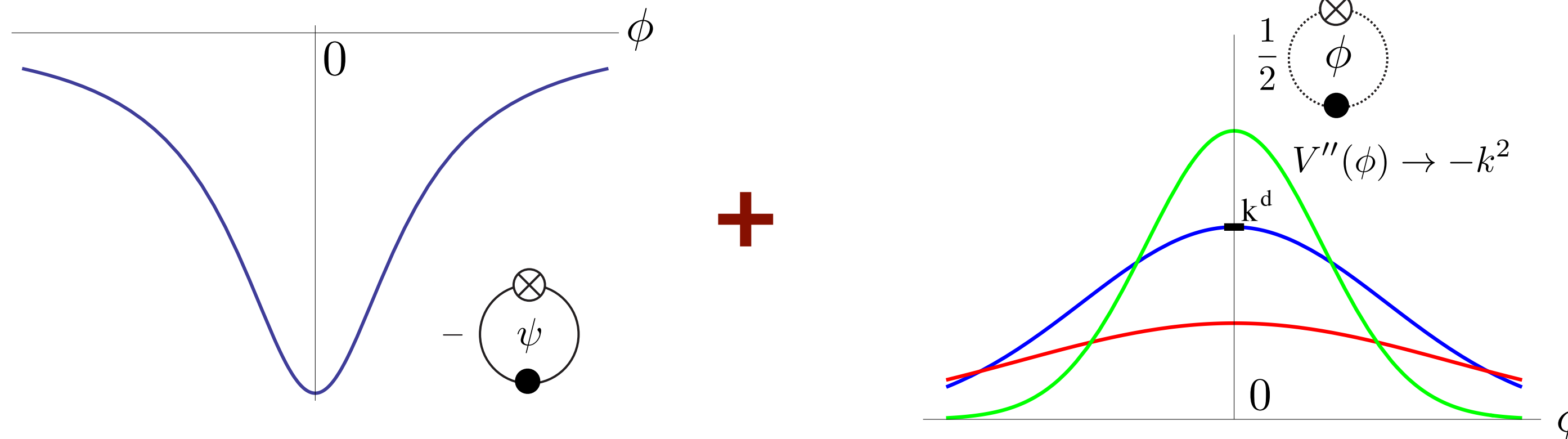


‘governs general phase structures’

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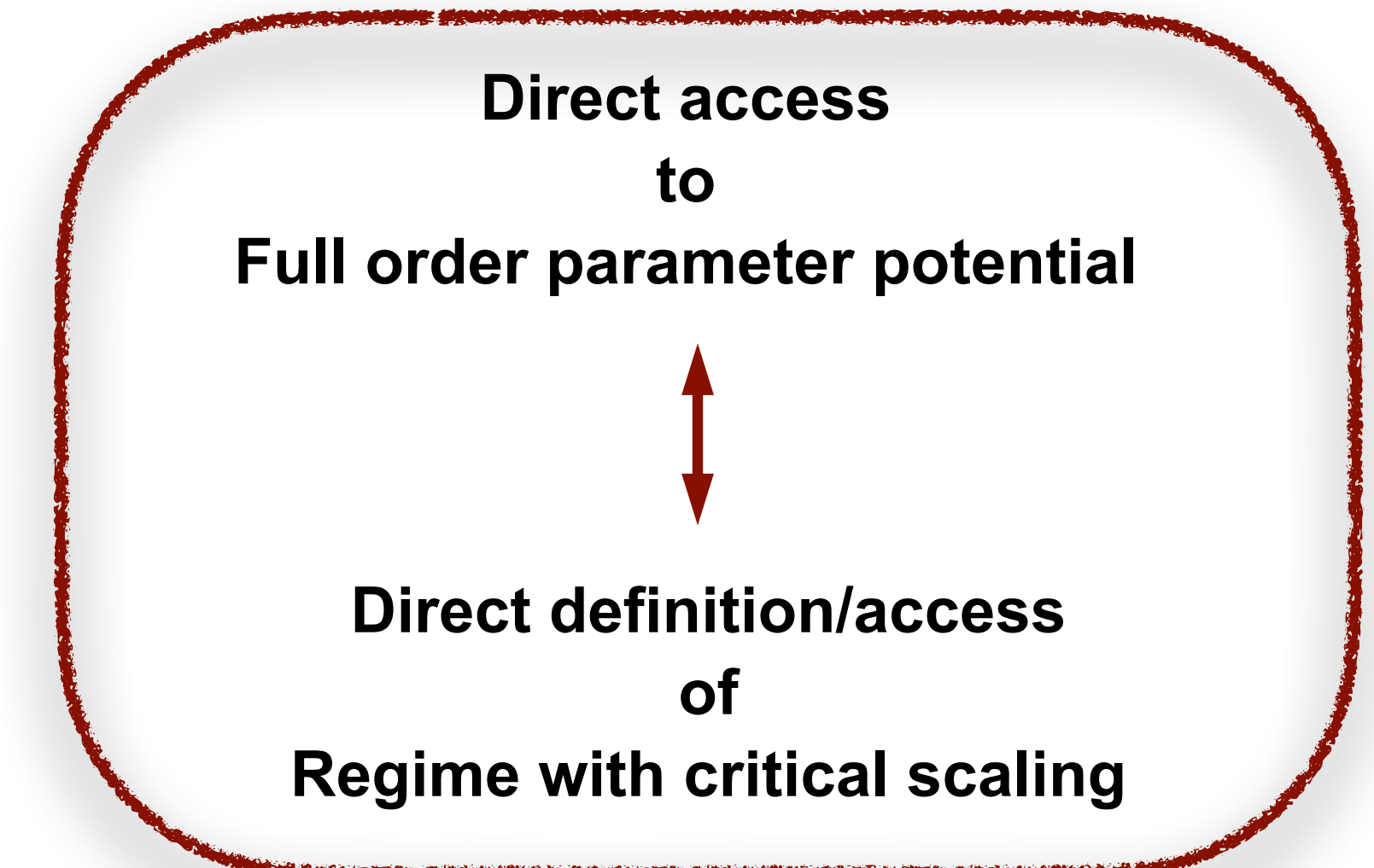
Spontaneous symmetry breaking

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‘governs general phase structures’

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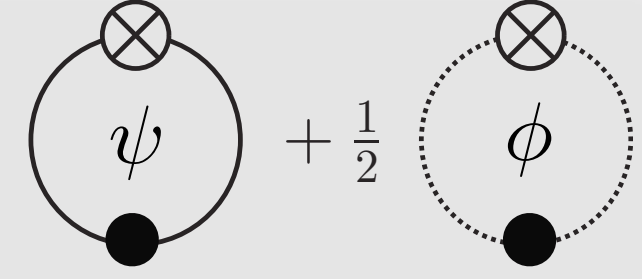
Spontaneous symmetry breaking

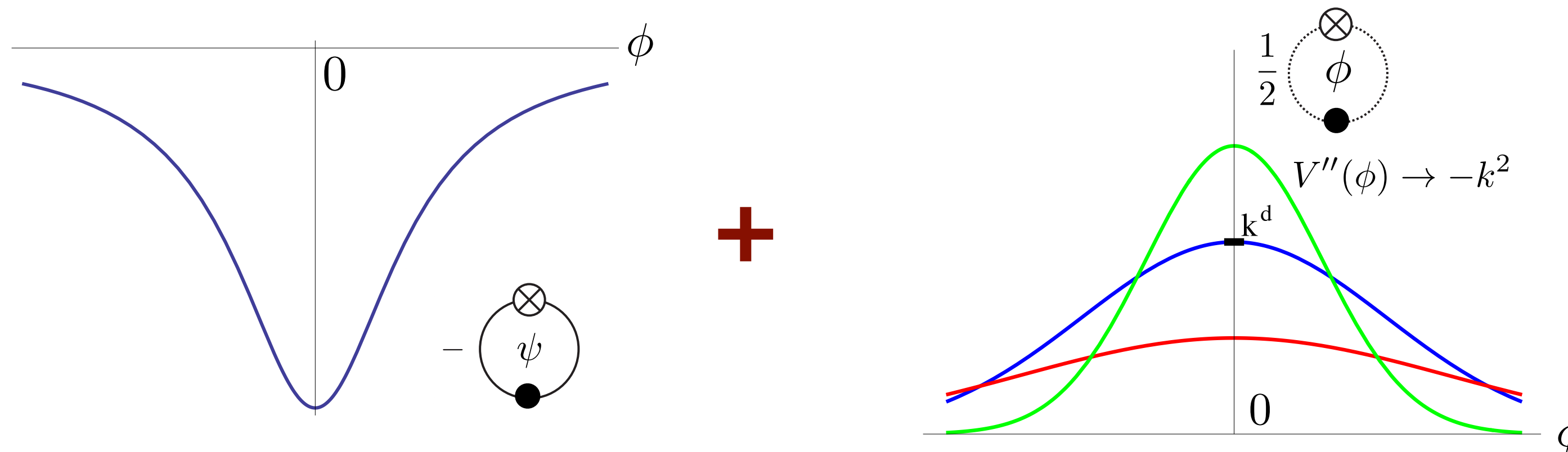
Initiating the hydro era (2019)

Grossi, Wink, SciPost Phys. Core 6 (2023)

State of the art time steppers

Ihssen, Sattler, Wink, CPC 300 (2024) 109182

$$\partial_t V_k(\phi) = - \text{diagram} + \frac{1}{2} \text{diagram}$$




‘governs general phase structures’

- bosonic flow is symmetry-restoring
- fermionic flow is symmetry-breaking
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Direct access
to
Full order parameter potential



Direct definition/access
of
Regime with critical scaling

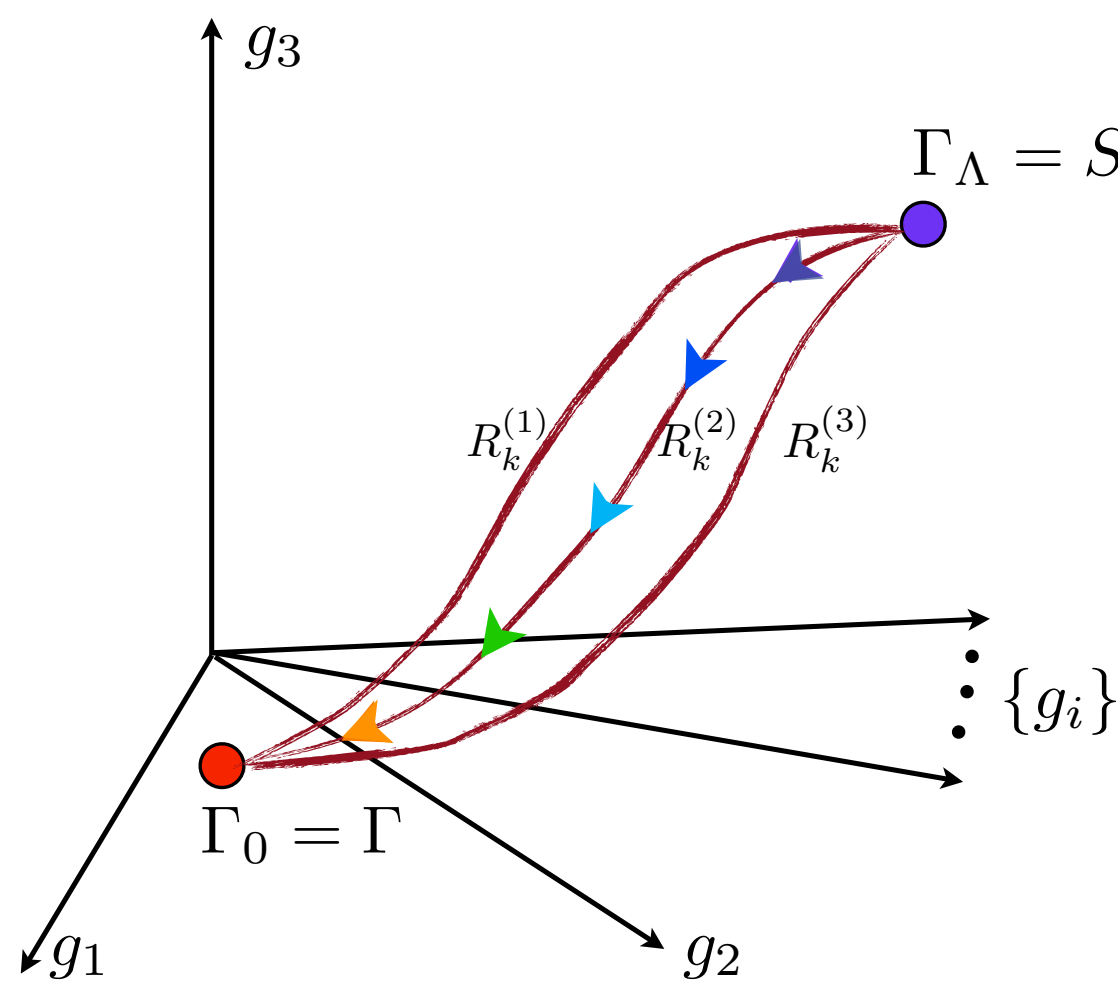
Jump forward to: Functional flows for QCD

Systematic error control & optimisation

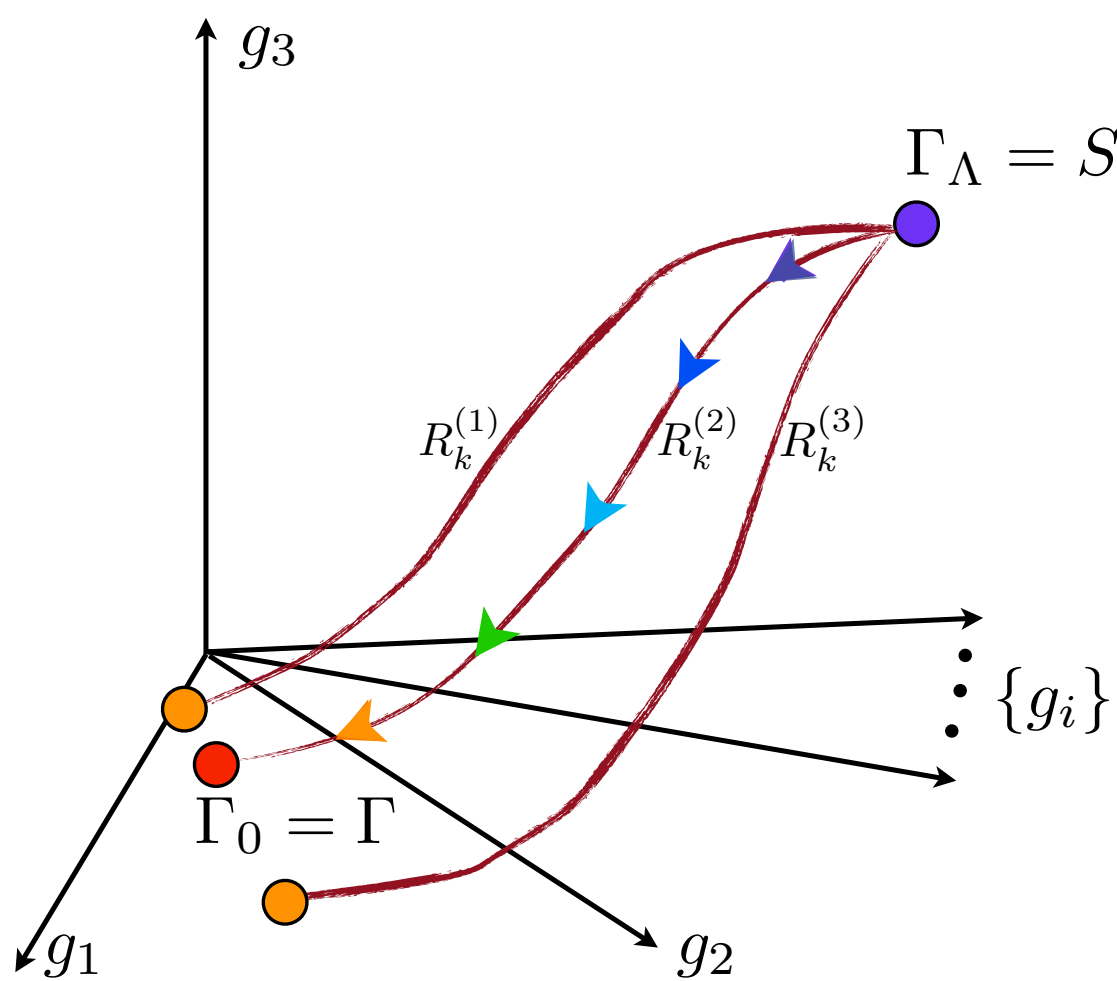
Systematic error control & optimisation

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$

Theory space



full flow



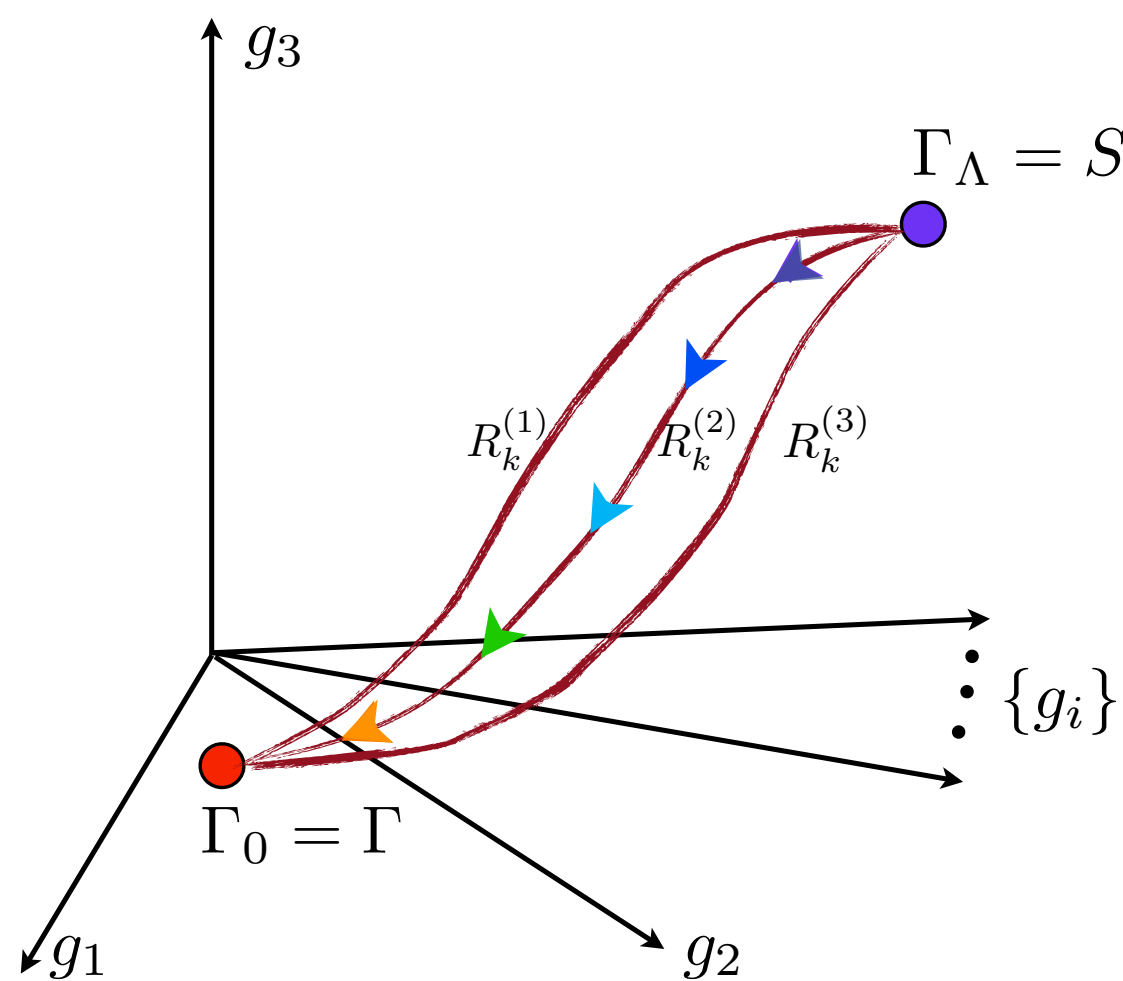
approximated flow

Optimisation: find $R_k^{(2)}$!

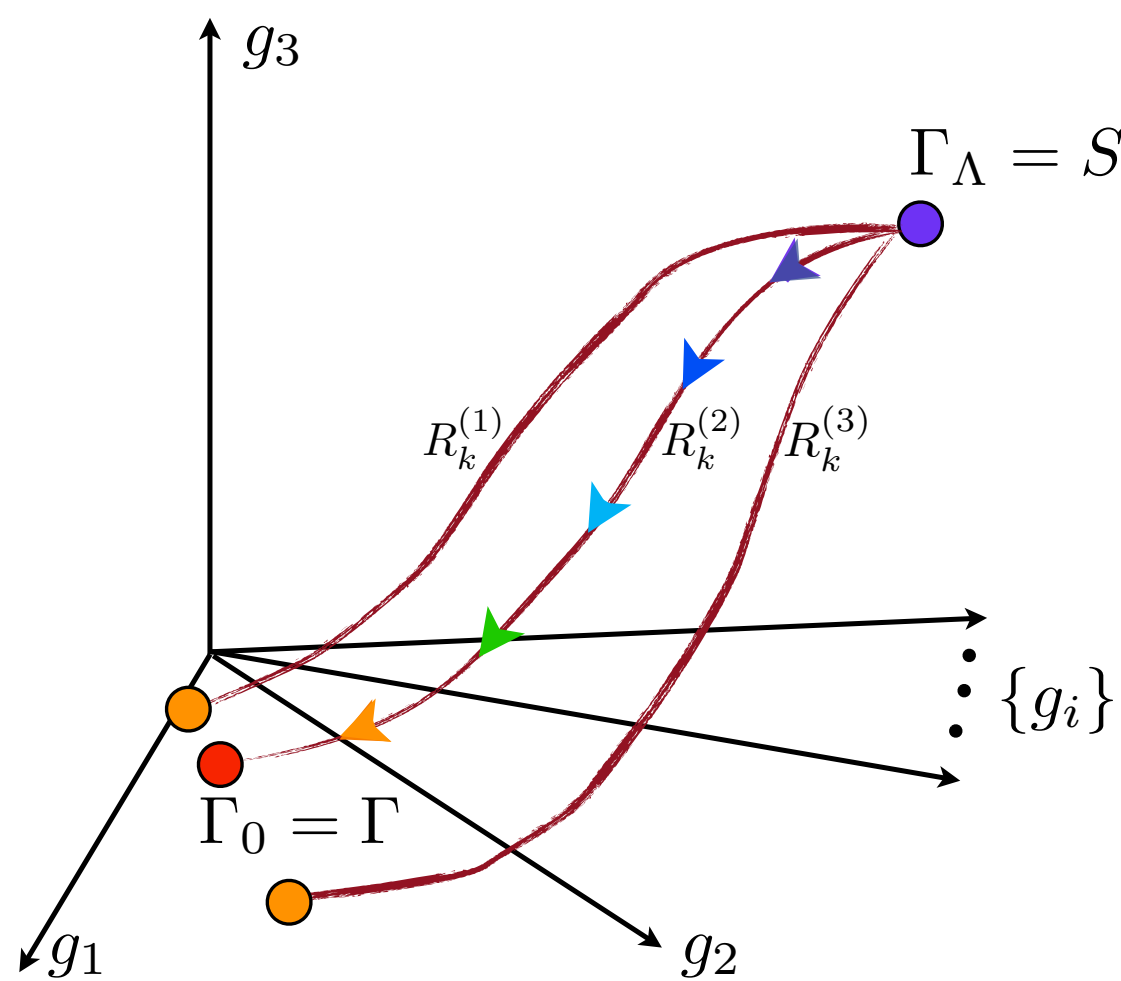
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Theory space



full flow



approximated flow

Optimisation: find $R_k^{(2)}$!

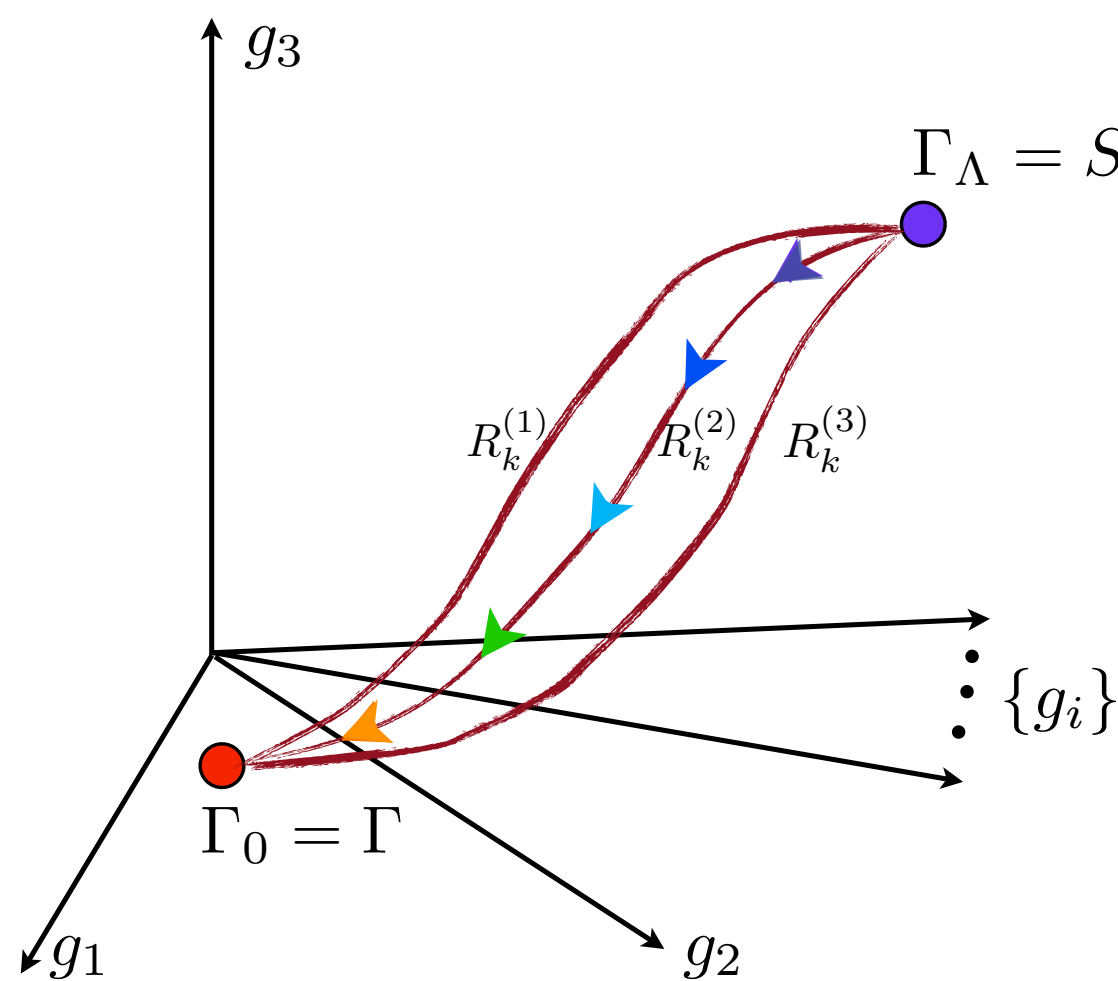
Principle of minimal sensitivity

eg. Liao, Polonyi, Strickland, NPB 567 (2000) 493-514
Canet, Delamotte, Mouhanna, Vidal, PRD 67 (2003) 065004

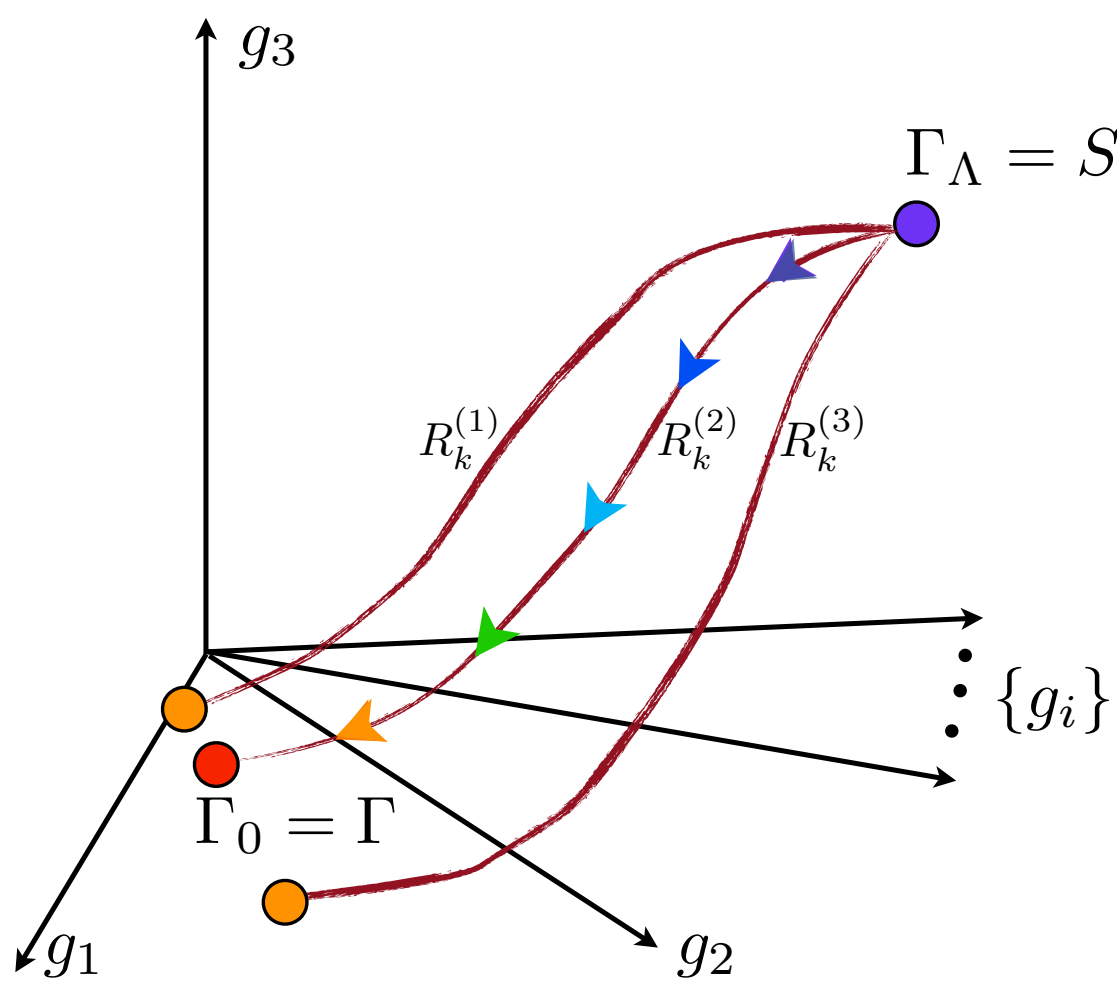
Systematic error control & optimisation

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$

Theory space



full flow



approximated flow

Optimisation: find $R_k^{(2)}$!

Most rapid convergence at fixed points

Litim, PLB 486 (2000) 92-99

Principle of minimal sensitivity

eg. Liao, Polonyi, Strickland, NPB 567 (2000) 493-514
Canet, Delamotte, Mouhanna, Vidal, PRD 67 (2003) 065004

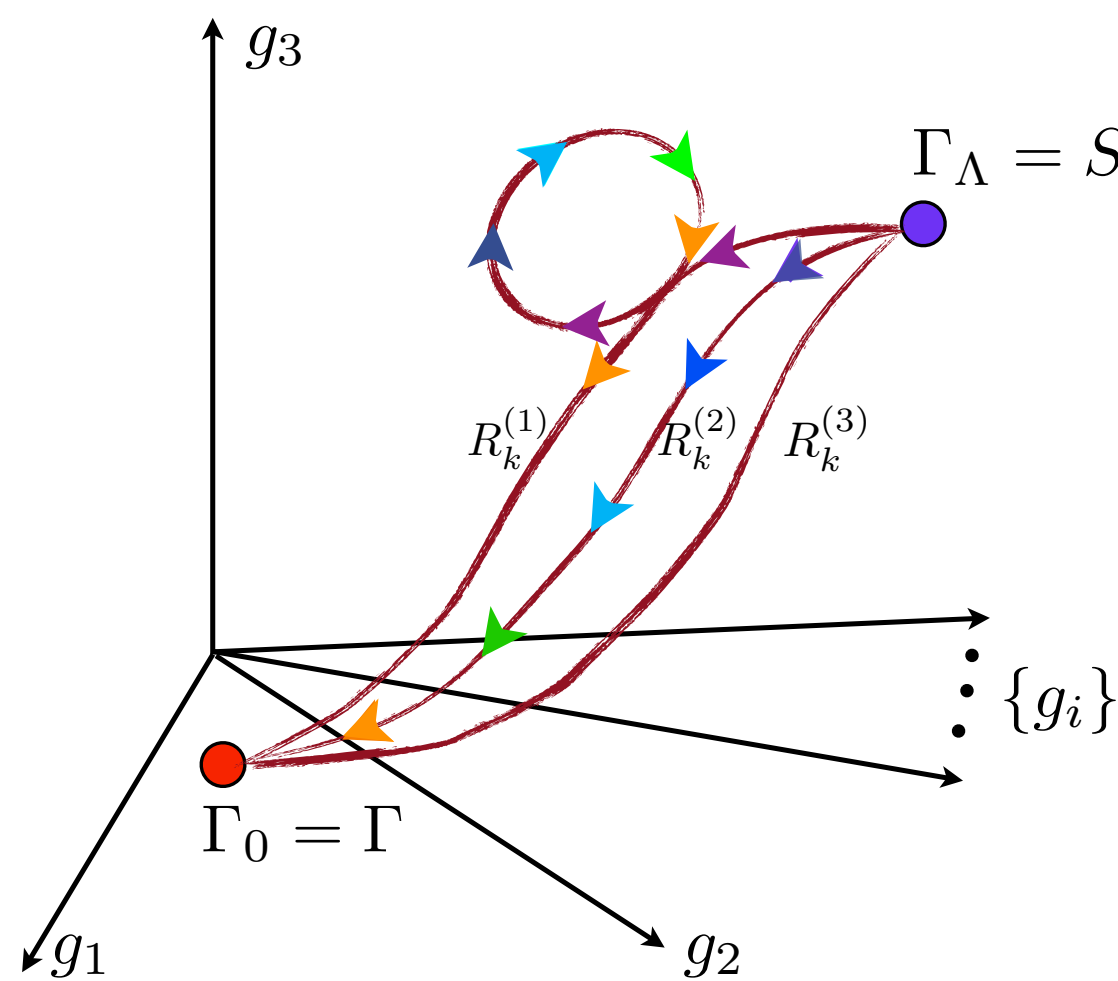
Functional optimisation: Integrability

JMP, AP 322 (2007) 2831
JMP, Scherer, Schmidt, Wetzel, AP 384 (2017) 165
RG-consistency: Braun, Leonhardt, JMP, SciPost Phys. 6 (2019) 5, 056

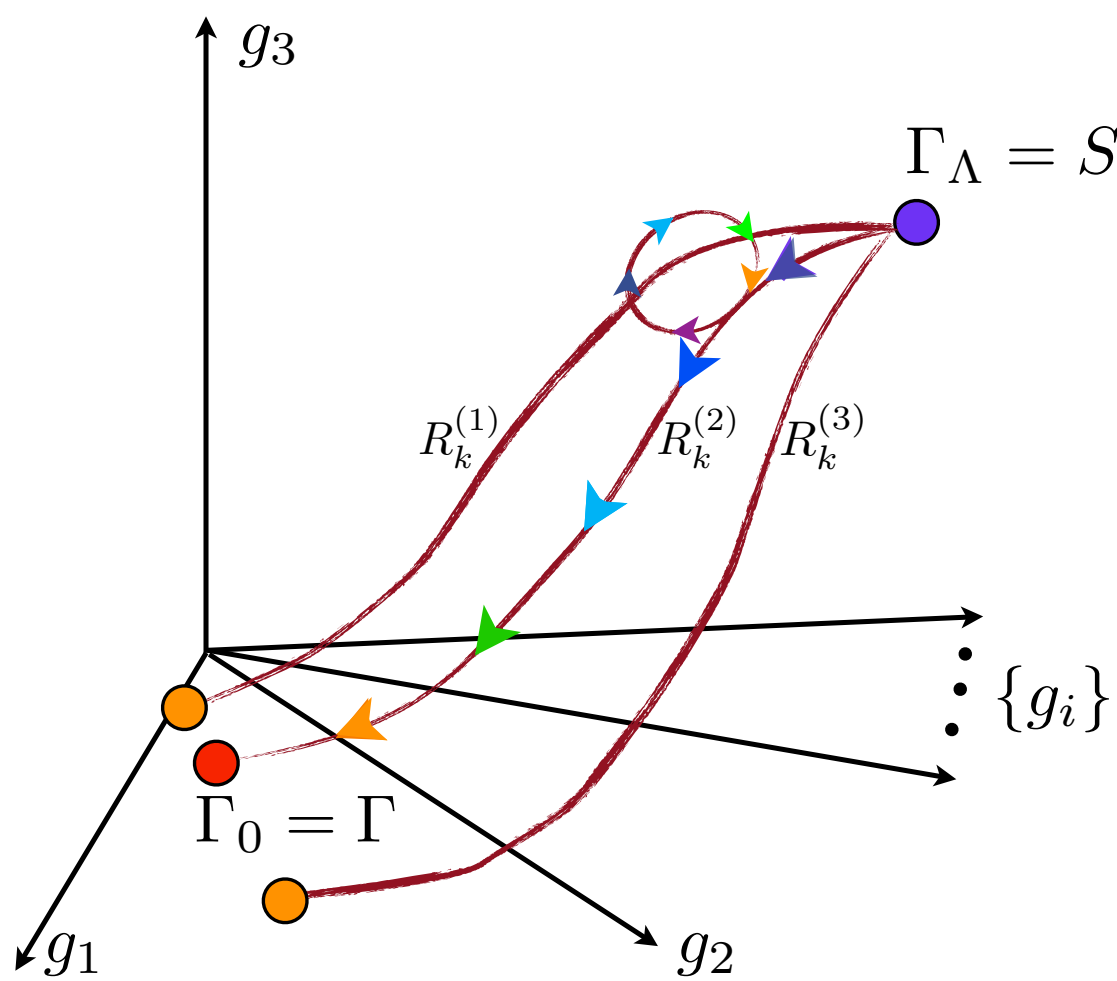
Systematic error control & optimisation

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k$$

Theory space



full flow



Optimised flow

Optimisation: find $R_k^{(2)}$!

$$\lim_{L \rightarrow 0} \frac{1}{L} \text{ (circular flow diagram) } \rightarrow 0$$

Functional optimisation: Integrability

JMP, AP 322 (2007) 2831

JMP, Scherer, Schmidt, Wetzel, AP 384 (2017) 165

RG-consistency: Braun, Leonhardt, JMP, SciPost Phys. 6 (2019) 5, 056

Just a reminder: It was your choice!

The phase structure of QCD

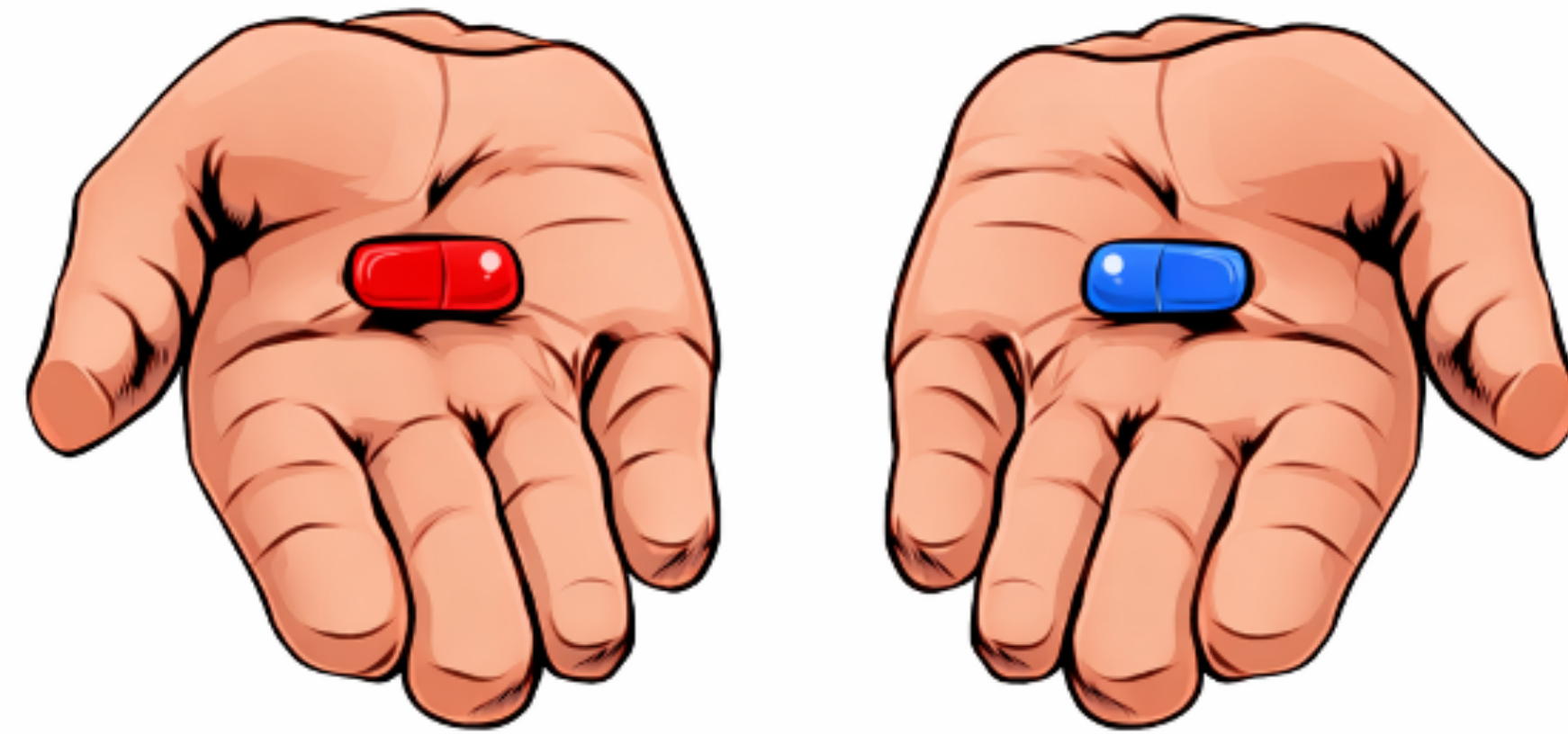
from functional the RG with all its g(l)ory details



How to compute

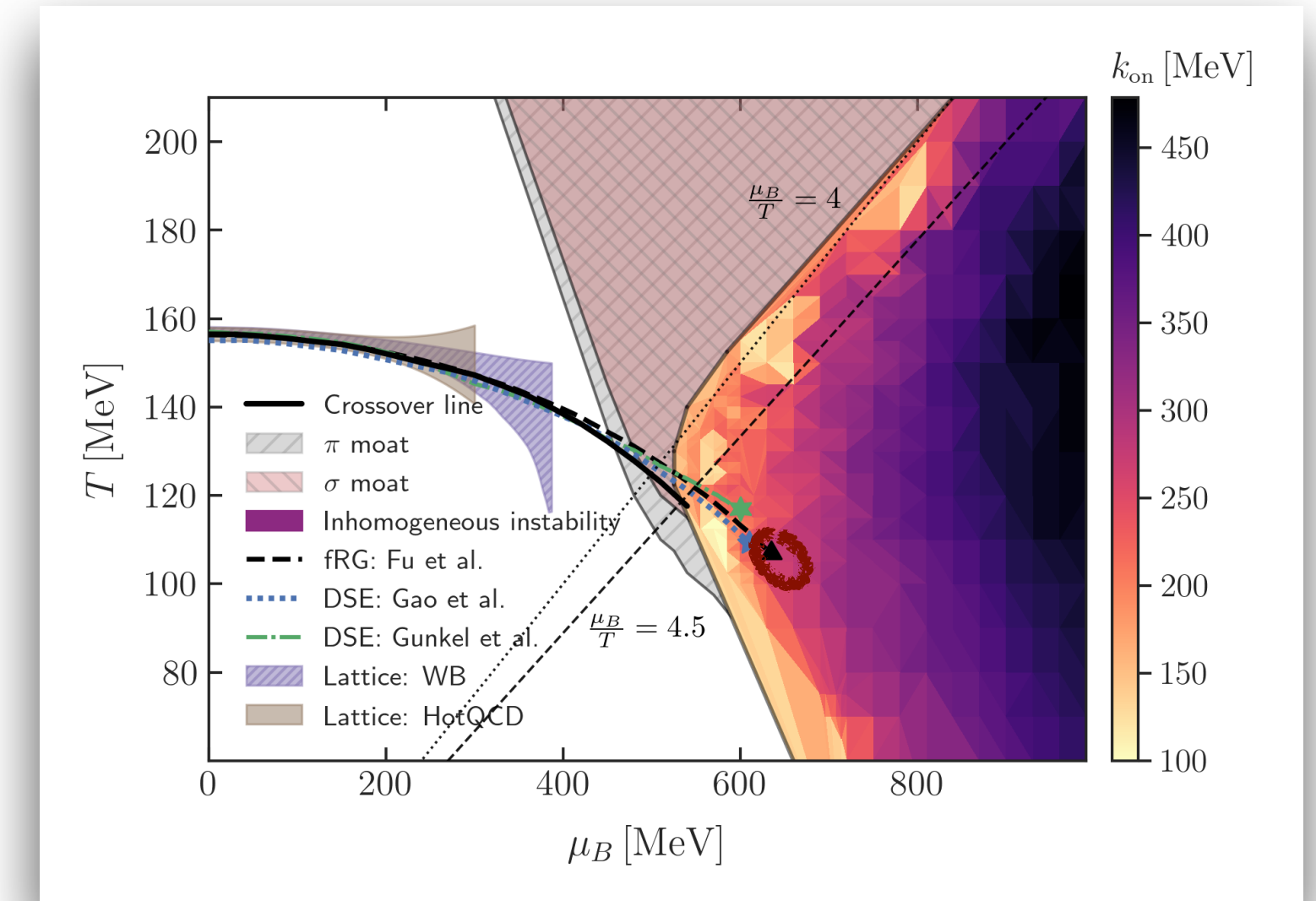
&

How to judge



Colloquium on the phase structure of QCD

from functional methods



Jump back to: Phase structure from functional QCD: how to

Functional flows for QCD

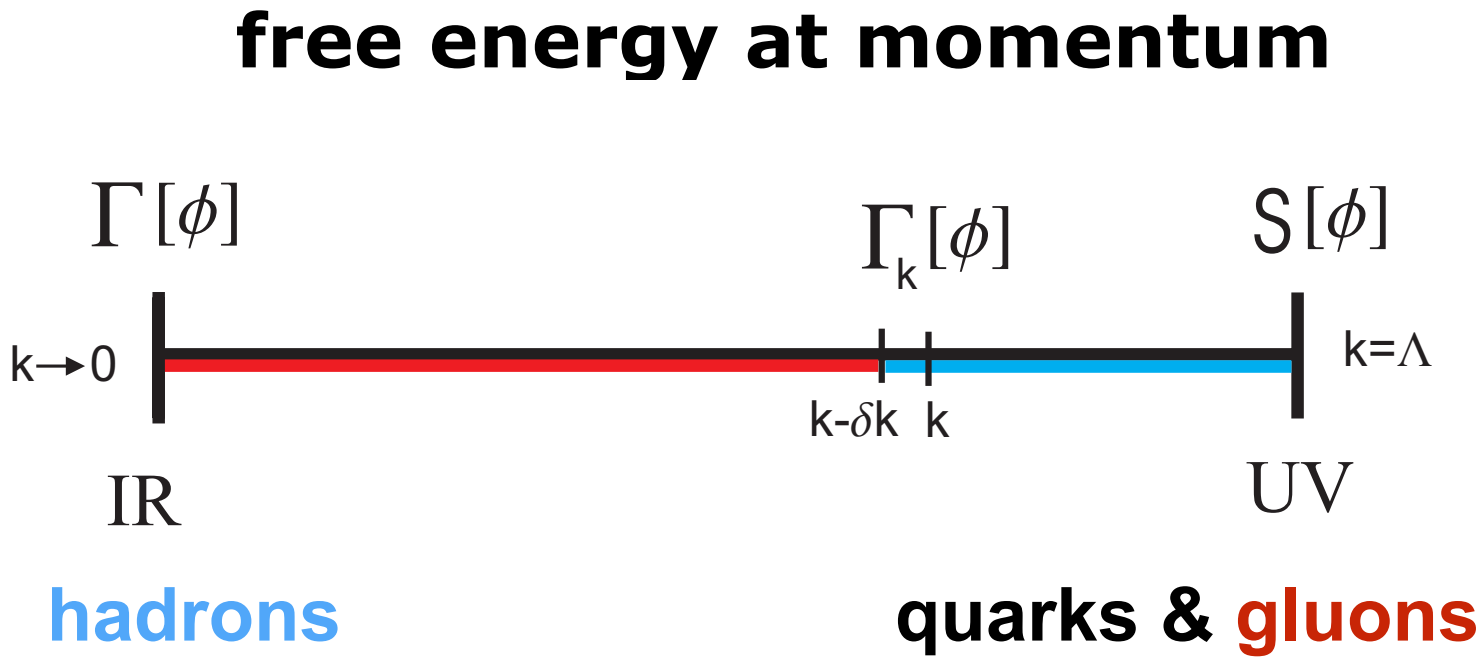
Jump back to: Phase structure from functional QCD: how to

Flows for correlation functions & chiral symmetry breaking

Functional flows for QCD

Dupuis et al, Phys.Rept. (2021)

Fu, Commun.Theor.Phys. 74 (2022) 9, 097304



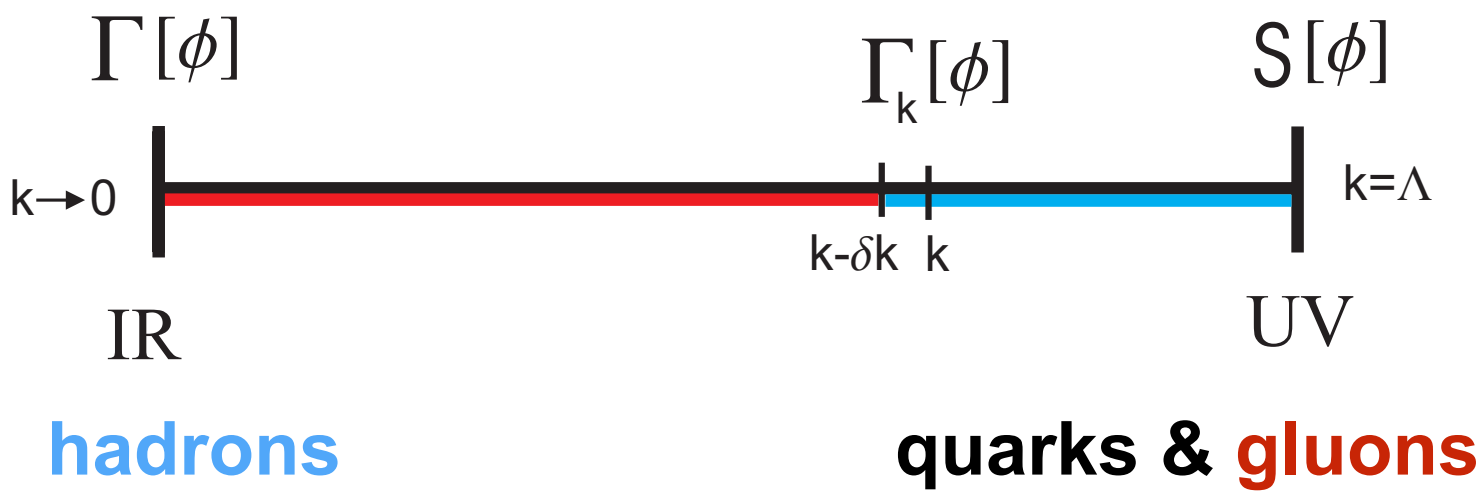
ab initio

Functional flows for QCD

Dupuis et al, Phys.Rept. (2021)

Fu, Commun.Theor.Phys. 74 (2022) 9, 097304

free energy at momentum



ab initio

functional RG:

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \left(\text{glue quantum fluctuations} - \text{quark quantum fluctuations} \right)$$

free energy/
grand potential

quark
quantum fluctuations

$$\Phi = (A_\mu, c, \bar{c}, q, \bar{q})$$

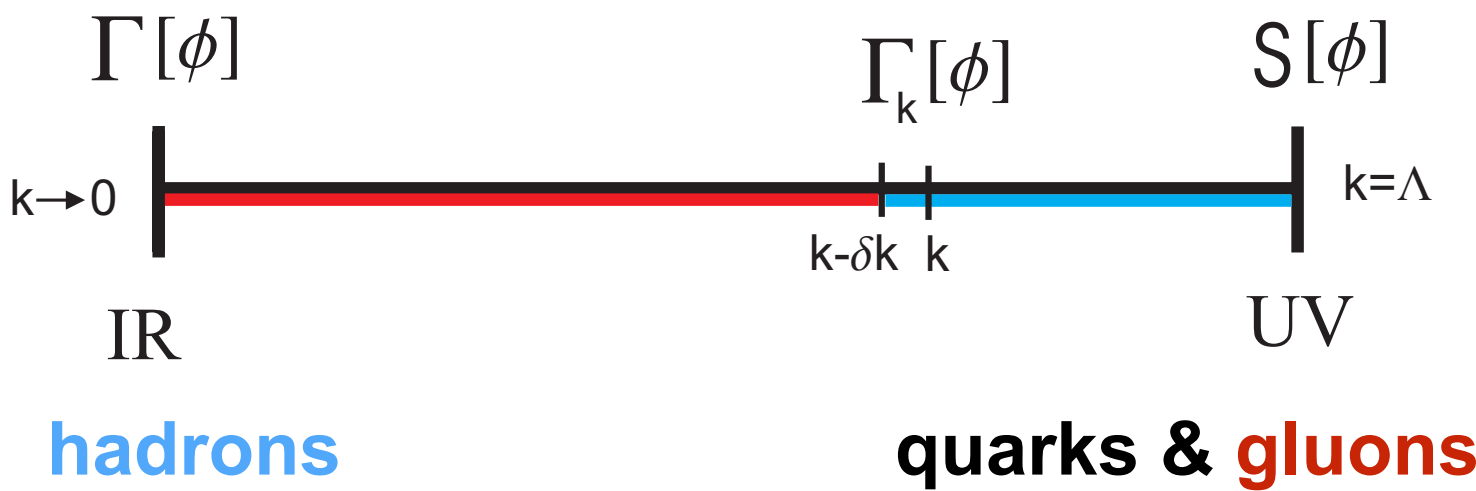
RG-scale k : $t = \ln k$

Functional flows for QCD

Dupuis et al, Phys.Rept. (2021)

Fu, Commun.Theor.Phys. 74 (2022) 9, 097304

free energy at momentum



ab initio

fRG approach with emergent composites/dynamical hadronisation

functional RG: $\left(\partial_t + \int_x \dot{\Phi} \frac{\delta}{\delta \Phi}\right) \Gamma_k[\Phi] =$

$\frac{1}{2}$

-

$\frac{1}{2}$

-

+

$\frac{1}{2}$

$\Phi = (A_\mu, c, \bar{c}, q, \bar{q}, \phi)$

glue
quantum fluctuations

quark
quantum fluctuations

hadronic
quantum fluctuations

RG-scale k: $t = \ln k$

$\partial_t R =$ $\Gamma^{(n)} =$

$\left(\partial_t + 2 \frac{\delta \dot{\Phi}}{\delta \Phi}\right) R =$ $G =$

ϕ hadronic composites

Flows for correlation functions

functional RG:

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \left(\text{glue quantum fluctuations} - \text{quark quantum fluctuations} \right)$$

free energy/
grand potential

glue
quantum fluctuations

quark
quantum fluctuations

The diagram illustrates the functional RG equation for the flow of the effective action Γ_k . The equation is: $\partial_t \Gamma_k[\Phi] = \frac{1}{2} \left(\text{glue quantum fluctuations} - \text{quark quantum fluctuations} \right)$. The terms are represented by Feynman diagrams: a red wavy loop for glue quantum fluctuations, a dashed black loop for quark quantum fluctuations, and a solid black loop for quark quantum fluctuations. The final term is a blue double-line loop, also representing quark quantum fluctuations. Each diagram has a purple diamond vertex at the top.

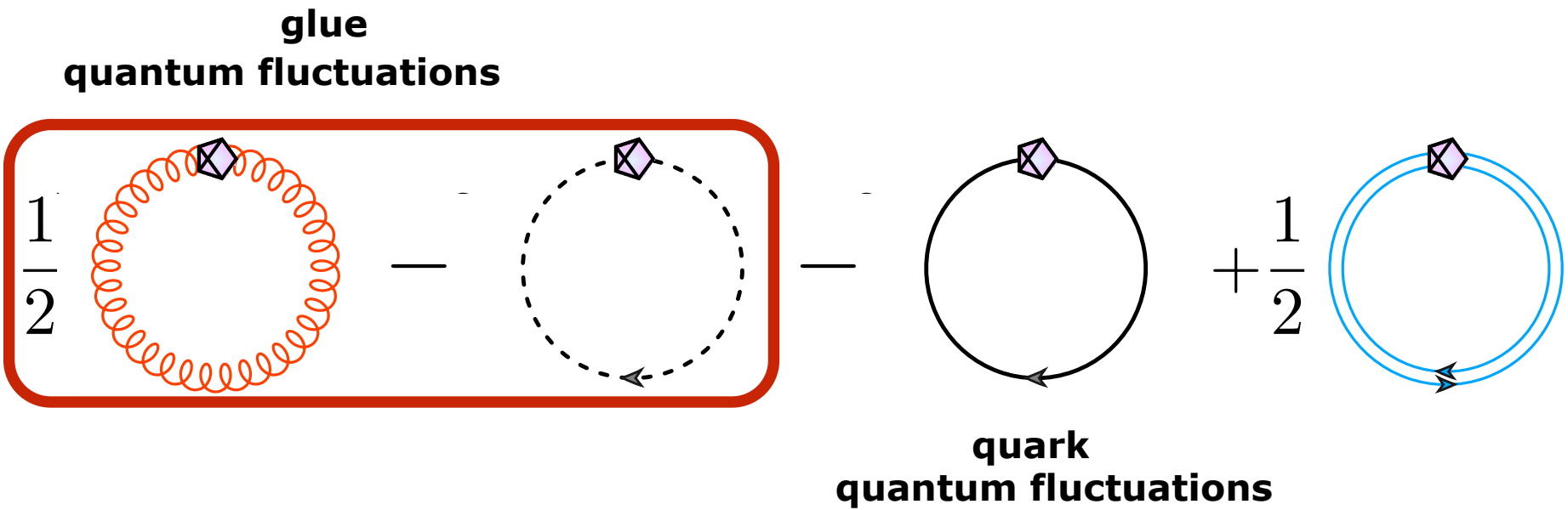
Correlation functions

Flows for correlation functions

functional RG:

$$\partial_t \Gamma_k[\Phi] =$$

free energy/
grand potential



gluon propagator

$$\langle A_\mu A_\nu \rangle(p)$$

Pure glue

Correlation functions

Flows for correlation functions

functional RG: $\partial_t \Gamma_k[\Phi] =$ **free energy/
grand potential**

**glue
quantum fluctuations**

**quark
quantum fluctuations**

gluon propagator

$$\langle A_\mu A_\nu \rangle(p)$$

Correlation functions

Pure glue

... 1-loop exact

Flows for correlation functions

no hadronic composites

$$\phi = 0$$

functional RG: $\partial_t \Gamma_k[\Phi] = \frac{1}{2}$

free energy/
grand potential

glue
quantum fluctuations

quark
quantum fluctuations

Correlation functions

gluon propagator

$$\langle A_\mu A_\nu \rangle(p)$$

quark propagator

$$\langle q \bar{q} \rangle(p)$$

Flows for correlation functions

no hadronic composites

$$\phi = 0$$

functional RG:

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \left(\text{glue quantum fluctuations} - \text{quark quantum fluctuations} \right)$$

free energy/
grand potential

glue quantum fluctuations

quark quantum fluctuations

Correlation functions

gluon propagator

$$\langle A_\mu A_\nu \rangle(p)$$

quark propagator

$$\langle q \bar{q} \rangle(p)$$

quark-gluon vertex

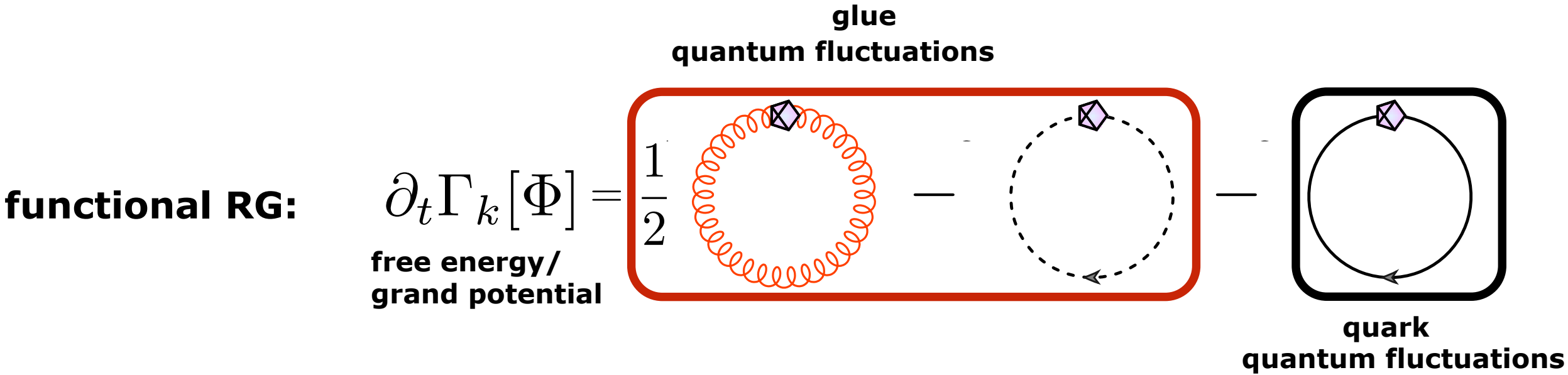
$$\langle q \bar{q} A_\mu \rangle(p_1, p_2)$$

Eight tranverse tensor structures

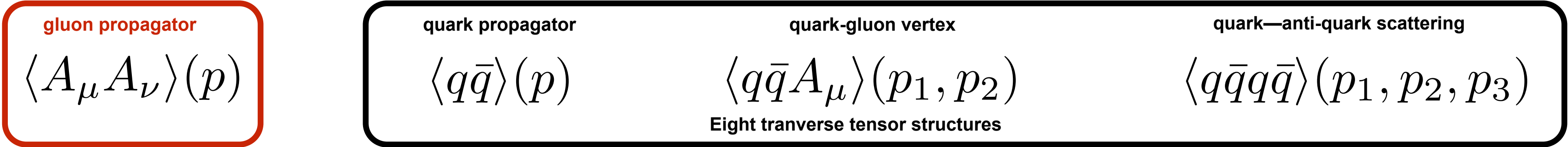
Flows for correlation functions

no hadronic composites

$$\phi = 0$$



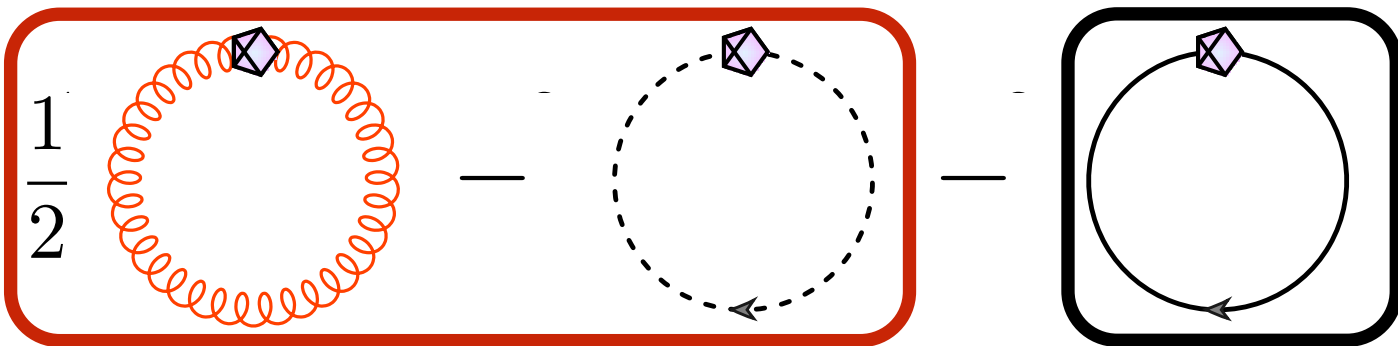
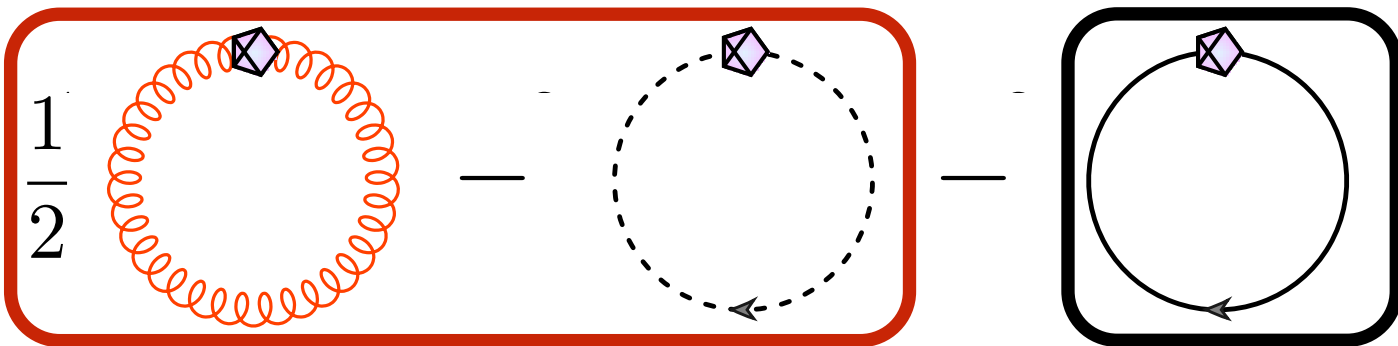
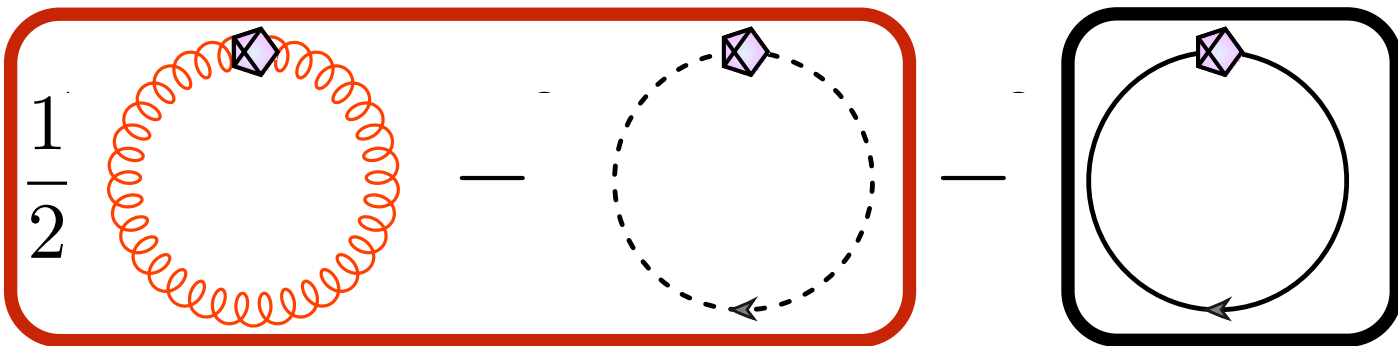
Correlation functions



Flows for correlation functions

no hadronic composites

$$\phi = 0$$

functional RG: $\partial_t \Gamma_k[\Phi] = \frac{1}{2}$  $-$  $-$ 

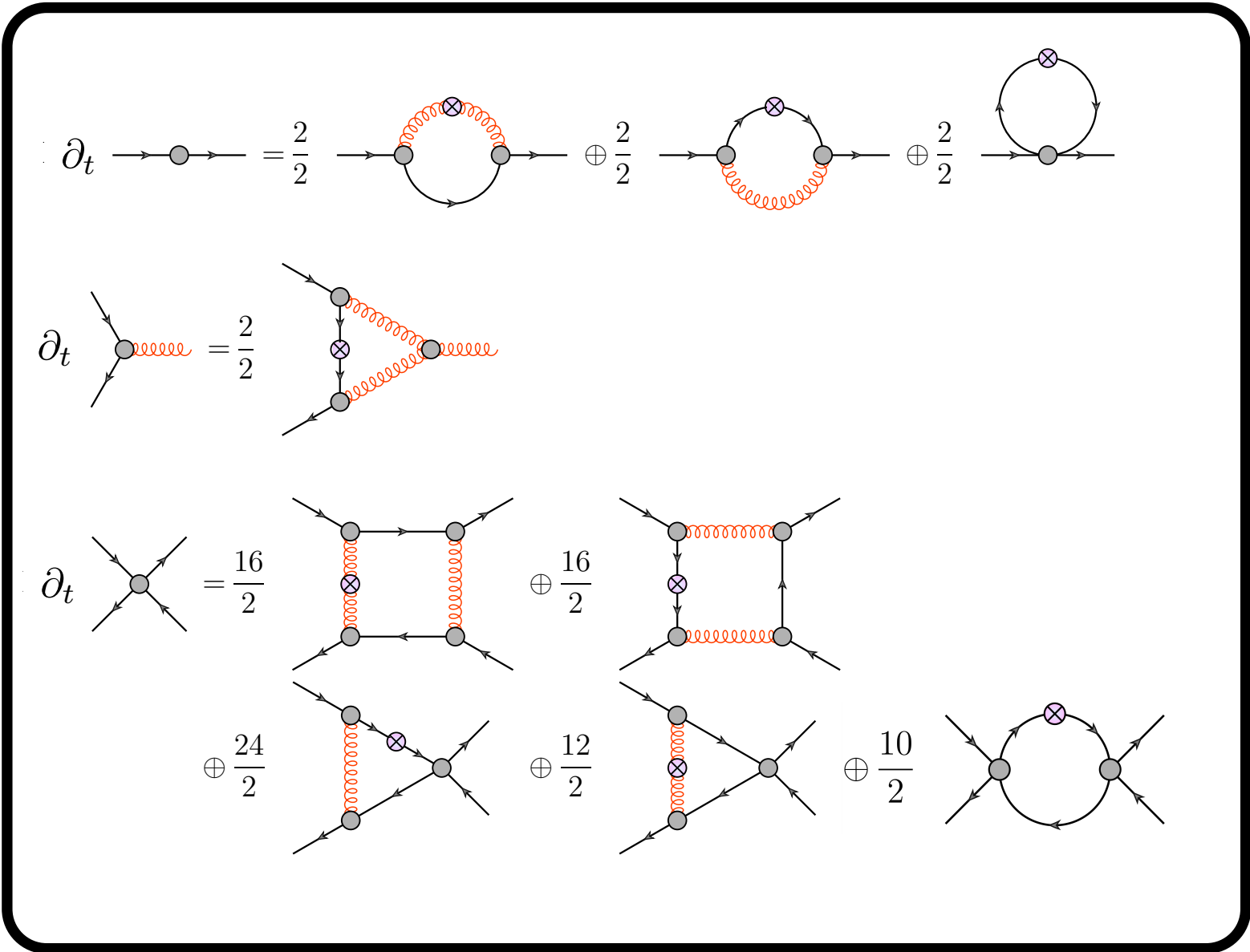
glue quantum fluctuations

quark quantum fluctuations

free energy/
grand potential

Correlation functions

gluon propagator	quark propagator	quark-gluon vertex	quark—anti-quark scattering
$\langle A_\mu A_\nu \rangle(p)$	$\langle q \bar{q} \rangle(p)$	$\langle q \bar{q} A_\mu \rangle(p_1, p_2)$	$\langle q \bar{q} q \bar{q} \rangle(p_1, p_2, p_3)$
	Eight tranverse tensor structures		



... 1-loop exact

Chiral symmetry breaking & mesons

no hadronic composites

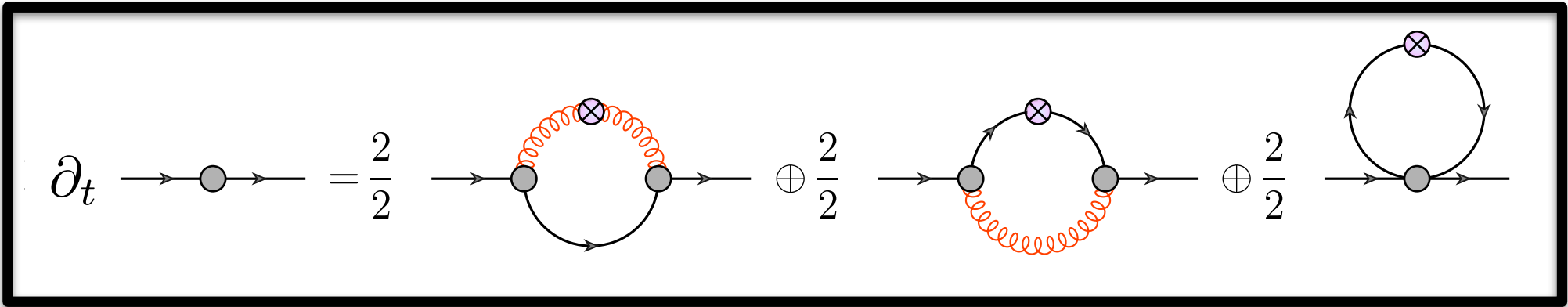
$$\phi = 0$$

functional RG: $\partial_t \Gamma_k[\Phi] = \frac{1}{2} \left(\text{glue quantum fluctuations} - \text{quark quantum fluctuations} \right)$

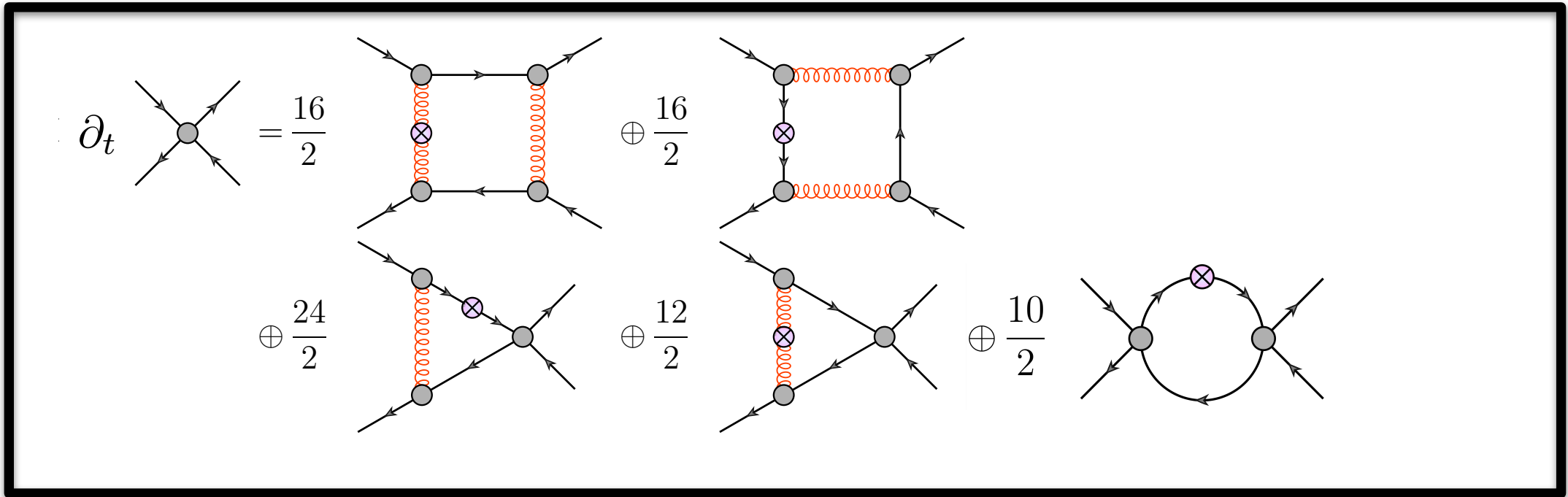
free energy/
grand potential

glue
quantum fluctuations

quark
quantum fluctuations



2 tensor structures

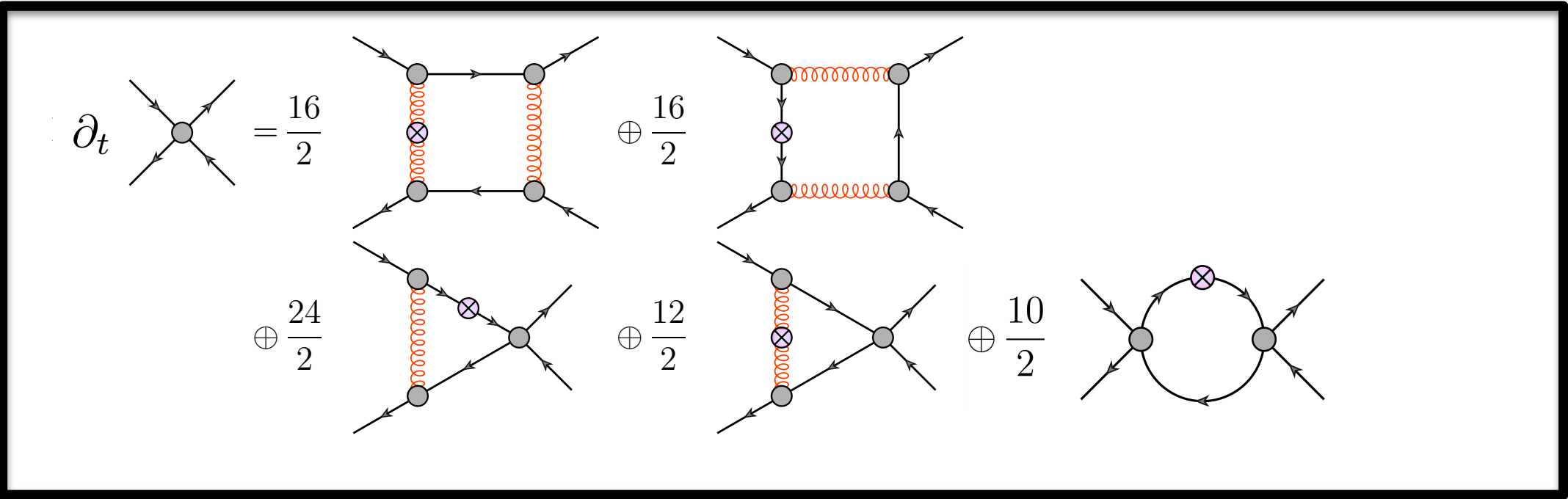


Two flavours: 10 momentum-independent tensor structures

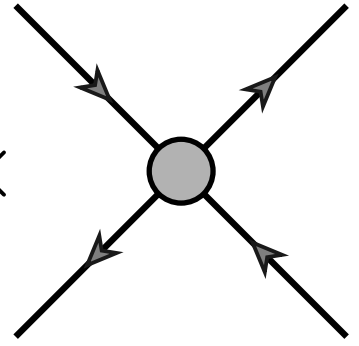
$$\Gamma_{\text{mat}} = \int_p \bar{q}(-p) \left[Z_q(p) i \not{p} + M_q(p) \right] q(p) + \sum_{i=1}^{10} \int_{\mathbf{p}} \lambda_{\bar{q}^2 q^2}^{(i)}(\mathbf{p}) \left(\bar{q}^2 \mathcal{T}_{\bar{q}^2 q^2}^{(i)} q^2 \right) (\mathbf{p}) + \dots$$

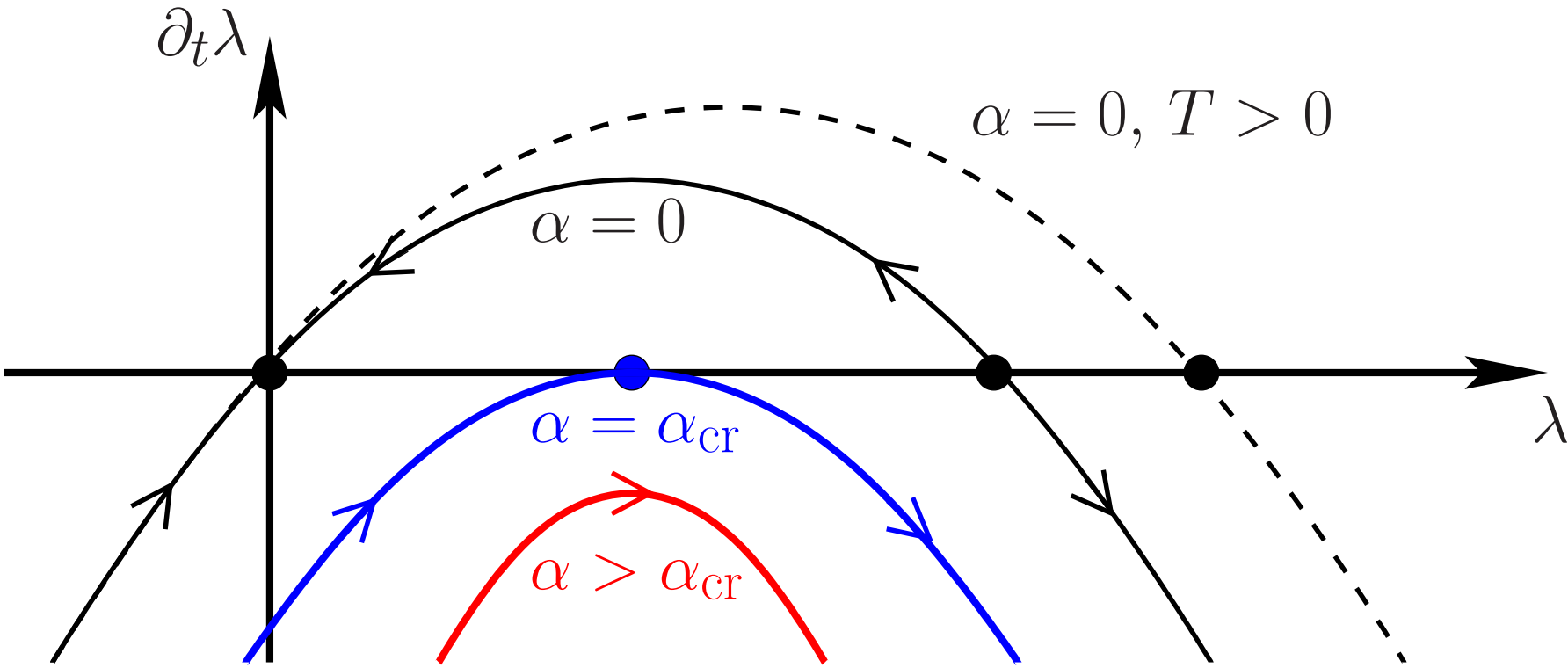
Chiral symmetry breaking & mesons

Chiral symmetry breaking in a nutshell



$$t = \log k / \Lambda_{\text{QCD}}$$

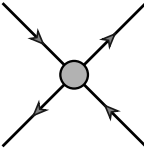
$\lambda = k^2 \times$  $| s - p s$



Chiral symmetry breaking & mesons

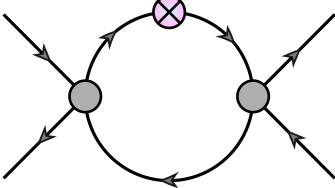
Chiral symmetry breaking in a nutshell

∂_t



=

$\oplus \frac{10}{2}$

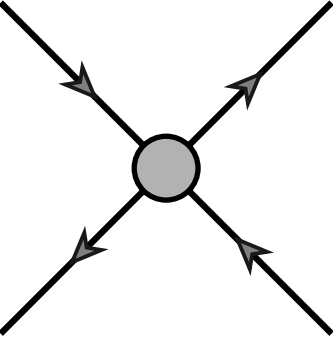


Beta-function of dimensionless scalar-pseudoscalar coupling

$$\partial_t \lambda = 2\lambda - A(k, M_q) \lambda^2$$

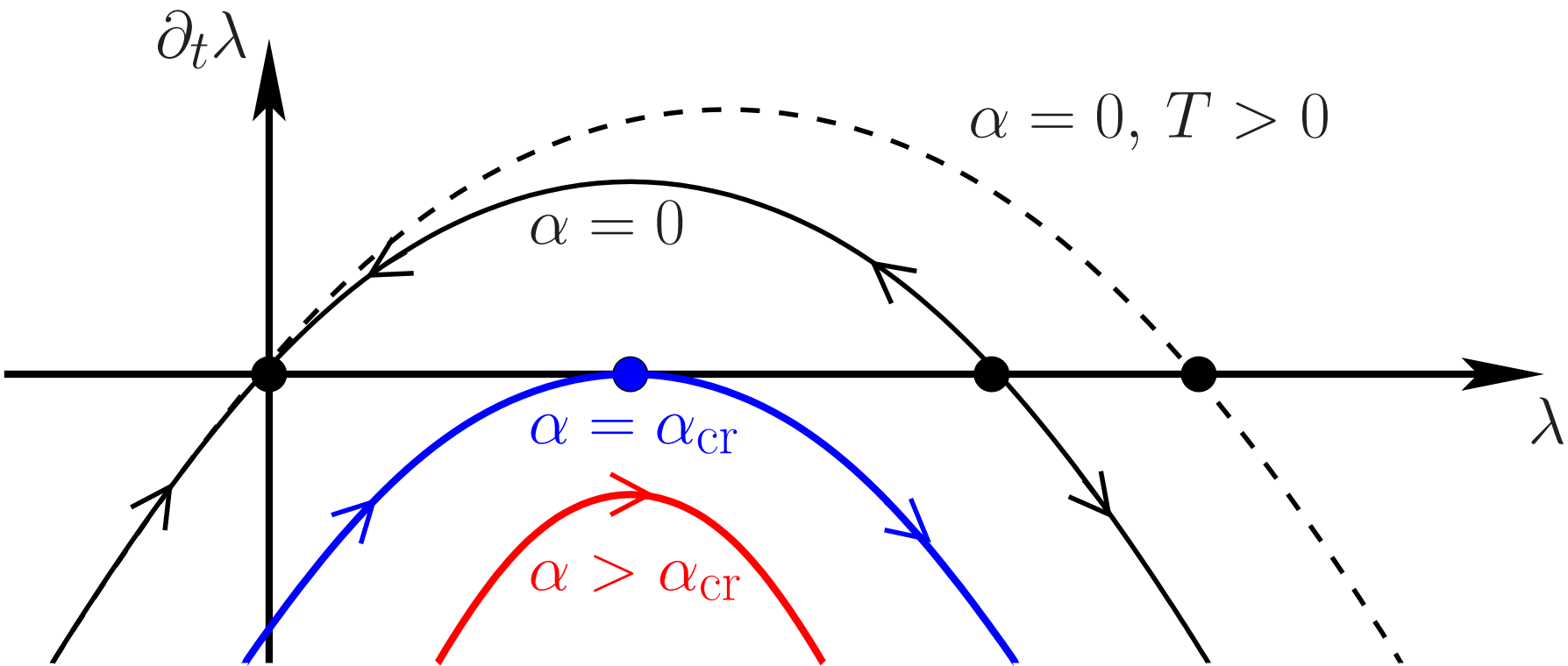
$$t = \log k / \Lambda_{\text{QCD}}$$

$\lambda = k^2 \times$



|

$s - ps$



Chiral symmetry breaking & mesons

Chiral symmetry breaking in a nutshell

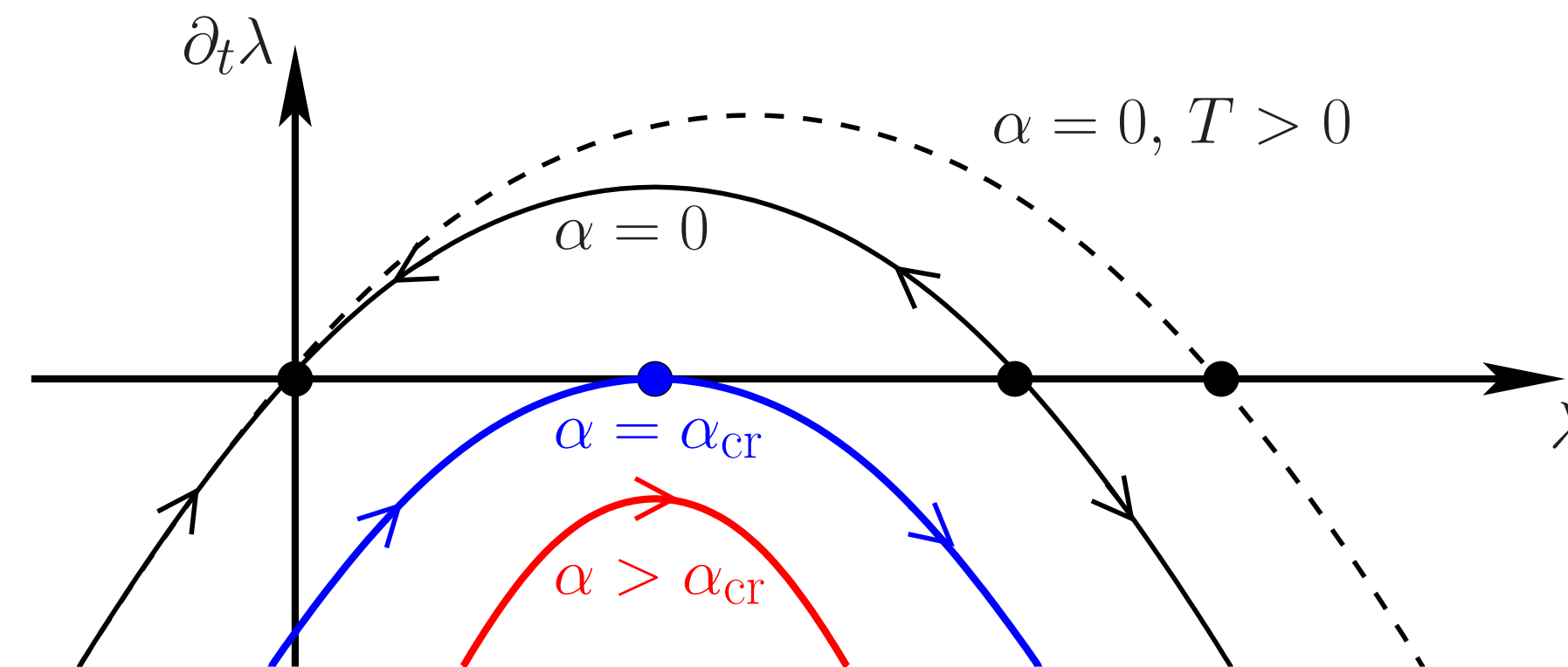
$$\partial_t \text{ (cross) } = \oplus \frac{24}{2} \text{ (diagram 1) } \oplus \frac{12}{2} \text{ (diagram 2) } \oplus \frac{10}{2} \text{ (diagram 3) }$$

Beta-function of dimensionless scalar-pseudoscalar coupling

$$\partial_t \lambda = 2\lambda - A(k, M_q) \lambda^2 - B(k, M_q, M_{\text{gap}}) \lambda \alpha_s$$

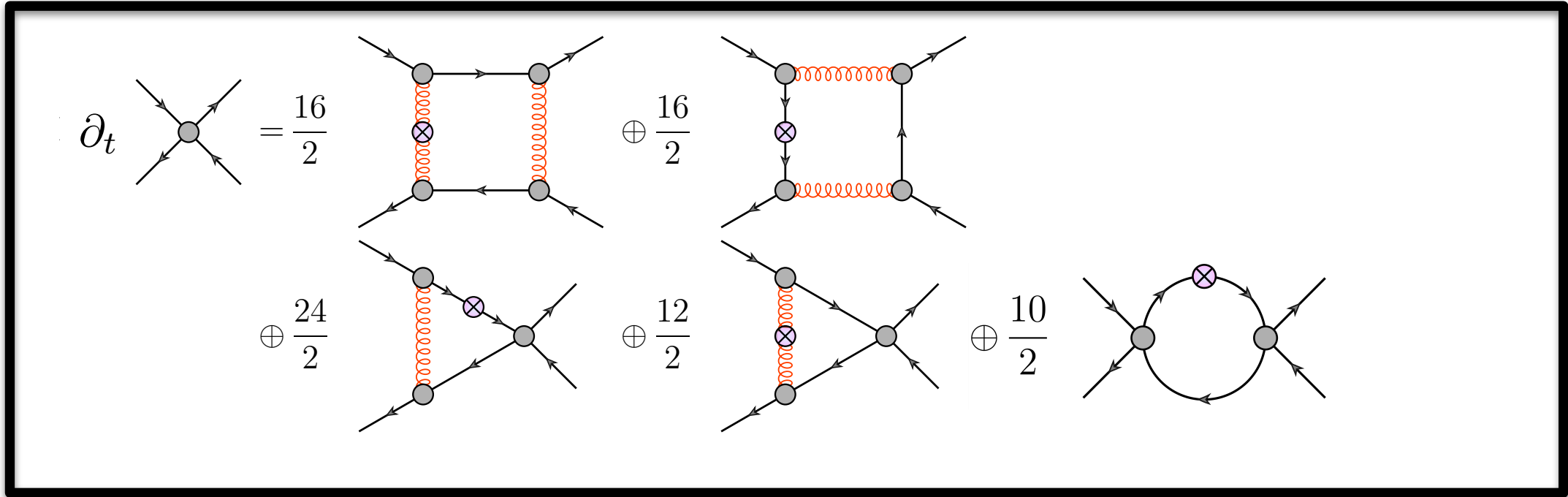
$$t = \log k / \Lambda_{\text{QCD}}$$

$$\lambda = k^2 \times \text{ (cross) } \quad | \quad s - ps$$



Chiral symmetry breaking & mesons

Chiral symmetry breaking in a nutshell

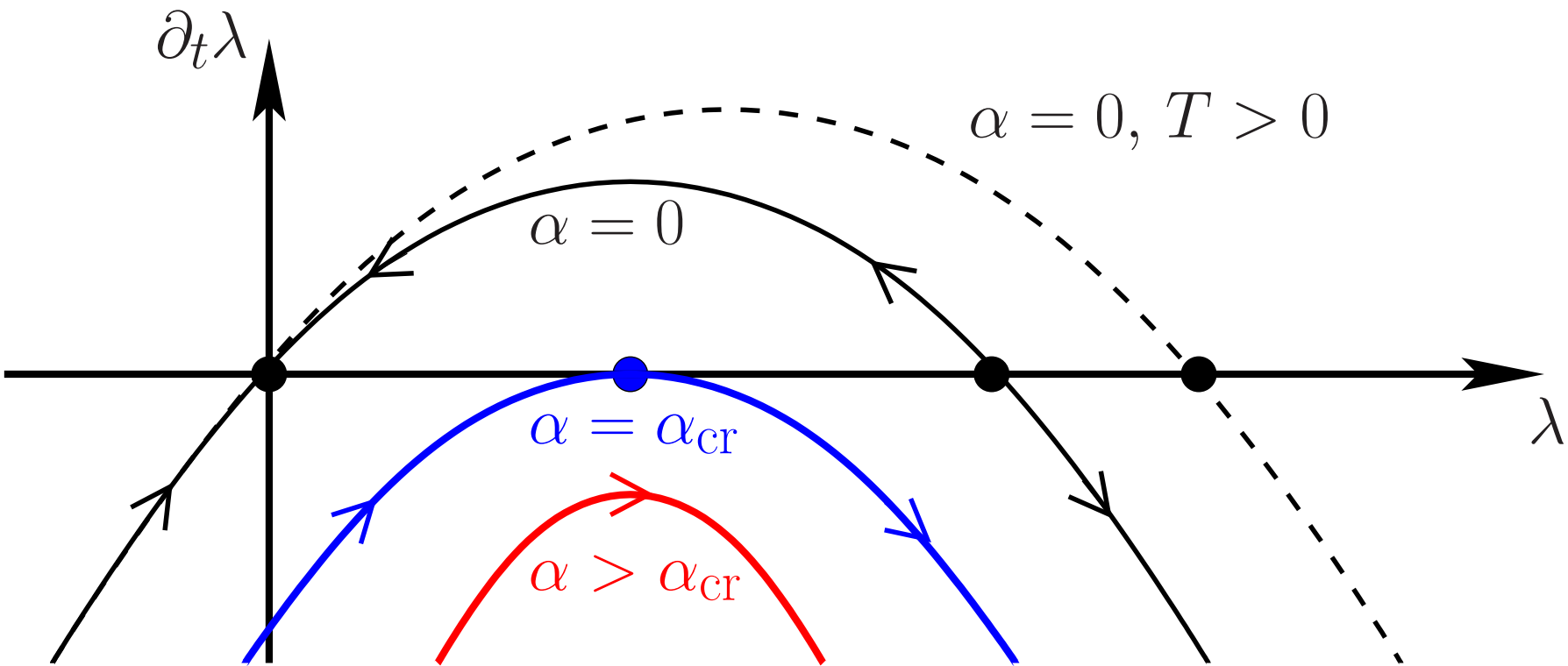


Beta-function of dimensionless scalar-pseudoscalar coupling

$$\partial_t \lambda = \left[2 - B(k, M_q, M_{\text{gap}}) \alpha_s \right] \lambda - A(k, M_q) \lambda^2 - C(k, M_q, M_{\text{gap}}) \alpha_s^2$$

$$t = \log k / \Lambda_{\text{QCD}}$$

$$\lambda = k^2 \times \text{diagram} \quad | \quad s-p$$



chiral symmetry breaking $\longleftrightarrow \alpha_s > \alpha_{s,cr}$

Jump forward to: Functional QCD & QCD phase structure

Getting dynamical: emergent hadrons & diquarks

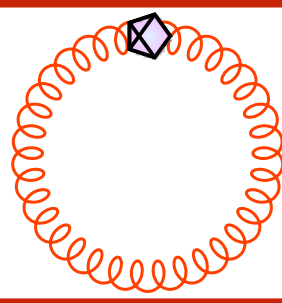
**Gies, Wetterich, PRD 65 (2002) 065001
PRD 69 (2004) 025001**

**JMP, AP 322 (2007) 2831-2915
Floerchinger, Wetterich, PLB 680 (2009) 371**

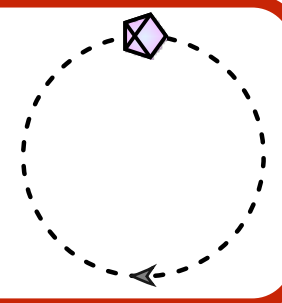
**Fu, JMP, Rennecke, PRD 101, (2020) 054032
Fukushima, JMP, Strodthoff, PRD 110 (2024) 076022**

Dynamical hadronisation: mesons & diquarks

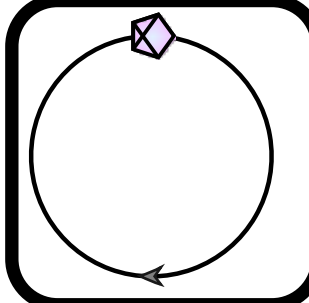
functional RG: $\left(\partial_t + \int_x \dot{\Phi} \frac{\delta}{\delta \Phi} \right) \Gamma_k[\Phi] =$

$\frac{1}{2}$


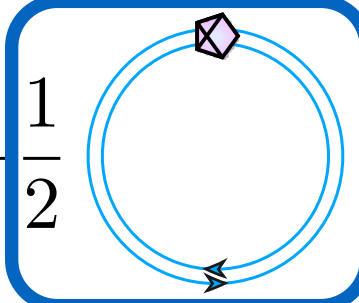
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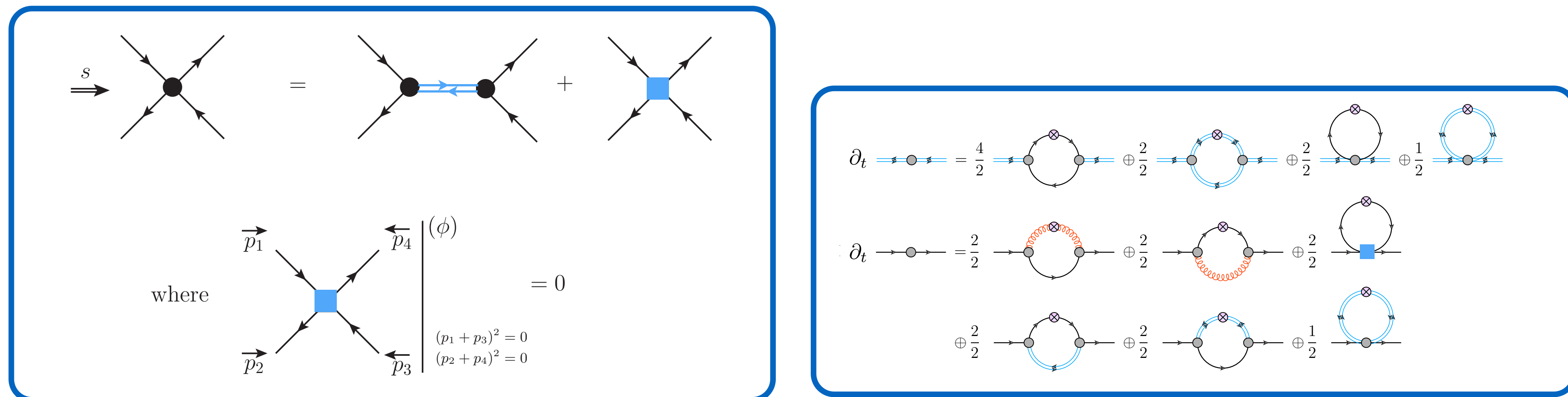
 $+$

$\frac{1}{2}$


glue
quantum fluctuations
quark
quantum fluctuations
hadronic
quantum fluctuations

‘DynHad for mesons & diquarks is BSE-DSE for QCD in a ‘unified’ effective action approach’

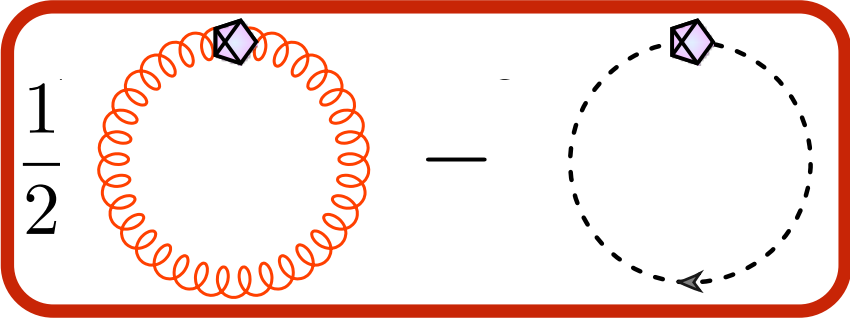
Dynamical hadronisation



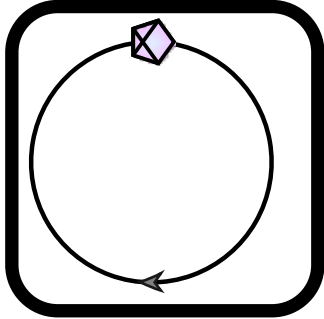
Functional flows for QCD

functional RG: $\left(\partial_t + \int_x \dot{\Phi} \frac{\delta}{\delta \Phi}\right) \Gamma_k[\Phi] =$

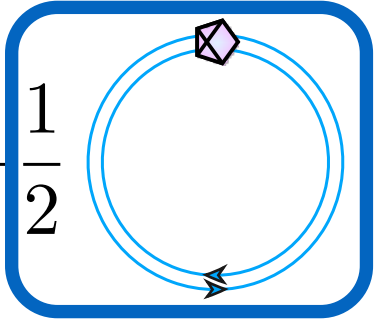
glue
quantum fluctuations



quark
quantum fluctuations



hadronic
quantum fluctuations



$-\frac{1}{2} \text{ (ghost loop)} + \frac{1}{2} \text{ (quark loop)} - \frac{1}{2} \text{ (quark loop)} + \frac{1}{2} \text{ (hadronic loop)}$

Correlation functions

gluon propagator

 $\langle A_\mu A_\nu \rangle(p)$

quark propagator

 $\langle q \bar{q} \rangle(p)$

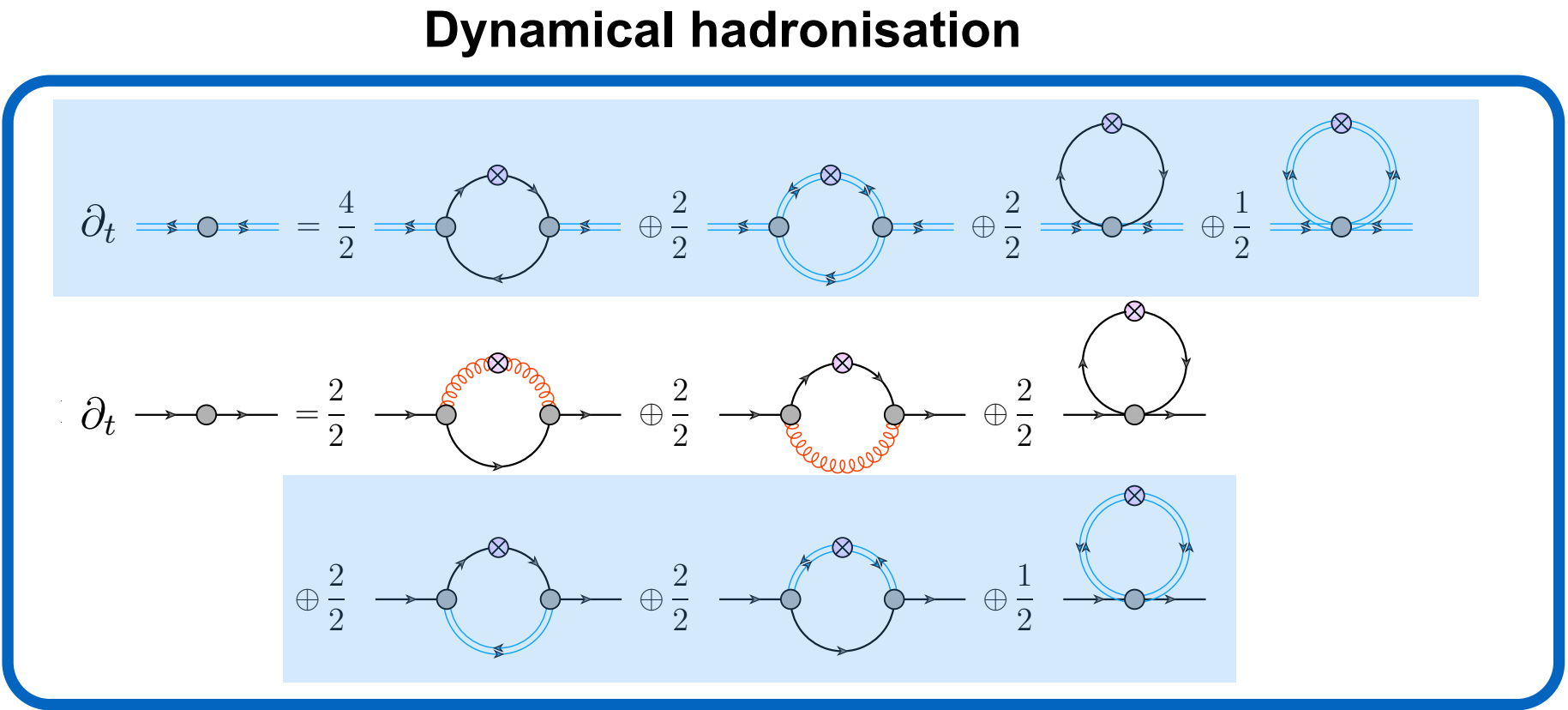
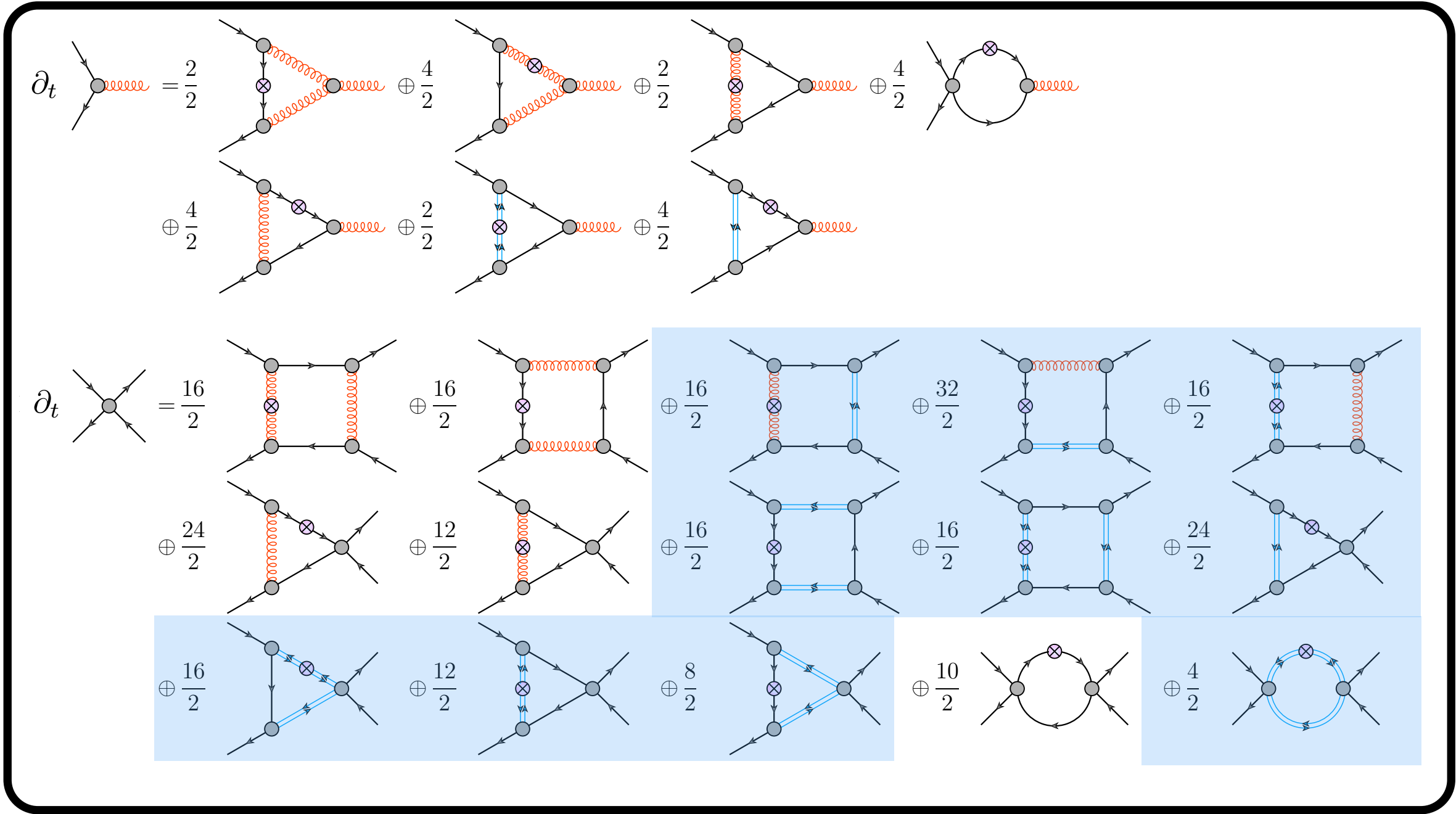
quark-gluon vertex

 $\langle q \bar{q} A_\mu \rangle(p_1, p_2)$

Eight transverse tensor structures

quark—anti-quark scattering

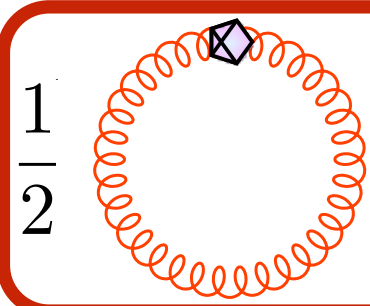
 $\langle q \bar{q} q \bar{q} \rangle(p_1, p_2, p_3)$

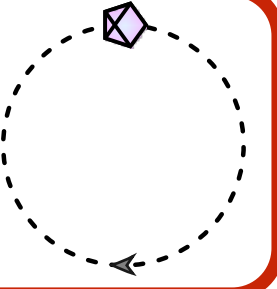


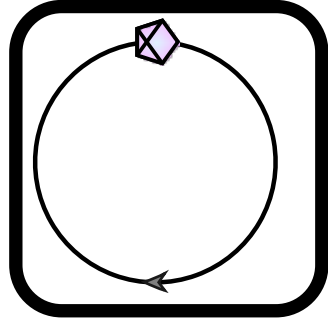
Dynamical hadronisation: mesons & diquarks

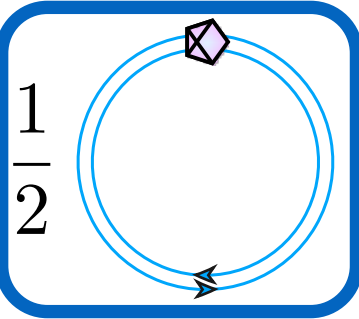
Implementation:

functional RG: $\left(\partial_t + \int_x \dot{\Phi} \frac{\delta}{\delta \Phi} \right) \Gamma_k[\Phi] =$

$\frac{1}{2}$


$-$


$-$


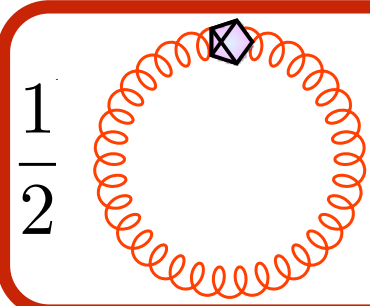
$+$
 $\frac{1}{2}$


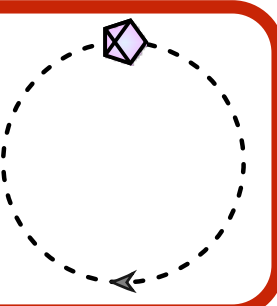
glue
quantum fluctuations
quark
quantum fluctuations
hadronic
quantum fluctuations

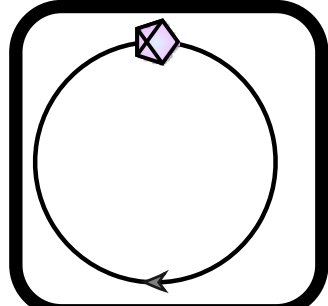
Dynamical hadronisation: mesons & diquarks

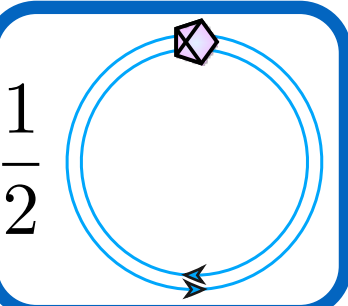
Implementation:

functional RG: $\left(\partial_t + \int_x \dot{\Phi} \frac{\delta}{\delta \Phi} \right) \Gamma_k[\Phi] =$

$\frac{1}{2}$


$-$


$-$


$+$
 $\frac{1}{2}$


glue quantum fluctuations
quark quantum fluctuations
hadronic quantum fluctuations

Consider path integral in the presence of sources for composite operators

JMP, AP 322 (2007) 2831-2915

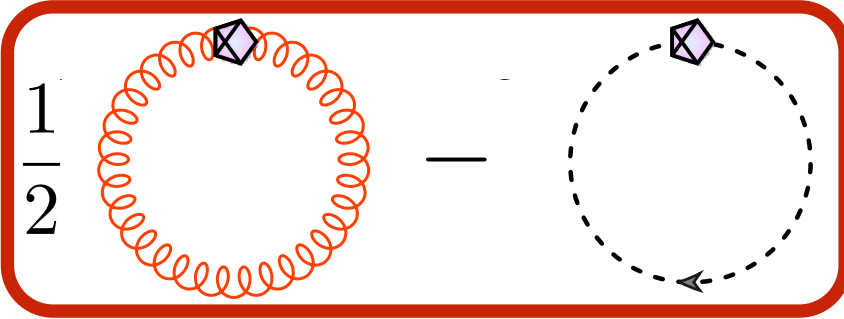
$$Z[J_q, J_{\bar{q}}, J_{\mathcal{O}}] = \int dq d\bar{q} e^{-S[q, \bar{q}] + \int J_q q - \bar{q} J_{\bar{q}} + \int J_{\mathcal{O}} \mathcal{O}[q, \bar{q}]}$$

Dynamical hadronisation: mesons & diquarks

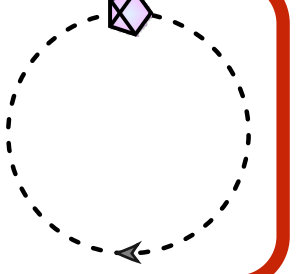
Implementation:

functional RG: $\left(\partial_t + \int_x \dot{\Phi} \frac{\delta}{\delta \Phi} \right) \Gamma_k[\Phi] =$

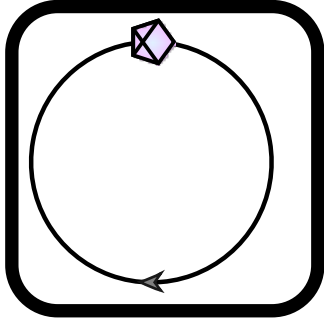
glue
quantum fluctuations



-



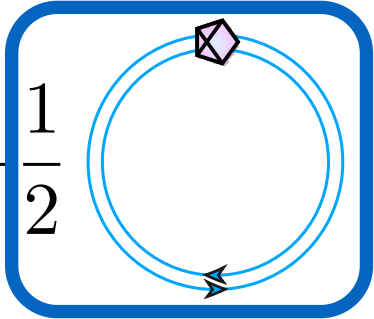
-



quark
quantum fluctuations

+

hadronic
quantum fluctuations



$\frac{1}{2} \quad \frac{1}{2}$

Consider path integral in the presence of sources for composite operators

JMP, AP 322 (2007) 2831-2915

$$Z[J_q, J_{\bar{q}}, J_{\mathcal{O}}] = \int dq d\bar{q} e^{-S[q, \bar{q}] + \int J_q q - \bar{q} J_{\bar{q}} + \int J_{\mathcal{O}} \mathcal{O}[q, \bar{q}]}$$

Choose scale-dependent $\mathcal{O}_k[q, \bar{q}]$ 'to optimise dynamics'!

$$\partial_t \Gamma_k[A_\mu, q, \bar{q}] \longrightarrow \partial_t \Gamma_k[\Phi] + \partial_t \mathcal{O}_k^{(i)}[\Phi] \frac{\delta \Gamma_k}{\delta \Phi_i}$$

$$\Phi = (A_\mu, q, \bar{q}, \langle \mathcal{O}^{(1)} \rangle, \dots)$$

Dynamical hadronisation: mesons & diquarks

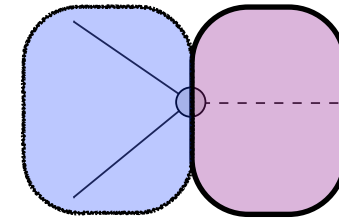
Implementation:

2001 - : Braun, Flörchinger, Fu Gies, JMP,
Rennecke, Wetterich, ...

$$\frac{\lambda_\psi}{2} [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\vec{\tau}\psi)^2] = \left[i h \bar{\psi}(\tau \cdot \Phi)\psi + \frac{1}{2}m_\phi^2\Phi^2 \right]_{\text{EoM}(\Phi)}$$

$$\lambda_\psi = \frac{h^2}{m_\phi^2}$$

Hubbard-Stratonovich



$$\Phi = (\sigma, \vec{\pi})$$

$$\tau \cdot \Phi = \sigma + i\gamma_5\vec{\sigma}\vec{\pi}$$

Consider path integral in the presence of sources for composite operators

JMP, AP 322 (2007) 2831-2915

$$Z[J_q, J_{\bar{q}}, J_{\mathcal{O}}] = \int dq d\bar{q} e^{-S[q, \bar{q}] + \int J_q q - \bar{q} J_{\bar{q}} + \int J_{\mathcal{O}} \mathcal{O}[q, \bar{q}]}$$

Common choices

$$T^i = (1, \gamma_5 \vec{\sigma})$$

Scalar-pseudoscalar channel

Dynamical hadronisation: mesons & diquarks

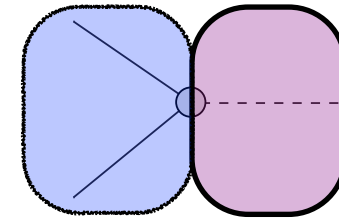
Implementation:

2001 - : Braun, Flörchinger, Fu Gies, JMP,
Rennecke, Wetterich, ...

$$\frac{\lambda_\psi}{2} [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\vec{\tau}\psi)^2] = \left[i h \bar{\psi}(\tau \cdot \Phi)\psi + \frac{1}{2}m_\phi^2\Phi^2 \right]_{\text{EoM}(\Phi)}$$

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Common choices

$$T^i = (1, \gamma_5 \vec{\sigma})$$

Scalar-pseudoscalar channel

$$T^i = \gamma_0$$

Density channel
(part of vector multiplet)

Dynamical hadronisation: mesons & diquarks

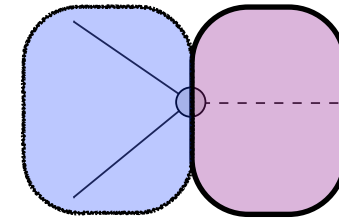
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Hubbard-Stratonovich



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Consider path integral in the presence of sources for composite operators

JMP, AP 322 (2007) 2831-2915

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Common choices

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Scalar-pseudoscalar channel

$$T^i = (\gamma_0, \vec{\gamma})$$

Density channel
(part of vector multiplet)

Dynamical hadronisation: mesons & diquarks

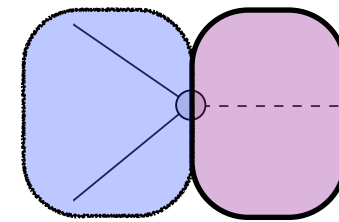
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Hubbard-Stratonovich



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Consider path integral in the presence of sources for composite operators

JMP, AP 322 (2007) 2831-2915

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Scalar-pseudoscalar channel

$$T^i = (\gamma_0, \vec{\gamma})$$

Density channel
(part of vector multiplet)

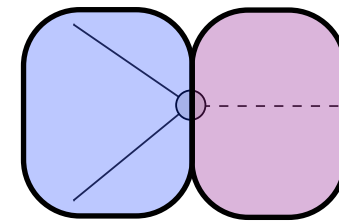
$$\psi \tilde{T}^i \psi$$

Diquark channels

Dynamical hadronisation: mesons & diquarks

Implementation:

$$\frac{\lambda_\psi}{2} [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\vec{\tau}\psi)^2] = \left[i h \bar{\psi}(\tau \cdot \Phi)\psi + \frac{1}{2}m_\phi^2\Phi^2 \right]_{\text{EoM}(\Phi)}$$



Hubbard-Stratonovich

$$\lambda_\psi = \frac{h^2}{m_\phi^2}$$

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(part of vector multiplet)

$$\psi\tilde{T}^i\psi$$

Diquark channels

Complete basis

$$N_f = 2 : 10$$

$$N_f = 3 : 26$$

Momentum-independent
tensor structures

Dynamical hadronisation: mesons & diquarks

Implementation:

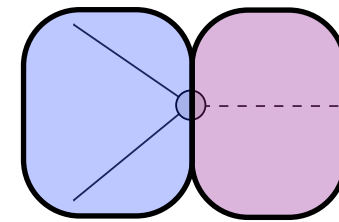
$$\frac{\lambda_\psi}{2} [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\vec{\tau}\psi)^2] = \left[i h \bar{\psi}(\tau \cdot \Phi)\psi + \frac{1}{2}m_\phi^2\Phi^2 \right]_{\text{EoM}(\Phi)}$$

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Hubbard-Stratonovich



Common choices

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Scalar-pseudoscalar channel

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Density channel
(part of vector multiplet)

$$\psi\tilde{T}^i\psi$$

Diquark channels

Complete basis

$$\mathbf{N}_f = 2 : 10$$

$$\mathbf{N}_f = 3 : 26$$

Momentum-independent
tensor structures

All tensor structures for $\mathbf{N}_f = 2 : 256$

Dynamical hadronisation: mesons & diquarks

Implementation:

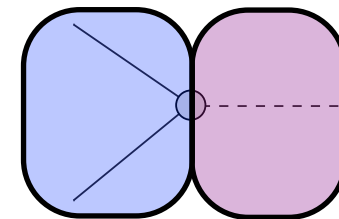
$$\frac{\lambda_\psi}{2} [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\vec{\tau}\psi)^2] = \left[i h \bar{\psi}(\tau \cdot \Phi)\psi + \frac{1}{2}m_\phi^2\Phi^2 \right]_{\text{EoM}(\Phi)}$$

$$\lambda_\psi = \frac{h^2}{m_\phi^2}$$

$$\Phi = (\sigma, \vec{\pi})$$

$$\tau \cdot \Phi = \sigma + i\gamma_5\vec{\sigma}\vec{\pi}$$

Hubbard-Stratonovich



General dynamical hadronisation

hadronised Flow

$$\frac{\partial}{\partial t} \Big|_\phi \Gamma_k[\phi] = \frac{1}{2} G_{k,\phi} \dot{R}_{k,\phi} + R_k G_{k,\phi} \frac{\delta \dot{\phi}}{\delta \phi} - \frac{\delta \Gamma}{\delta \phi} \dot{\phi}$$

$$\phi = (A_\mu, C, \bar{C}, q, \bar{q}, \Phi, \dots, n, \bar{n}, \dots)$$

mesons baryons

How to fix ϕ_k & $\dot{\phi}_k$?

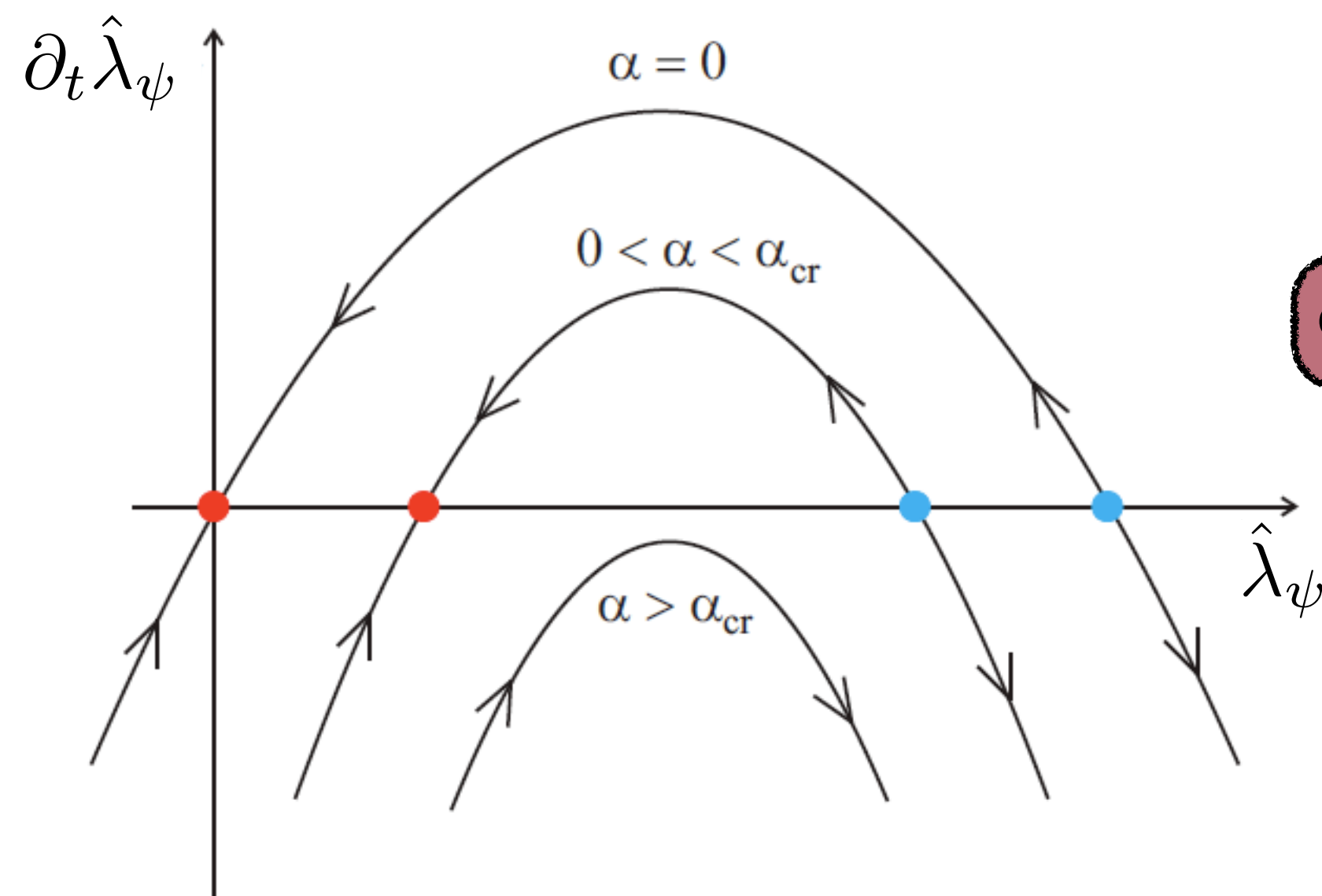
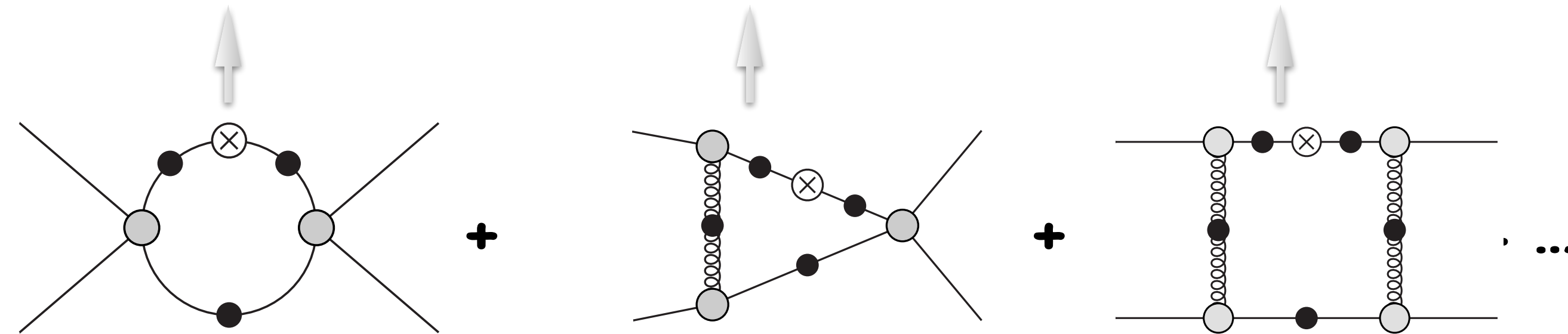
$$\dot{\Phi}_k \simeq \dot{A}_k \bar{\psi} \tau \psi + \dot{B}_k \Phi_k + \dot{C}_k$$

Dynamical hadronisation: mesons & diquarks

Implementation:

Flow for four-fermion coupling $\hat{\lambda}_\psi = \lambda_\psi k^2$ with infrared scale k

$$k \partial_k \hat{\lambda}_\psi = 2 \hat{\lambda}_\psi + A \left(\frac{T}{k} \right) \hat{\lambda}_\psi^2 + B \left(\frac{T}{k} \right) \hat{\lambda}_\psi \alpha_s + C \left(\frac{T}{k} \right) \alpha_s^2 + \dots$$

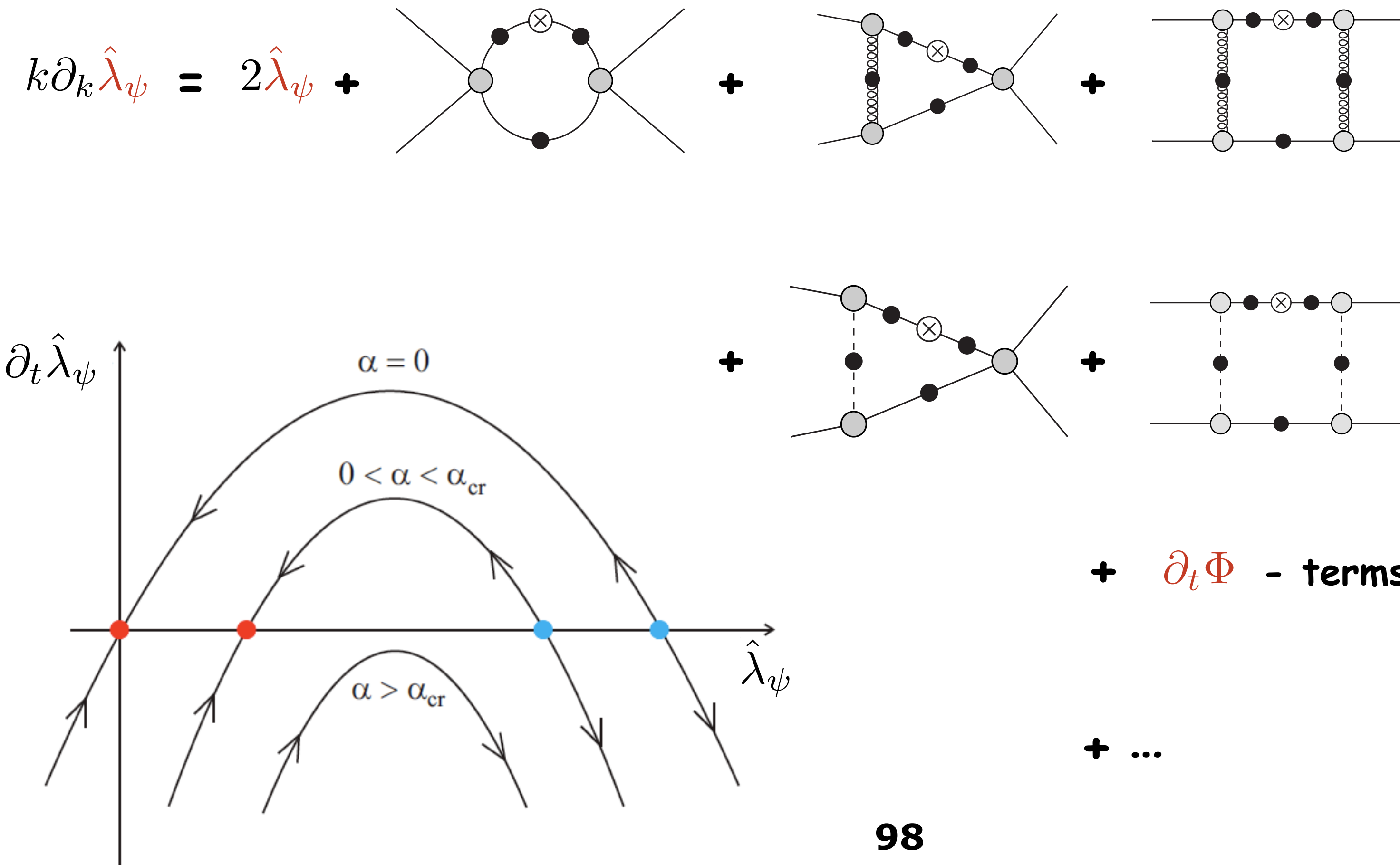


chiral symmetry breaking $\longleftrightarrow \alpha_s > \alpha_{s,cr}$

Dynamical hadronisation: mesons & diquarks

Implementation:

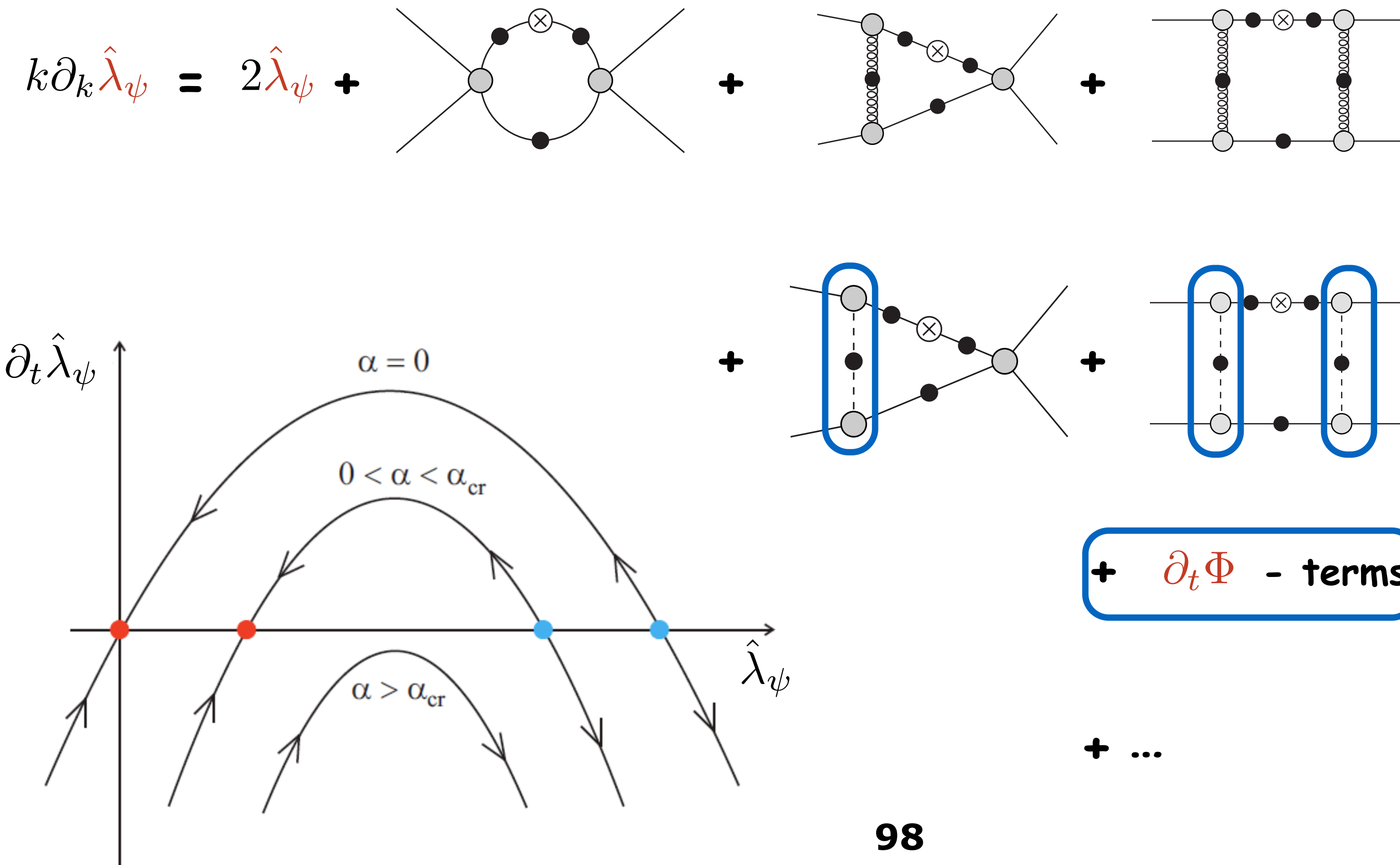
Flow for four-fermion coupling $\hat{\lambda}_\psi = \lambda_\psi k^2$ with infrared scale k



Dynamical hadronisation: mesons & diquarks

Implementation:

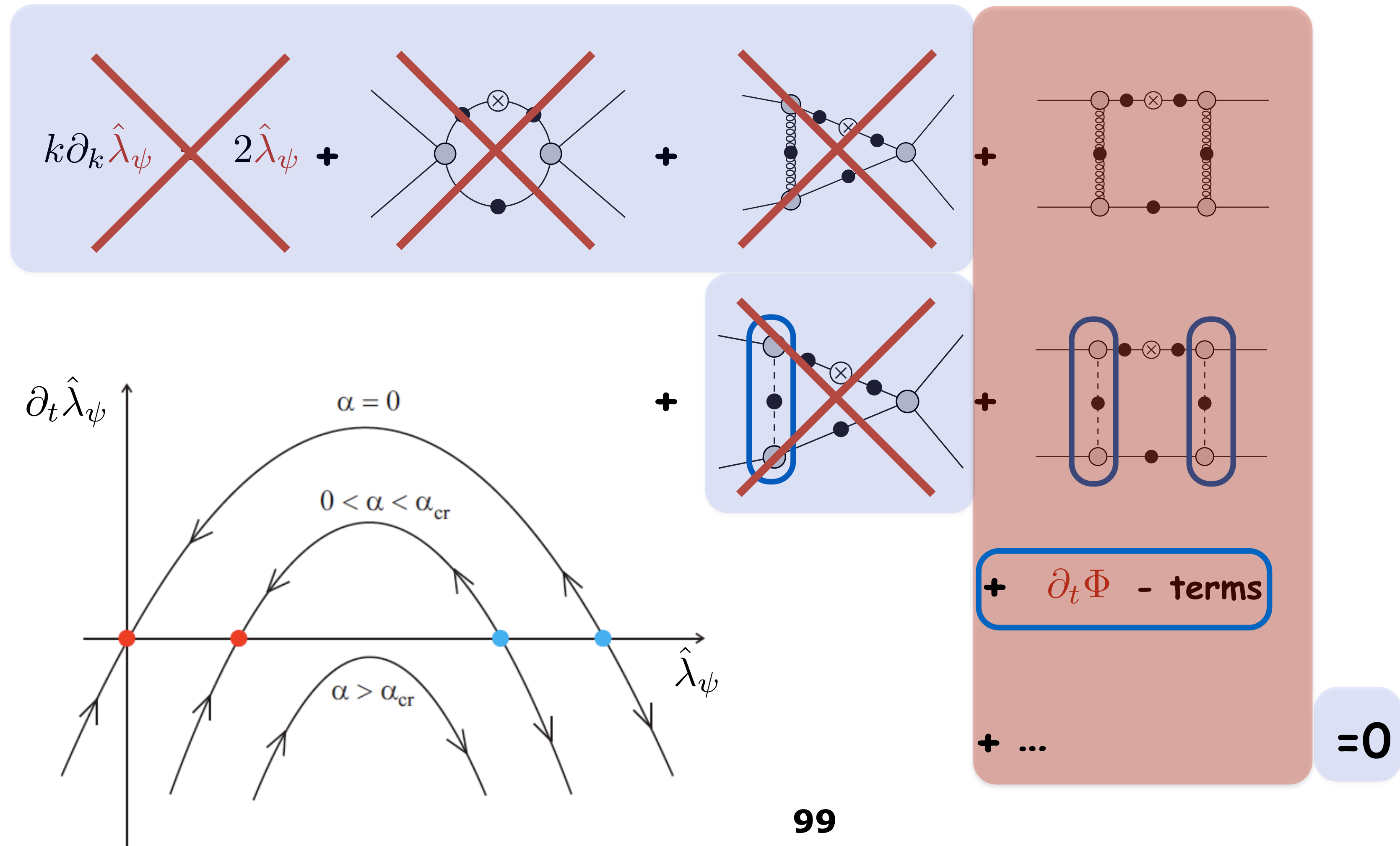
Flow for four-fermion coupling $\hat{\lambda}_\psi = \lambda_\psi k^2$ with infrared scale k



Dynamical hadronisation: mesons & diquarks

Implementation:

Full bosonisation $\hat{\lambda}_\psi = 0$



Dynamical hadronisation: mesons & diquarks

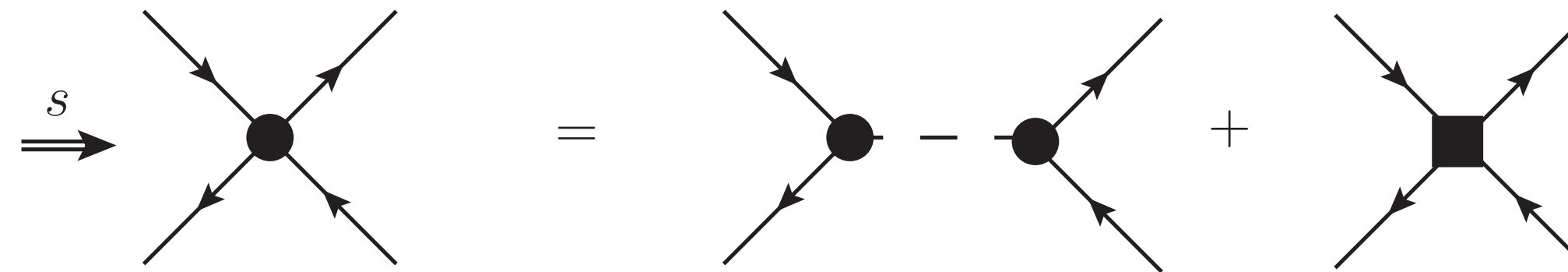
! Reminder !

Full bosonisation $\hat{\lambda}_\psi = 0$ Really?

Dynamical hadronisation: mesons & diquarks

! Reminder !

Full bosonisation $\hat{\lambda}_\psi = 0$ Really?



where

The diagram shows a black square vertex with four external lines. The top-left line is labeled $\vec{p}_1$ and points away from the vertex. The bottom-left line is labeled $\vec{p}_2$ and points towards the vertex. The bottom-right line is labeled $\vec{p}_3$ and points away from the vertex. The top-right line is labeled $\vec{p}_4$ and points towards the vertex. To the right of the diagram is a vertical line, and to its right are the conditions (ϕ) , $(p_1 + p_3)^2 = 0$, and $(p_2 + p_4)^2 = 0$, followed by an equals sign and a zero.

(i) Complete dynamical hadronisation of one tensor channel removes one momentum channel!

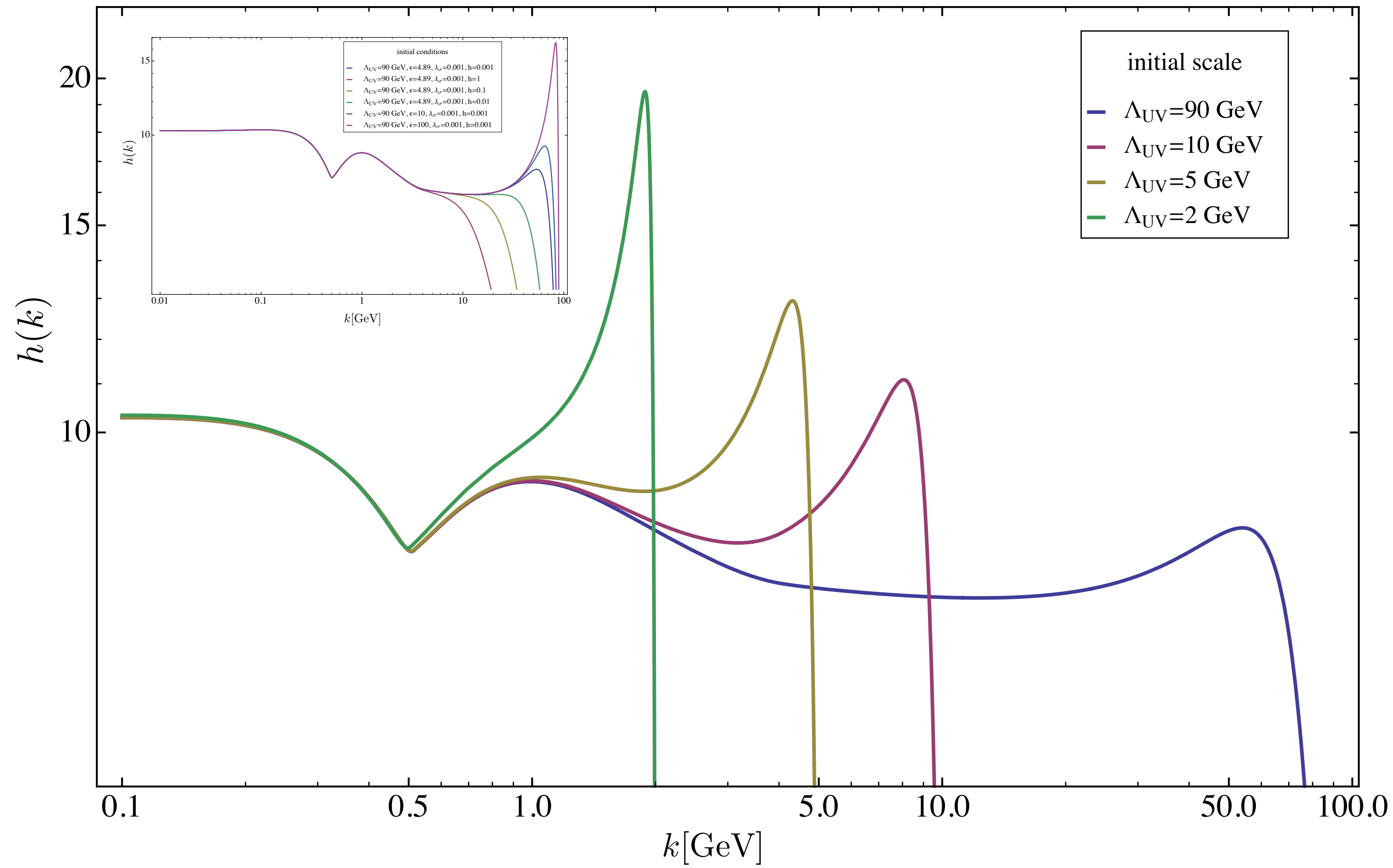
(ii) Residual four-quark vertex left!

Dynamical hadronisation at work

Jump back to: Phase structure from functional QCD: how to

Dynamical hadronisation

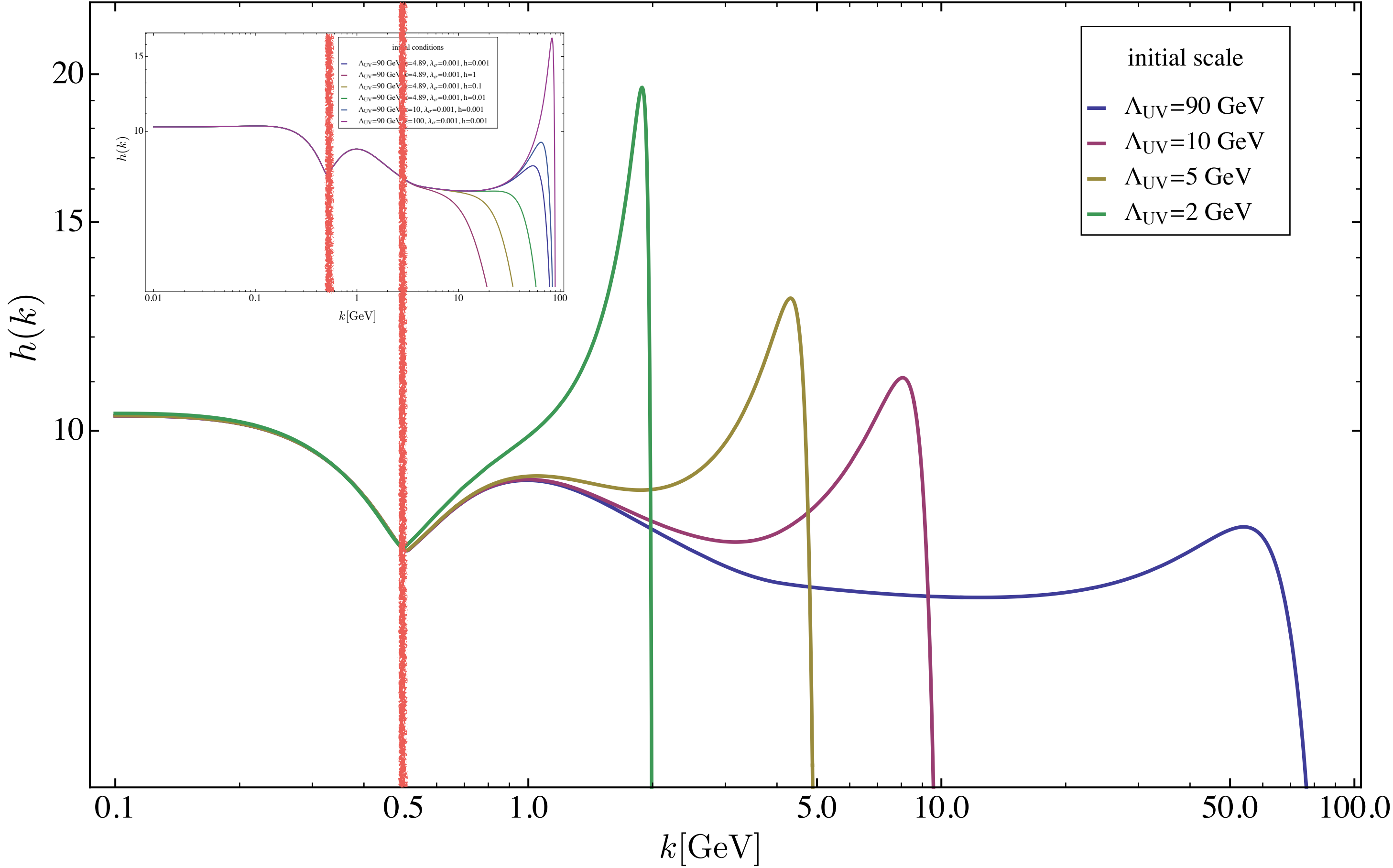
Stability & decoupling



Dynamical hadronisation

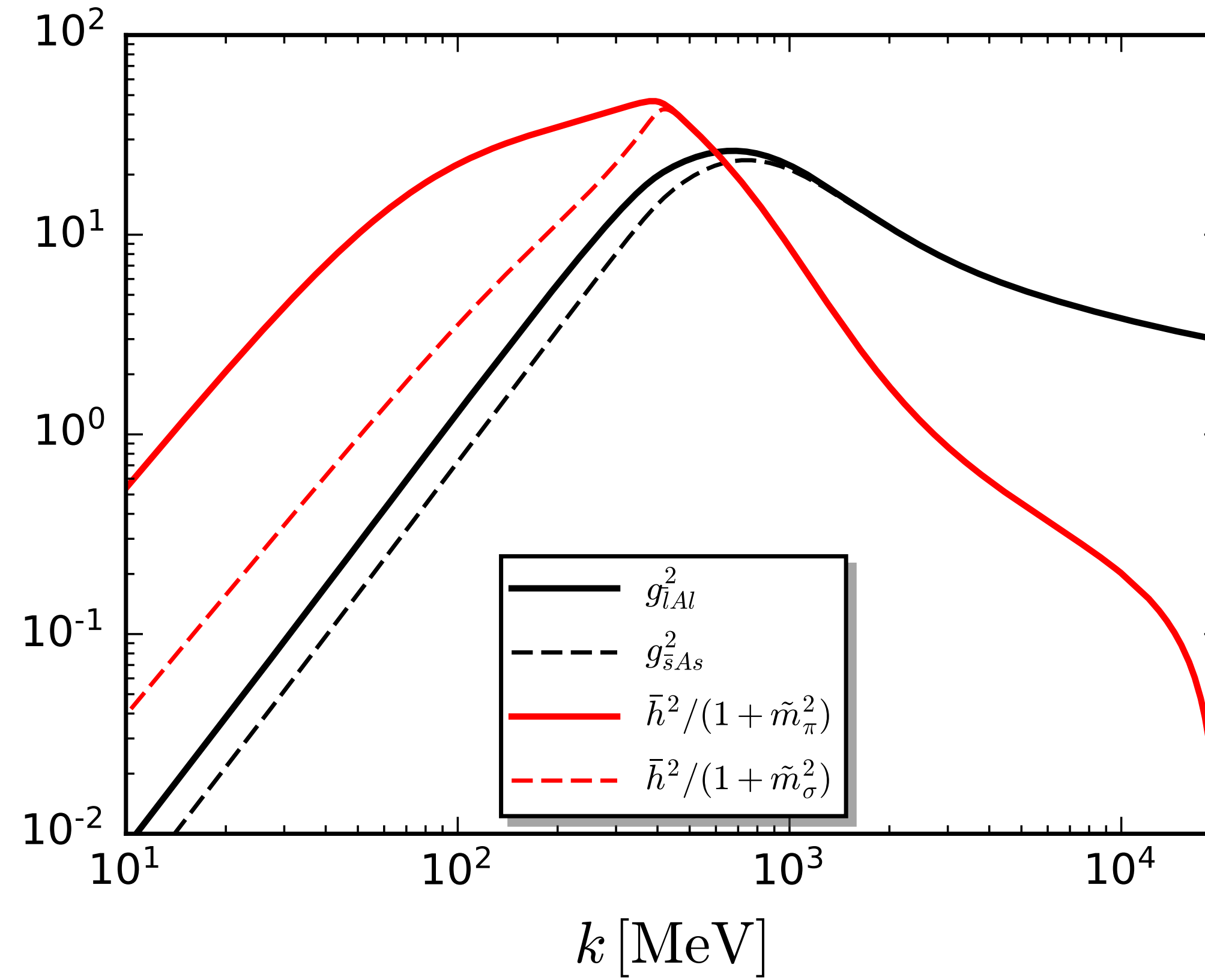
Cutoff scale of dynamical
chiral symmetry breaking

Stability & decoupling



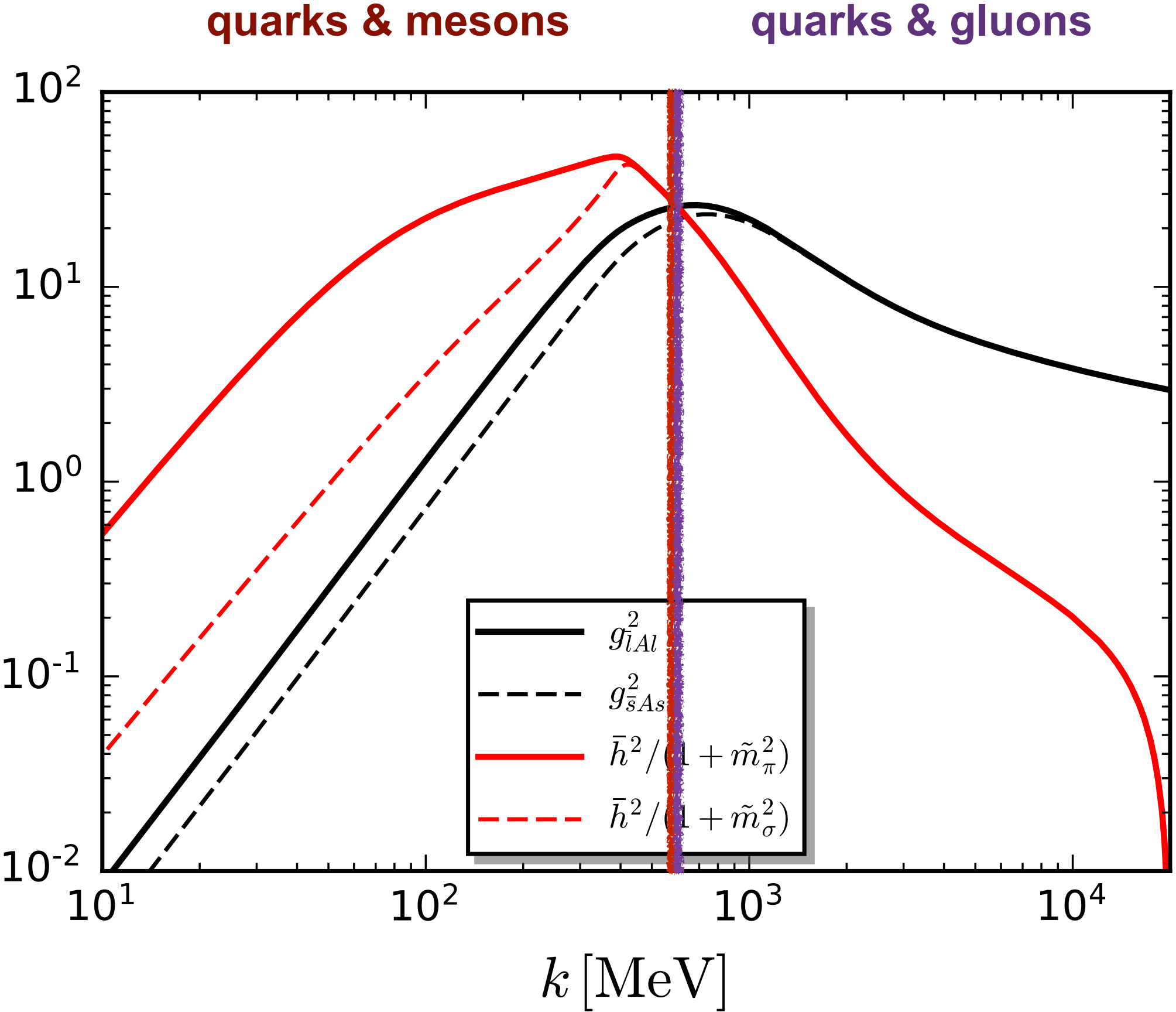
Dynamical hadronisation

Stability & decoupling



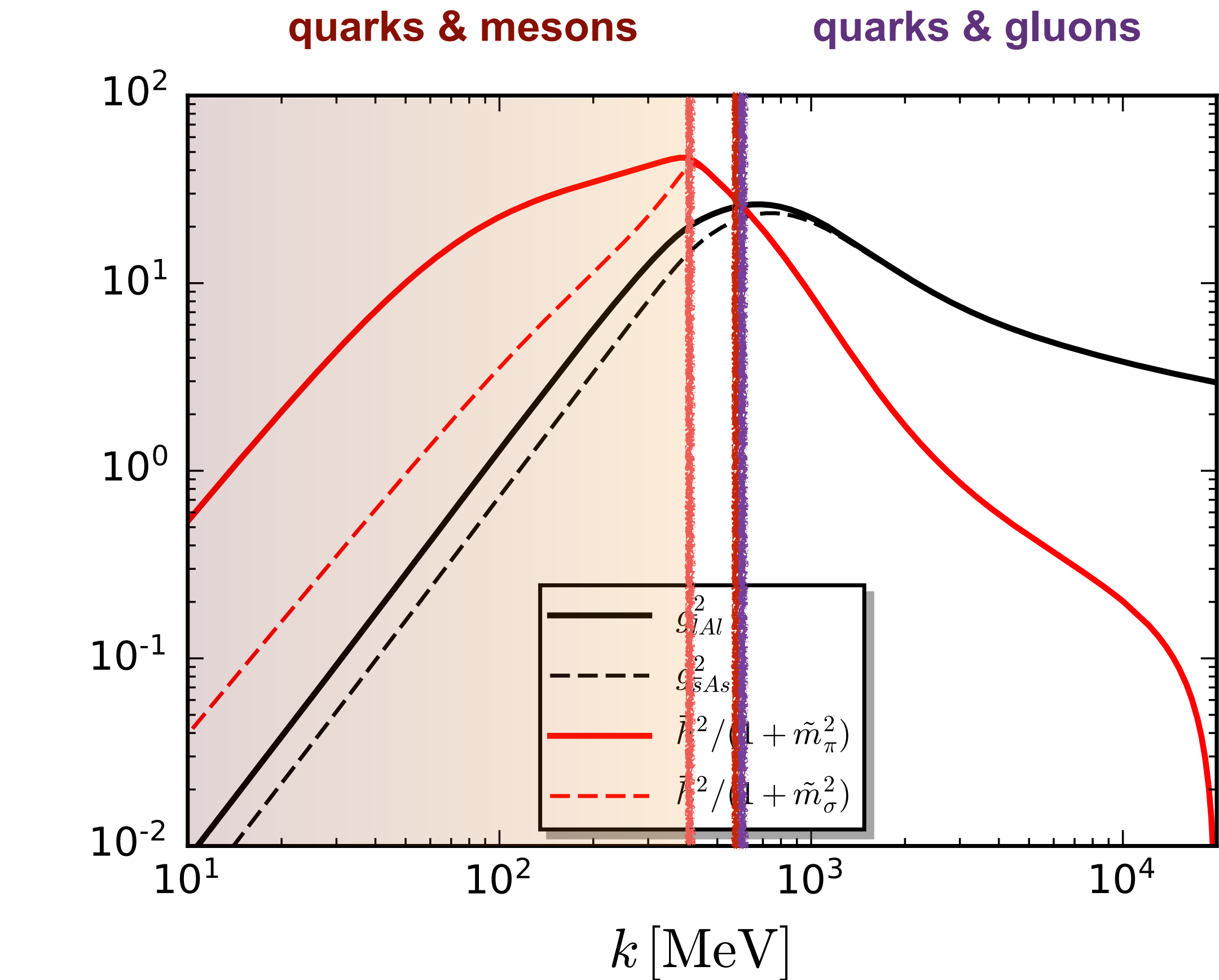
Dynamical hadronisation

Stability & decoupling



Dynamical hadronisation

Stability & decoupling

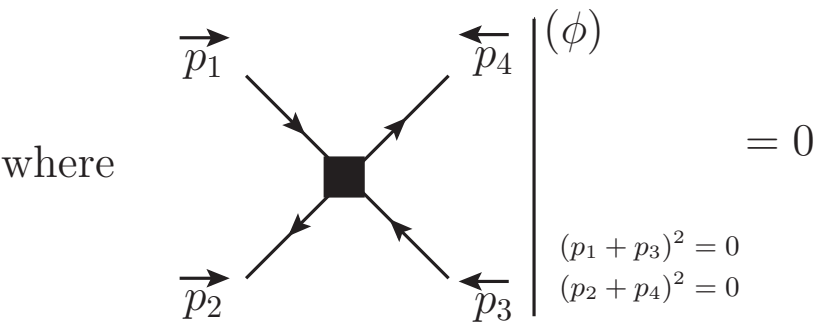
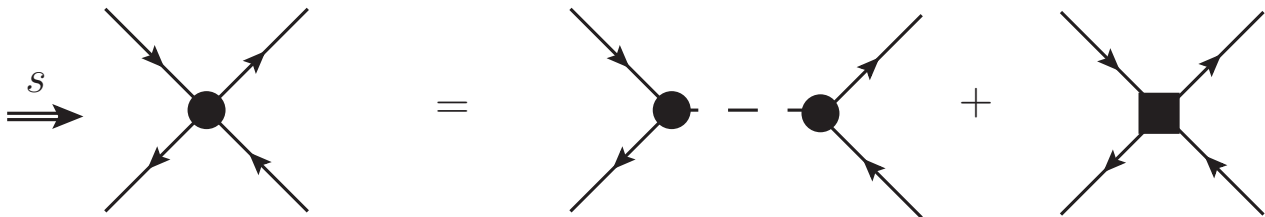


Pions: Chiral perturbation theory

Dynamical hadronisation at work

Mesons & diquarks:

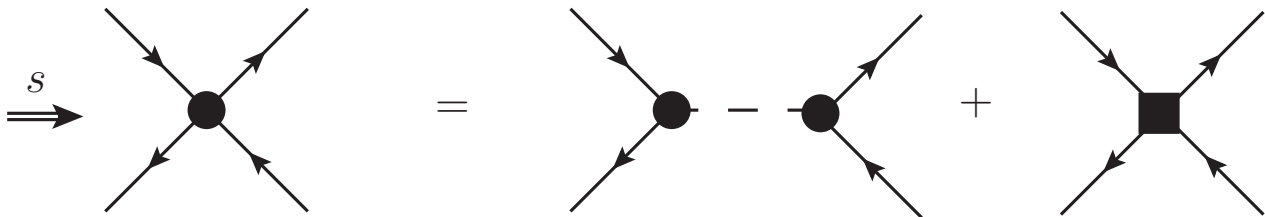
(i)



Dynamical hadronisation at work

Mesons & diquarks:

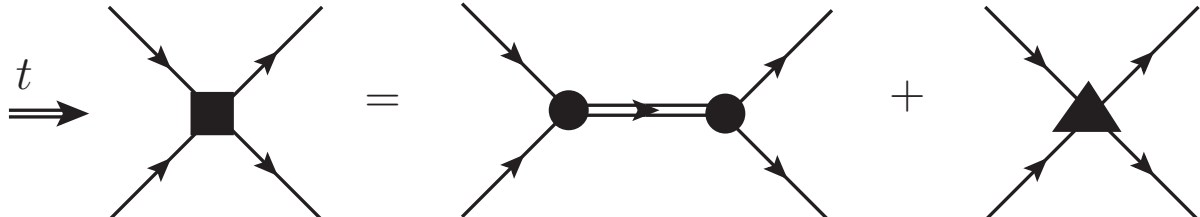
(i)



where

A diagram showing a square vertex with four external lines labeled $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4$. To the right of the diagram is a vertical line with (ϕ) at the top, followed by $= 0$ and two equations: $(p_1 + p_3)^2 = 0$ and $(p_2 + p_4)^2 = 0$.

(ii)



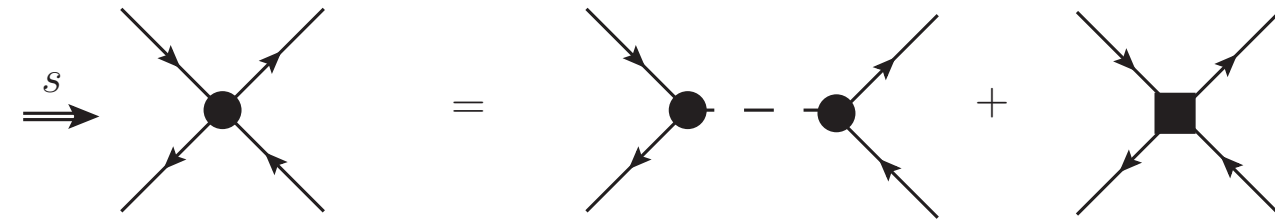
where

A diagram showing a triangle vertex with four external lines labeled $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4$. To the right of the diagram is a vertical line with (d) at the top, followed by $= 0$ and two equations: $(p_1 + p_2)^2 = 0$ and $(p_3 + p_4)^2 = 0$.

Dynamical hadronisation at work

Mesons & diquarks:

(i)

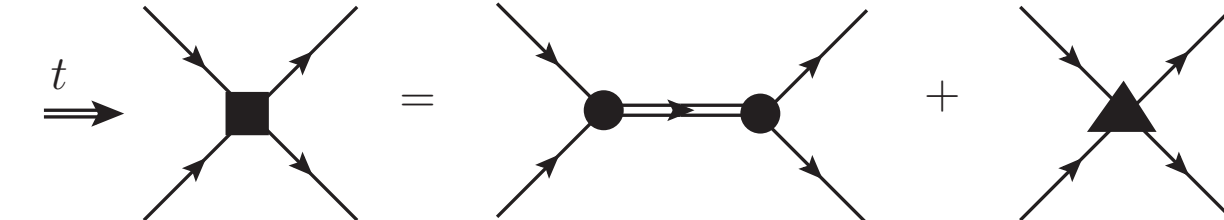


where

$$\left. \begin{array}{c} \vec{p}_1 \quad \vec{p}_4 \\ \vec{p}_2 \quad \vec{p}_3 \end{array} \right| (\phi) = 0$$

$$\begin{array}{l} (p_1 + p_3)^2 = 0 \\ (p_2 + p_4)^2 = 0 \end{array}$$

(ii)

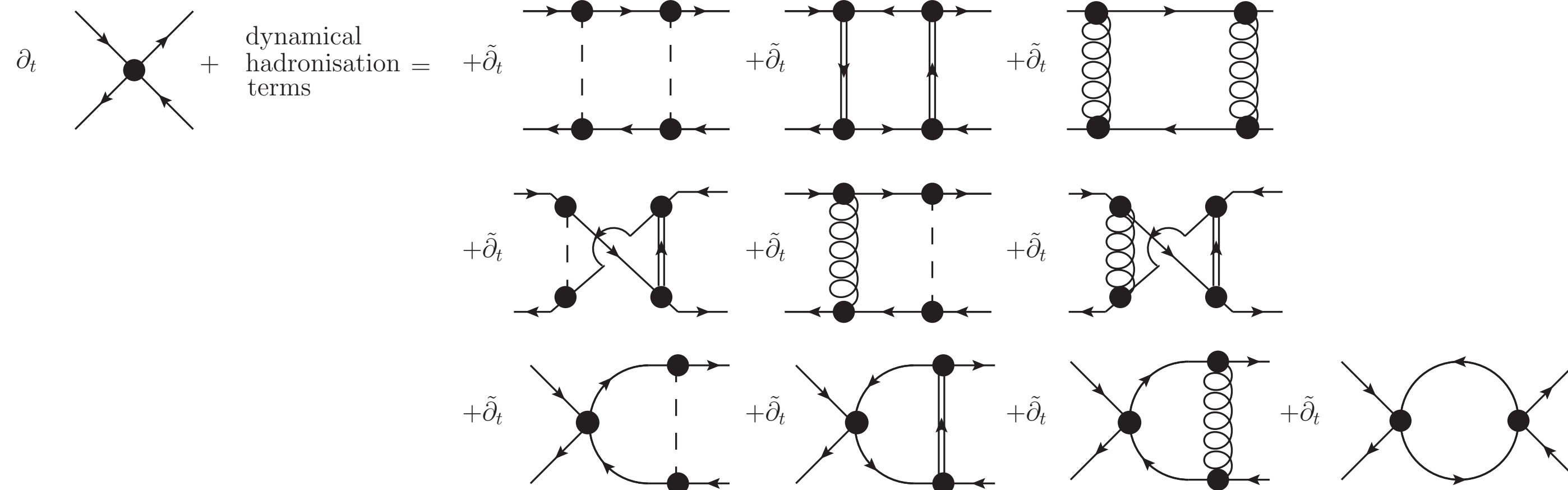


where

$$\left. \begin{array}{c} p_1 \quad p_2 \\ p_3 \quad p_4 \end{array} \right| (d) = 0$$

$$\begin{array}{l} (p_1 + p_2)^2 = 0 \\ (p_3 + p_4)^2 = 0 \end{array}$$

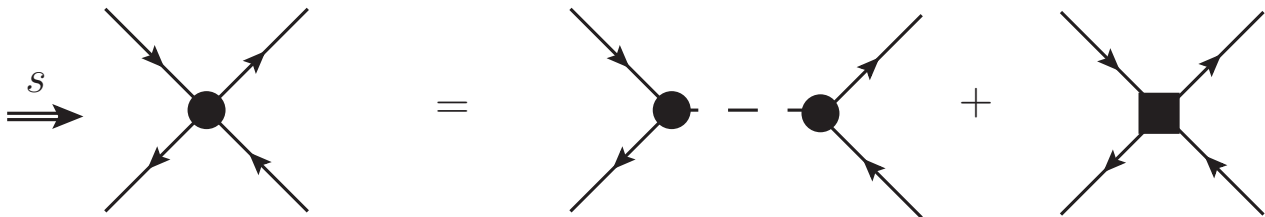
Schematical flow



Dynamical hadronisation at work

Mesons & diquarks:

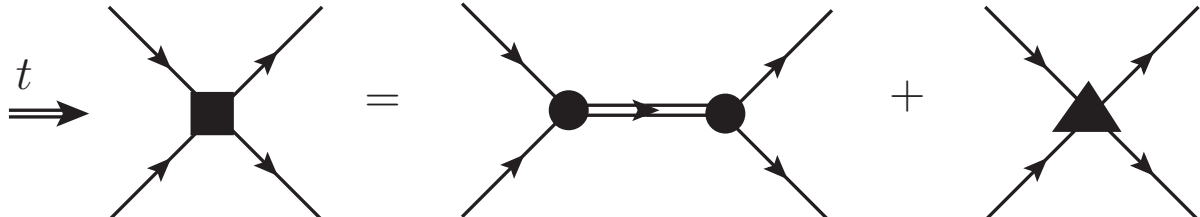
(i)



where

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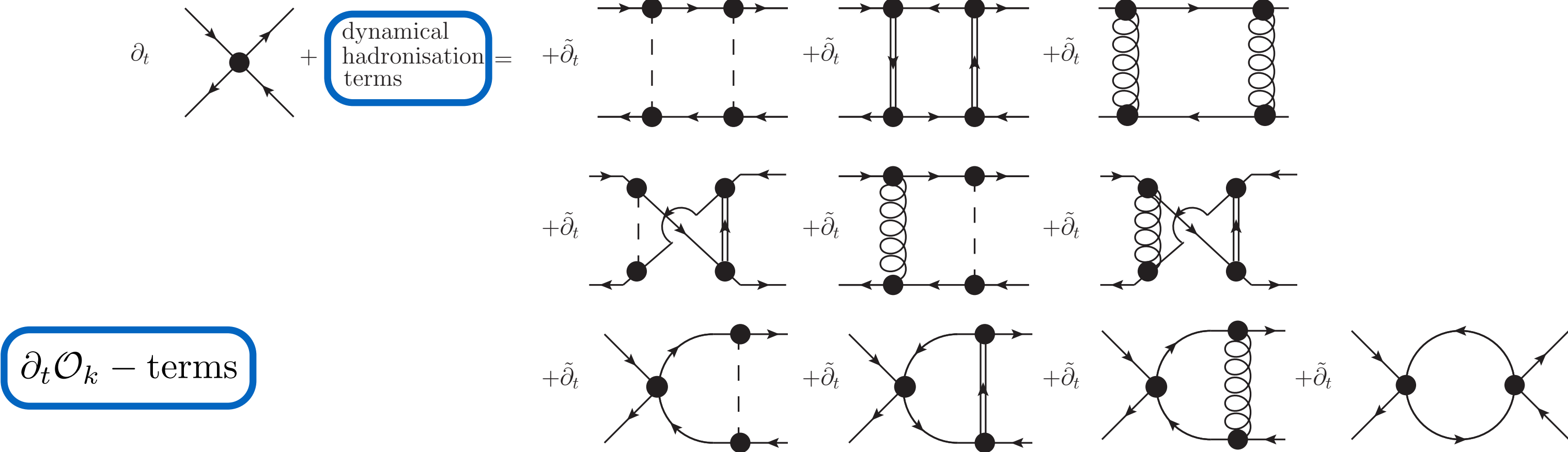
(ii)



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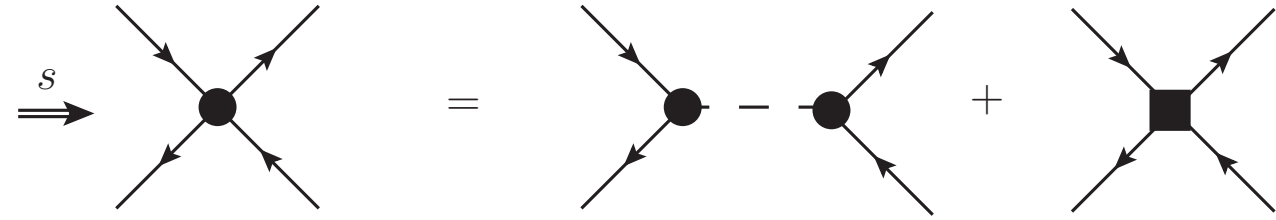
Schematical flow



Dynamical hadronisation at work

Mesons & diquarks:

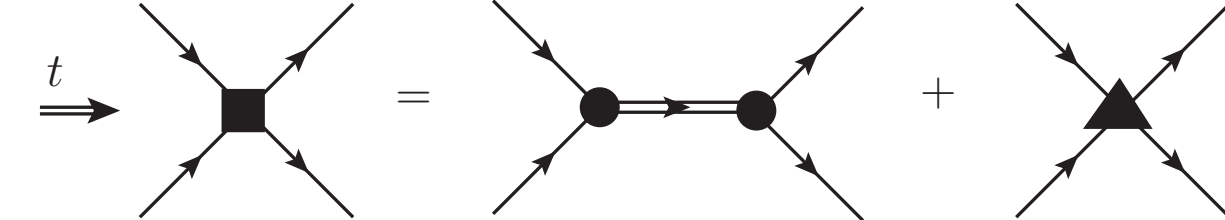
(i)



where

$$(\phi) \left. \begin{array}{l} (p_1 + p_3)^2 = 0 \\ (p_2 + p_4)^2 = 0 \end{array} \right\} = 0$$

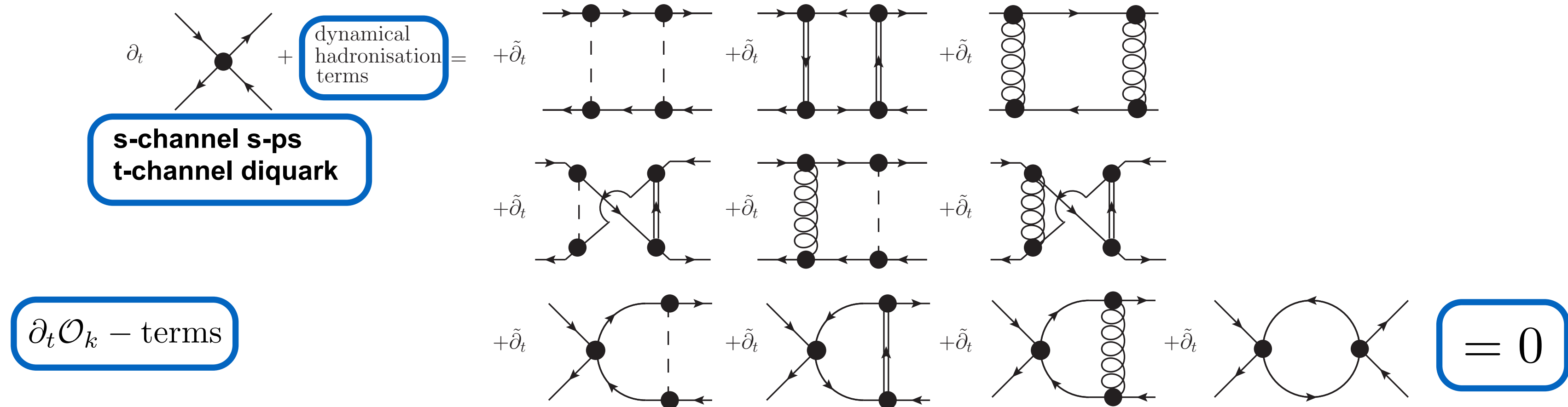
(ii)



where

The diagram shows a central black triangle representing a four-point vertex. Four external lines meet at this vertex, each with an arrow pointing away from the vertex. The top-left line is labeled p_1 , the top-right line is labeled p_2 , the bottom-left line is labeled p_3 , and the bottom-right line is labeled p_4 . To the right of the vertex, a vertical line extends upwards and is labeled (d) . Below the vertex, the text $= 0$ is written. At the bottom of the diagram, two equations are listed: $(p_1 + p_2)^2 = 0$ and $(p_3 + p_4)^2 = 0$.

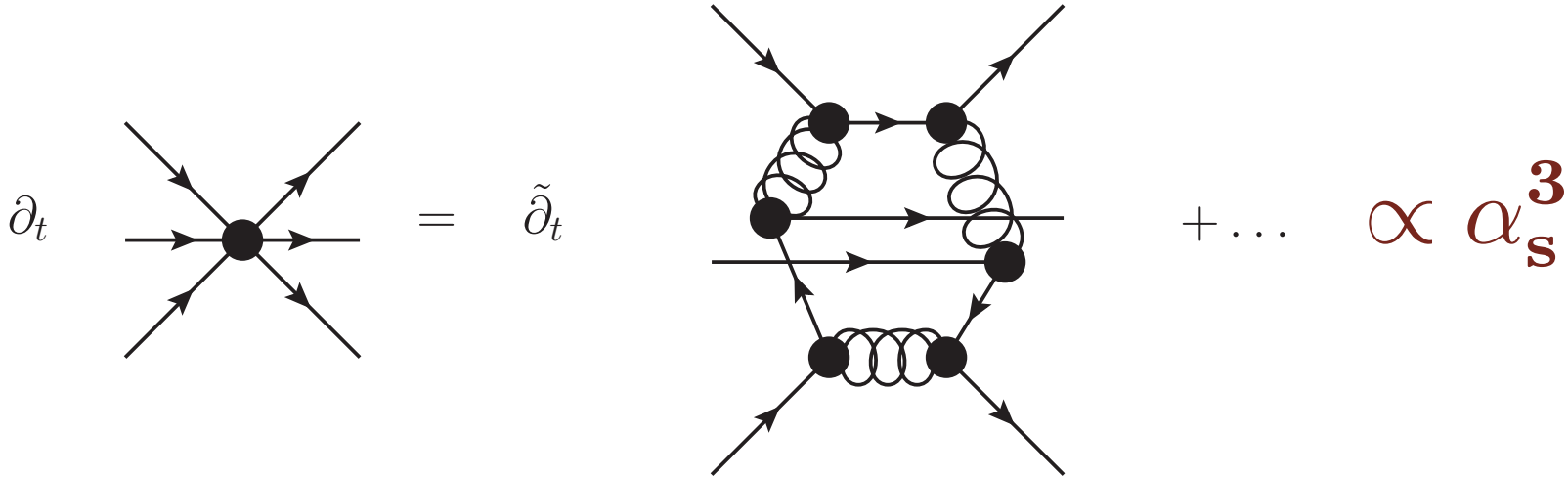
Schematical flow



Dynamical hadronisation at work

baryons:

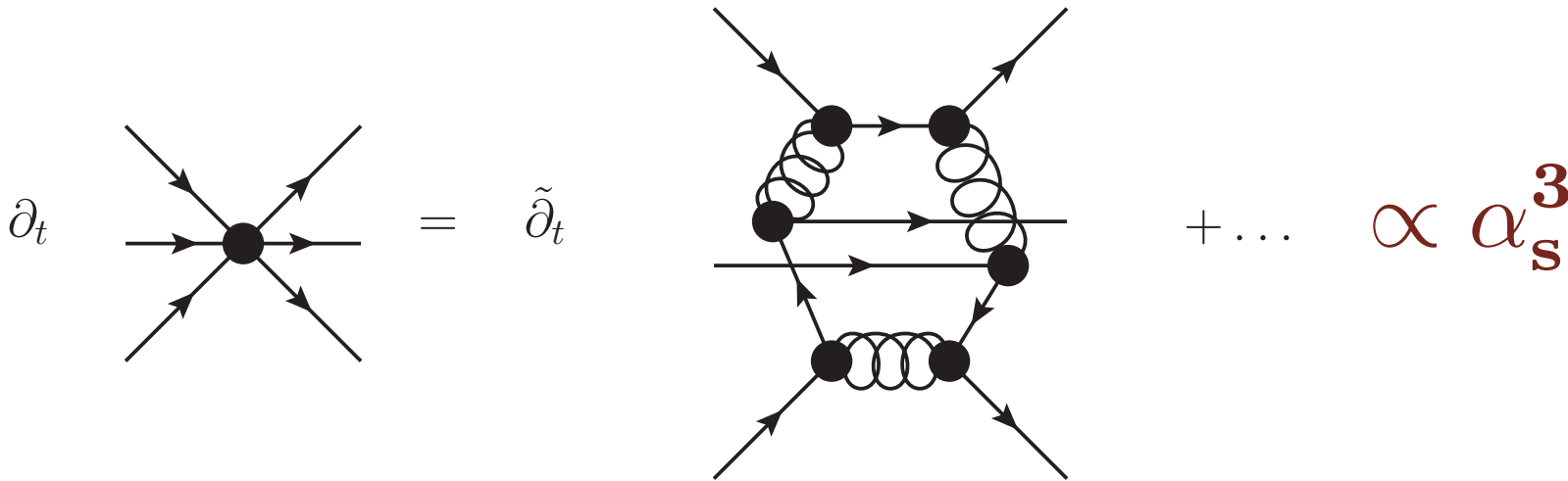
Dominant UV-process:



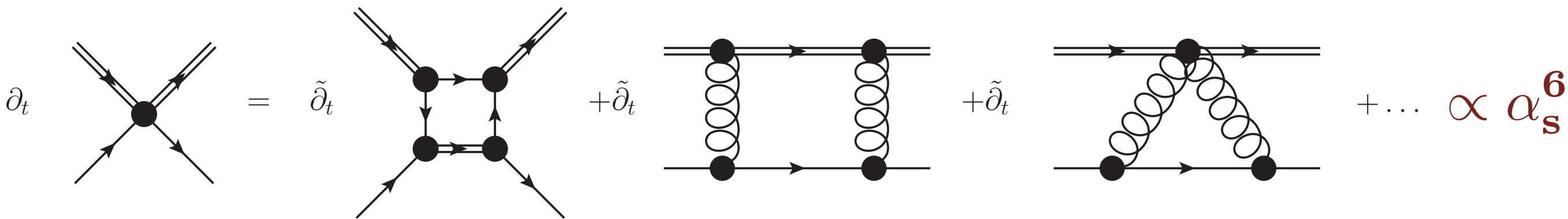
Dynamical hadronisation at work

baryons:

Dominant UV-process:



UV-subdominant:

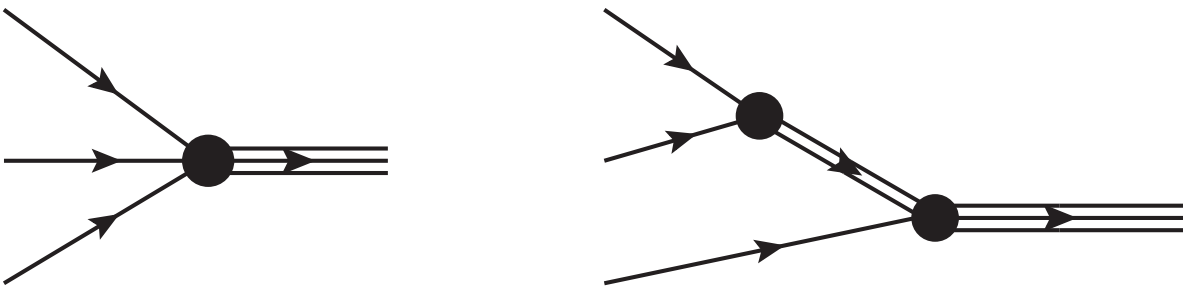


Dynamical hadronisation at work

baryons:

Baryonisation

Baryon formation processes

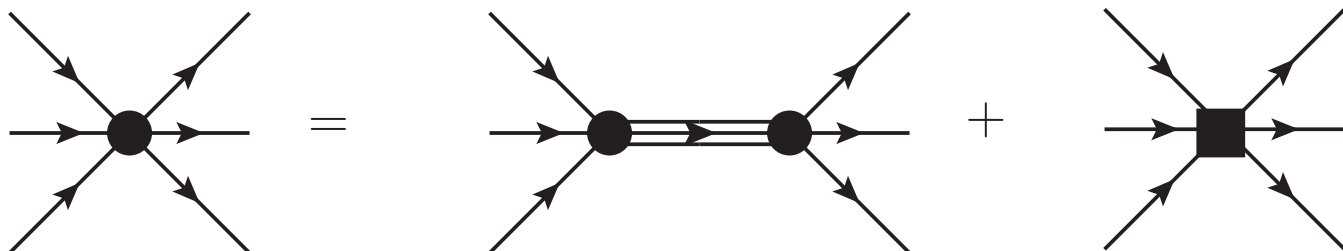


Dynamical hadronisation at work

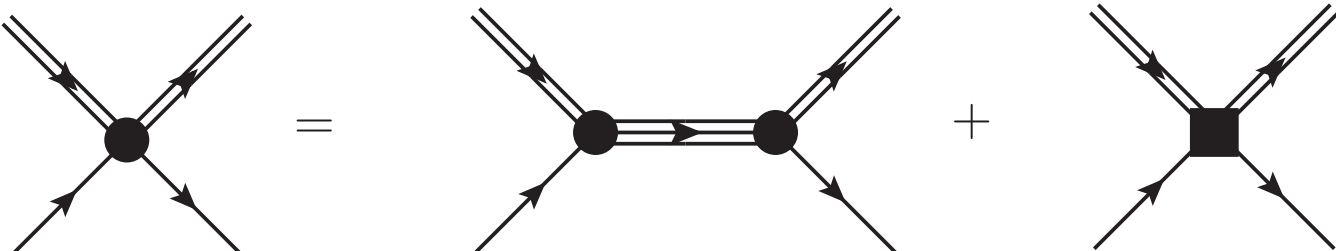
baryons:

Baryonisation

three-quark scattering



quark-diquark scattering



Dynamical hadronisation at work

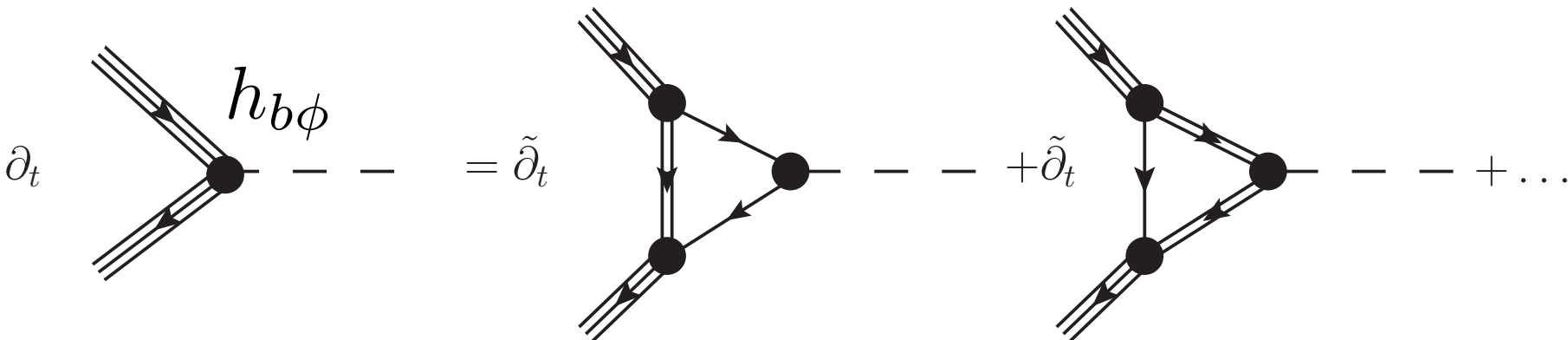
baryons:

Baryonisation

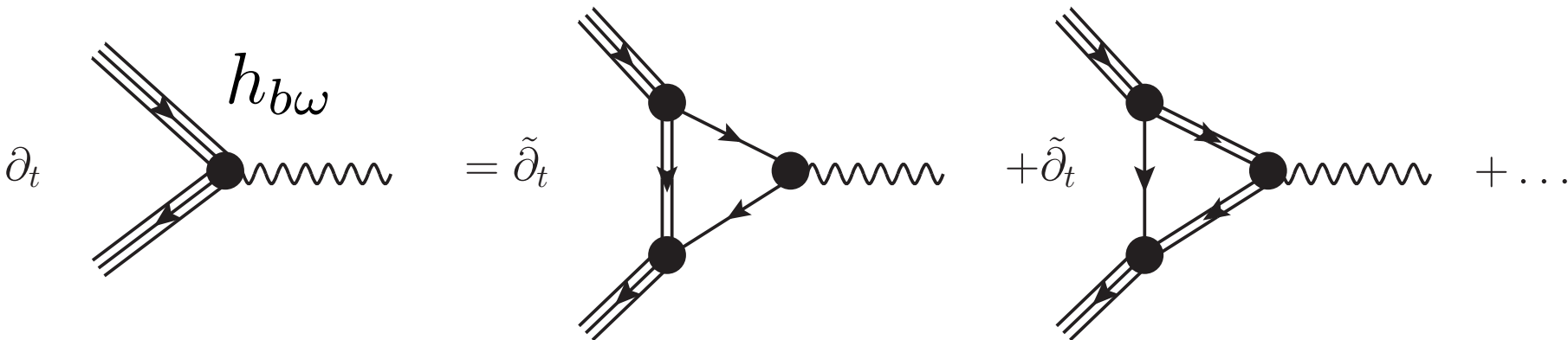


Yukawa-flows with baryonisation

nucleon-nucleon — $(\vec{\pi}, \sigma)$ scattering:



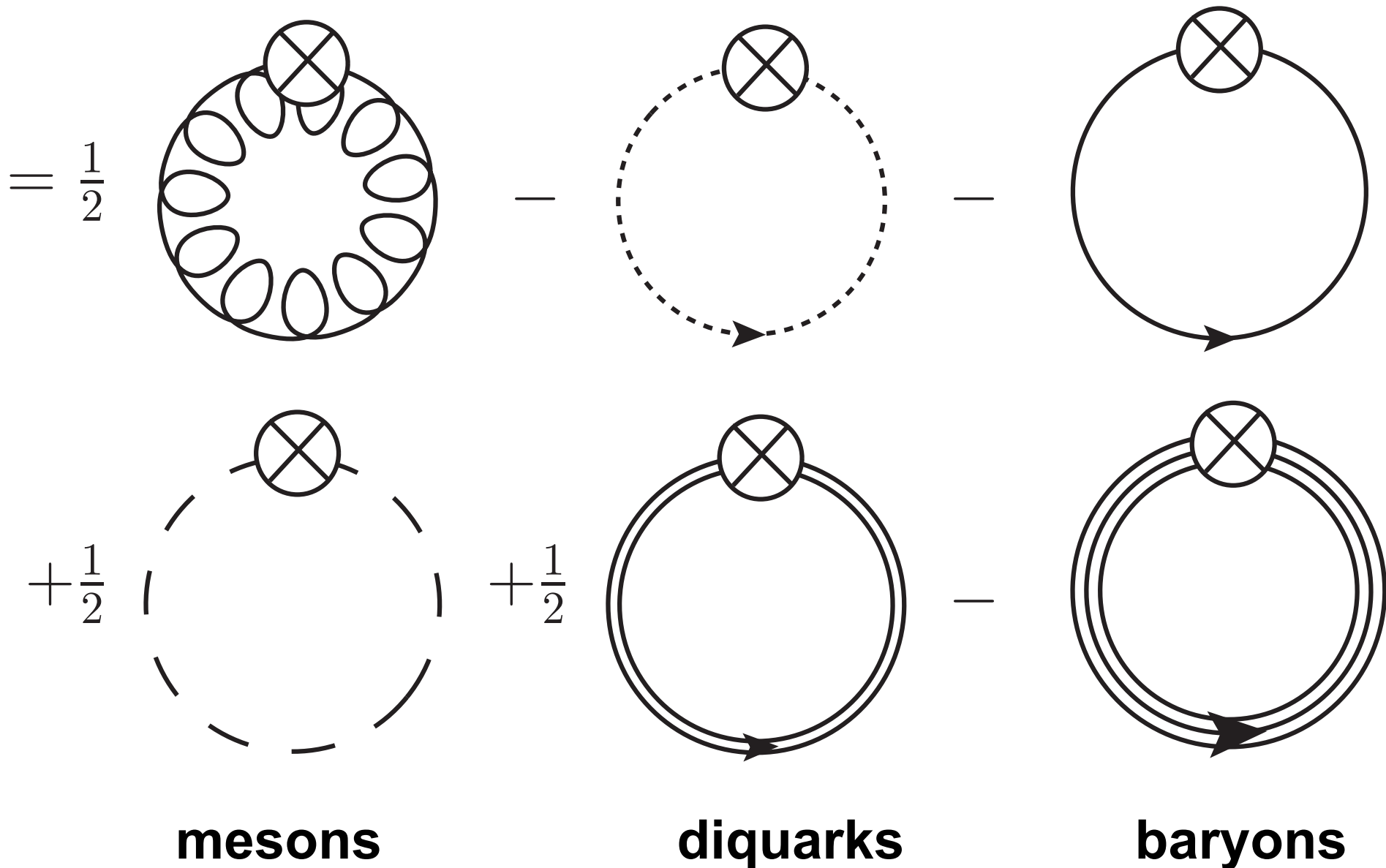
nucleon-nucleon — ω_μ scattering:



Dynamical hadronisation at work

‘DynHad for mesons, diquarks & baryons is Faddeev-BSE-DSE for QCD in a ‘unified’ effective action approach’

$$\left(\partial_t + \partial_t \Phi_{i,k}[\Phi] \frac{\delta}{\delta \Phi_i}\right) (\Gamma_k[\Phi] + c_{\Phi_i} \Phi_i)$$



(II) Functional QCD and the QCD phase structure

- **QCD at finite temperature and density**

- Benchmarks in the vacuum X
- Correlation functions at finite temperature X
- Polyakov loop from functional approaches X

▪ Phase structure from functional QCD: how to ✓

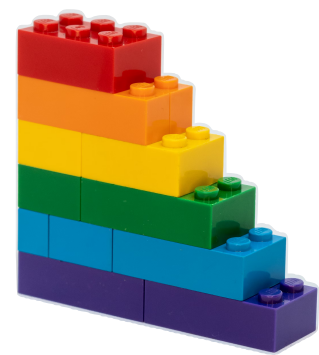
- **QCD phase structure**

- Locating the QCD phase boundary and the critical end point ✓
- Fluctuations of conserved charges: Ripples of the critical end point X

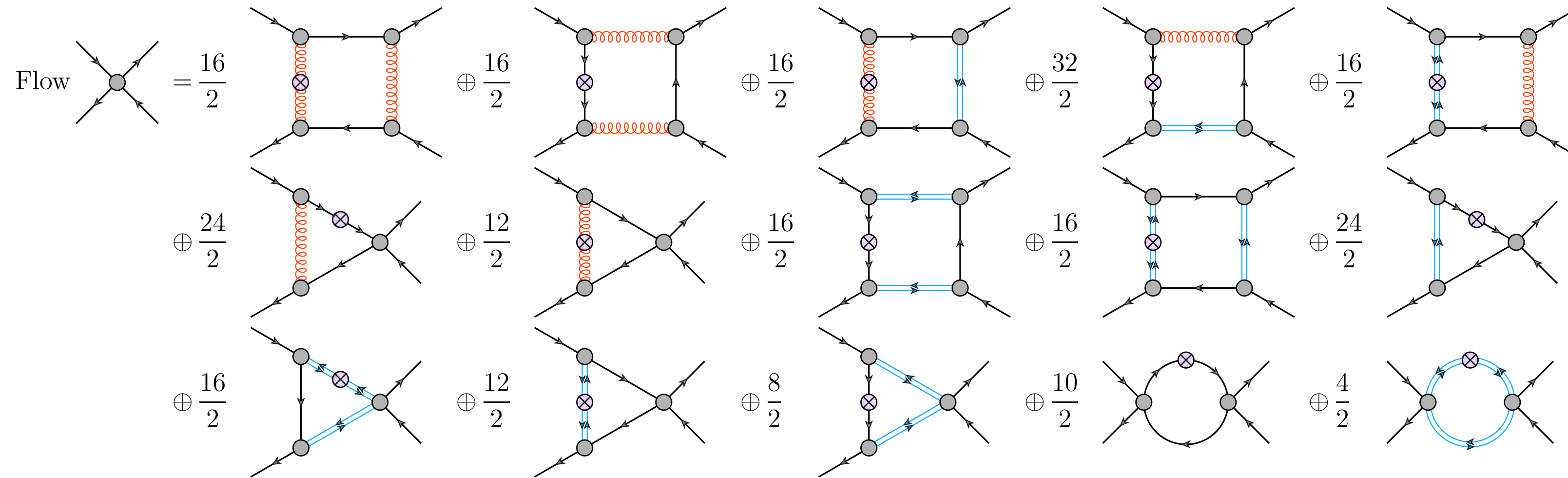
Jump back to: Getting dynamical: emergent hadrons & diquarks

Jump back to: Critical end point and new phases: where do we stand?

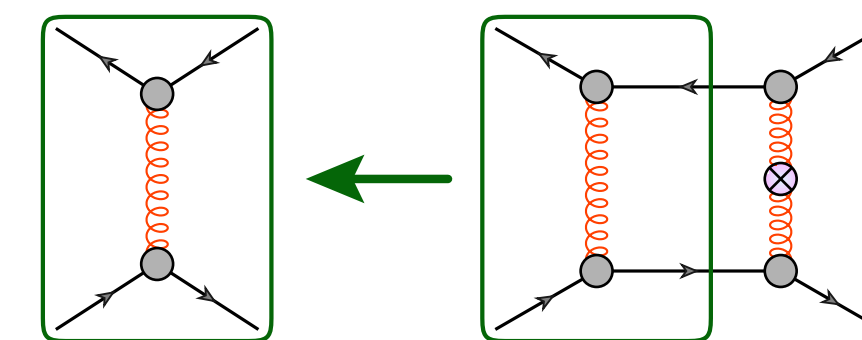
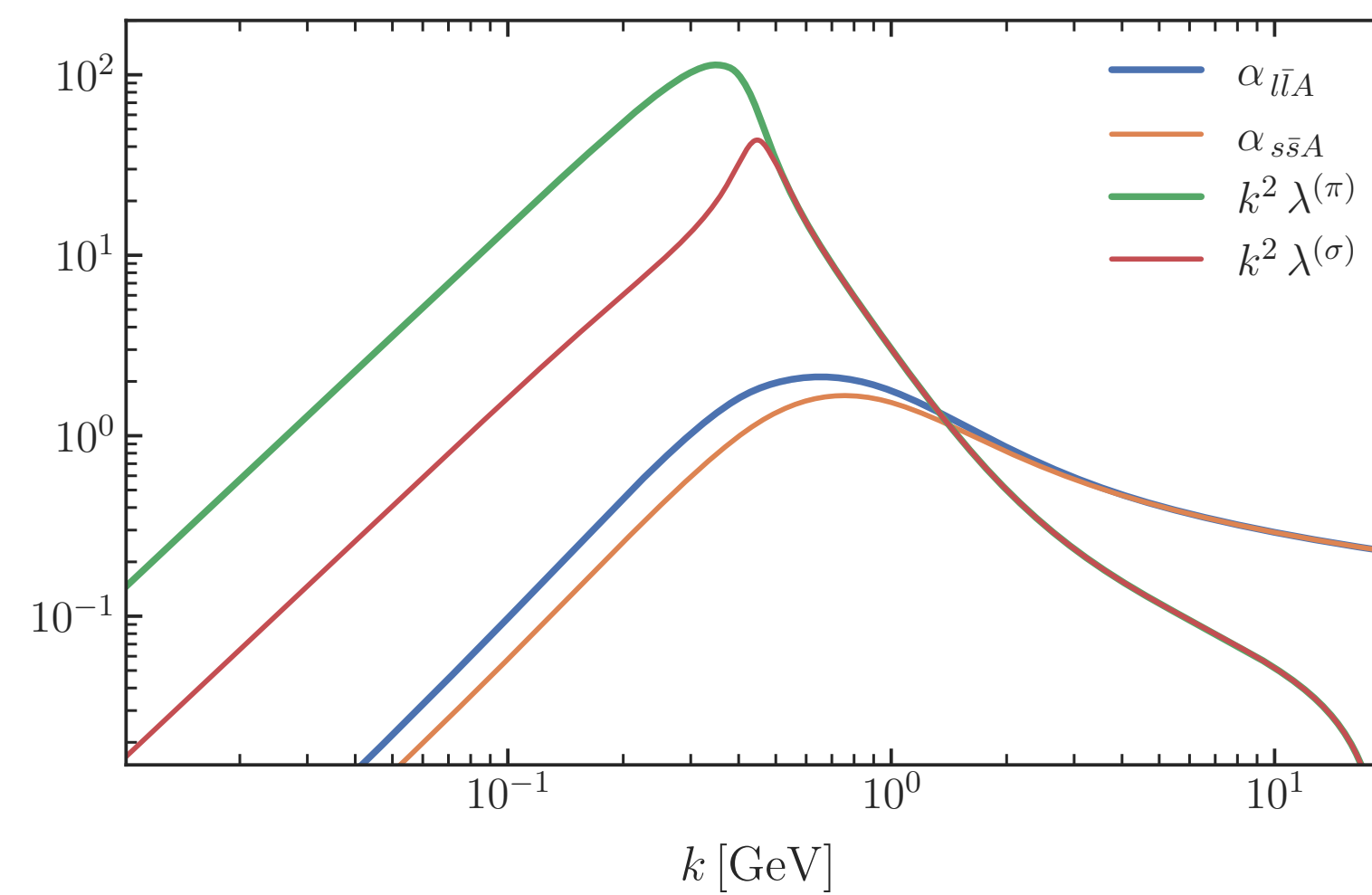
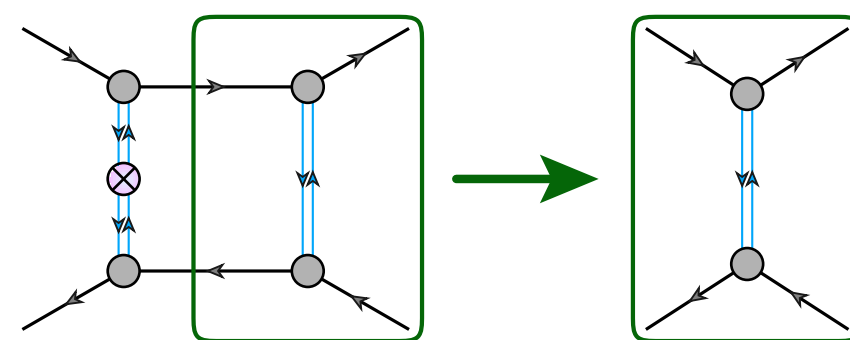
Phase structure from functional QCD: how to

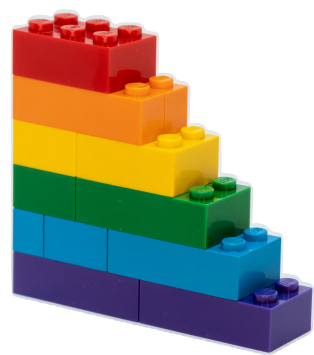


How to: systematic error estimates & the LEGO[®] principle

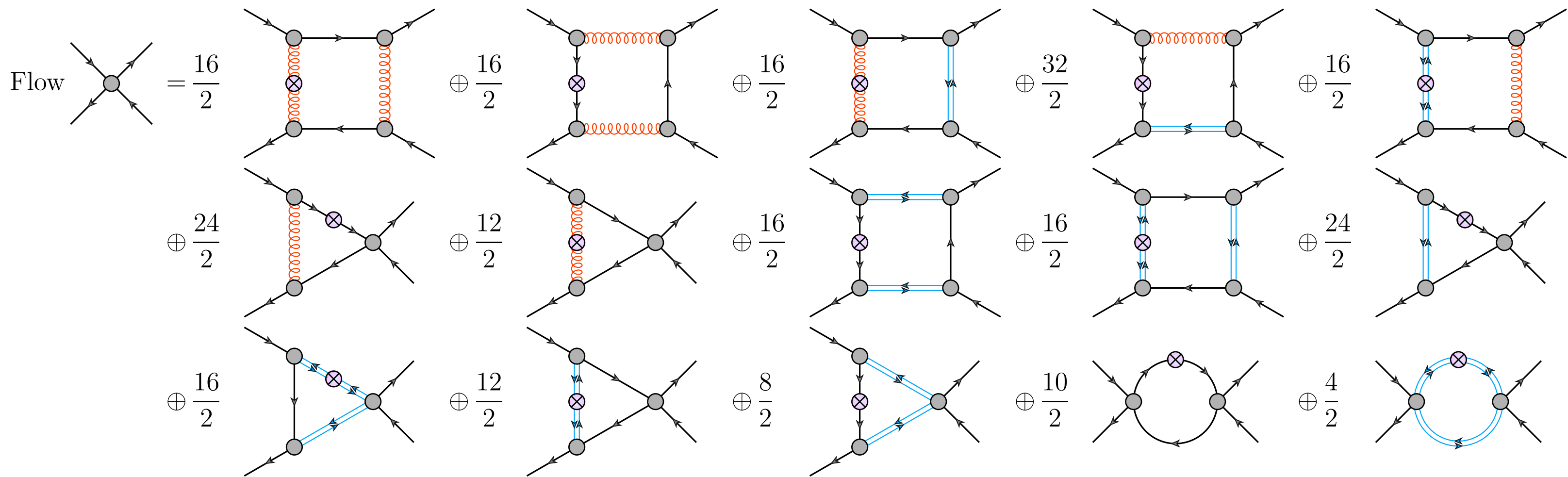


Example: 4-quark scattering vertex

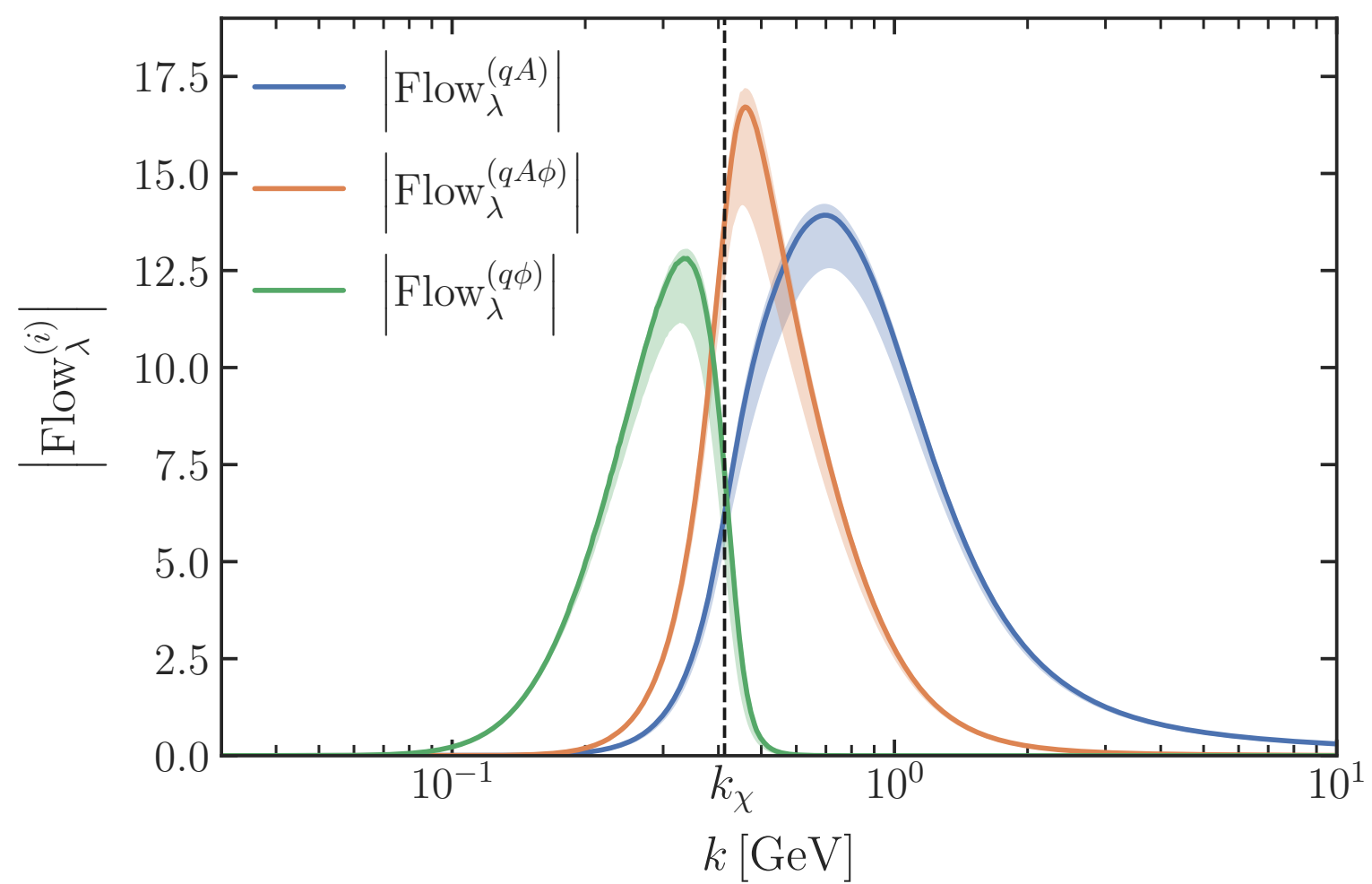


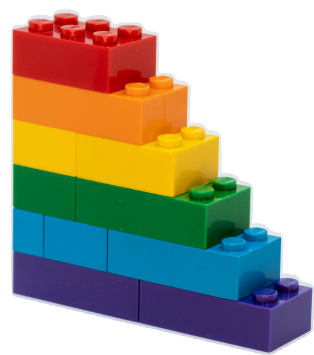


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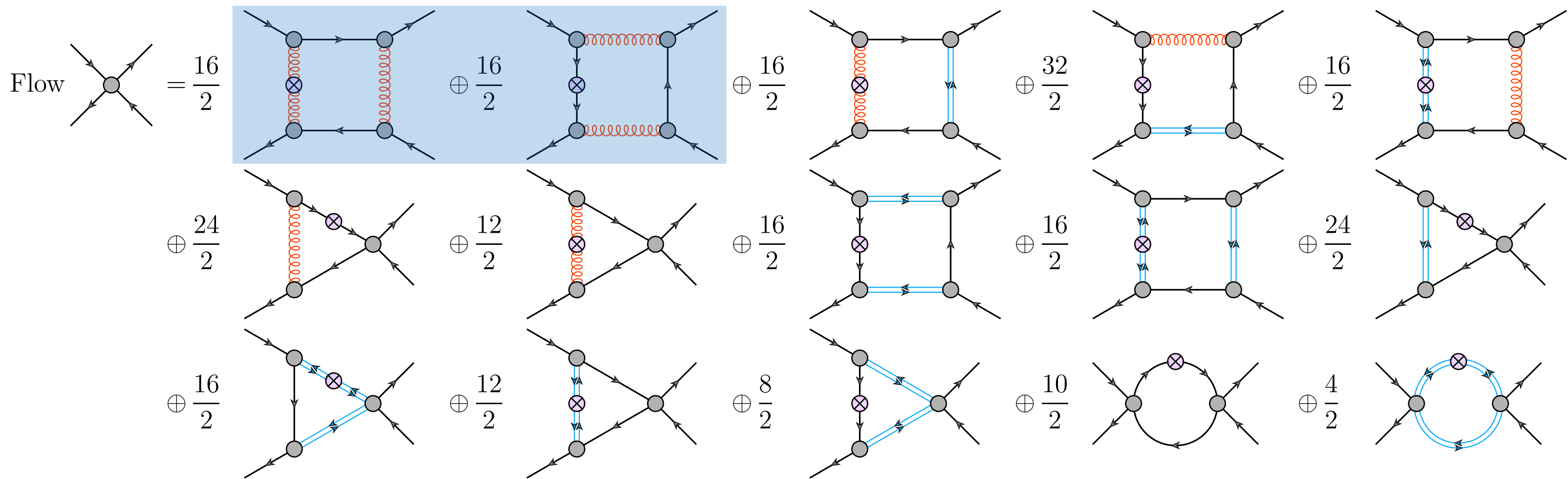


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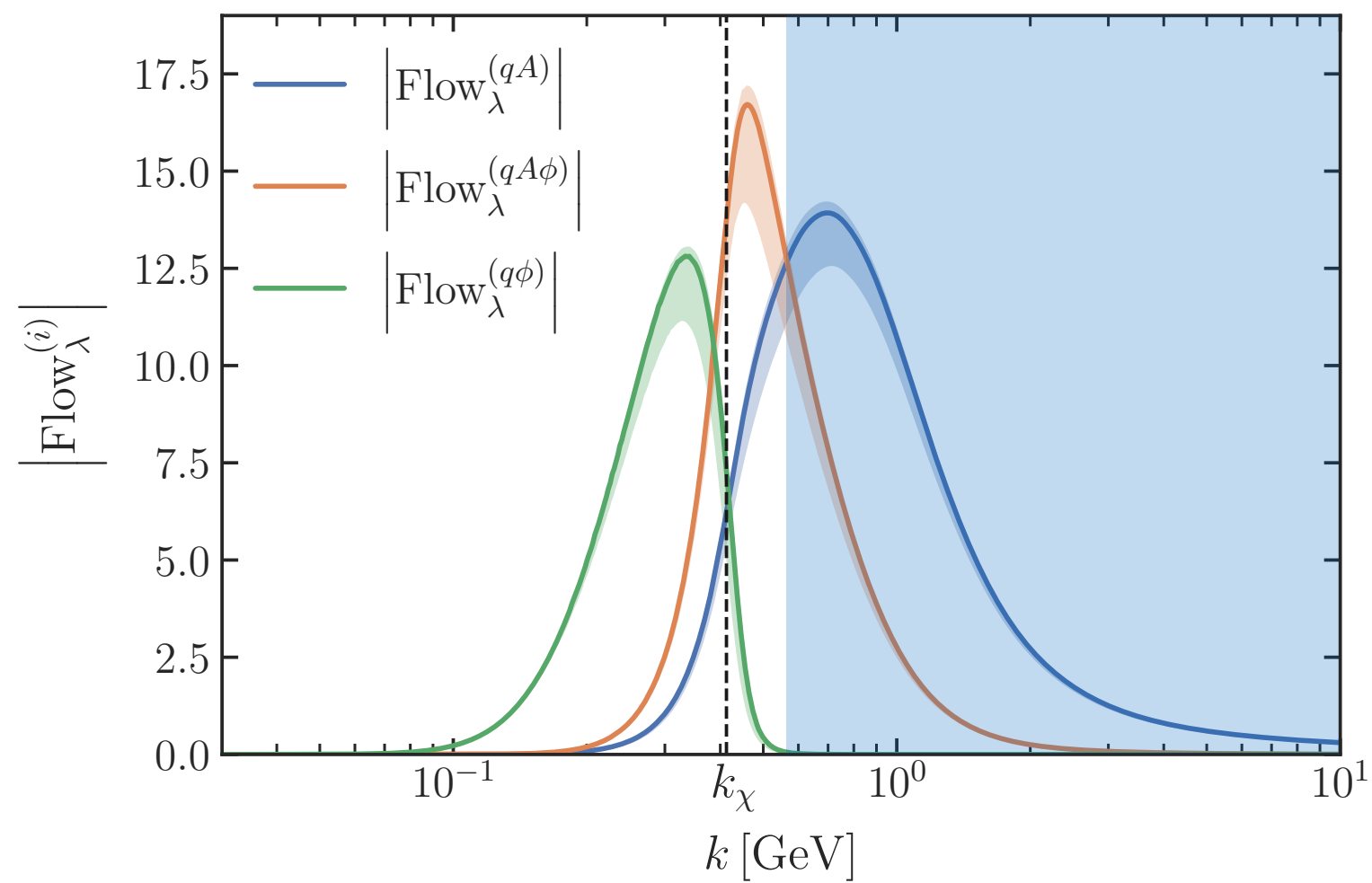


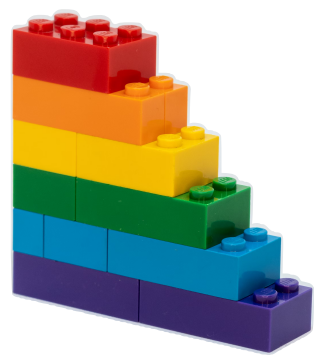


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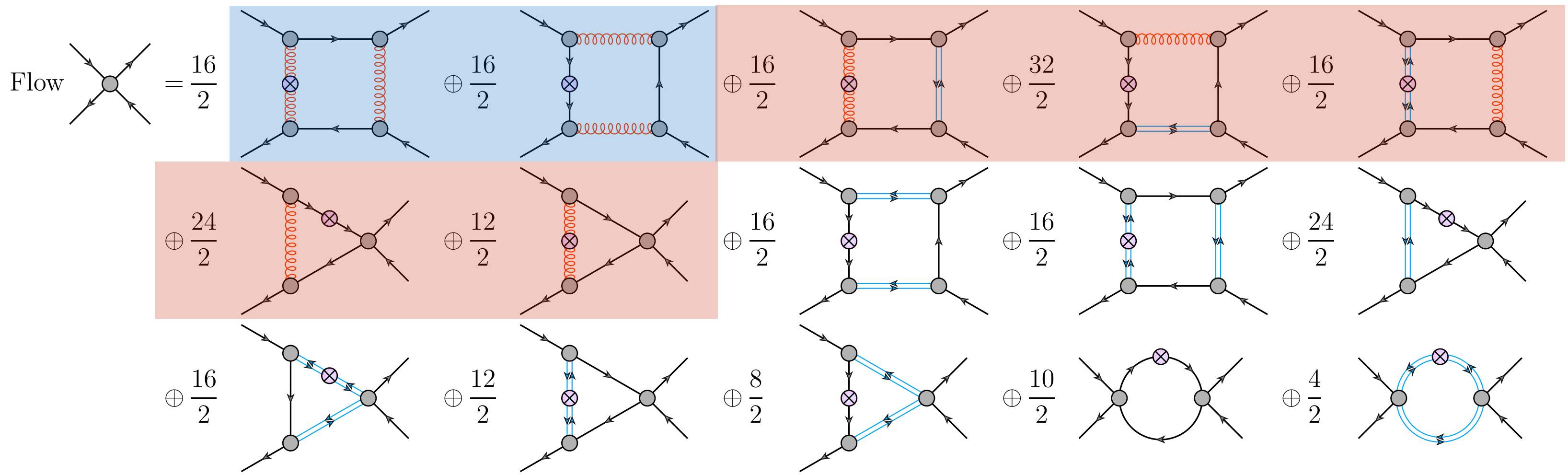


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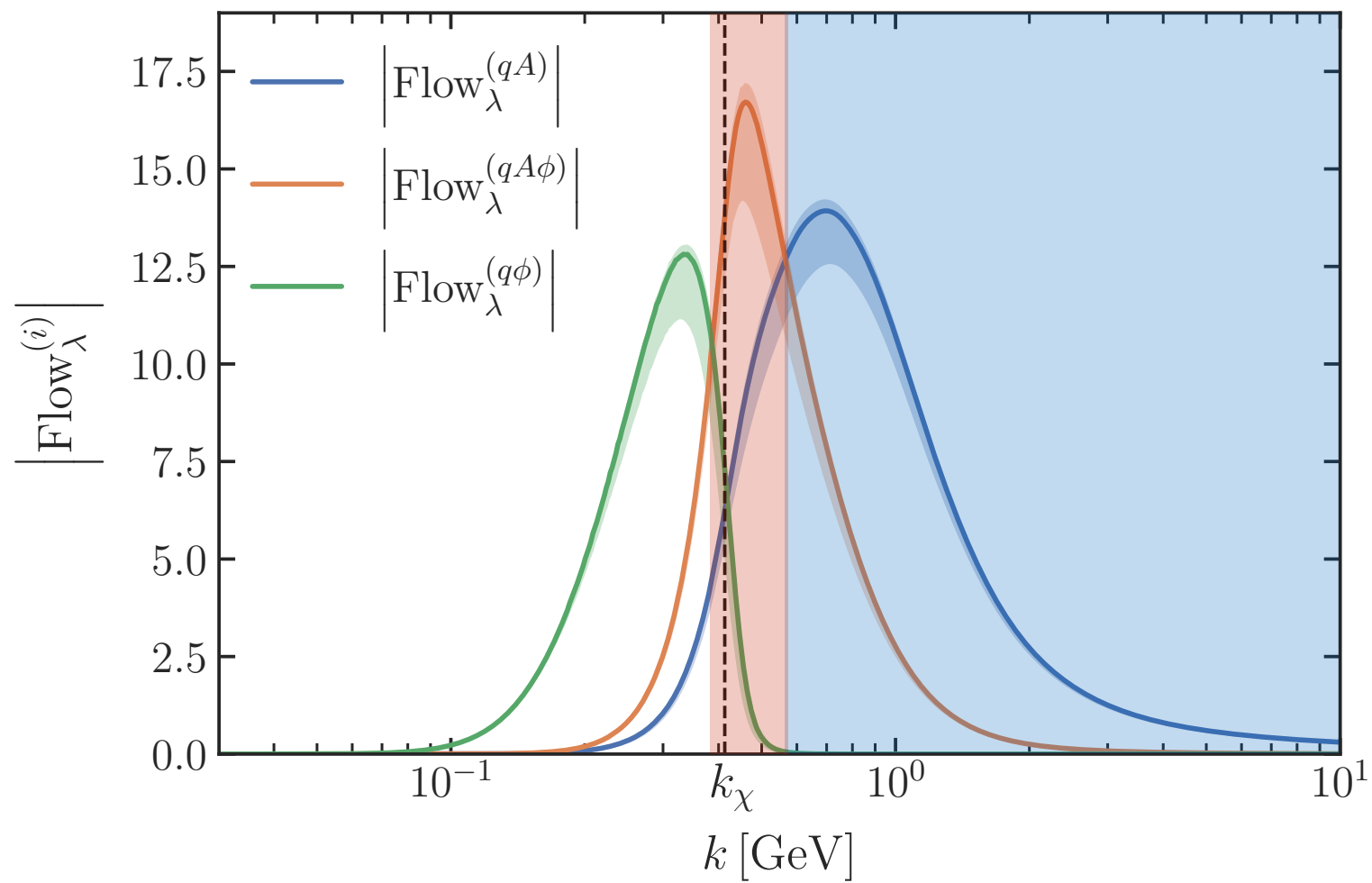


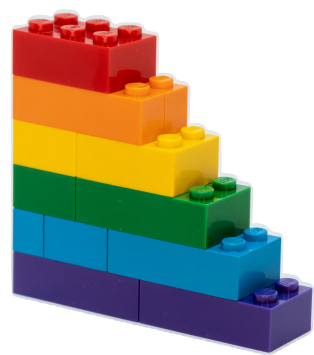


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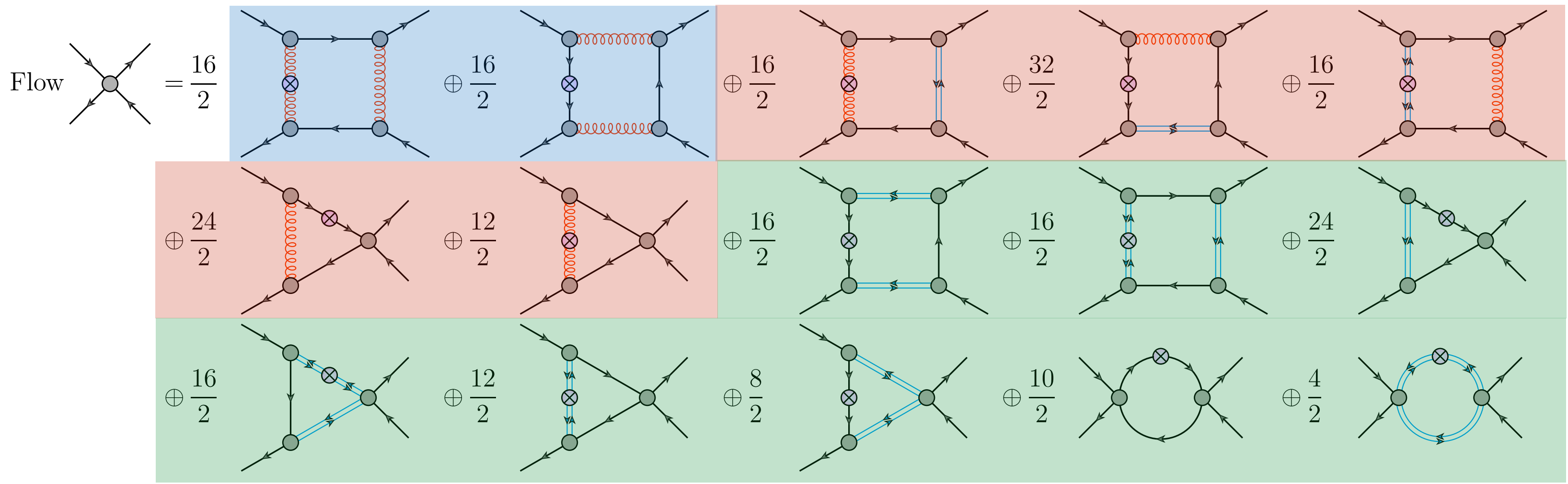


Example: 4-quark scattering vertex

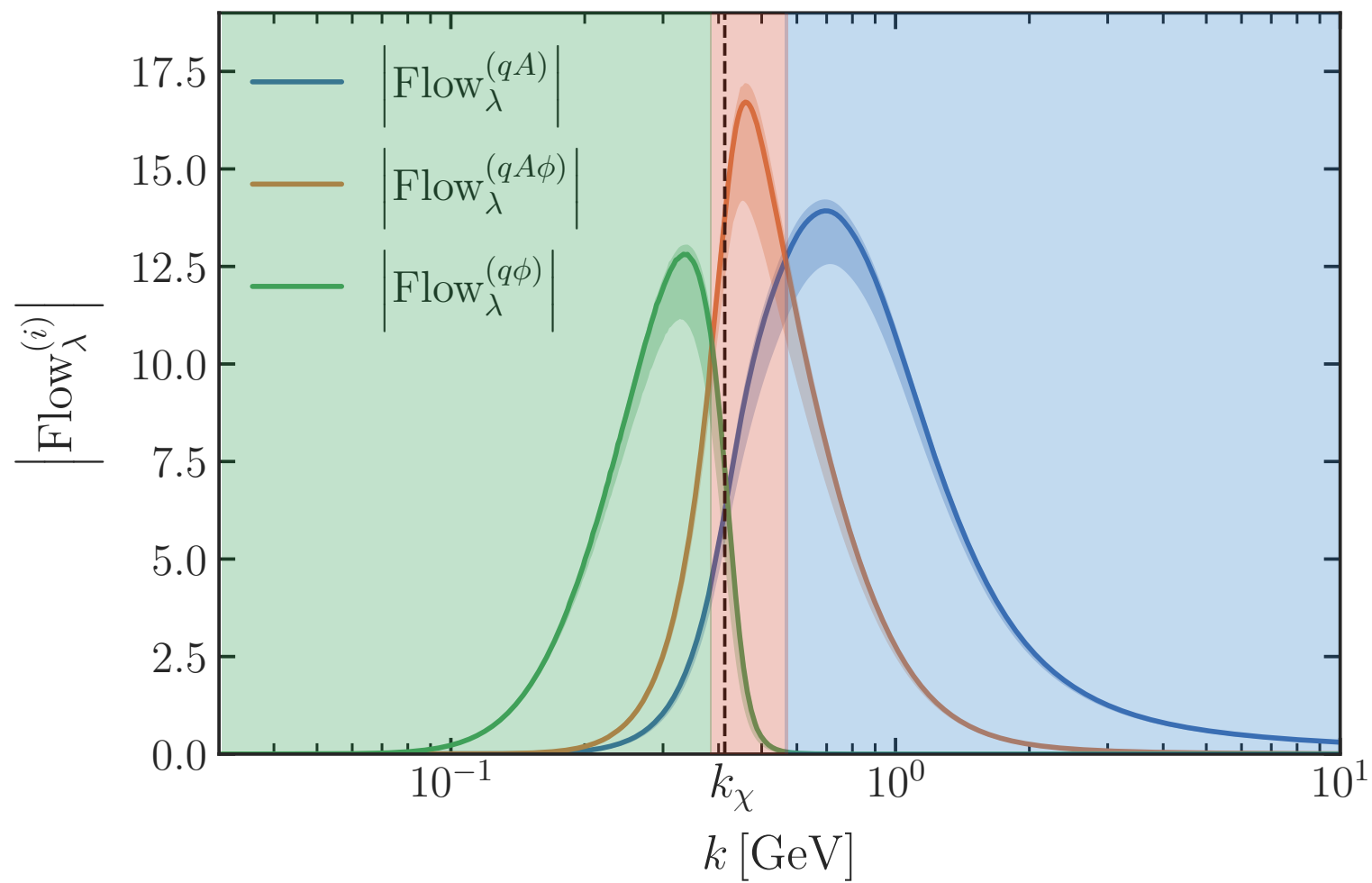


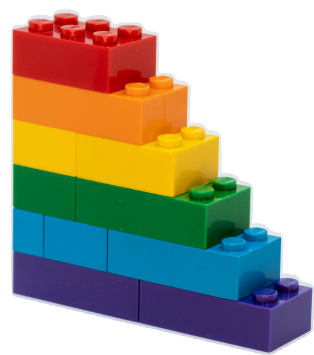


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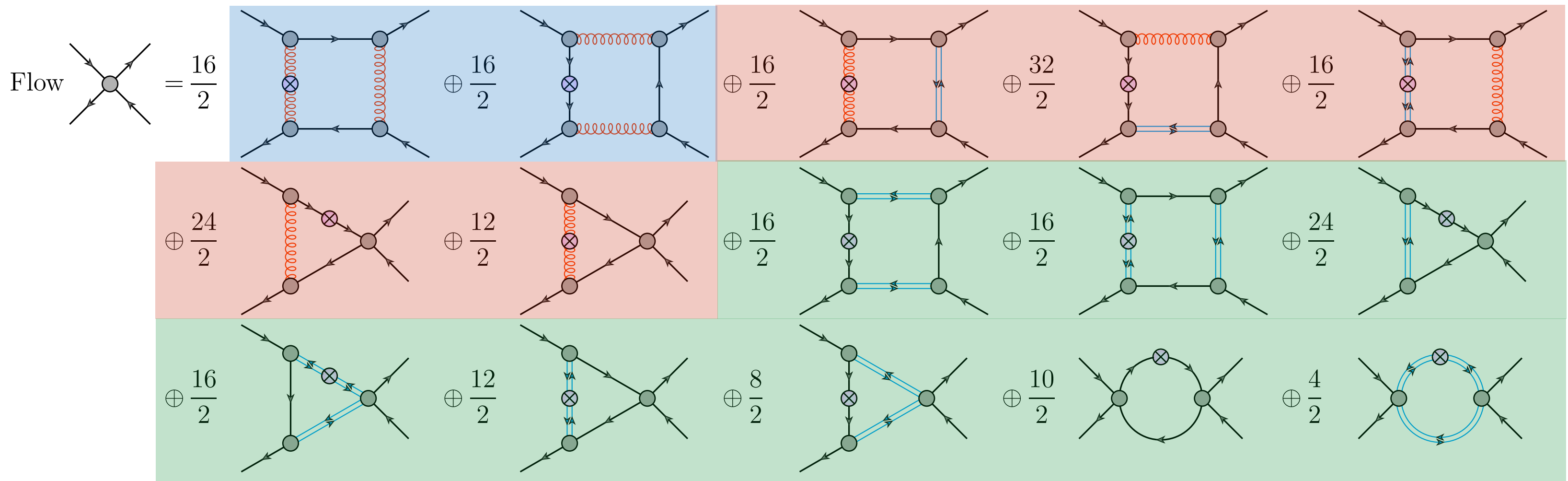


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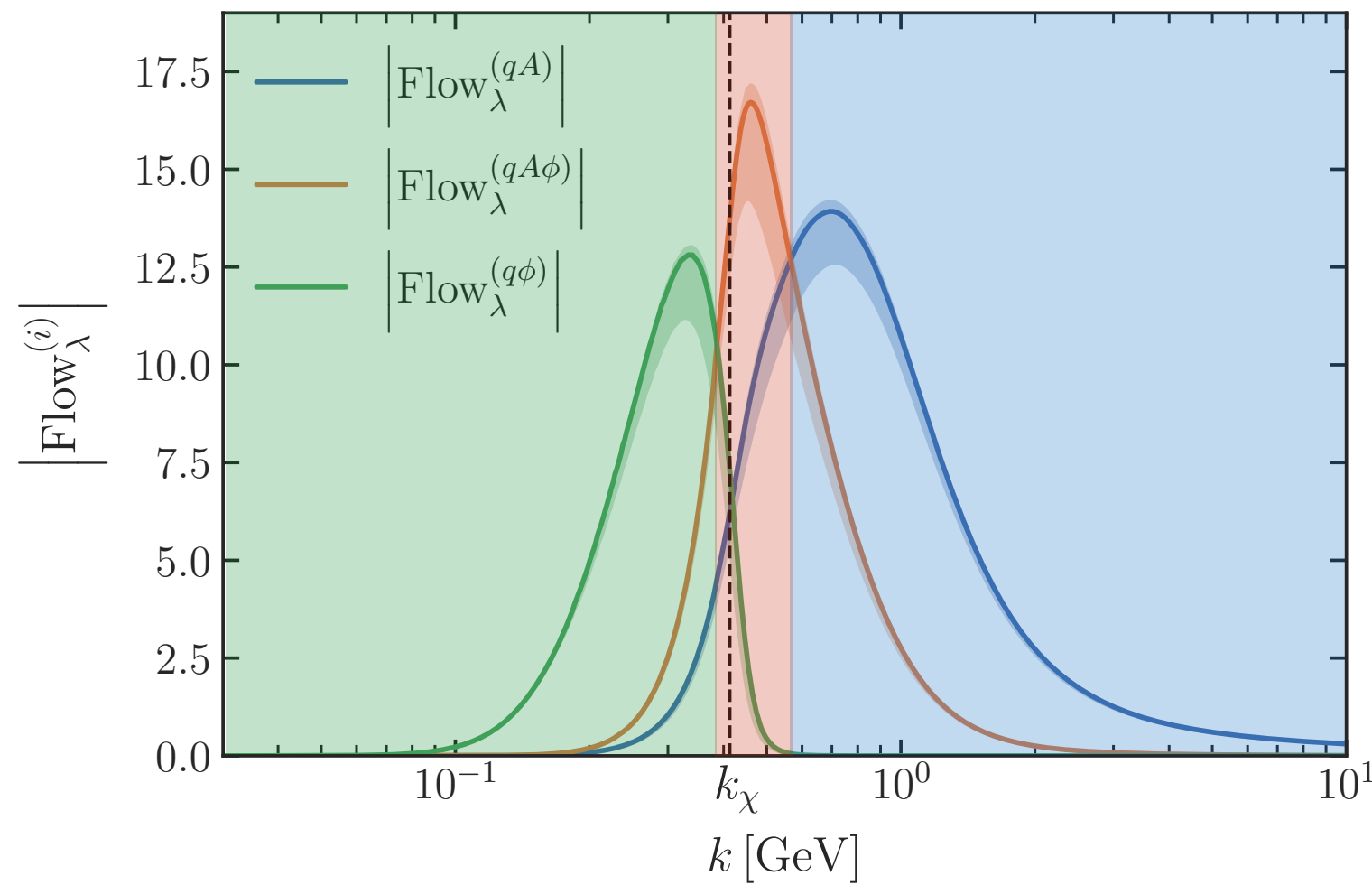




How to: systematic error estimates & the LEGO[®] principle



The unreasonable effectiveness of low energy effective theories



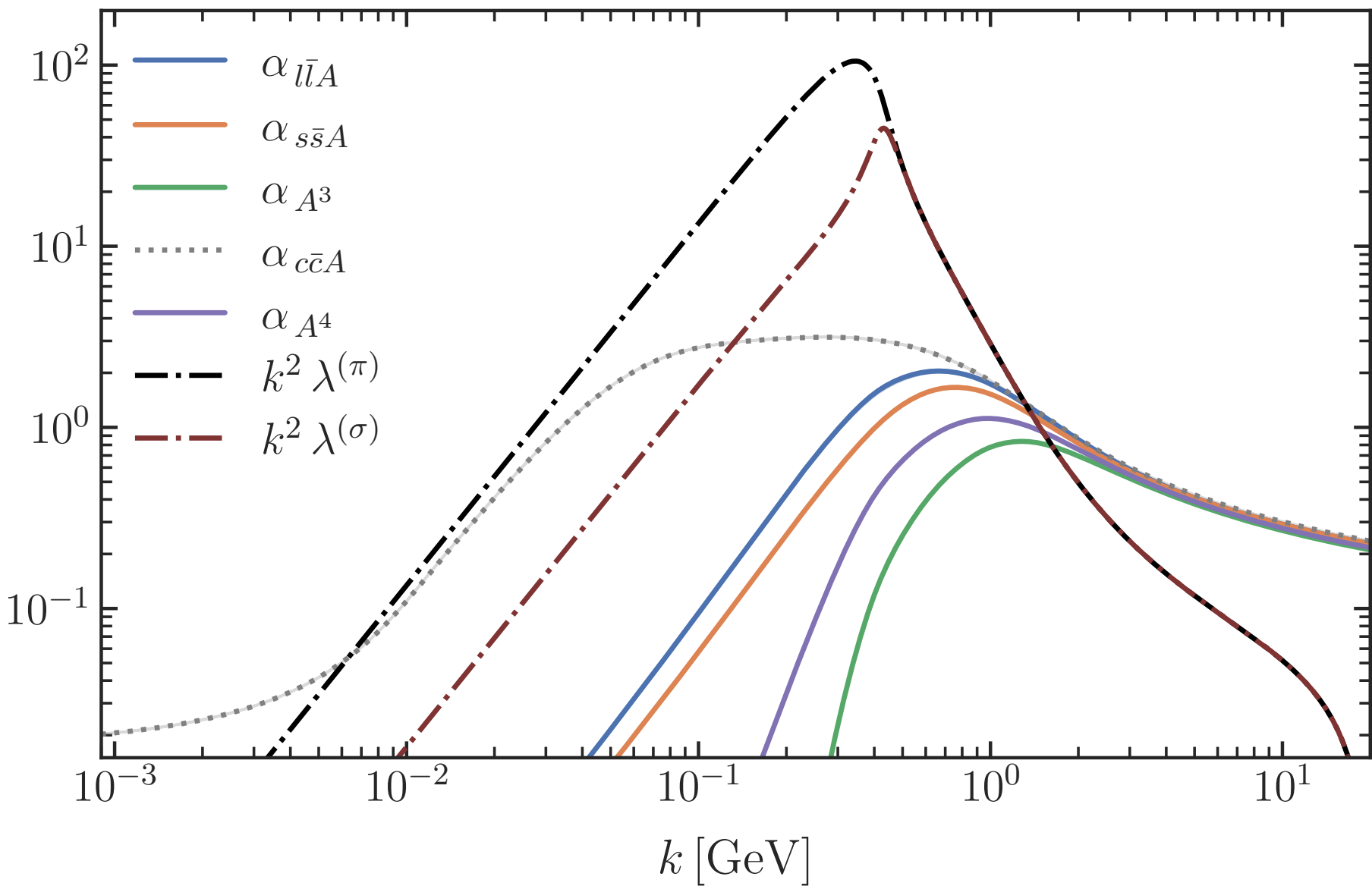
Access and combined use of
error estimates
from functional QCD & LEFTs

How to: the unreasonable effectiveness of low energy effective theories

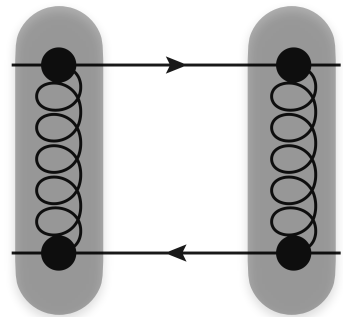
$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \left(\text{orange loop} - \text{dashed loop} - \text{solid loop} + \frac{1}{2} \text{blue loop} \right)$$

fQCD

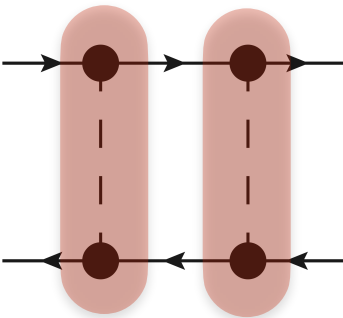
Sequential decoupling of gluon, quark, sigma, pion fluctuations



$$\frac{g_{lAl}^2}{g_{\bar{s}As}^2}$$



$$k^2 \lambda^{(\pi, \sigma)}$$



Vacuum: Ihssen, JMP, Sattler, Wink, arXiv:2408.08413

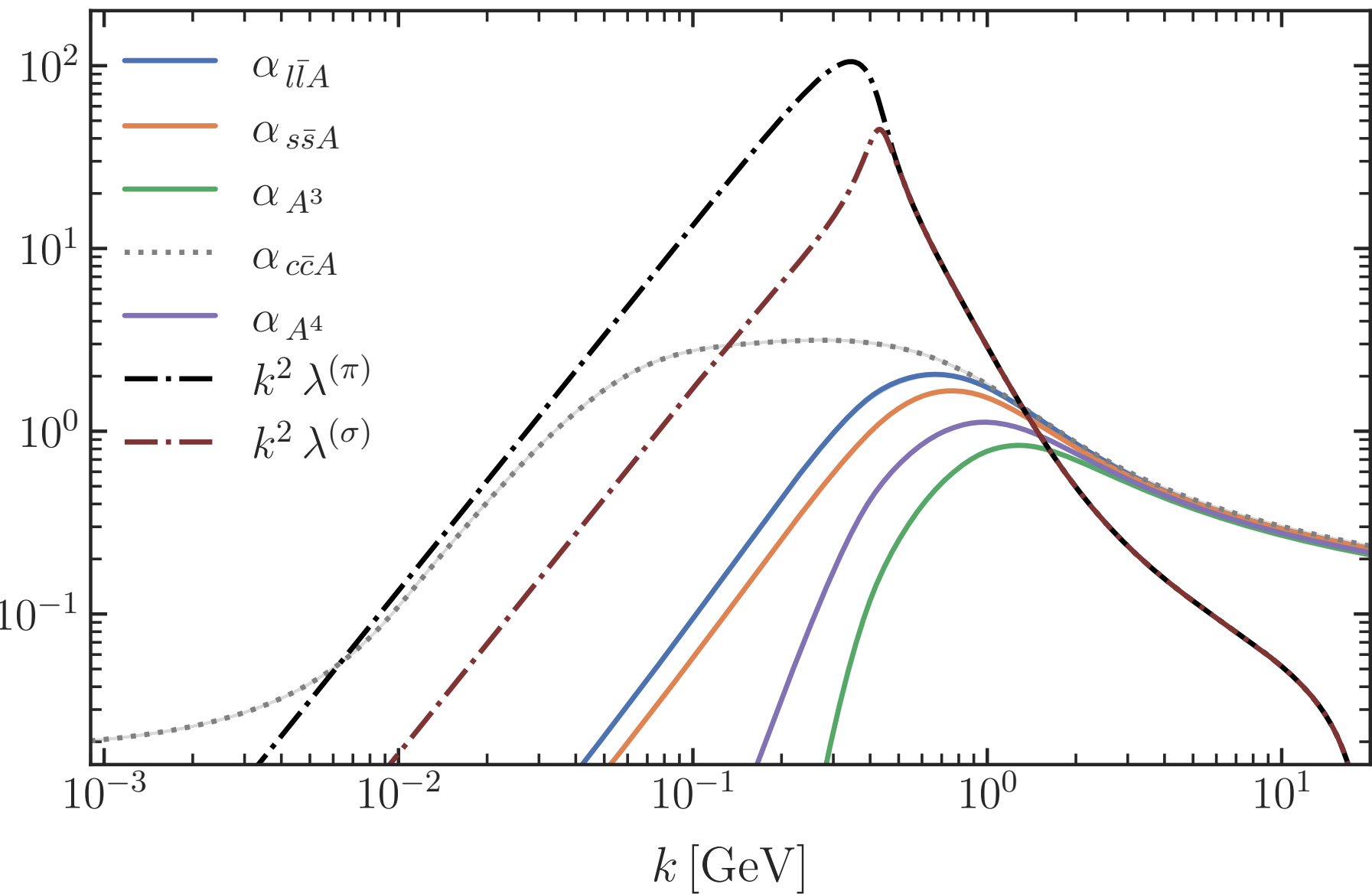
Phase structure: Fu, JMP, Rennecke, PRD 101, (2020) 054032

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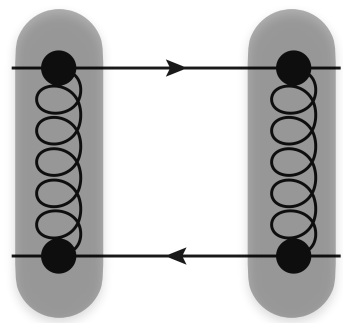
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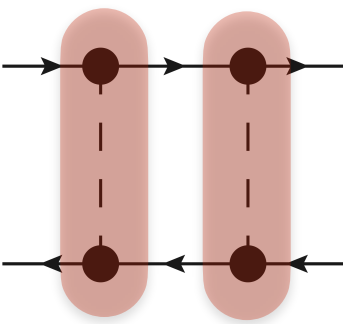
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PQM-model

$$\frac{1}{2} \left(\text{orange loop} - \text{dashed loop} + \text{solid loop} + \frac{1}{2} \text{blue loop} \right)$$

PNJL-model

$$\frac{1}{2} \left(\text{orange loop} - \text{dashed loop} + \text{solid loop} \right)$$

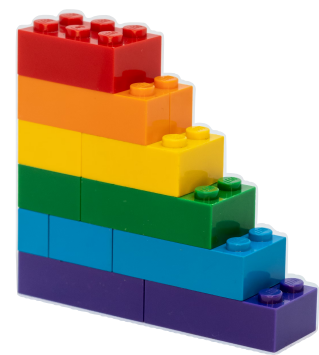
QM-model

$$- \text{solid loop} + \frac{1}{2} \text{blue loop}$$

NJL-model

$$- \text{solid loop}$$

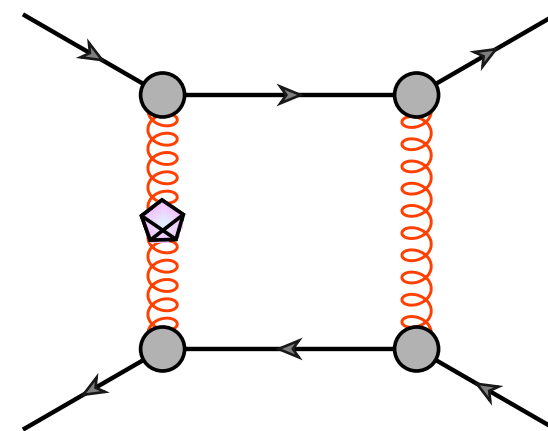
QCD-assisted low energy effective theories

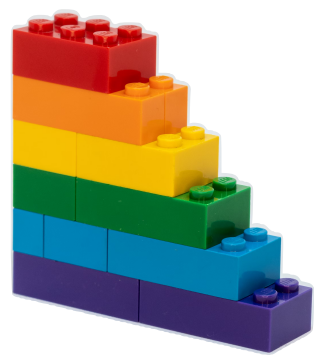


How to: systematic error estimates & the LEGO[®] principle



Example: Disect quark-gluon box

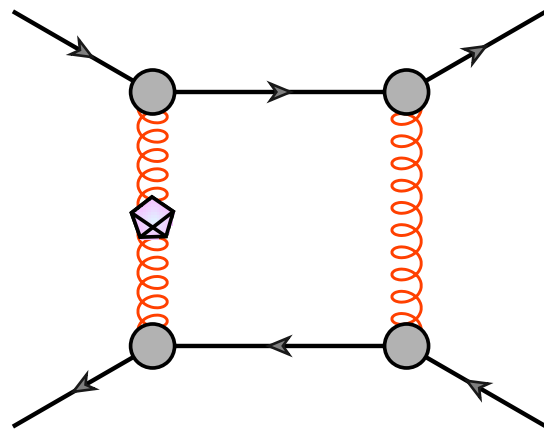




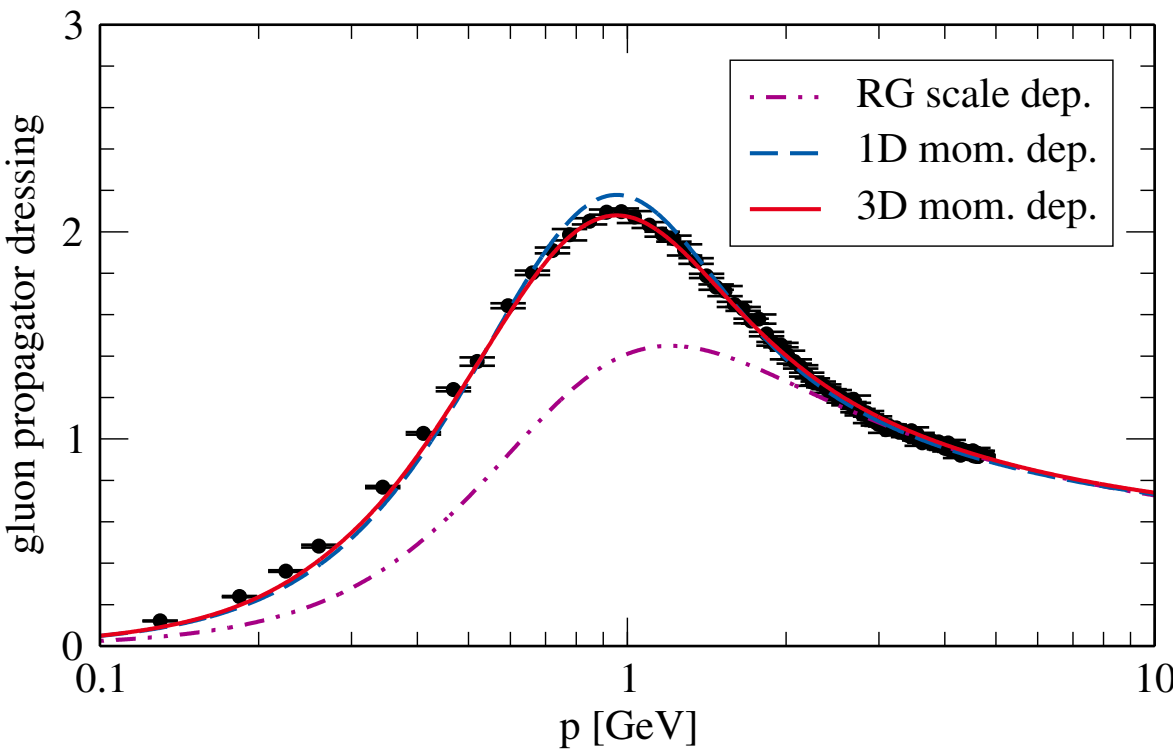
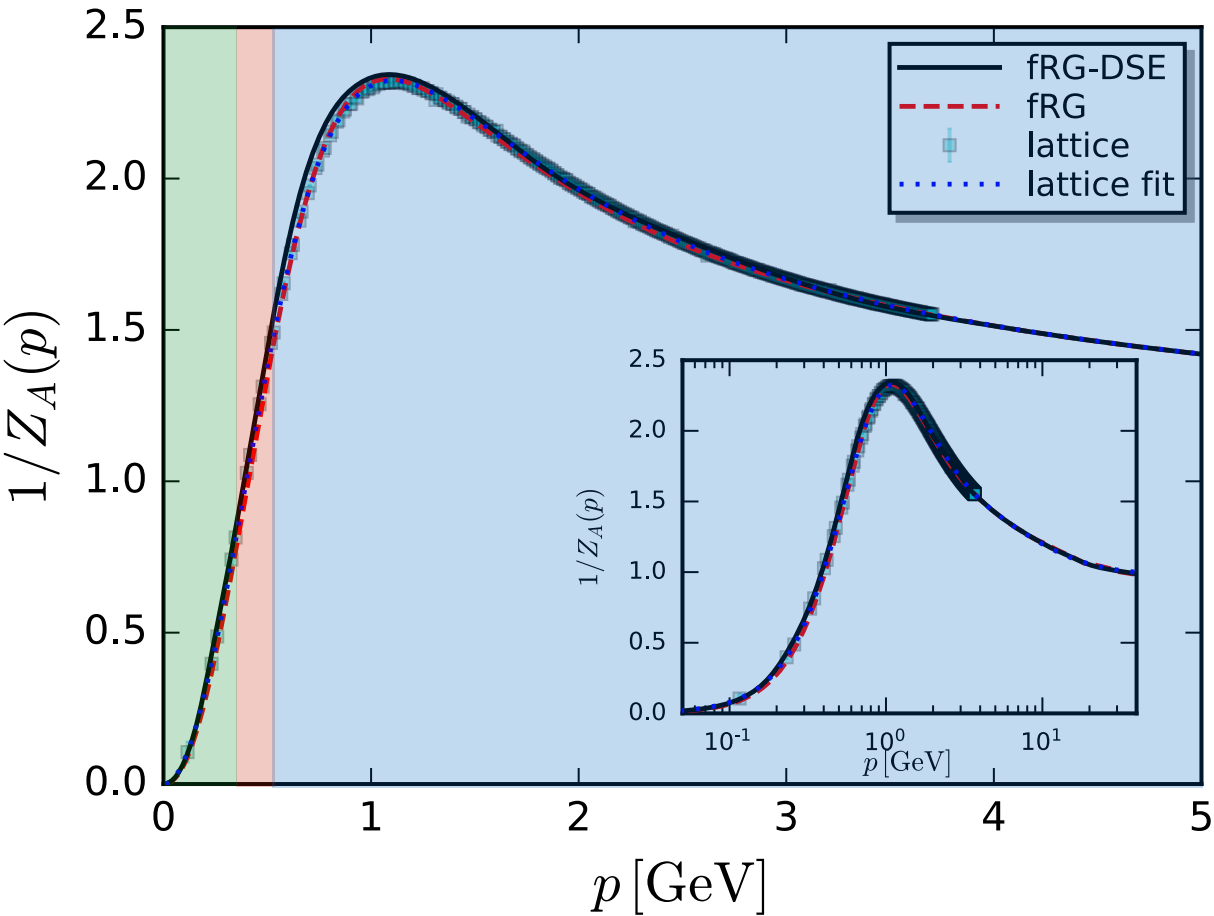
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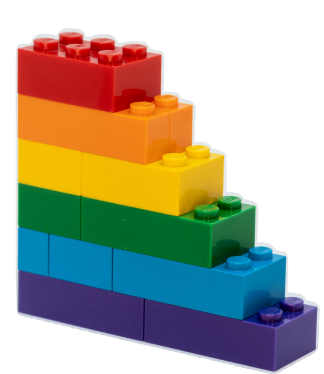


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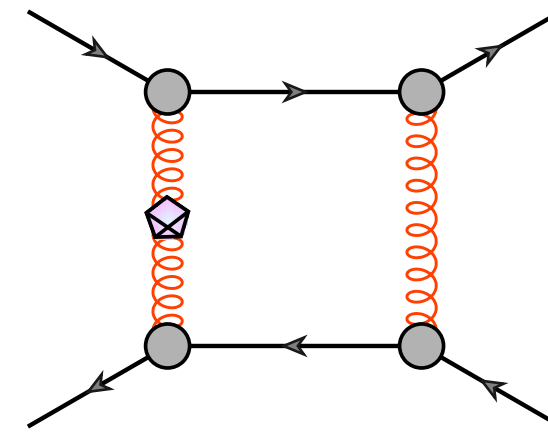
gluon two-point correlator



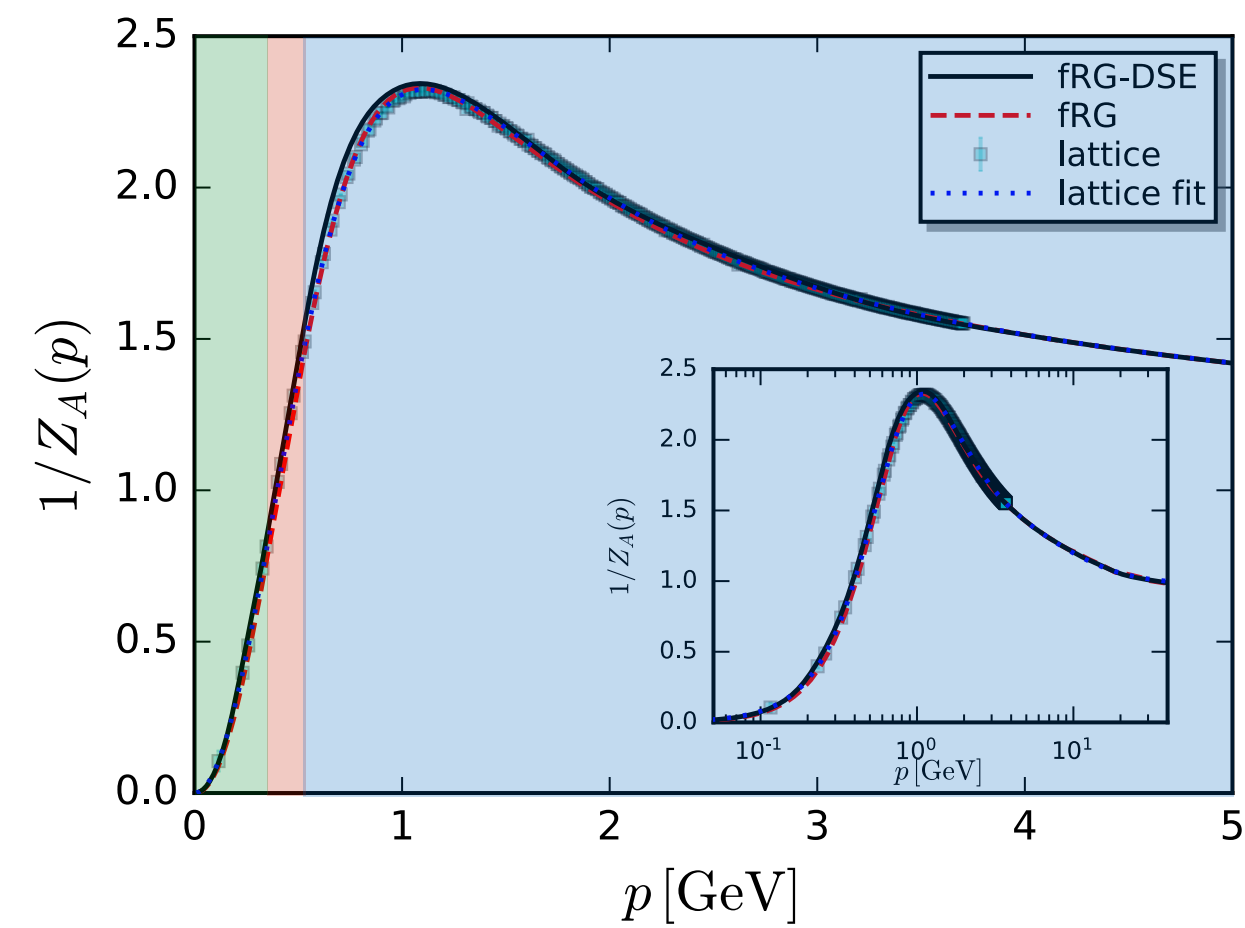


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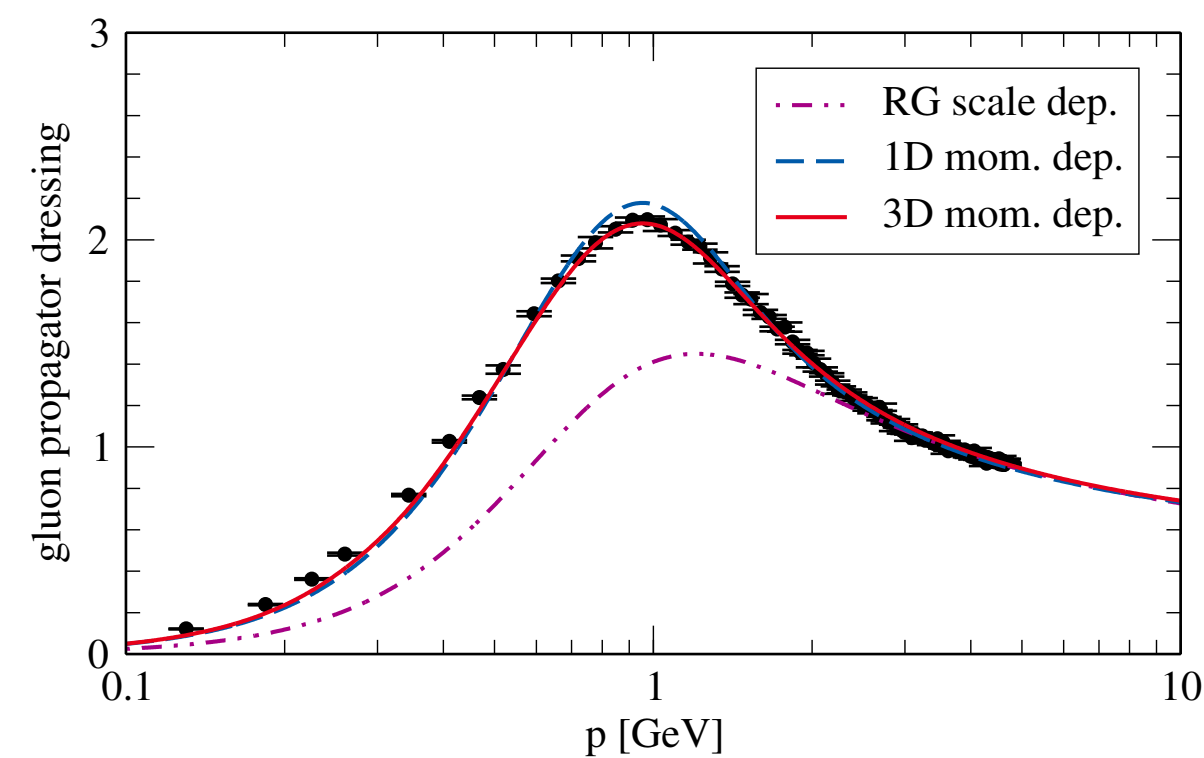
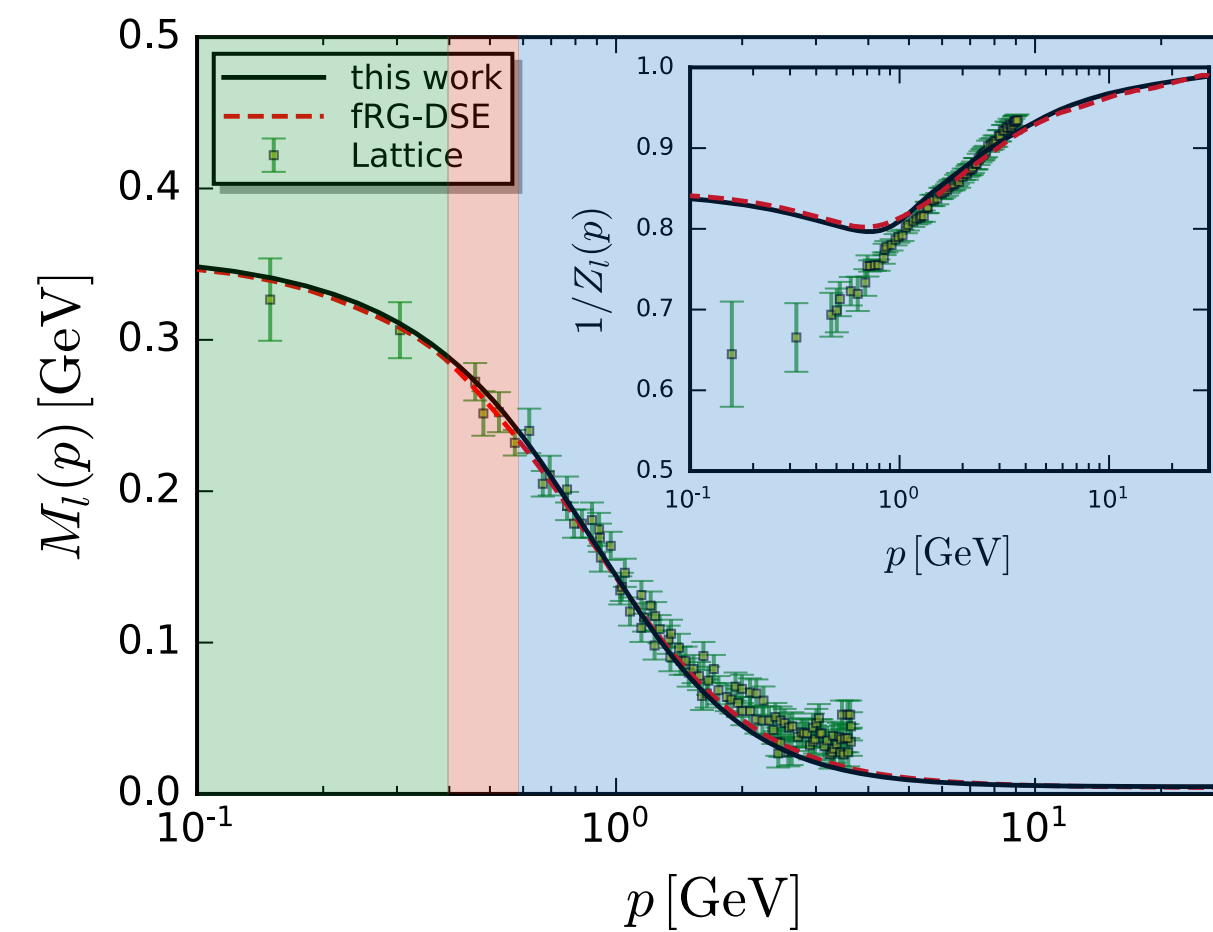
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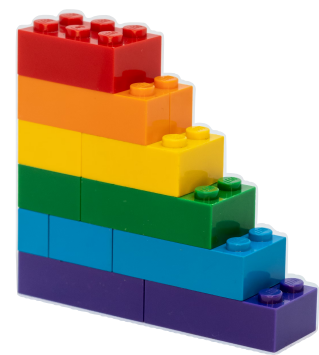


gluon two-point correlator



quark two-point correlator

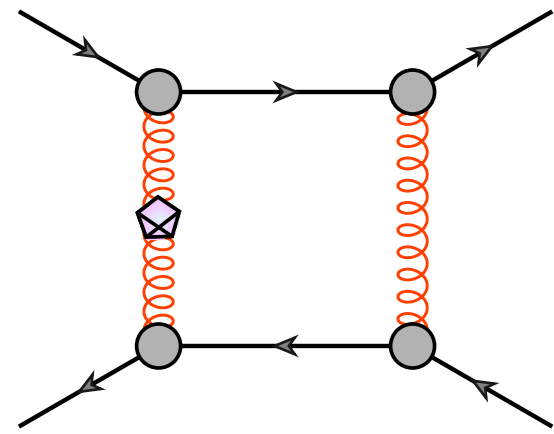




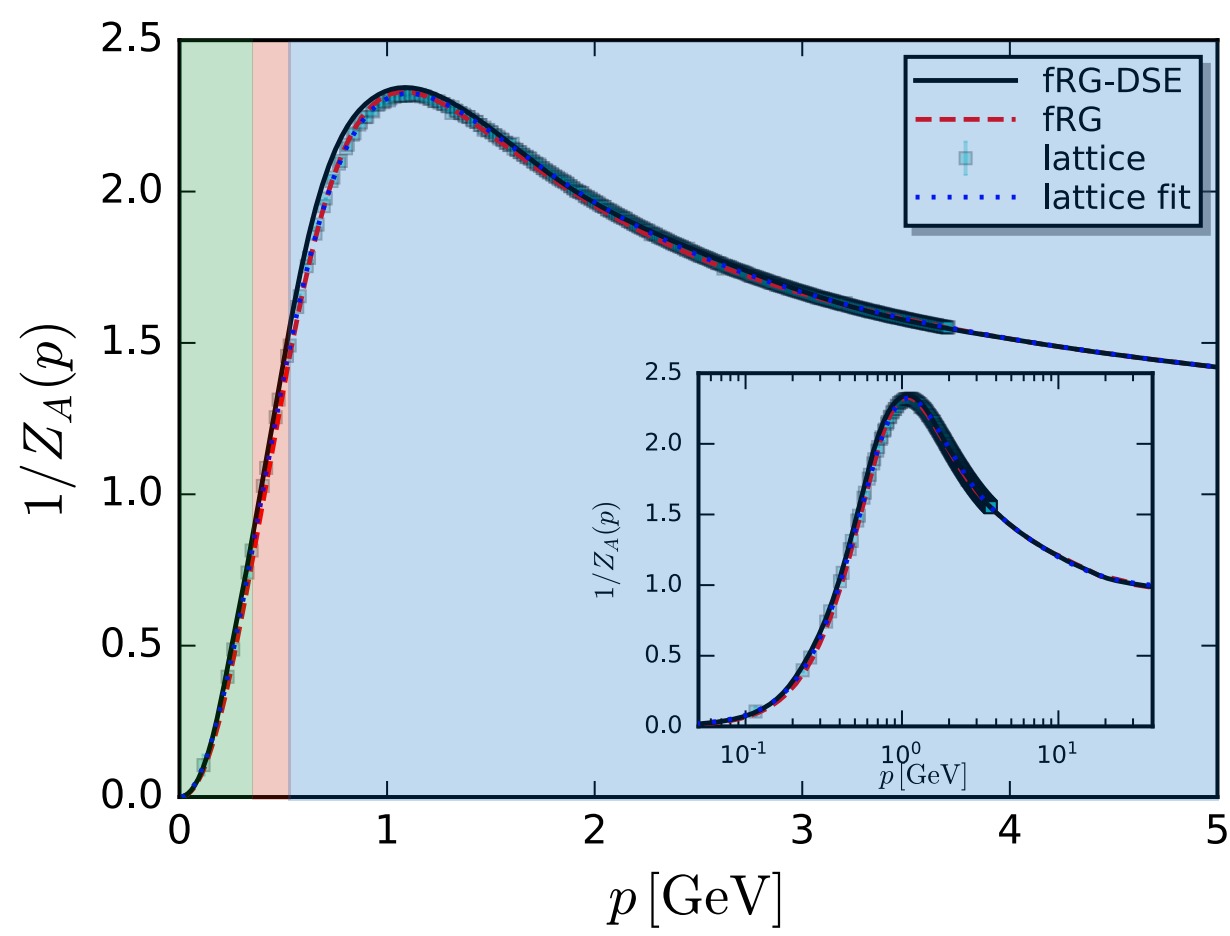
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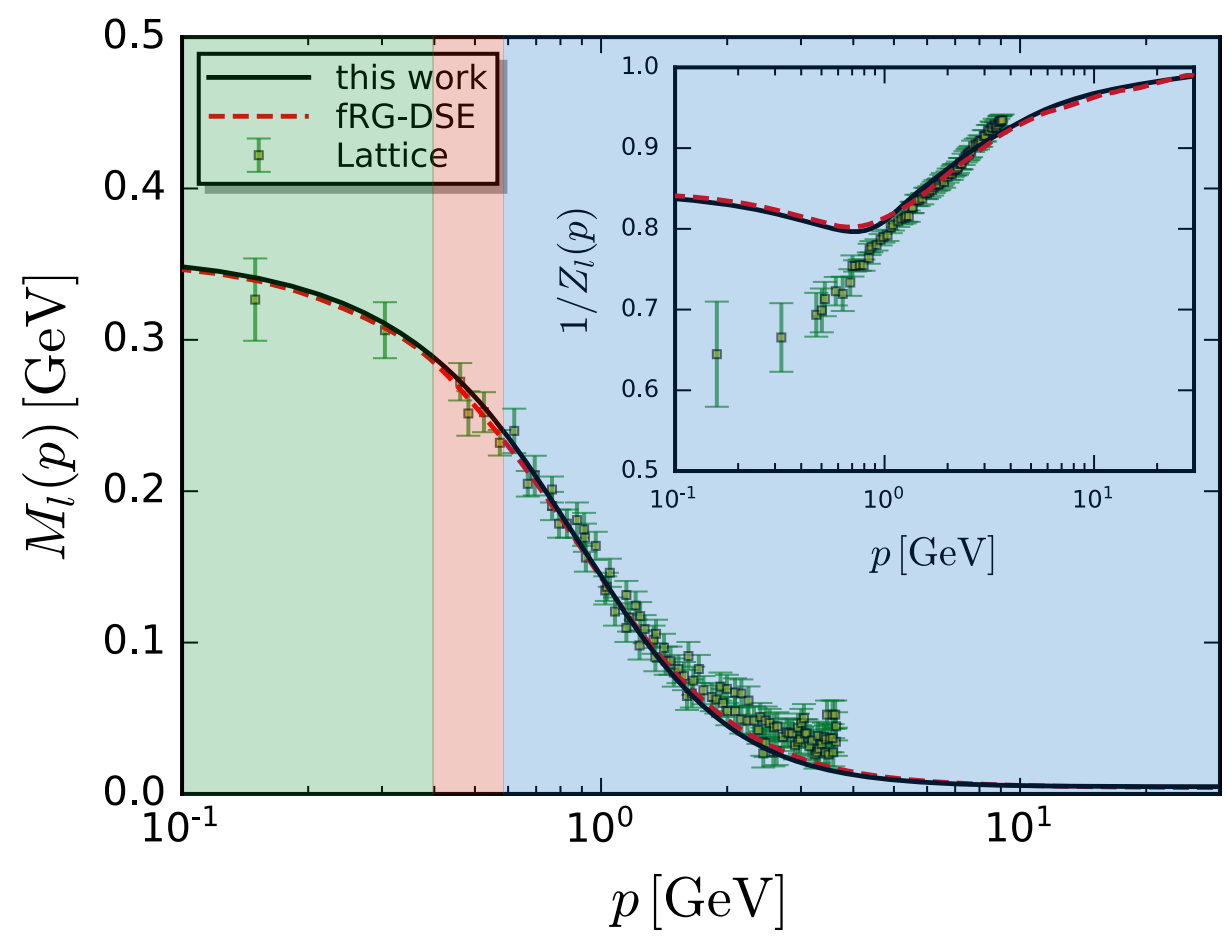
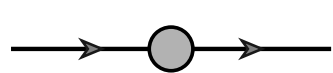
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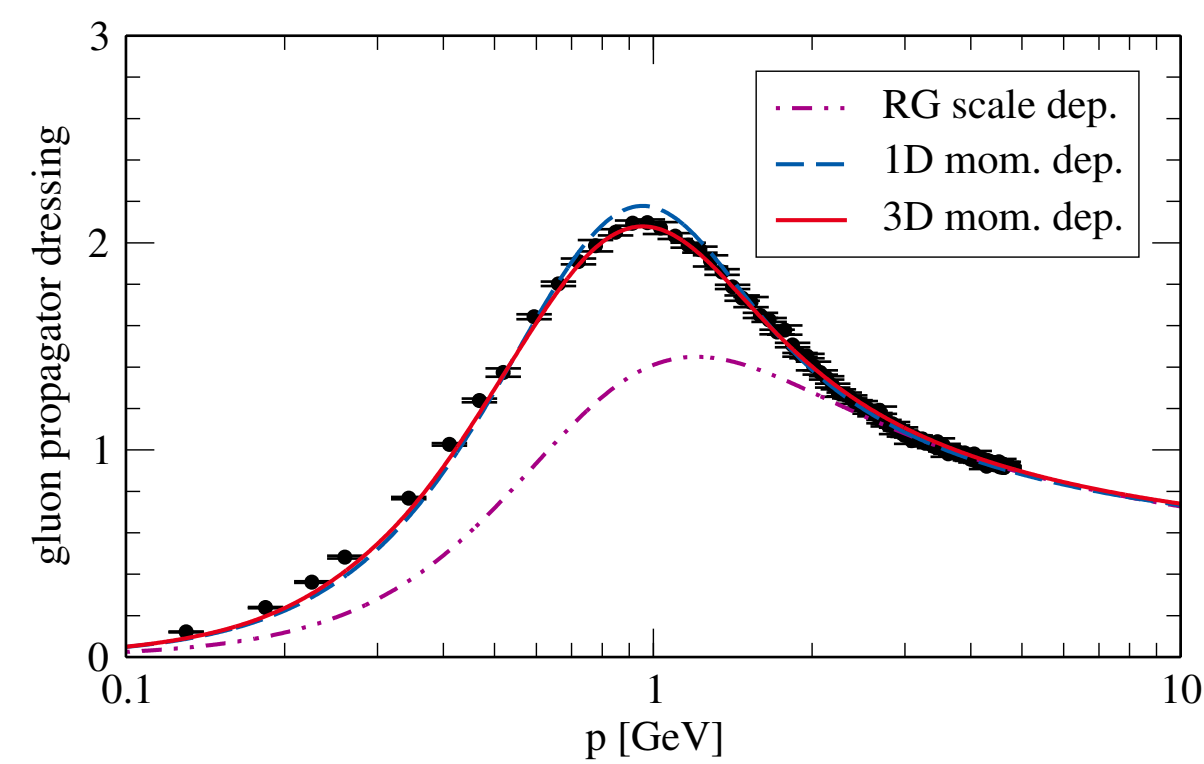
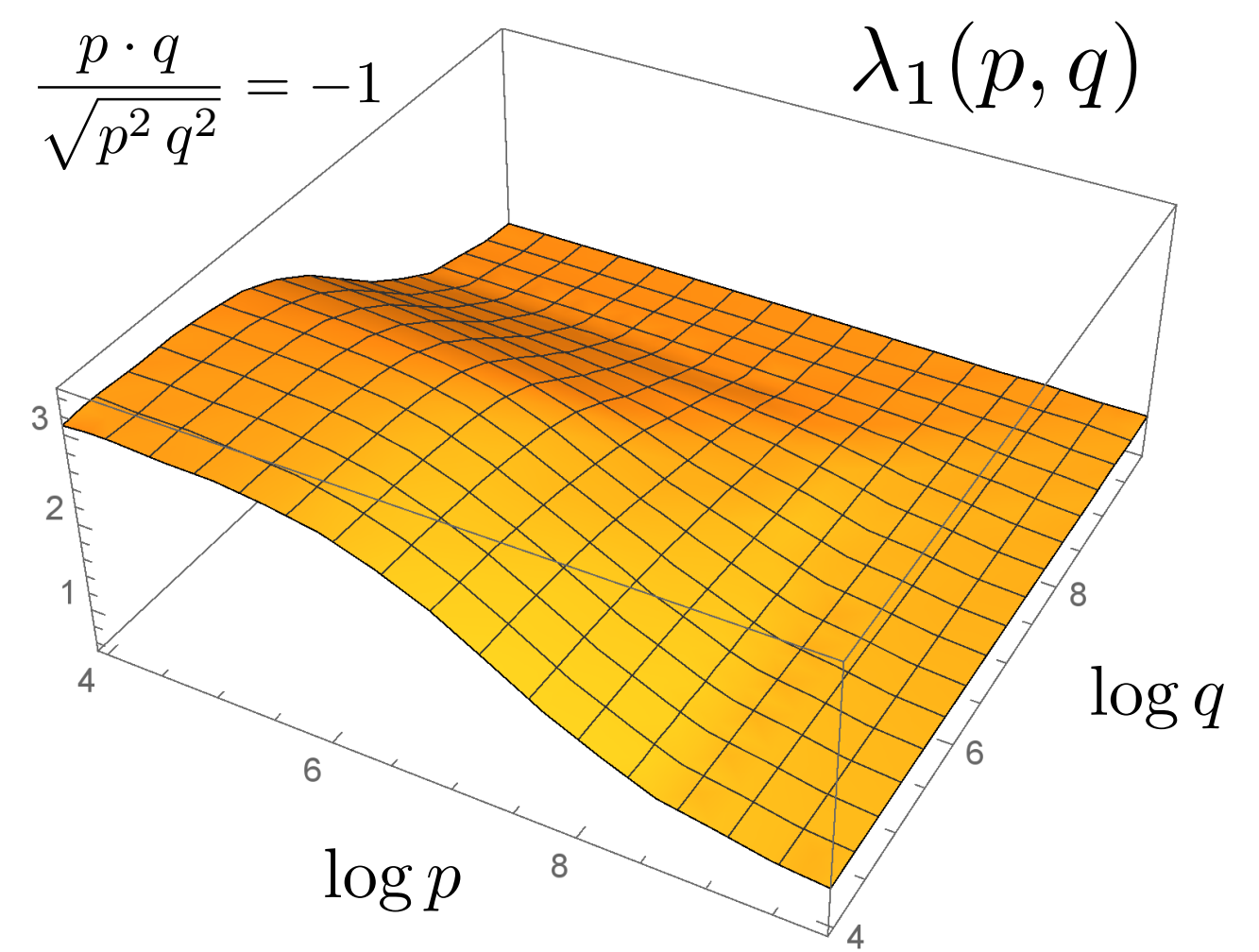
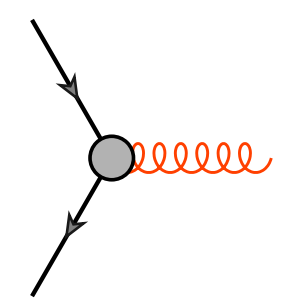
gluon two-point correlator



quark two-point correlator



quark-gluon scattering



$$\lambda_1(p, q), \dots, \lambda_8(p, q)$$

Relevant: $\lambda_1, \lambda_4, \lambda_7$

VertEXpand

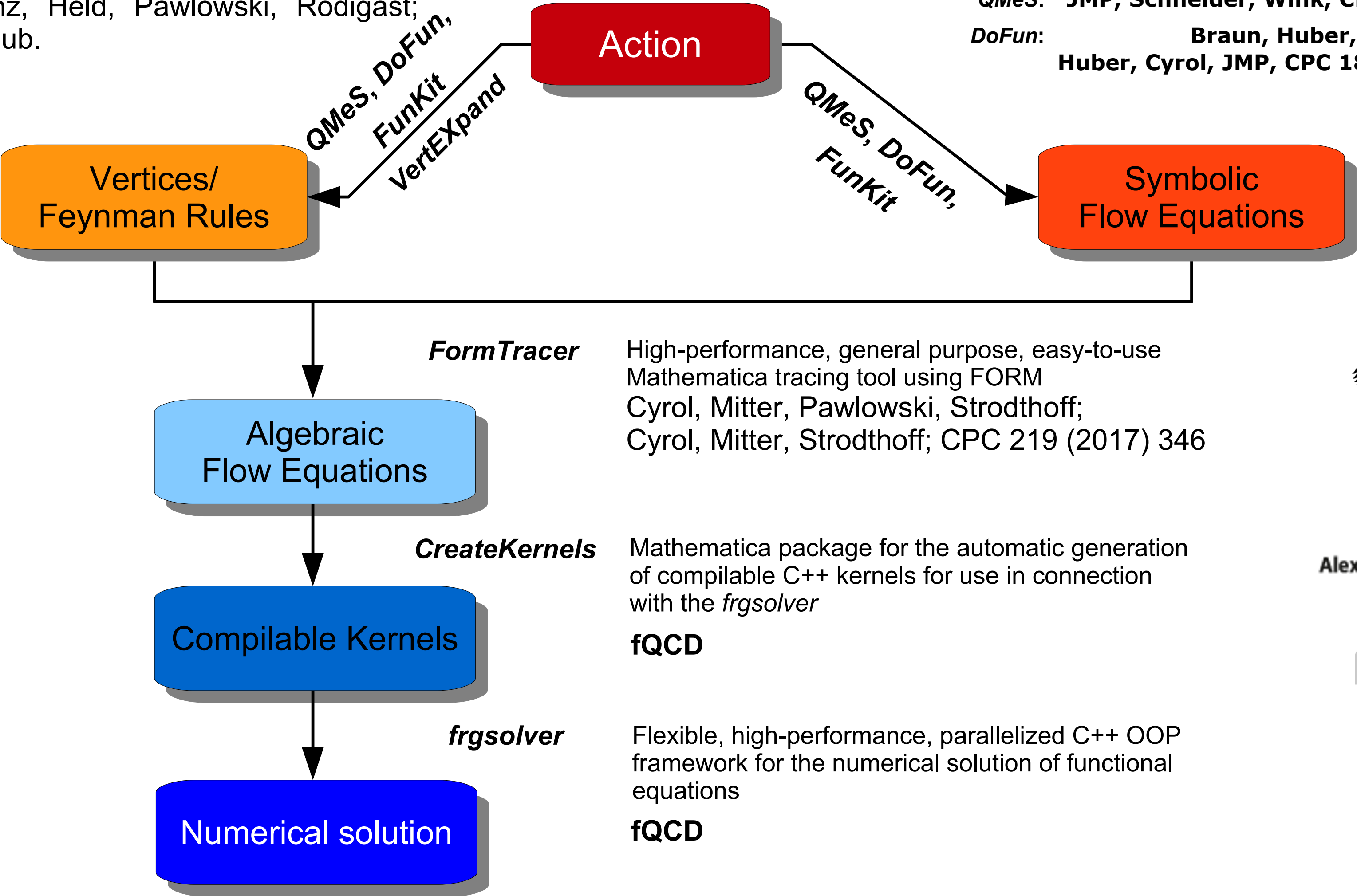
Mathematica package for the derivation of vertices from a given action using FORM
Denz, Held, Pawlowski, Rodigast; unpub.

QMeS, DoFun

Mathematica packages for the derivation of functional equations

QMeS: JMP, Schneider, Wink, CPC287 (2023) 108711

DoFun: Braun, Huber, CPC 183 (2012) 1290
Huber, Cyrol, JMP, CPC 183 248 (2020) 107058



FormTracer

High-performance, general purpose, easy-to-use Mathematica tracing tool using FORM
Cyrol, Mitter, Pawlowski, Strodthoff;
Cyrol, Mitter, Strodthoff; CPC 219 (2017) 346

CreateKernels

Mathematica package for the automatic generation of compilable C++ kernels for use in connection with the *frgsolver*

fQCD

frgsolver

Flexible, high-performance, parallelized C++ OOP framework for the numerical solution of functional equations

fQCD

GEFÖRDERT VOM



Bundesministerium
für Bildung
und Forschung



Alexander von Humboldt
Stiftung/Foundation

FWF

Der Wissenschaftsfonds.

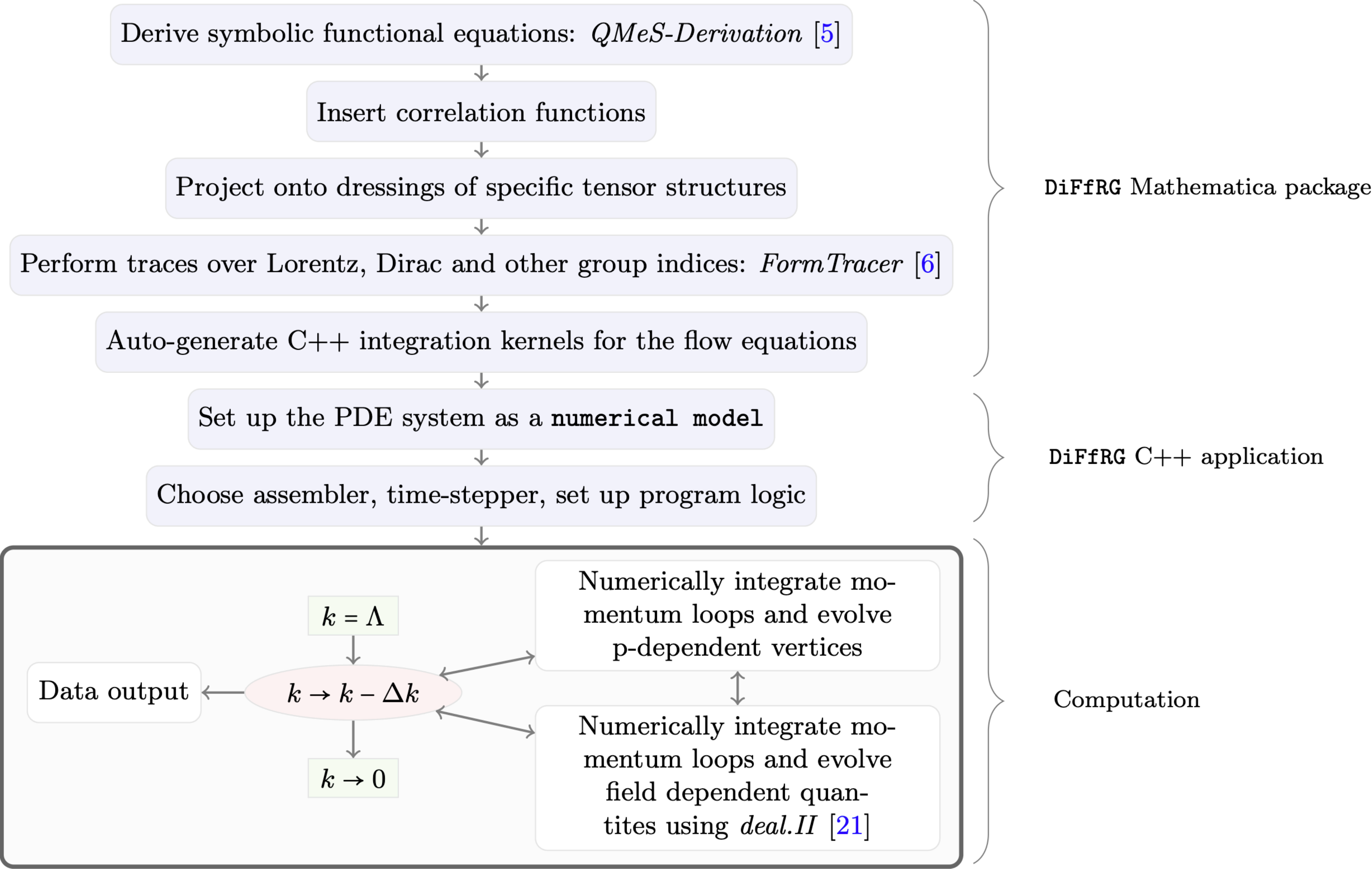


European Research Council

Established by the European Commission

fQCD: workflow

DiFfRG
github



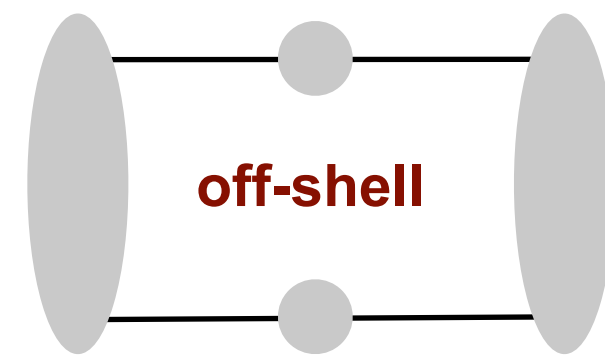


How to: three remarks on functional methods

'... and now for something
completely different...'

- off-shell representation of thermodynamic observables

e.g. $\text{Tr} \langle q(x) \bar{q}(x) \rangle$



pressure, trace anomaly,
fluctuations, volume flucs., ...

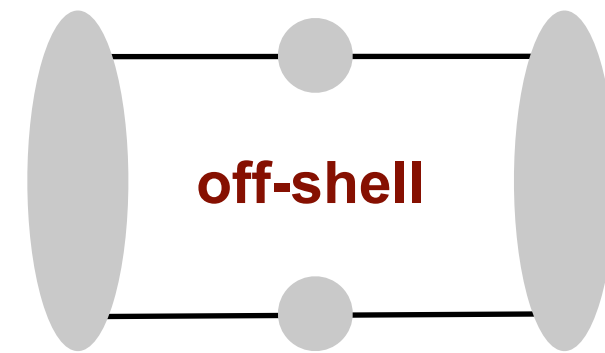


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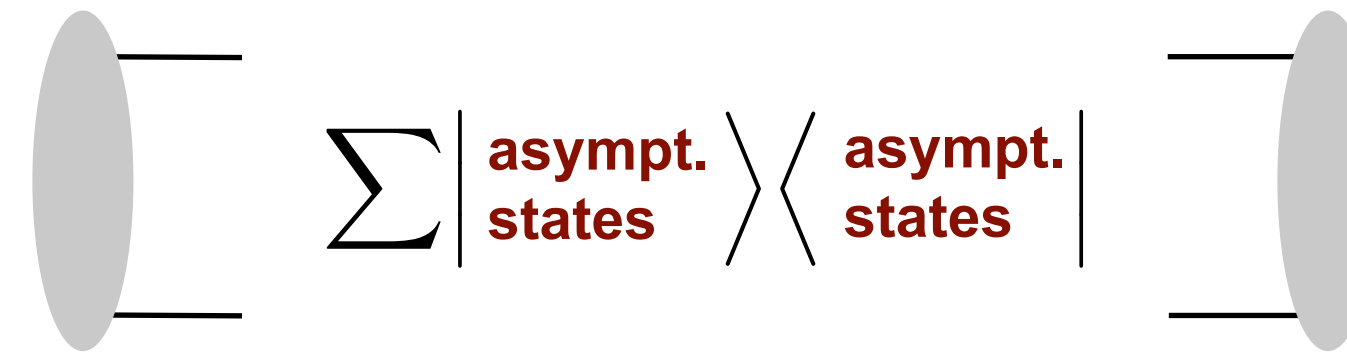
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on-shell



e.g. hadron resonances

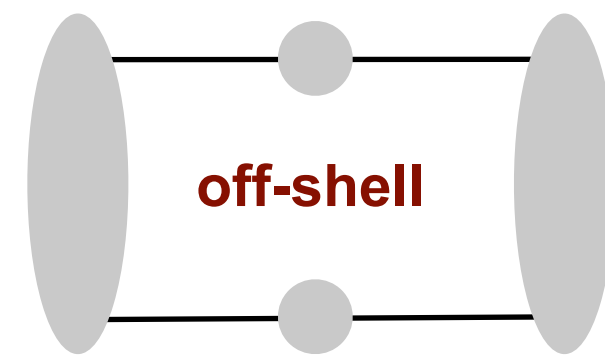


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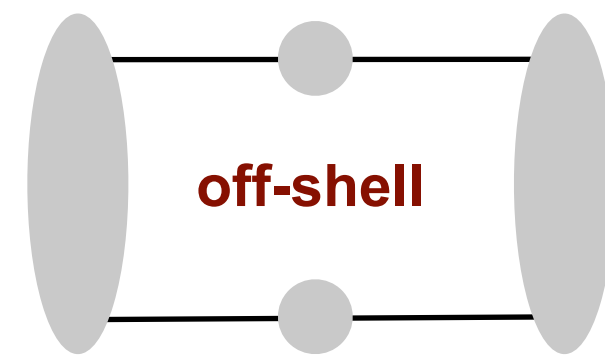


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e.g. hadron resonances

▪ gauge fixing = parameterisation

$$\langle q(x_1) \cdots \bar{q}(x_{2n}) A_\mu(y_1) \cdots A_\mu(y_m) h(z_1) \cdots h(z_l) \rangle$$

Consequences

I: simple correlations

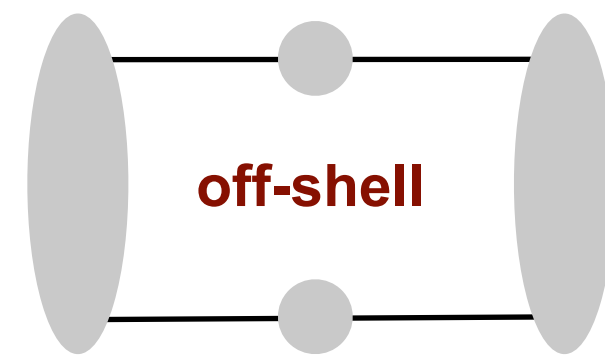


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Consequences

I: simple correlations

II: Difficult access to some observables
'No free lunch theorem'

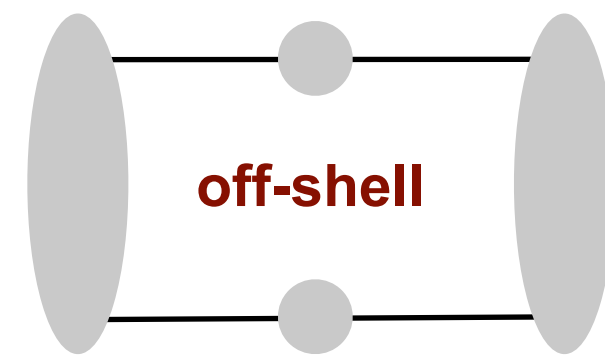


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Consequences

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▪ 'Your mean field is not my mean field'

$$\left. \frac{\delta S_{\text{cl}}[\phi]}{\delta \phi} \right|_{\phi=\bar{\phi}} = 0$$

$$\left. \frac{\delta \Gamma[\phi]}{\delta \phi} \right|_{\phi=\bar{\phi}_{\text{quant}}} = 0$$



How to: direct computations vs extrapolations

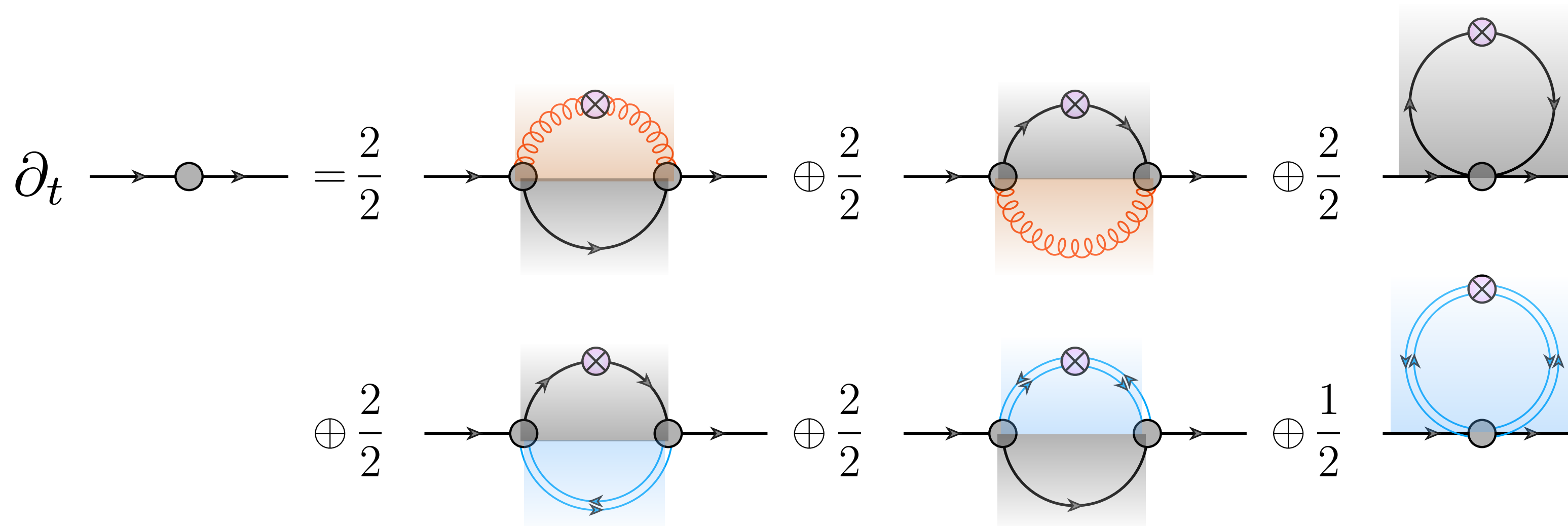
'... and now for something completely different...'

To QCD or not to QCD....a minimal point of view

Those are my interpretations,
and if you don't like them....
well, I have others

- **Self-consistent truncations to functional relations define analytic functions in μ_B , eg:**

$$\partial_t \left\langle q(x) \bar{q}(y) \right\rangle^{-1}(\mu_B) = \text{Loop} \left[\left\langle q(x) \bar{q}(y) \right\rangle(\mu_B), \left\langle q(x) A_\mu(y) \bar{q}(z) \right\rangle(\mu_B), \dots; \mu_B \right]$$





'... and now for something
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Extrapolations & von Neumann's Elephant

Extrapolations are ill-conditioned problems

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'There are two ways of doing calculations in theoretical physics.

One way is to have a clear physical picture of the process that you are calculating.

The other way is to have a precise and self-consistent mathematical formalism.

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Fermi to Dyson



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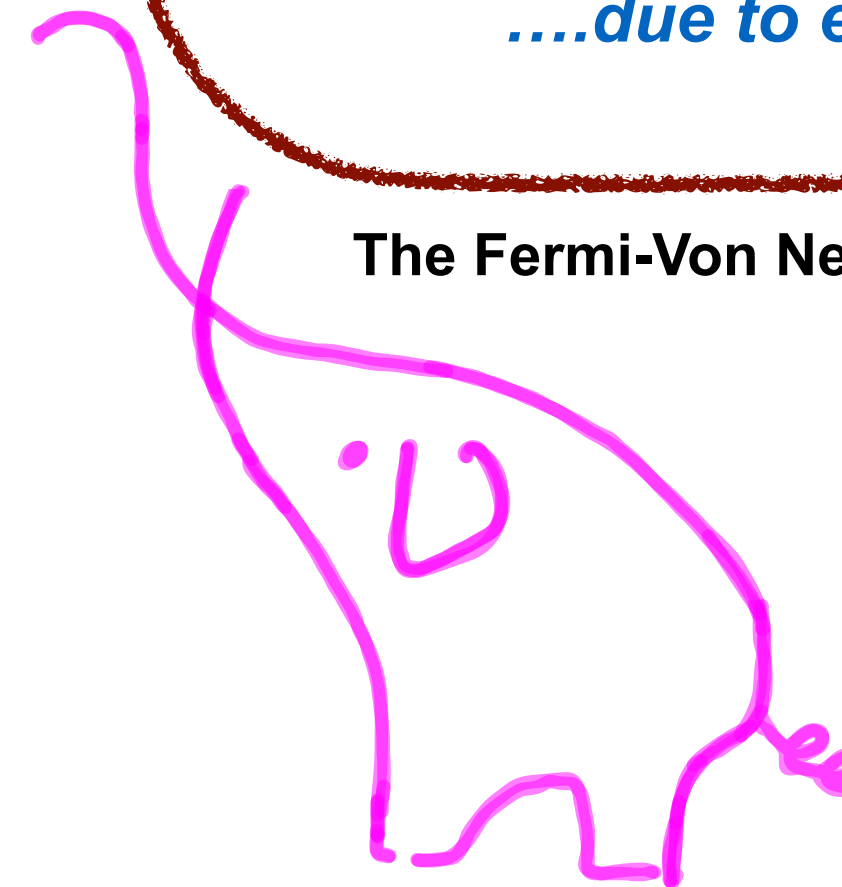
Von Neumann

*With four parameter models we can fit many data
but we cannot extrapolate without external information*

**In particular, extrapolations cannot predict the existence
and location of the CEP without external information**

*The other way is to have a precise and self-consistent
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The Fermi-Von Neumann QCD phase structure elephant





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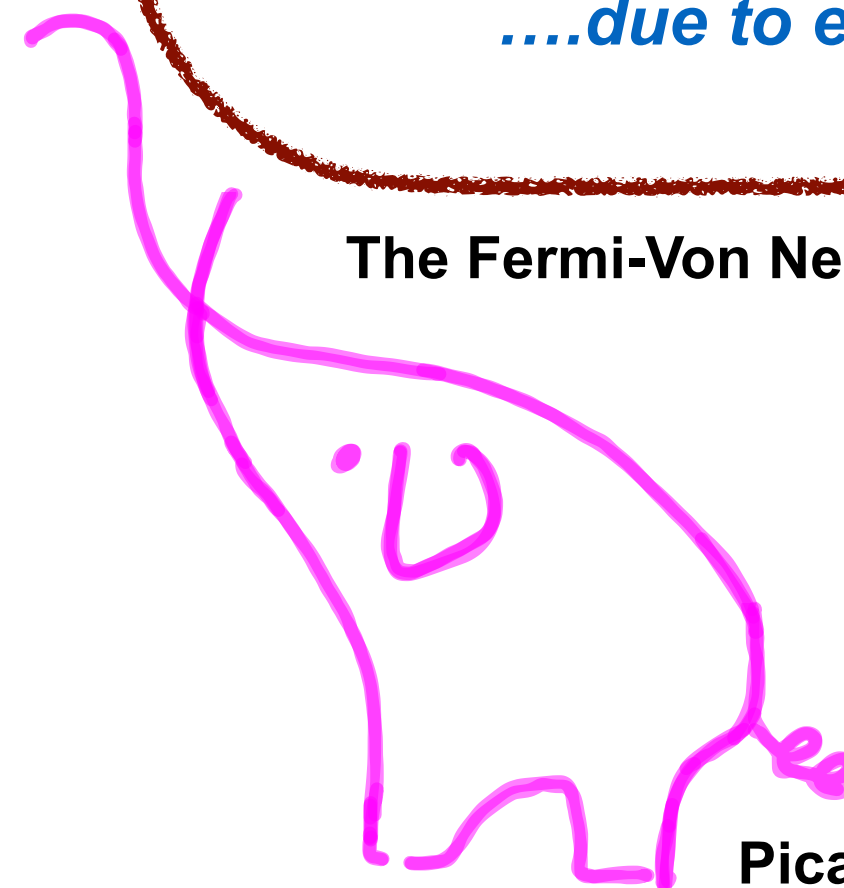
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Picasso-assisted Pawlowski elephant



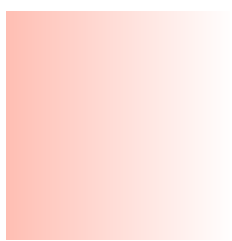
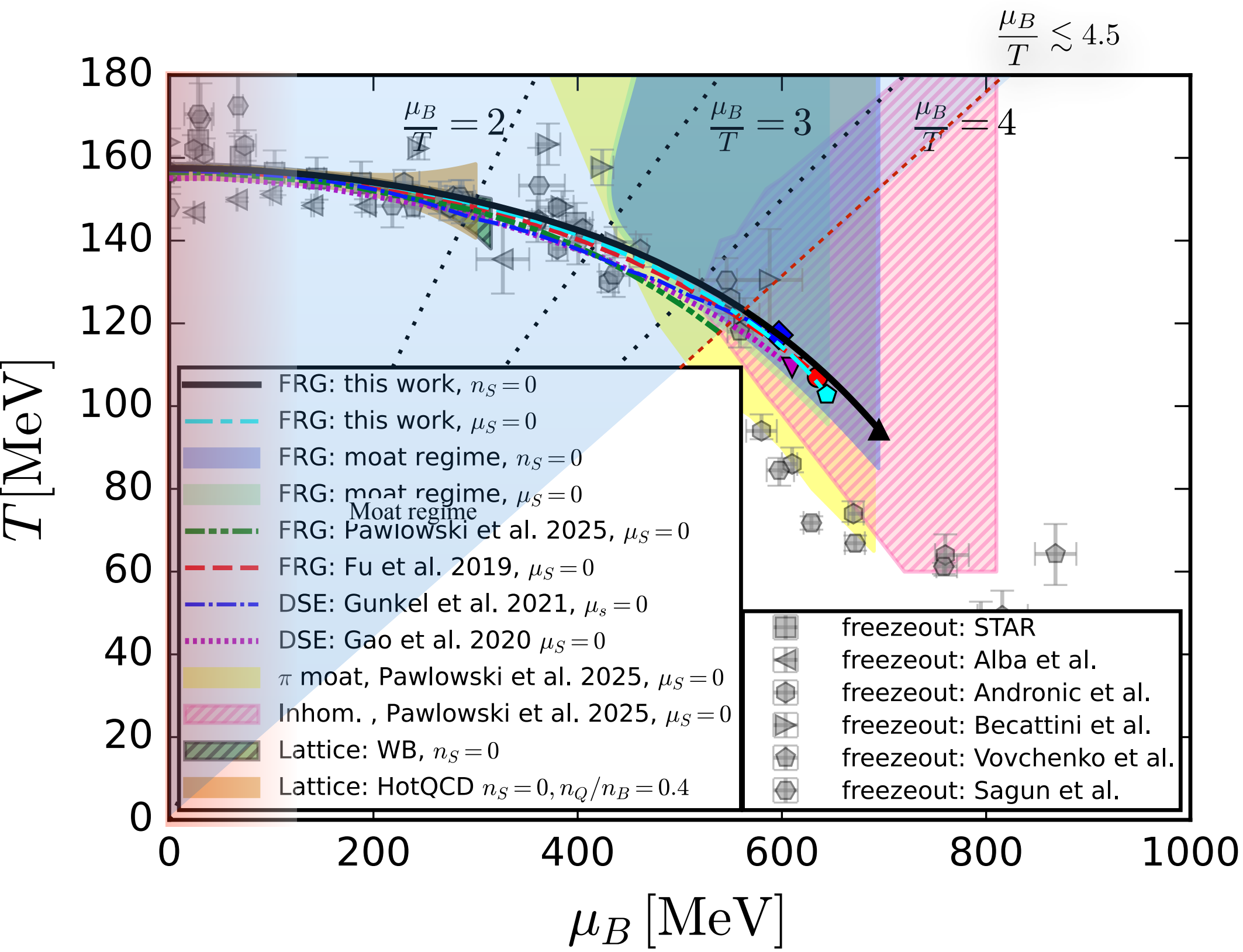
How to: direct computations and the minimal point of view

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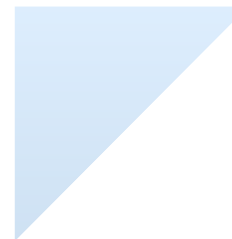
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- Consequences for functional QCD predictions at finite density



By now the best truncations to functional QCD pass lattice benchmark tests at vanishing and small chemical potential



Regime of quantitative reliability of current best truncation

$$\mu_B/T \lesssim 4.5$$

Unique: QCD-based analytic continuations that satisfy the lattice benchmarks at small chemical potential.



Great opportunity for a combined analysis of high density QCD (Exp. data + lattice QCD + functional QCD + LEFTs)

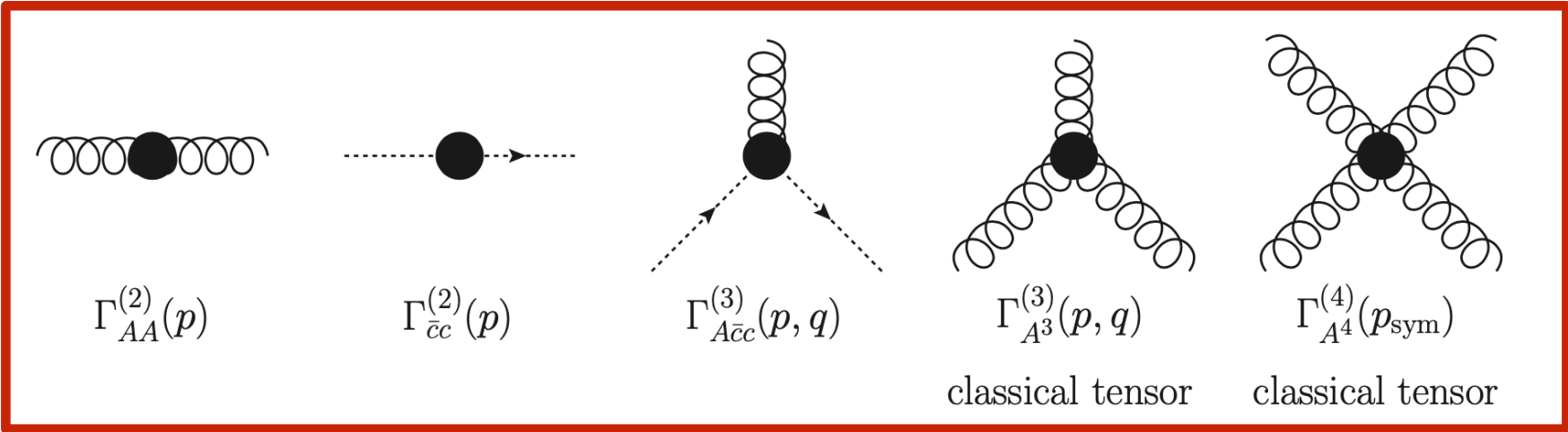
Jump forward to: QCD phase structure

QCD at finite temperature and density

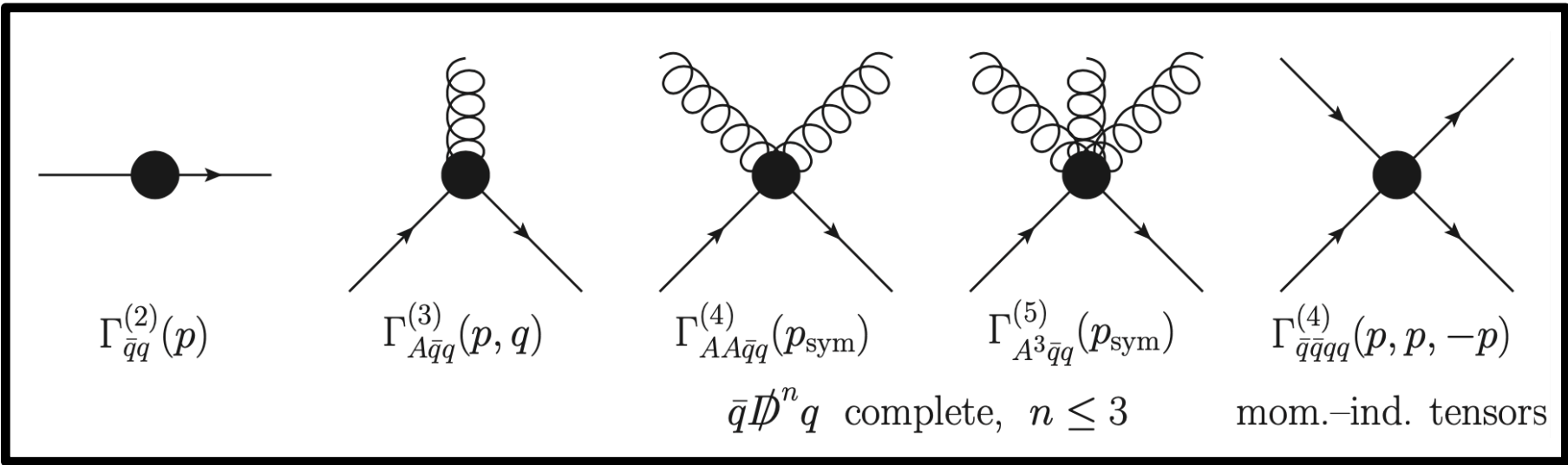
Jump back to: Critical end point and new phases: where do we stand?

Benchmarks in the vacuum

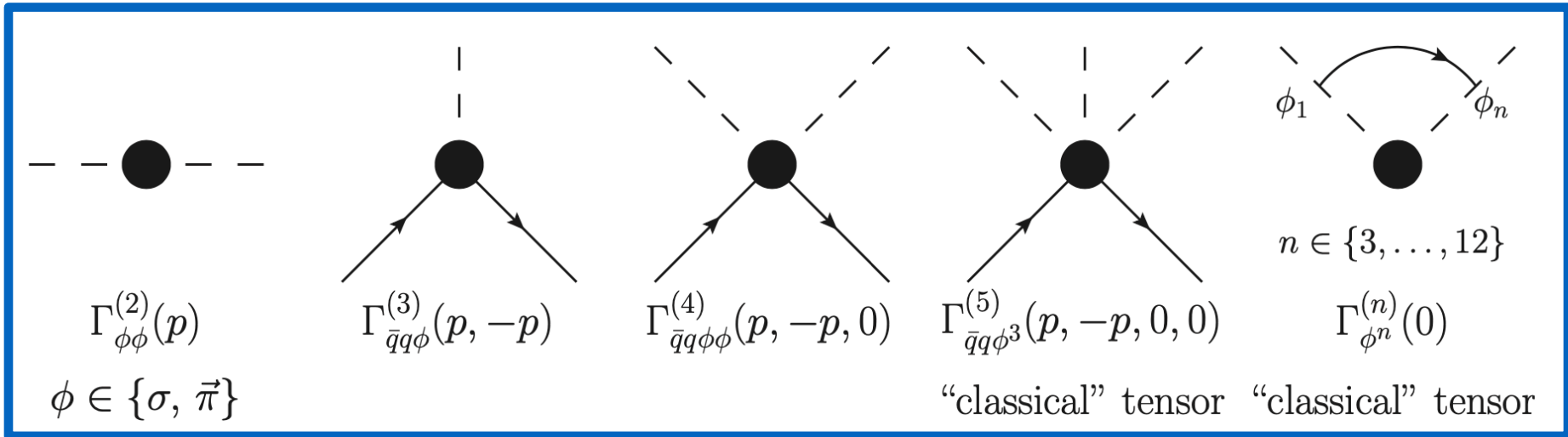
Current set of correlation functions



glue sector



quark-gluon sector



quark-meson sector

2 flavours

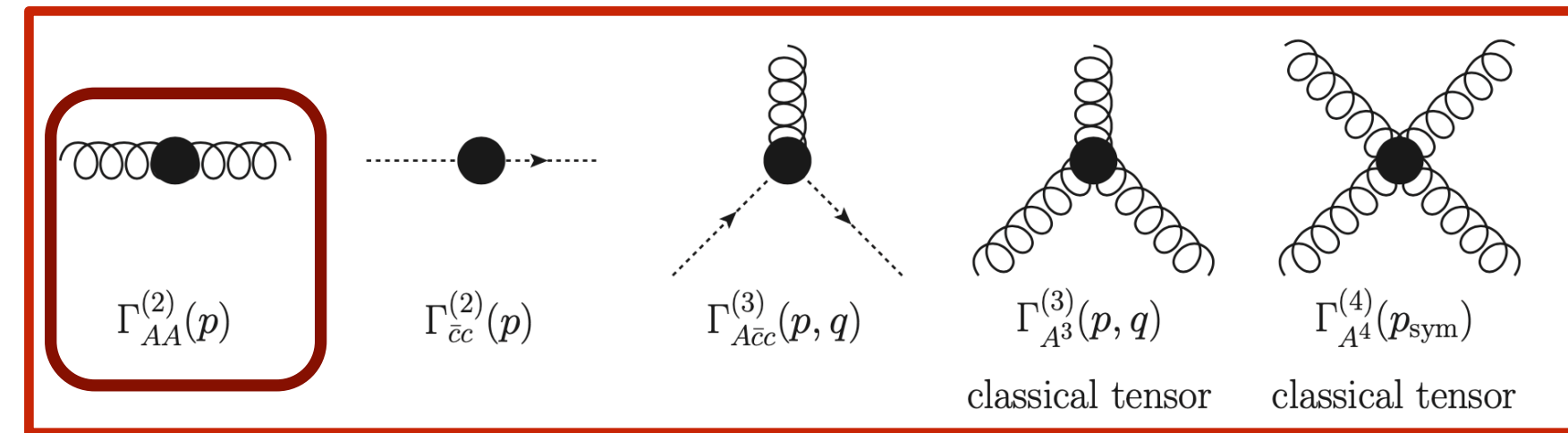
Aiming at apparent convergence

Cyrol, Mitter, JMP, Strodthoff, PRD 97 (2018) 054006,
PRD 97 (2018) 054015

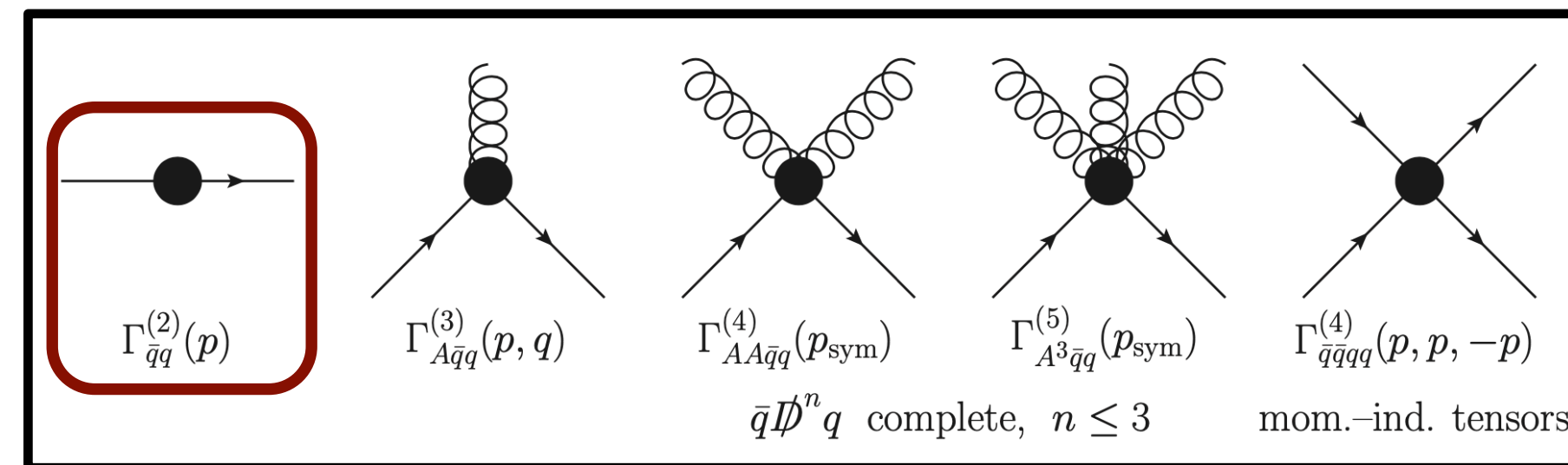
Cyrol, Fister, Mitter, JMP, Strodthoff, PRD 94 (2016) 054005

Mitter, JMP, Strodthoff, PRD 91 (2015) 054035

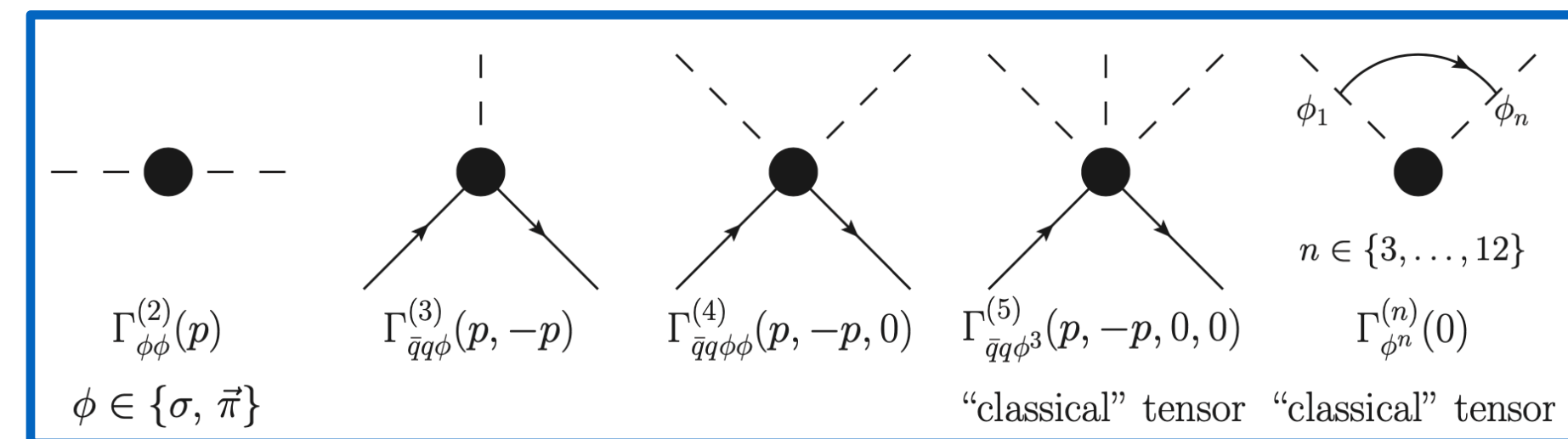
Current set of correlation functions



glue sector



quark-gluon sector



quark-meson sector

2 flavours

Aiming at apparent convergence

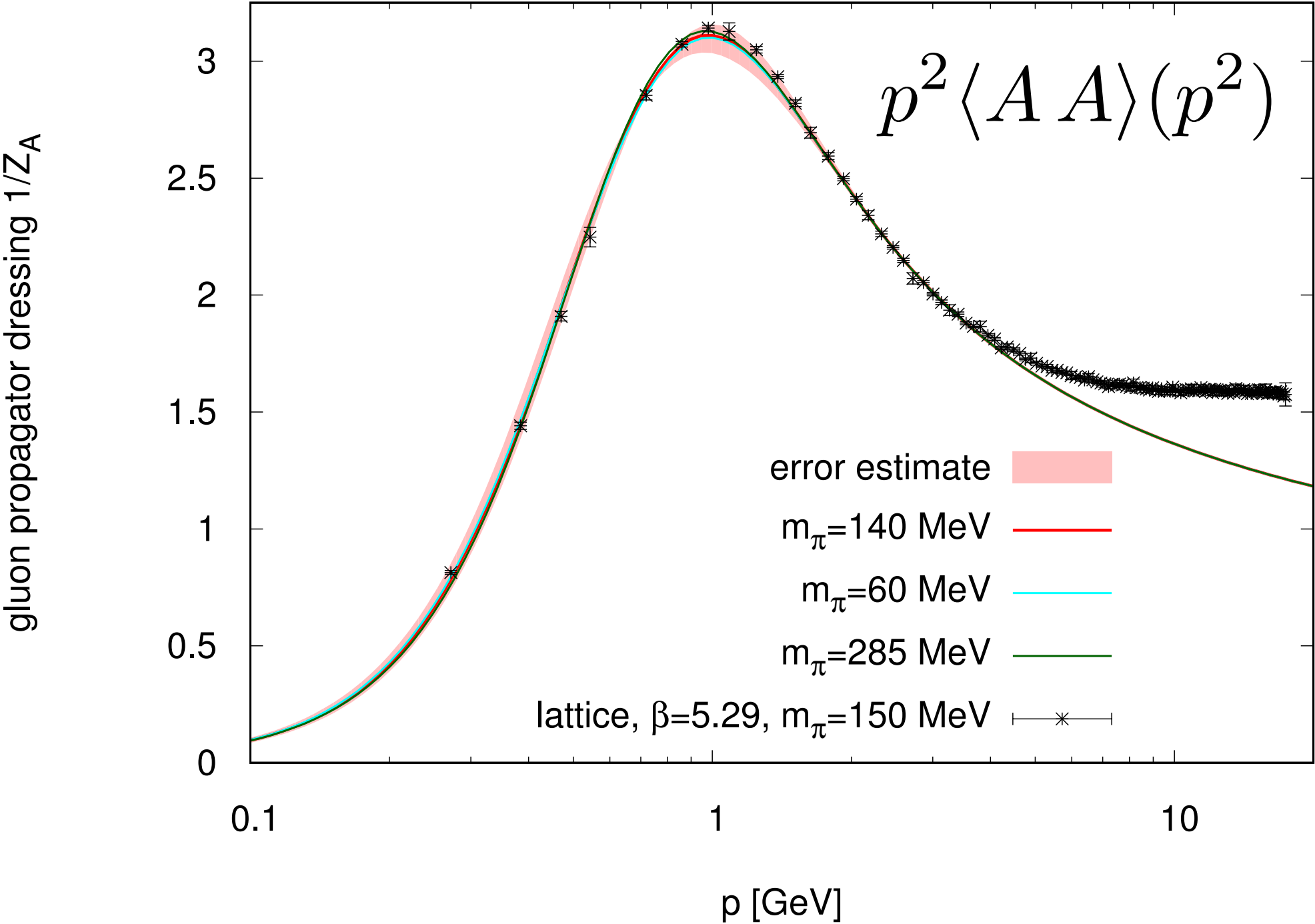
Cyrol, Mitter, JMP, Strodthoff, PRD 97 (2018) 054006,
PRD 97 (2018) 054015

Cyrol, Fister, Mitter, JMP, Strodthoff, PRD 94 (2016) 054005

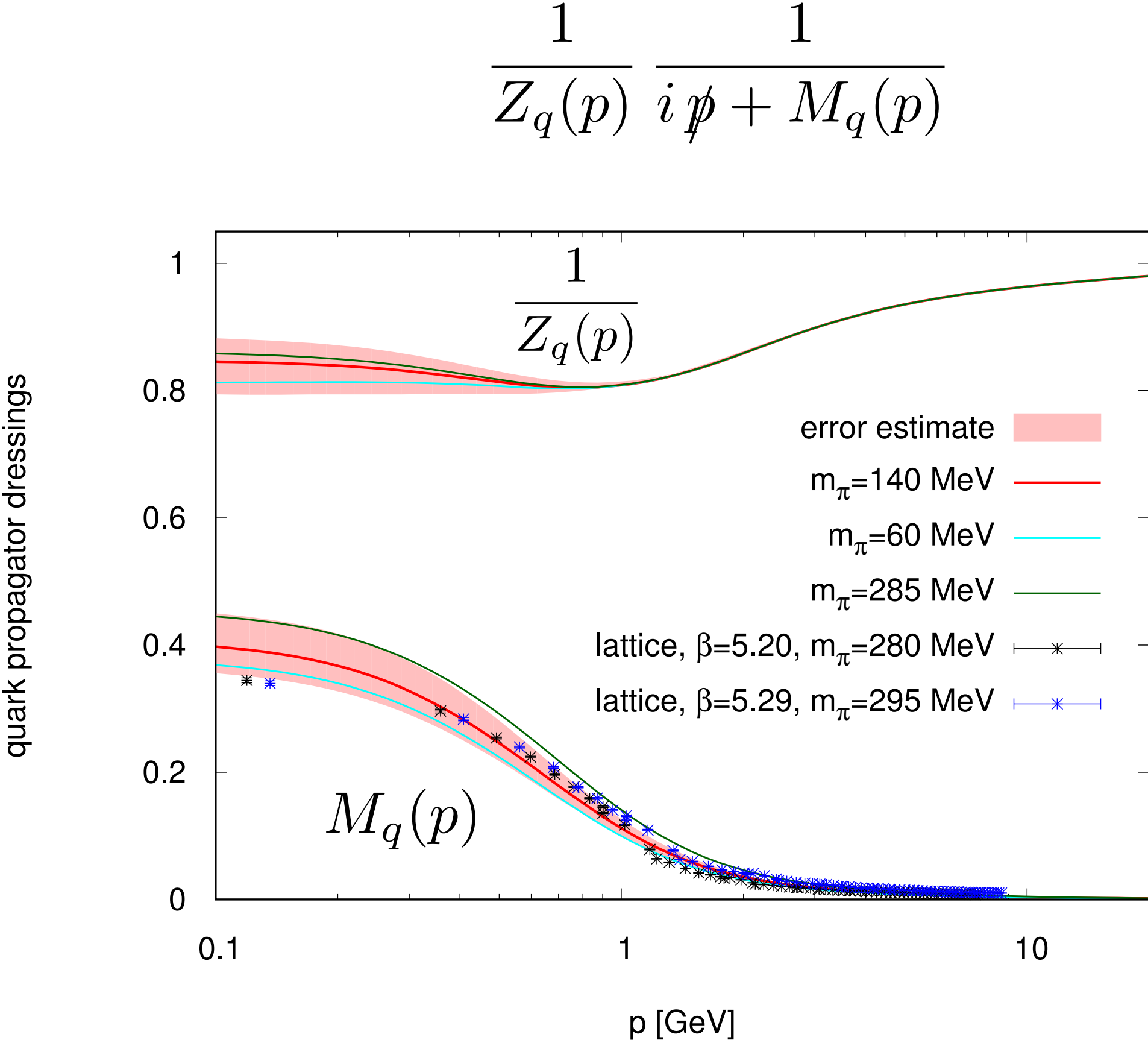
Mitter, JMP, Strodthoff, PRD 91 (2015) 054035

Euclidean propagators

Two-flavour QCD

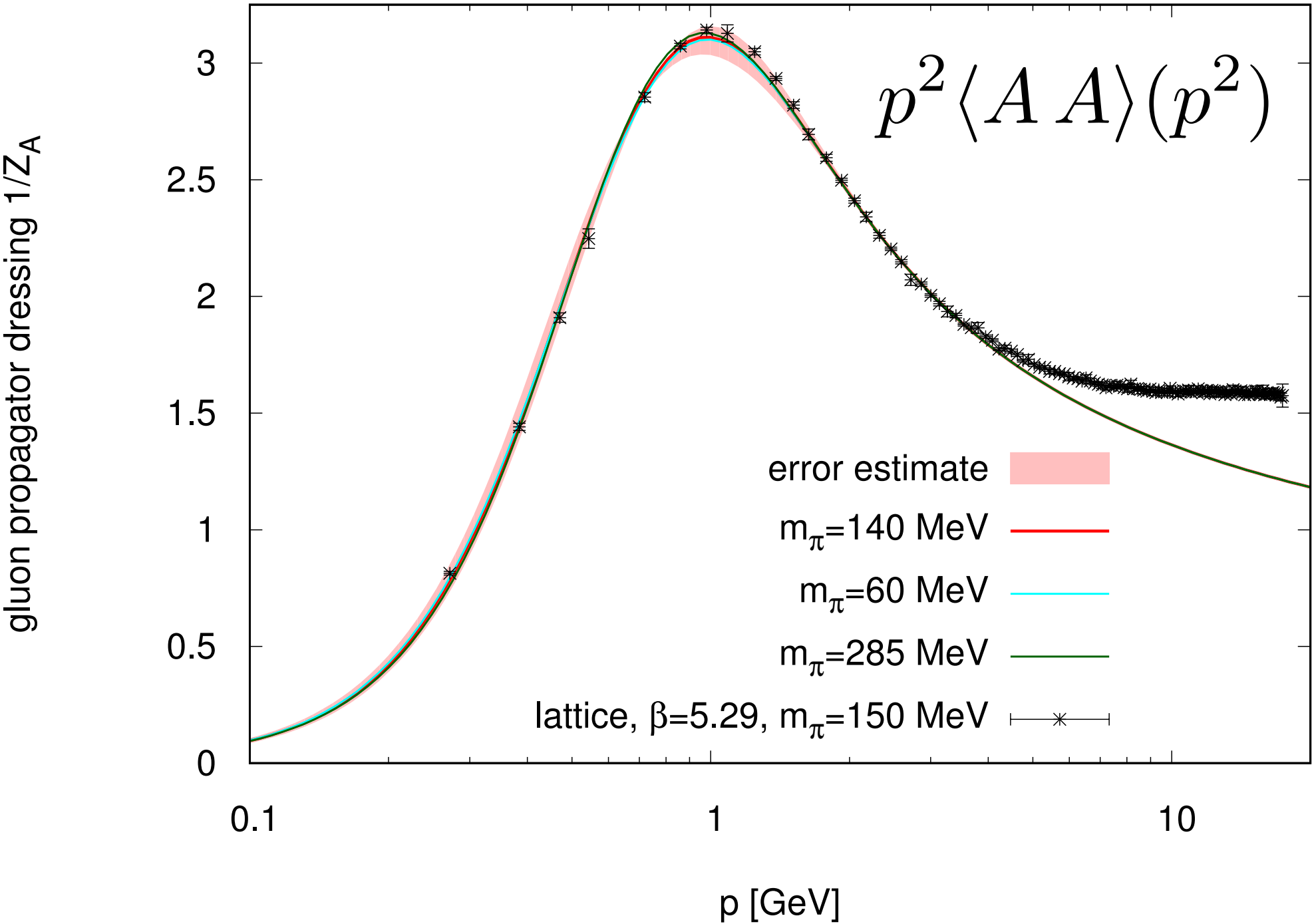


lattice, e.g.: Oliveira et al, Acta Phys.Polon.Supp. 9 (2016) 363
Sternbeck et al, PoS LATTICE2016 (2017)
A. Athenodorou et al, PLB 761 (2016) 444



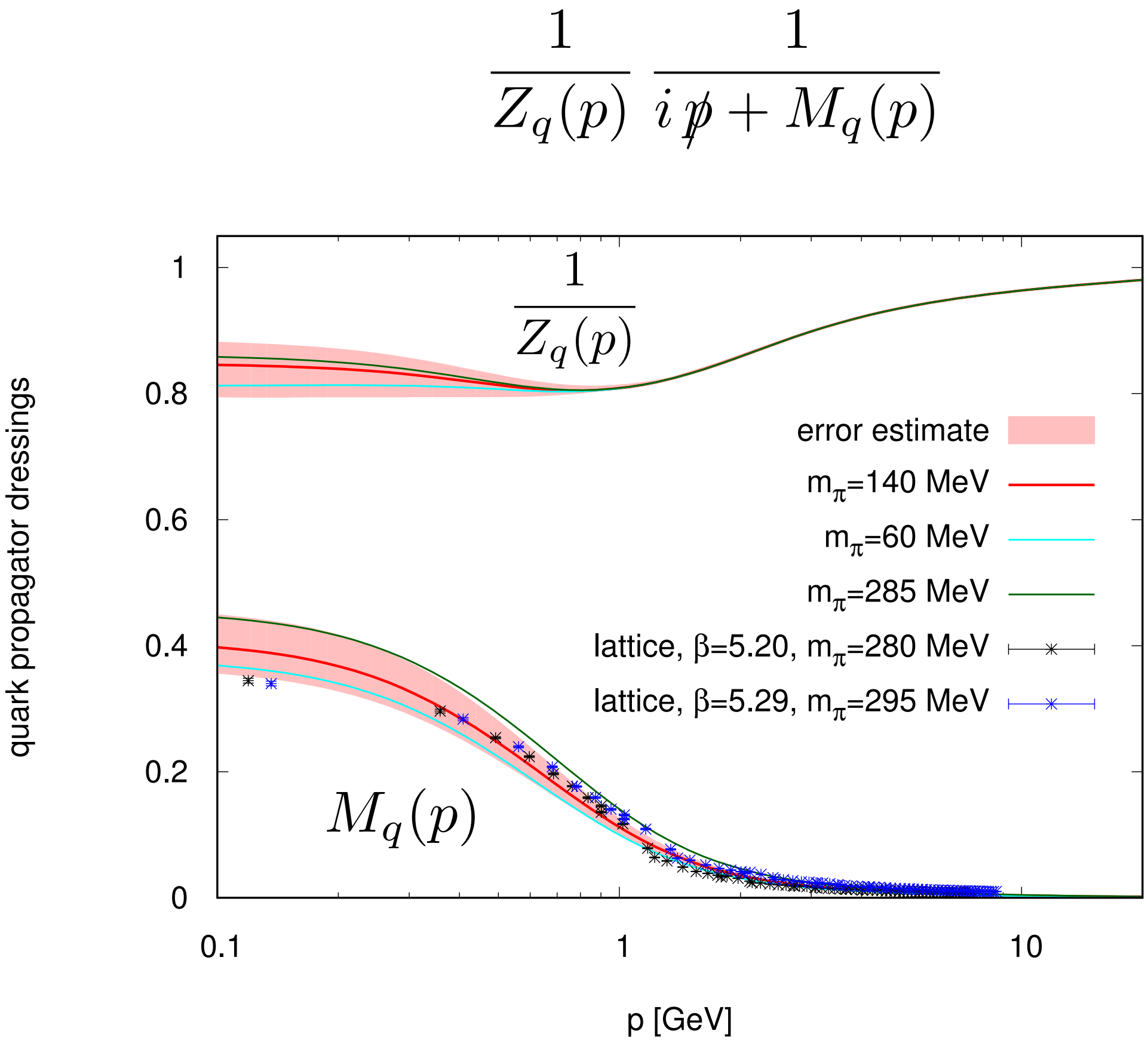
Euclidean propagators

Two-flavour QCD

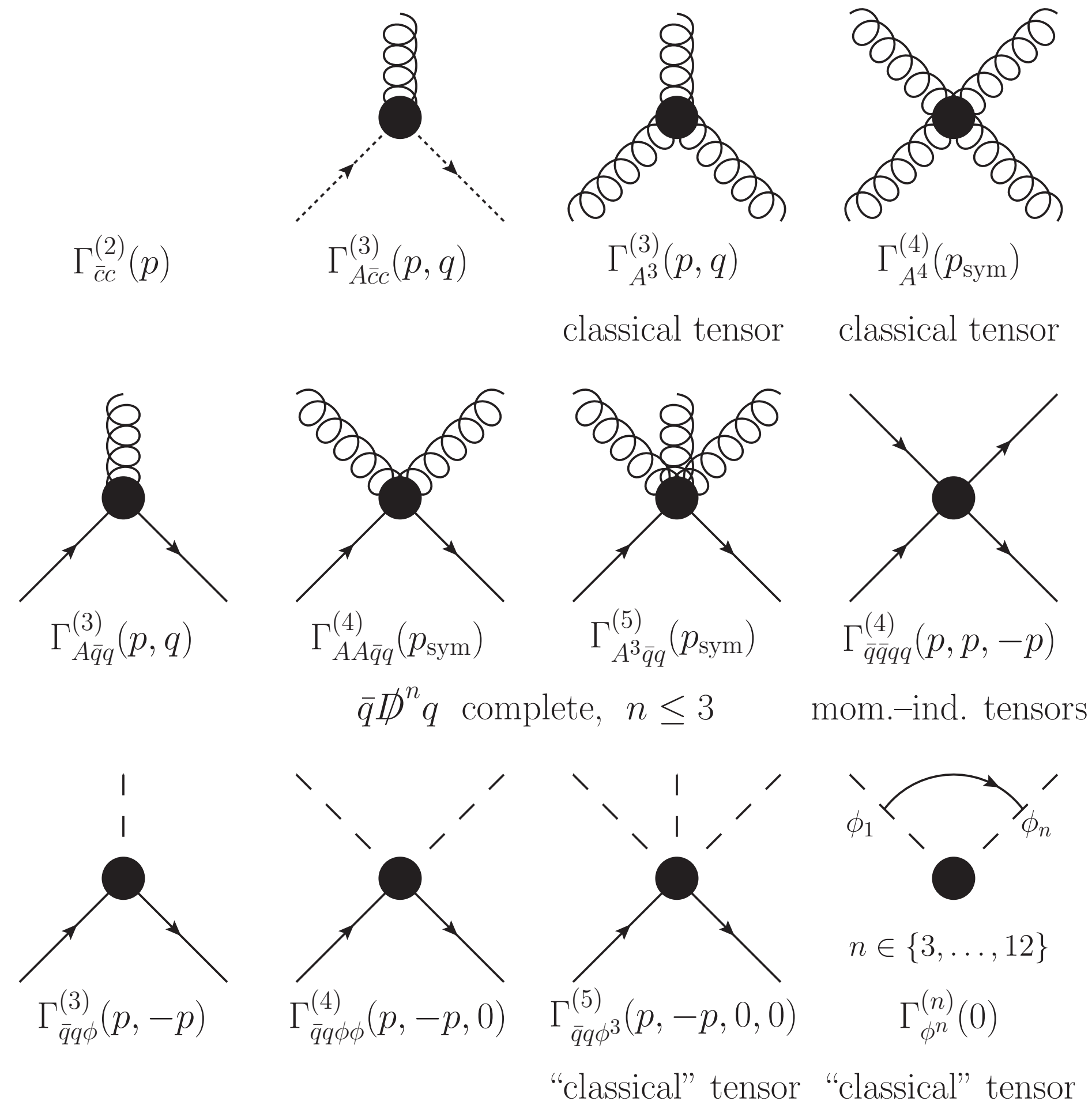


lattice, e.g.: Oliviera et al, Acta Phys.Polon.Supp. 9 (2016) 363
Sternbeck et al, PoS LATTICE2016 (2017)
A. Athenodorou et al, PLB 761 (2016) 444

simple correlations

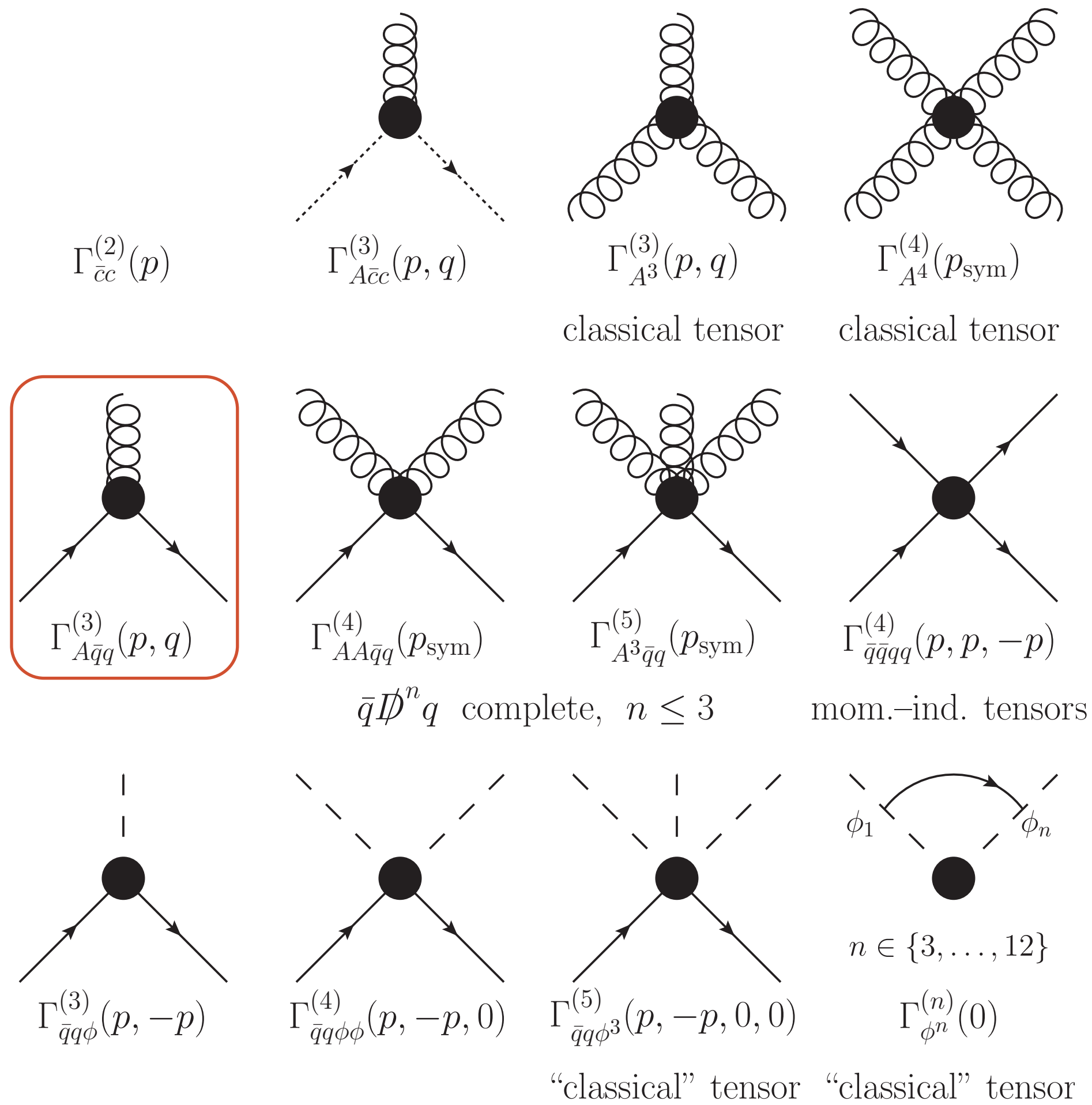


Vertices



Aiming at apparent convergence

Vertices



Aiming at apparent convergence

Quark-gluon vertex

$$\left[\Gamma_{\bar{q}qA}^{(3)} \right]_{\mu}^a (p, q) = 1_{2 \times 2}^{\text{flav}} T^a \sum_{i=1}^8 \lambda_i(p, q) \left[\mathcal{T}_{\bar{q}qA}^{(i)} \right]_{\mu} (p, q)$$

covariant expansion scheme

$$\bar{q} \not{D} q : \quad \left[\mathcal{T}_{\bar{q}qA}^{(1)} \right]_{\mu} (p, q) = -i \gamma_{\mu}$$

$$\bar{q} \not{D}^2 q : \quad \left[\mathcal{T}_{\bar{q}qA}^{(2)} \right]_{\mu} (p, q) = (p - q)_{\mu} 1_{4 \times 4}$$

$$\bar{q} \not{D}^3 q : \quad \left[\mathcal{T}_{\bar{q}qA}^{(5)} \right]_{\mu} (p, q) = i (\not{p} + \not{q})(p - q)_{\mu}$$

$$\left[\mathcal{T}_{\bar{q}qA}^{(3)} \right]_{\mu} (p, q) = (\not{p} - \not{q}) \gamma_{\mu}$$

$$\left[\mathcal{T}_{\bar{q}qA}^{(6)} \right]_{\mu} (p, q) = i (\not{p} - \not{q})(p - q)_{\mu}$$

$$\left[\mathcal{T}_{\bar{q}qA}^{(4)} \right]_{\mu} (p, q) = (\not{p} + \not{q}) \gamma_{\mu}$$

$$\left[\mathcal{T}_{\bar{q}qA}^{(7)} \right]_{\mu} (p, q) = \frac{i}{2} [\not{p}, \not{q}] \gamma_{\mu}$$

Aiming at apparent convergence

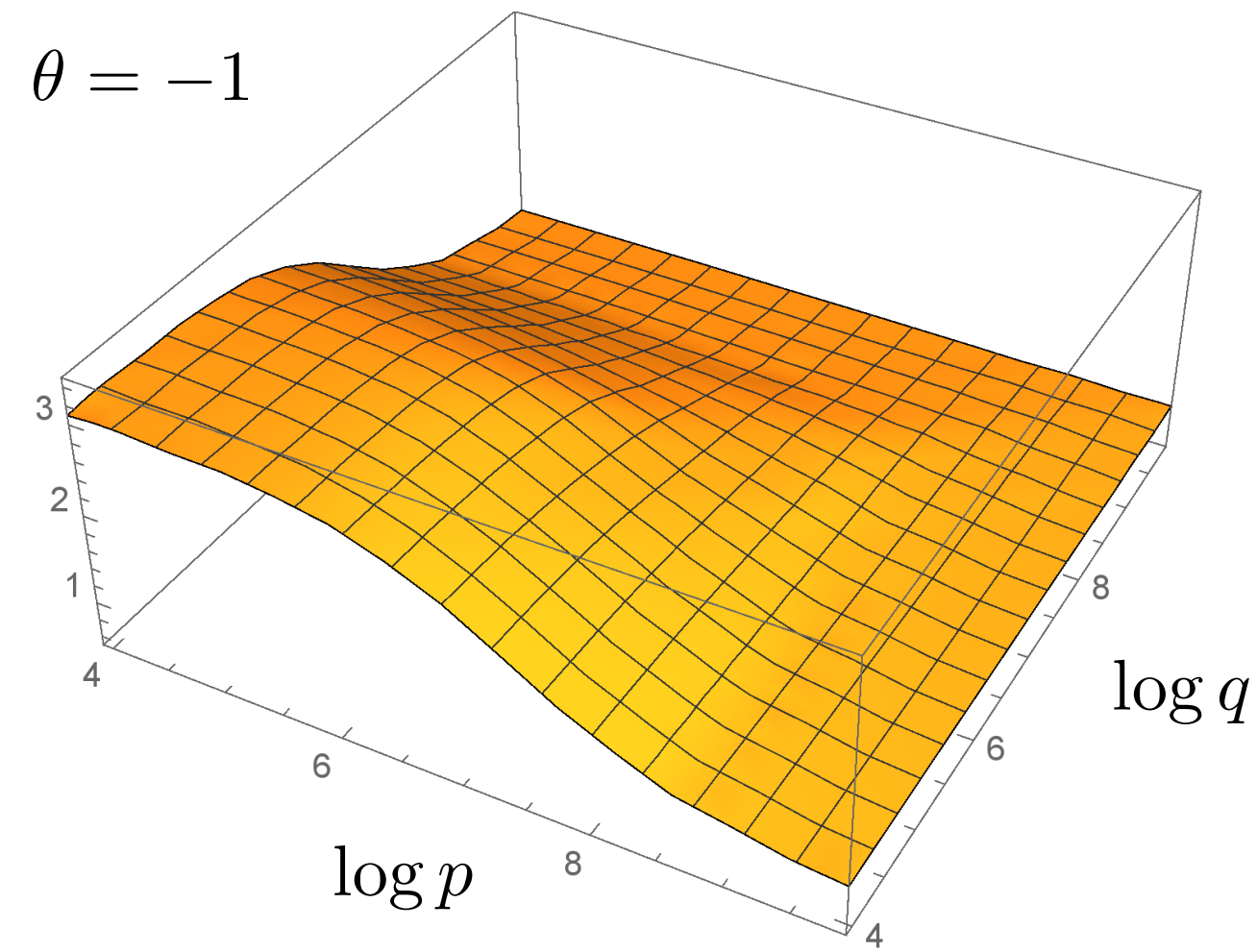
quenched: Mitter, JMP, Strodthoff, PRD 91 (2015) 054035

Quark-gluon vertex

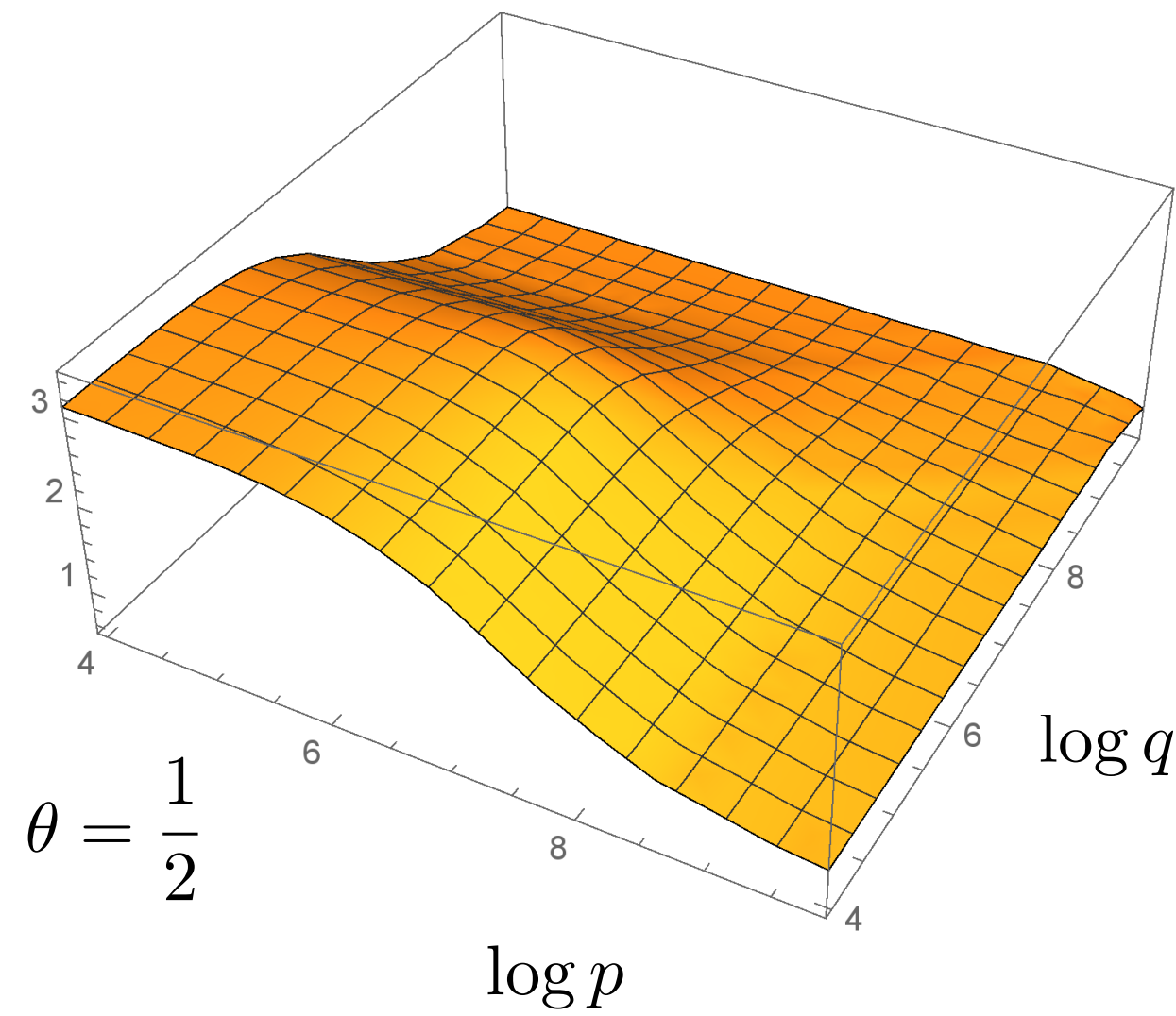
$$\theta = \frac{p \cdot q}{\sqrt{p^2 q^2}}$$

p,q in MeV

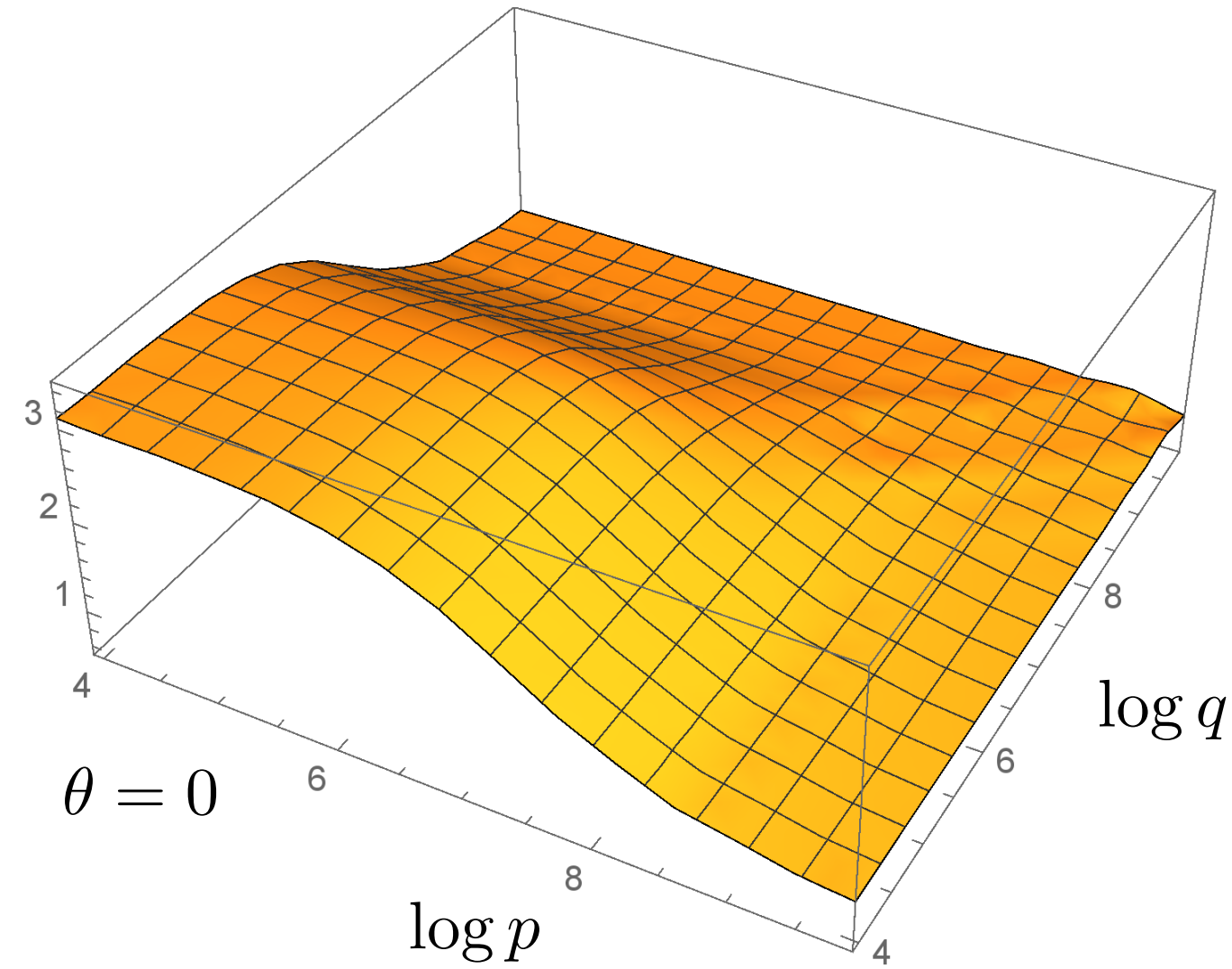
$$\theta = -1$$



$$\lambda_1(p, q)$$



$$\theta = \frac{1}{2}$$



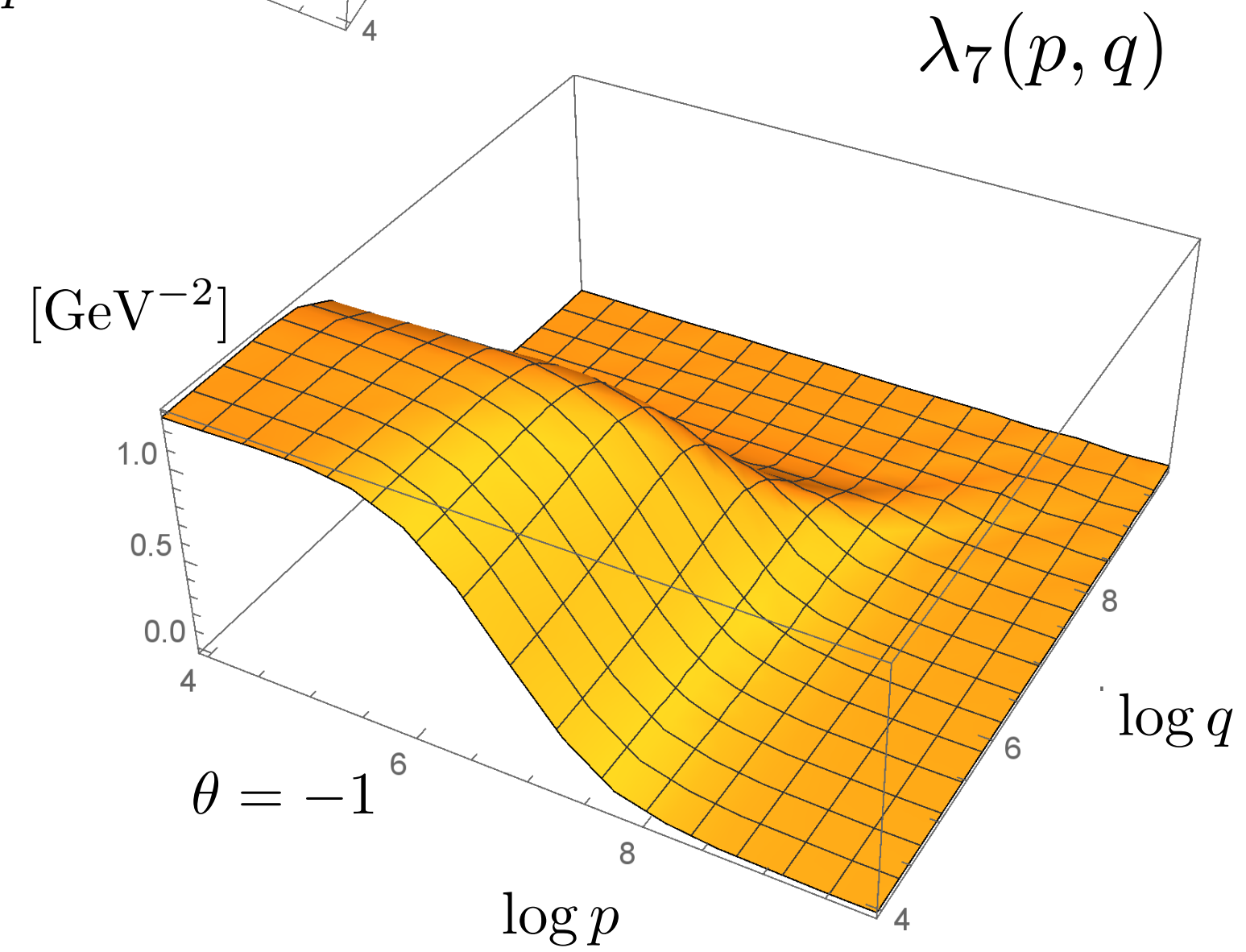
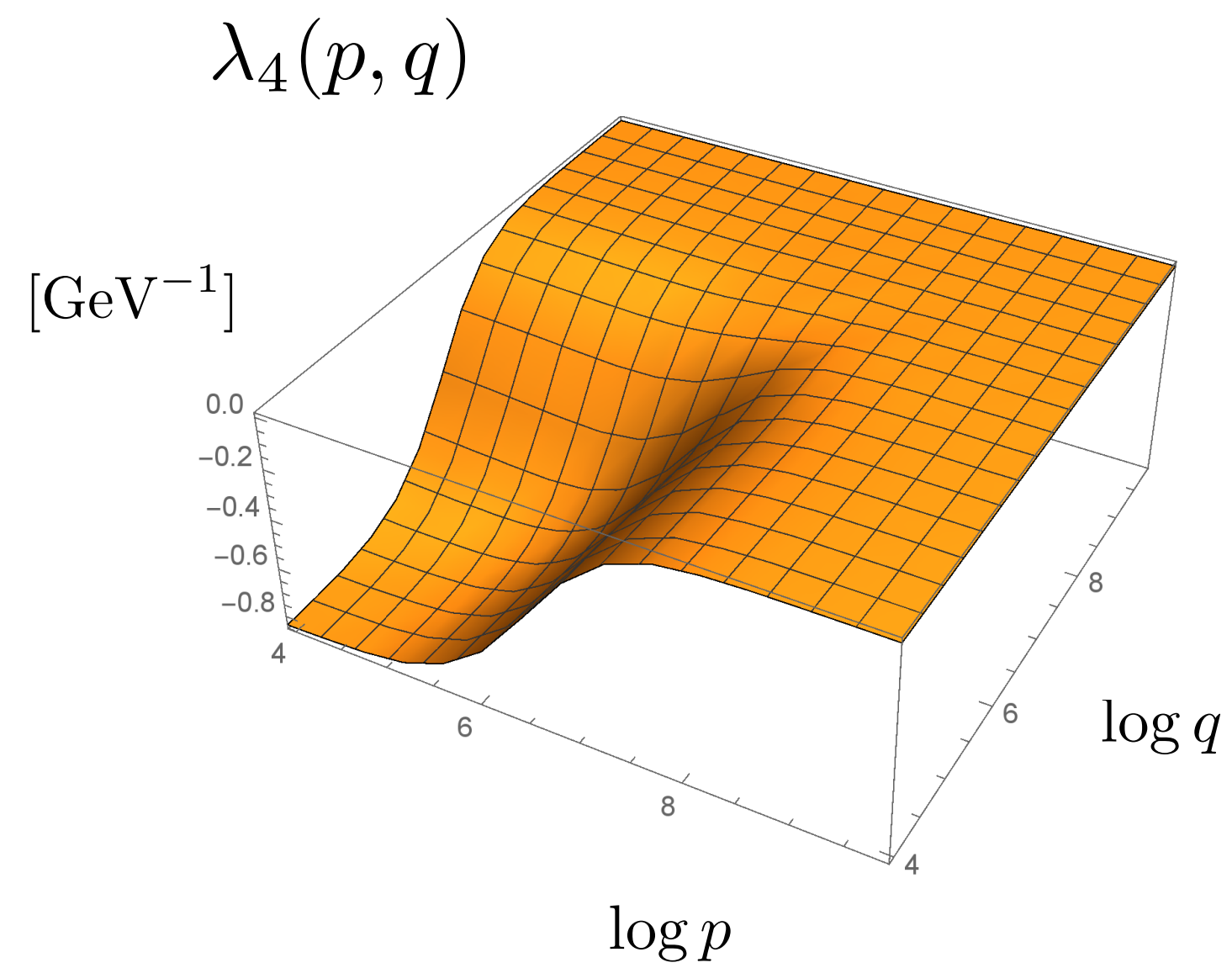
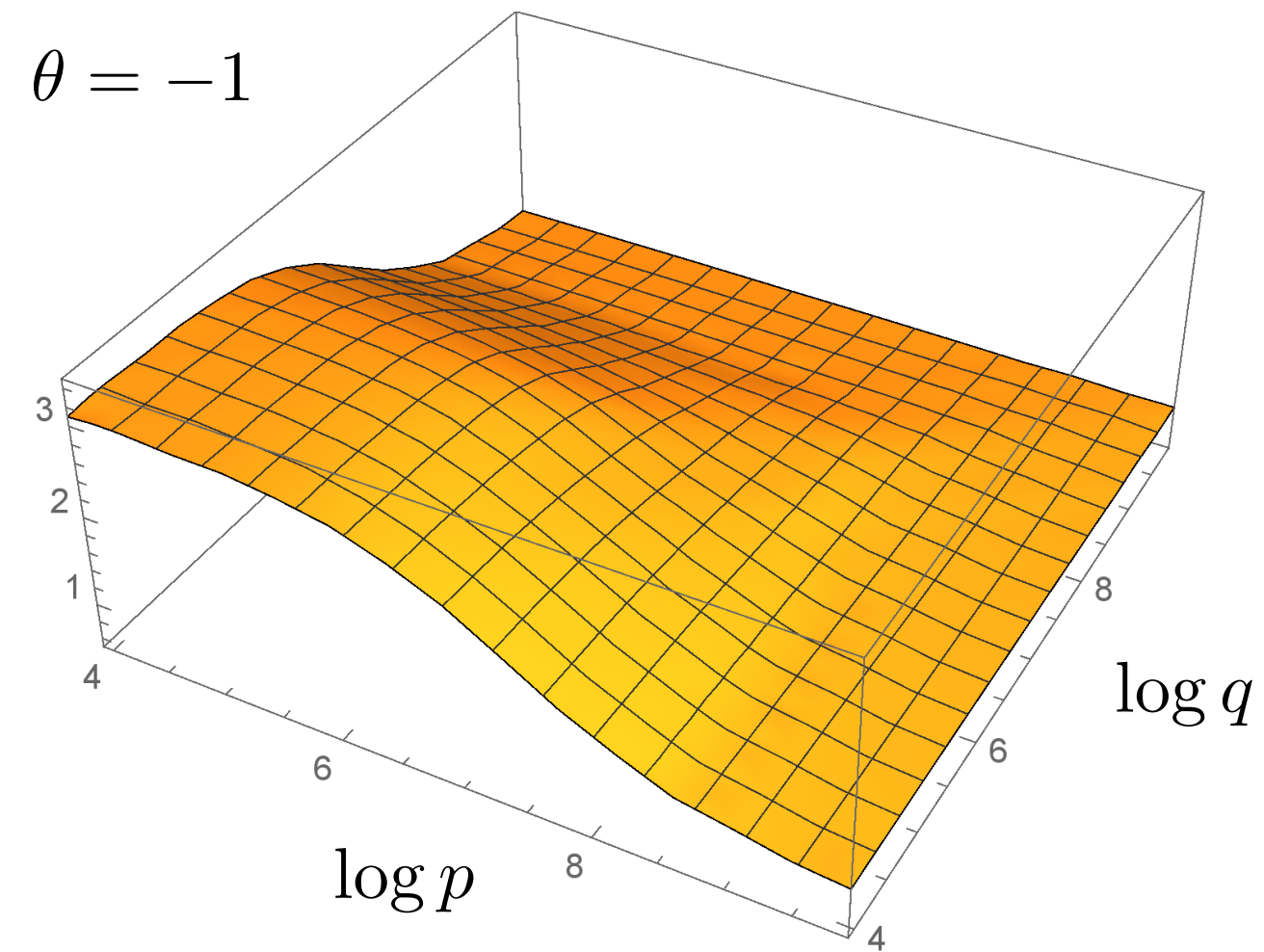
$$\theta = 0$$

Aiming at apparent convergence

Quark-gluon vertex

$$\theta = \frac{p \cdot q}{\sqrt{p^2 q^2}}$$

p,q in MeV



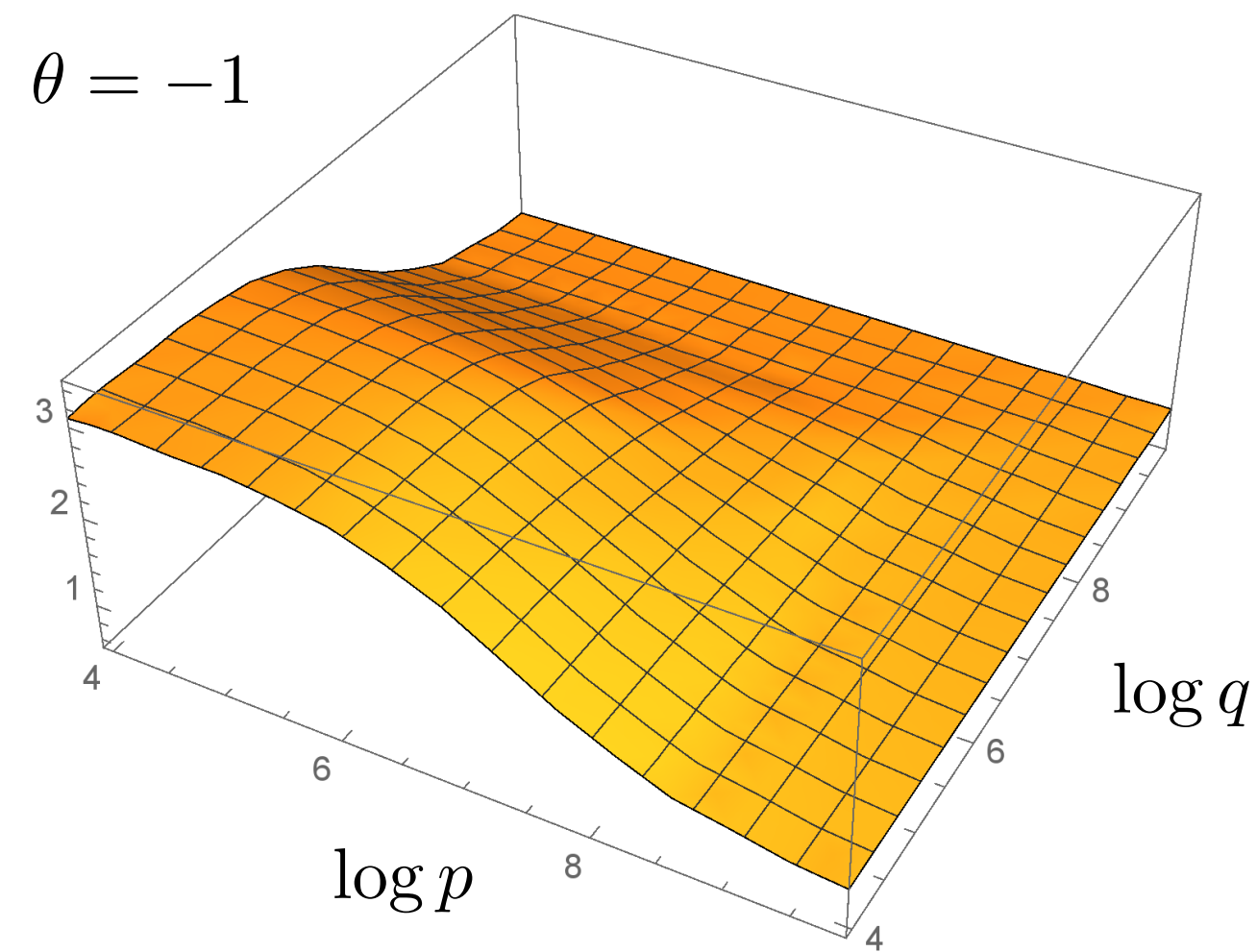
Aiming at apparent convergence



Quark-gluon vertex

$$\theta = \frac{p \cdot q}{\sqrt{p^2 q^2}}$$

p,q in MeV



$$\lambda_1(p, q)$$

up-to-date 1st principles works:

FunMethods: Williams, EPJ A51 (2015) 57
Sanchis-Alepuz, Williams, PLB 749 (2015) 592
Williams, Fischer, Heupel, PRD 93 (2016) 034026

Aguilar, Binosi, Ibanez, Papavassiliou, PRD 89 (2014) 065027
Binosi, Chang, Papavassiliou, Qin, Roberts, PRD 95 (2017) 031501
Aguilar, Cardona, Ferreira, Papavassiliou, arXiv:1610.06158

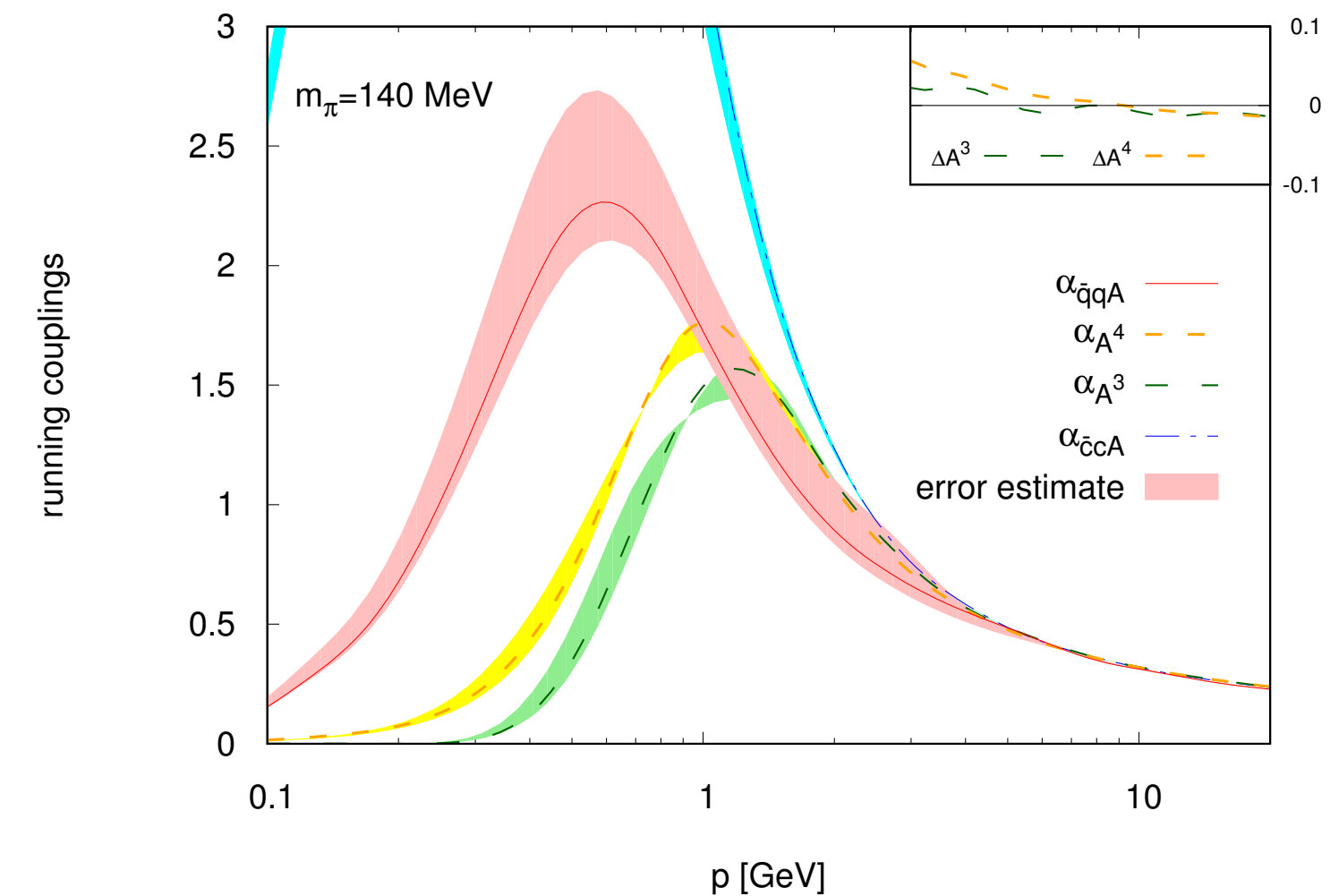
Mitter, JMP, Strodthoff, PRD 91 (2015) 054035

Pelaez, Tissier, Wschebor, PRD 92 (2015) 045012

Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, PPNP 91 (2016) 1

lattice: Oliveira, Kizilersü, Silva, Skullerud, Sternbeck, Williams, APP Suppl. 9 (2016) 363

Beware of BRST



Aiming at apparent convergence

fRG/DSE-assisted DSE/fRG

fRG/DSE-assisted DSE/fRG

More generally: X-assisted Y

X=fRG, DSE, nPI,
lattice, exp. data
Y=fRG, DSE, nPI

fRG/DSE-assisted DSE/fRG

More generally: X-assisted Y

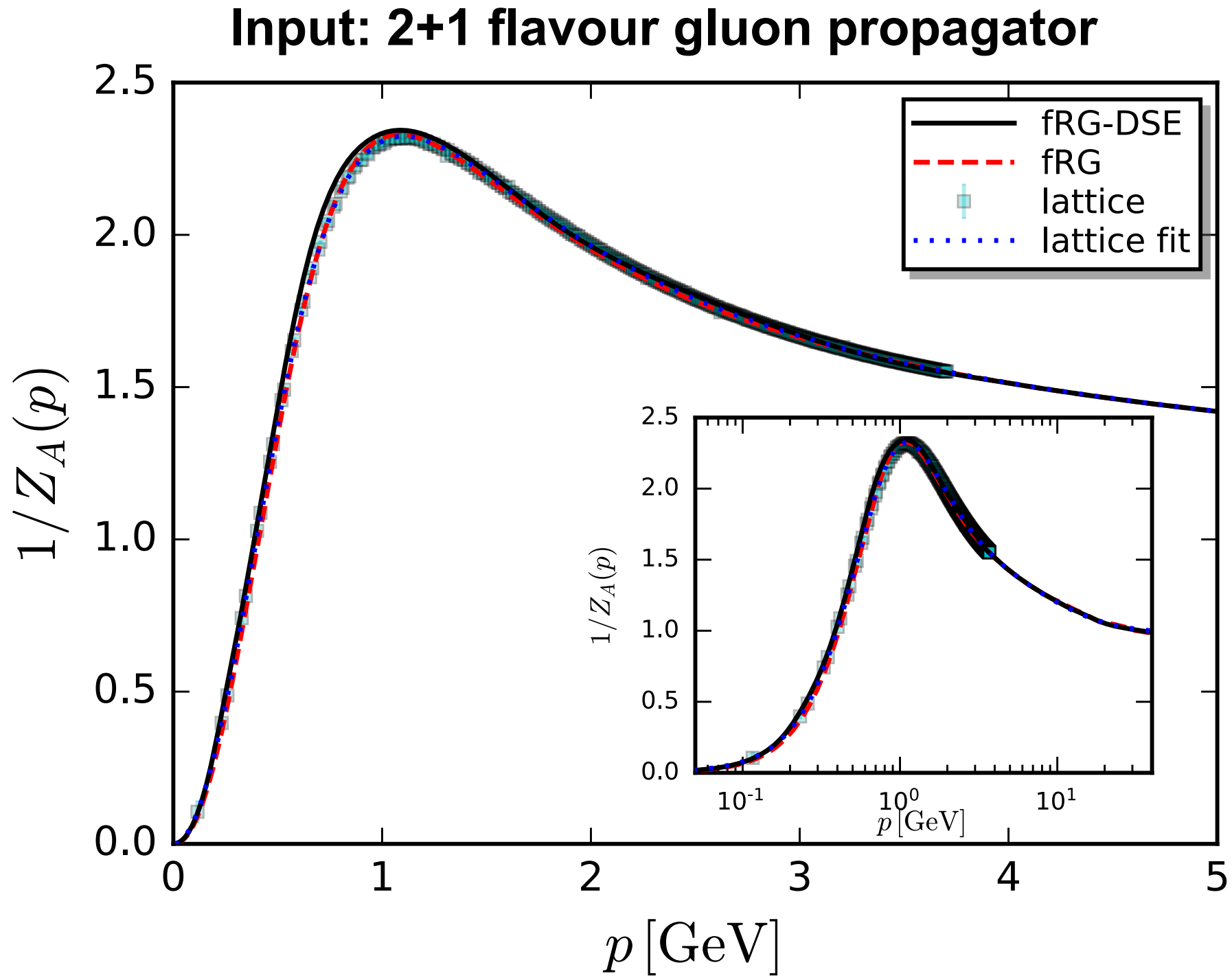
X=fRG, DSE, nPI,
lattice, exp. data
Y=fRG, DSE, nPI

Example: use (a) 2+1 fRG-assisted gluon
(b) optional: two-flavour fRG quark-gluon vertex in 2+1 flavour DSE quark gap eq.

fRG/DSE-assisted DSE/fRG

More generally: X-assisted Y X=fRG, DSE, nPI,
lattice, exp. data
Y=fRG, DSE, nPI

Example: use (a) 2+1 fRG-assisted gluon in 2+1 flavour DSE quark gap eq.
 (b) optional: two-flavour fRG quark-gluon vertex



fRG: Fu, JMP, Rennecke, PRD 97 (2018) 054006

fRG-DSE: Gao, JMP, PLB 820 (2021) 136584
PRD 102 (2020) 034027

lattice: Zafeiropoulos et al, PRL 122 (2019) 16, 162002
Cui et al, CPC 44 (2020) 8, 083102

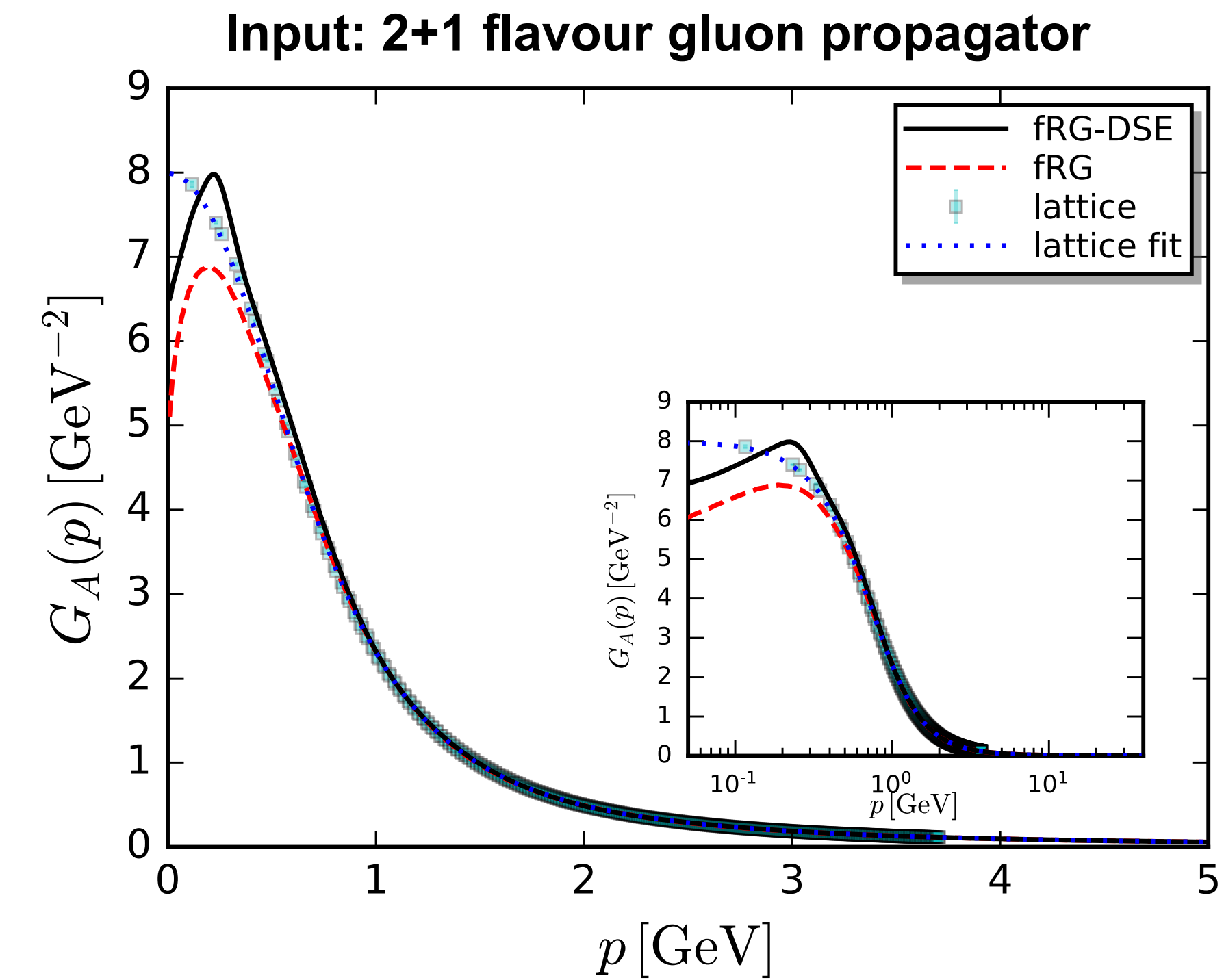
FunResults based on:

Cyrol, Mitter, JMP, Strodthoff, PRD 97 (2018) 054006

fRG/DSE-assisted DSE/fRG

More generally: X-assisted Y X=fRG, DSE, nPI,
lattice, exp. data
Y=fRG, DSE, nPI

Example: use (a) 2+1 fRG-assisted gluon in 2+1 flavour DSE quark gap eq.
 (b) optional: two-flavour fRG quark-gluon vertex



fRG: Fu, JMP, Rennecke, PRD 97 (2018) 054006

fRG-DSE: Gao, JMP, PLB 820 (2021) 136584
PRD 102 (2020) 034027

lattice: Zafeiropoulos et al, PRL 122 (2019) 16, 162002
Cui et al, CPC 44 (2020) 8, 083102

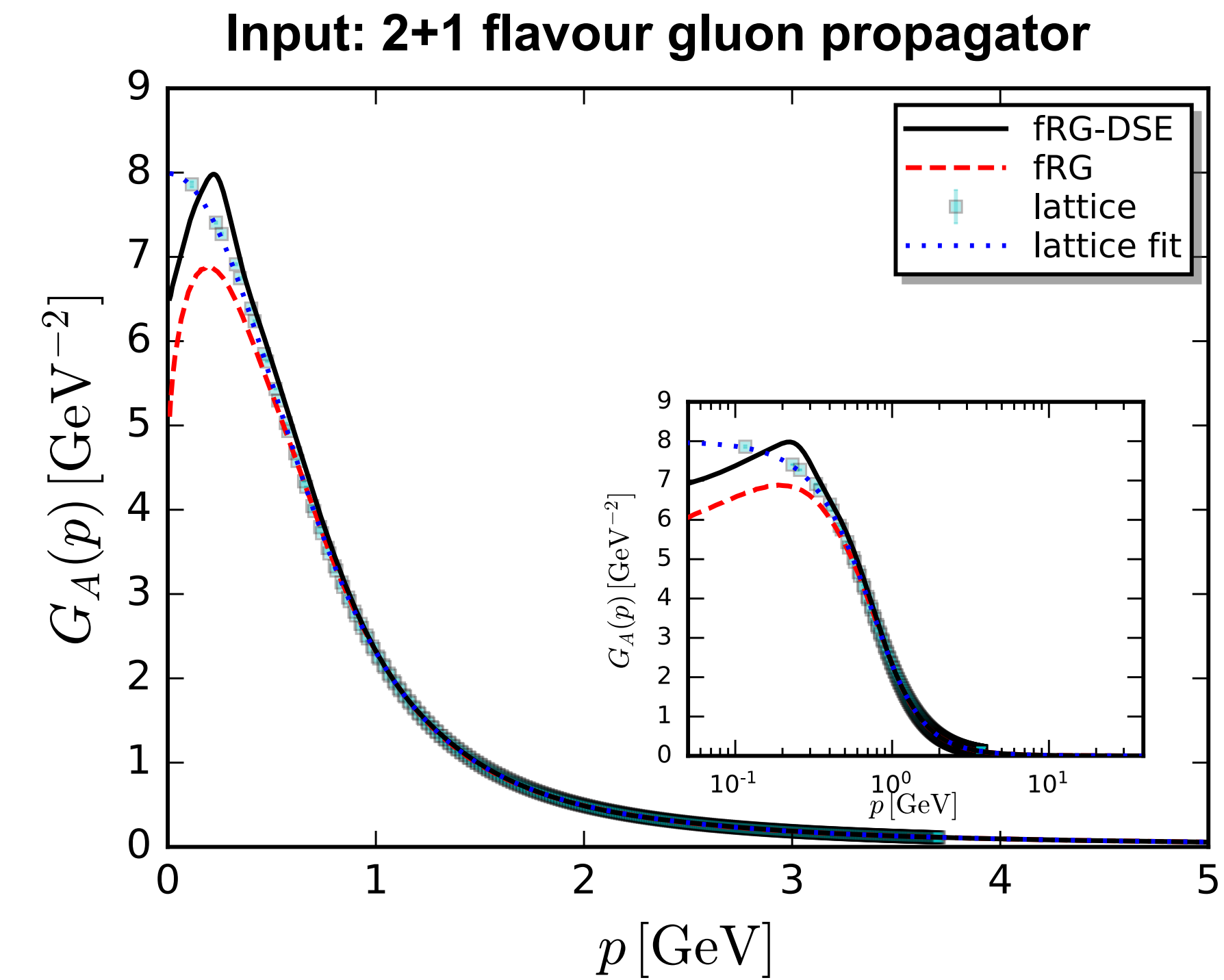
FunResults based on:

Cyrol, Mitter, JMP, Strodthoff, PRD 97 (2018) 054006

fRG/DSE-assisted DSE/fRG

More generally: X-assisted Y X=fRG, DSE, nPI,
lattice, exp. data
Y=fRG, DSE, nPI

Example: use (a) 2+1 fRG-assisted gluon in 2+1 flavour DSE quark gap eq.
 (b) optional: two-flavour fRG quark-gluon vertex



fRG: Fu, JMP, Rennecke, PRD 97 (2018) 054006

fRG-DSE: Gao, JMP, PLB 820 (2021) 136584
PRD 102 (2020) 034027

lattice: Zafeiropoulos et al, PRL 122 (2019) 16, 162002
Cui et al, CPC 44 (2020) 8, 083102

FunResults based on:

Cyrol, Mitter, JMP, Strodthoff, PRD 97 (2018) 054006

Further example, e.g. lattice-assisted 2+1 flavour DSE

Aguilar et al, EPC 80 (2020) 2, 154

fRG/DSE-assisted DSE/fRG

Example: use

(a) 2+1 fRG-assisted gluon

(b) optional: two-flavour fRG quark-gluon vertex

in 2+1 flavour DSE quark gap eq.

Full quark gluon

$$\left(\text{line with grey circle} \right)^{-1} = \left(\text{line} \right)^{-1} + \text{line with black, grey, and blue circles and a loop}.$$

The diagram illustrates the Dyson equation for the photon propagator. On the left, the full propagator is shown as a wavy line with a blue circle at the vertex, representing the sum of all orders of perturbation theory. This is equated to the sum of the tree-level term and a series of loop corrections. The first loop correction is a triangle diagram with two blue circles and one grey circle, labeled A_μ . The second loop correction is a more complex diagram with two blue circles and two grey circles, labeled B_μ . The series continues with ellipses, indicating higher-order terms.

fRG/DSE-assisted DSE/fRG

Example: use (a) 2+1 fRG-assisted gluon in 2+1 flavour DSE quark gap eq.
 (b) optional: two-flavour fRG quark-gluon vertex

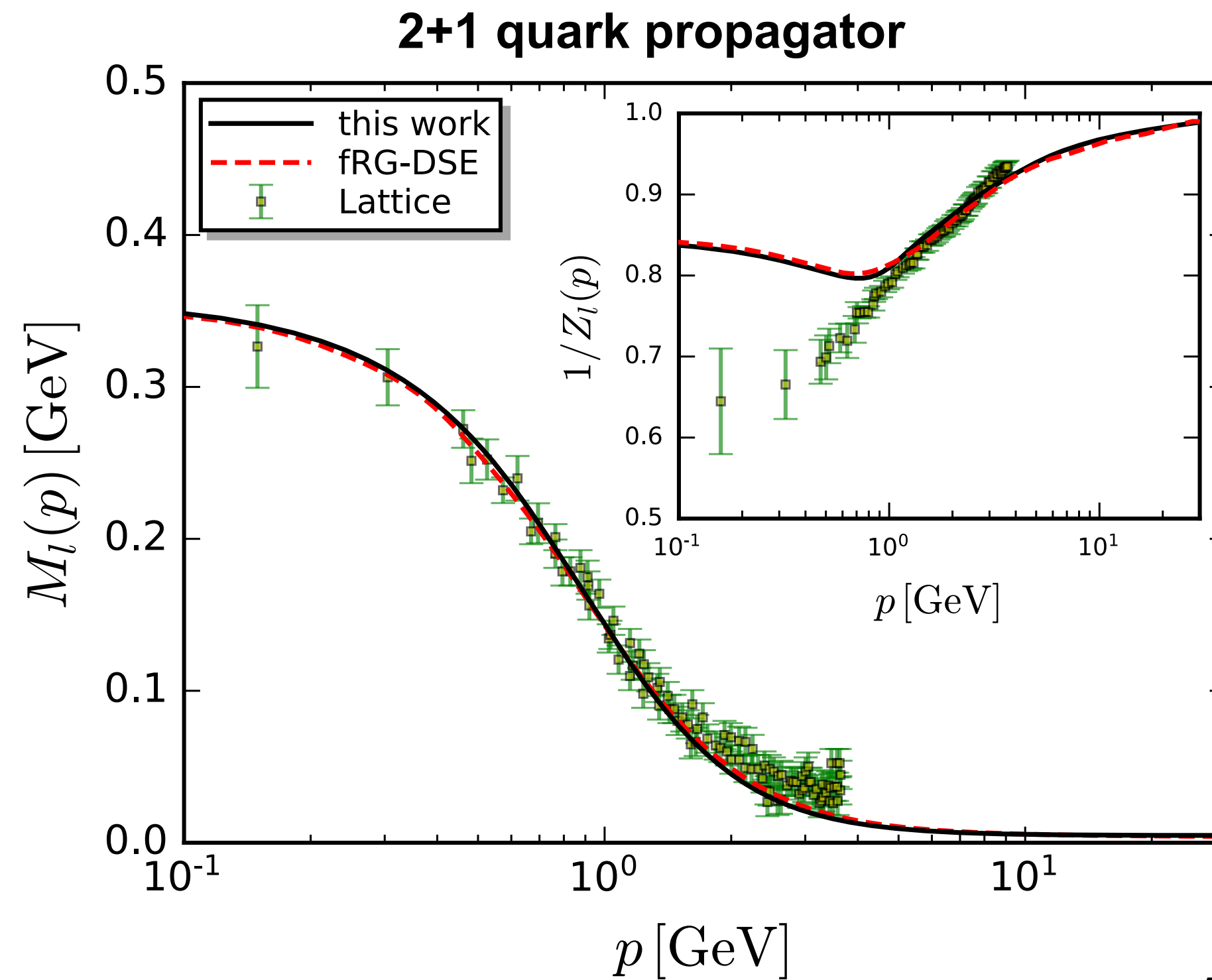
Full quark
gluon

$$\left(\text{gluon line with blob} \right)_p^{-1} = \left(\text{bare gluon line} \right)_p^{-1} + \text{self-energy diagram}$$

$$\text{quark-gluon vertex} = \text{tree-level vertex} + \text{A}_\mu \text{ diagram} + \text{B}_\mu \text{ diagram} + \dots$$

fRG/DSE-assisted DSE/fRG

Results



Chiral condensate

DSE: $\Delta_{l,\chi}(2 \text{ GeV}) = (269.3(7) \text{ MeV})^3$

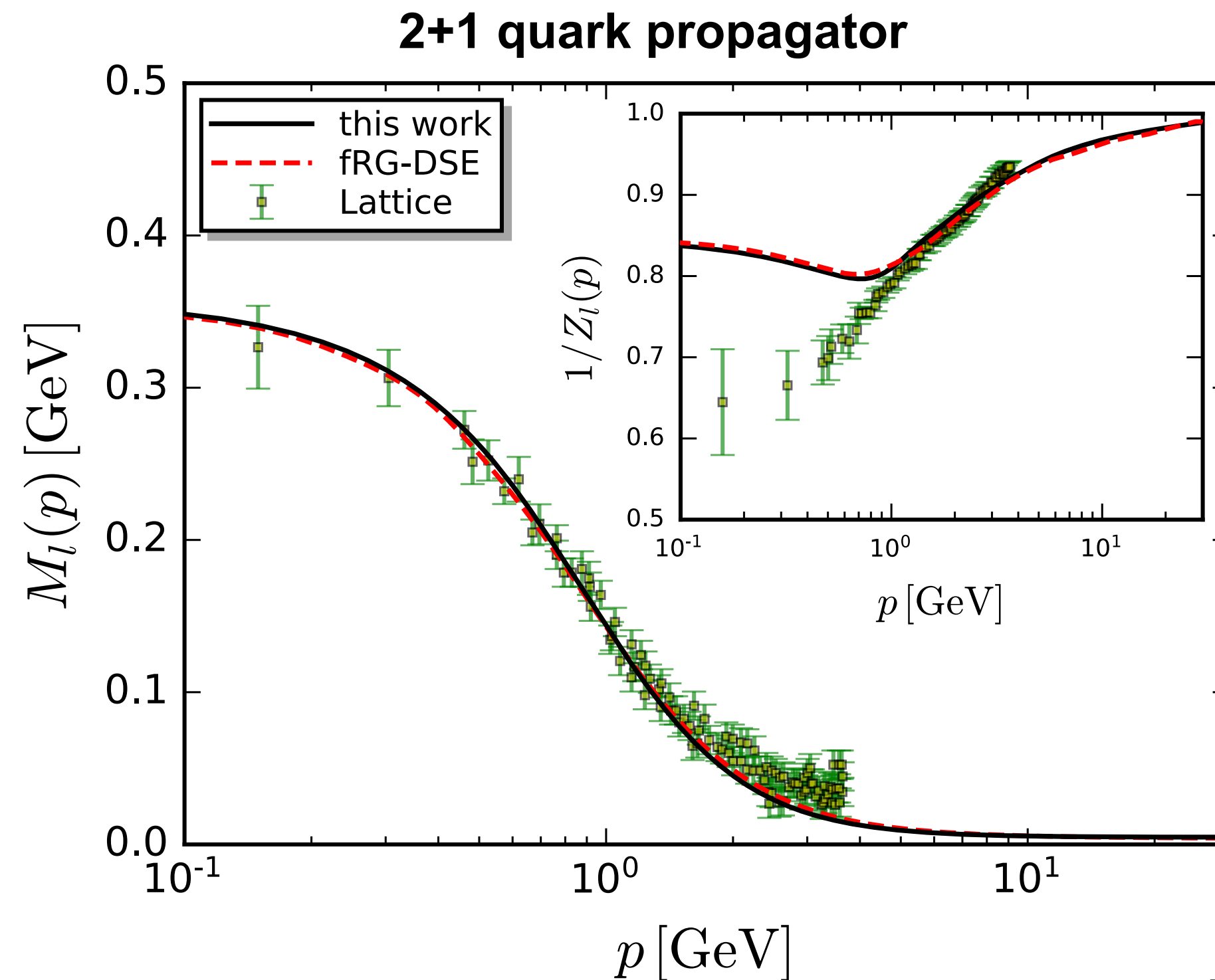
Lattice: $\Delta_{l,\chi}(2 \text{ GeV}) = (272(5) \text{ MeV})^3$

FLAG: Aoki et al, EPJC 80 (2020) 2, 113

fRG/DSE-assisted DSE/fRG

Example: use (a) 2+1 fRG-assisted gluon (b) optional: two-flavour fRG quark-gluon vertex in 2+1 flavour DSE quark gap eq.

Results



Chiral condensate

DSE: $\Delta_{l,\chi}(2 \text{ GeV}) = (269.3(7) \text{ MeV})^3$

Lattice: $\Delta_{l,\chi}(2 \text{ GeV}) = (272(5) \text{ MeV})^3$

FLAG: Aoki et al, EPJC 80 (2020) 2, 113

Correlation functions at finite temperature

YM-theory: gluonic correlation functions

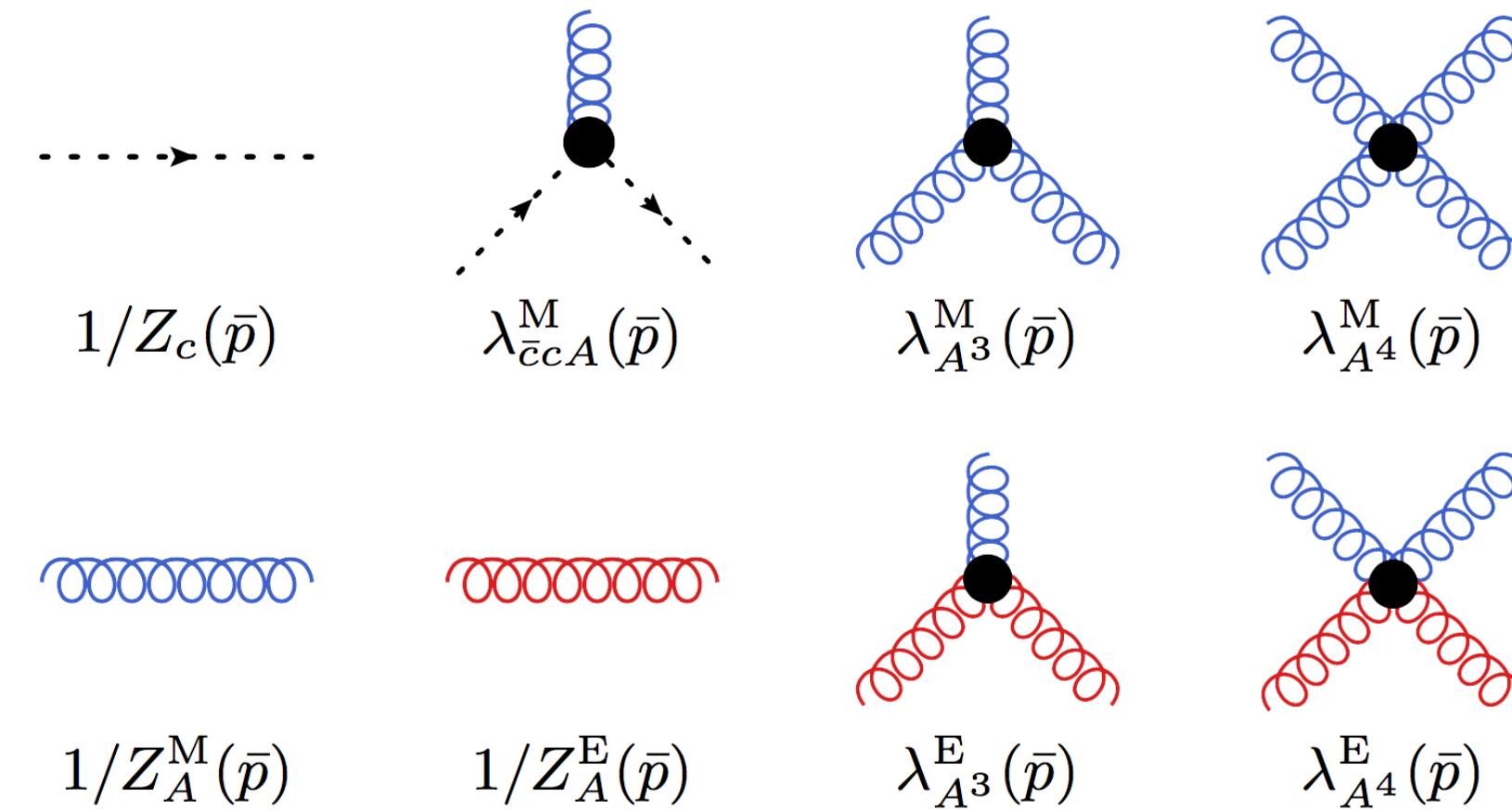
$$\partial_t \text{---}^{-1} = \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---}$$

$$\partial_t \text{---}^{-1} = \text{---} \text{---} \text{---} - 2 \text{---} \text{---} \text{---} + \frac{1}{2} \text{---} \text{---} \text{---}$$

$$\partial_t \text{---} = - \text{---} \text{---} \text{---} - \text{---} \text{---} \text{---} + \text{perm.}$$

$$\partial_t \text{---} = - \text{---} \text{---} \text{---} + 2 \text{---} \text{---} \text{---} - \text{---} \text{---} \text{---} + \text{perm.}$$

$$\partial_t \text{---} = - \text{---} \text{---} \text{---} - \text{---} \text{---} \text{---} + 2 \text{---} \text{---} \text{---} - \text{---} \text{---} \text{---} + \text{perm.}$$



Aiming at apparent convergence

YM-theory: gluonic correlation functions

$$\partial_t \text{---}^{-1} = \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---}$$

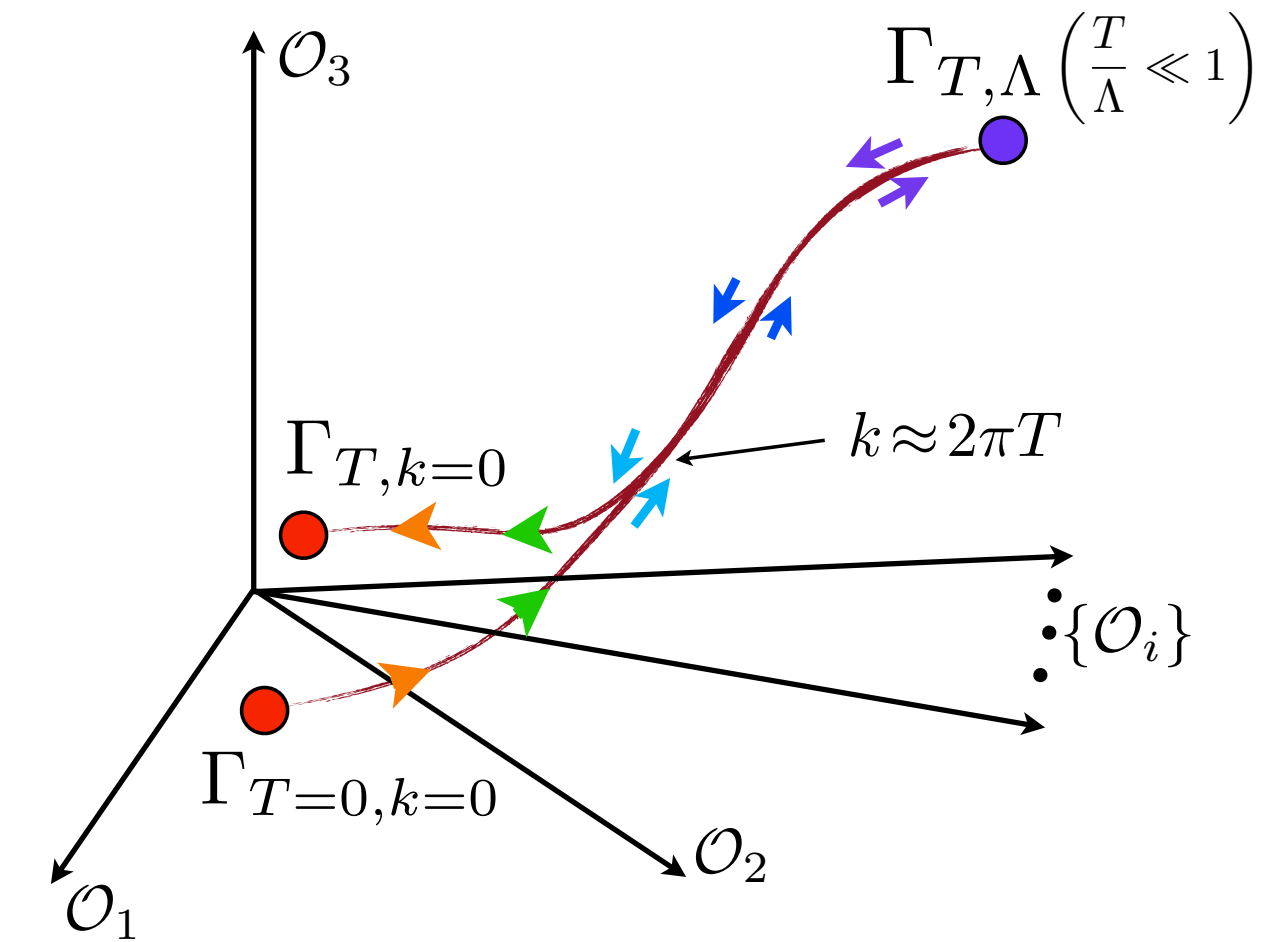
$$\partial_t \text{---}^{-1} = \text{---} \text{---} \text{---} - 2 \text{---} \text{---} \text{---} + \frac{1}{2} \text{---} \text{---} \text{---}$$

$$\partial_t \text{---} = - \text{---} \text{---} \text{---} - \text{---} \text{---} \text{---} + \text{perm.}$$

$$\partial_t \text{---} = - \text{---} \text{---} \text{---} + 2 \text{---} \text{---} \text{---} - \text{---} \text{---} \text{---} + \text{perm.}$$

$$\partial_t \text{---} = - \text{---} \text{---} \text{---} - \text{---} \text{---} \text{---} + 2 \text{---} \text{---} \text{---} - \text{---} \text{---} \text{---} + \text{perm.}$$

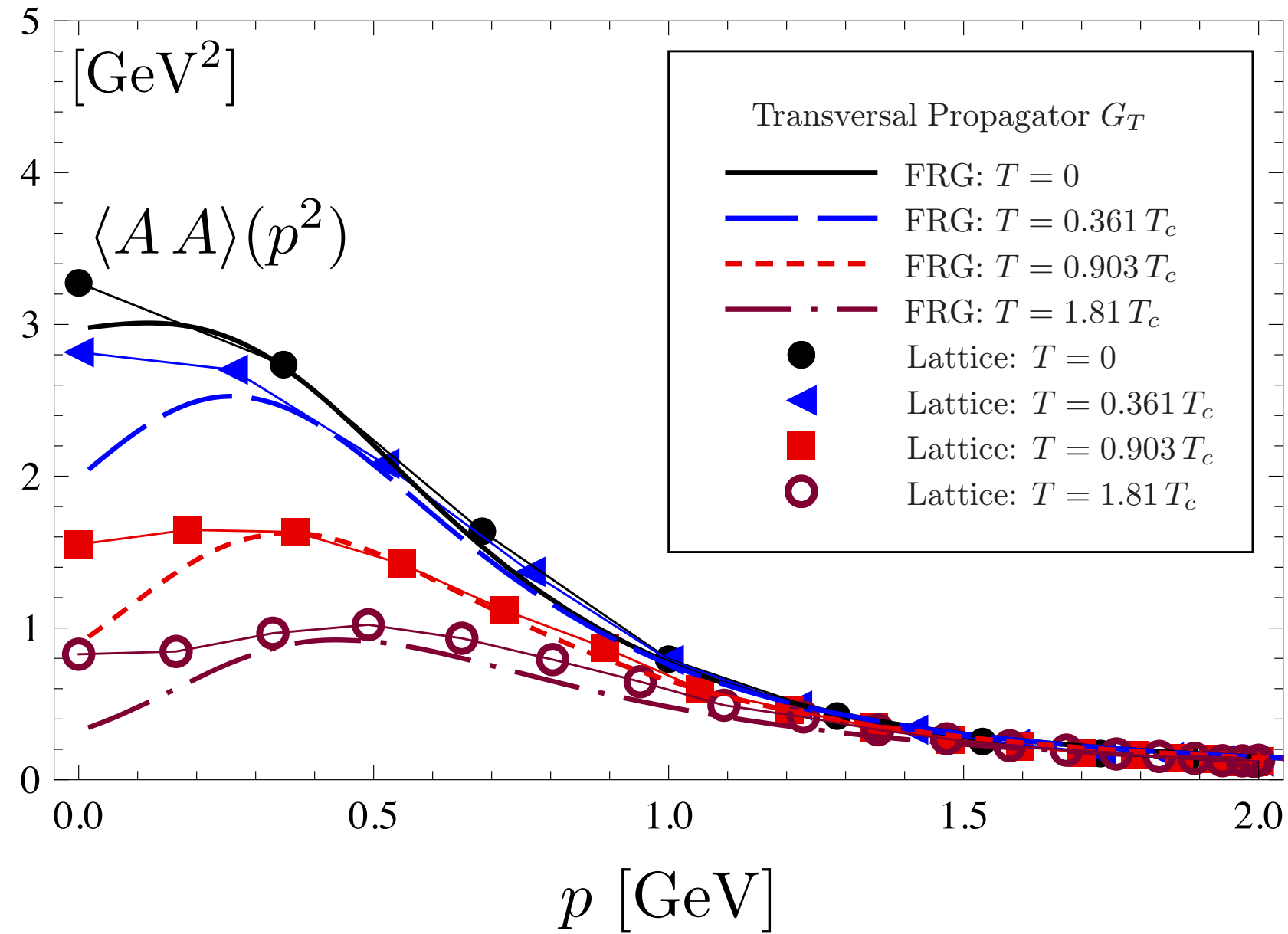
Thermal flows



Aiming at apparent convergence

Euclidean gluon propagator at finite T

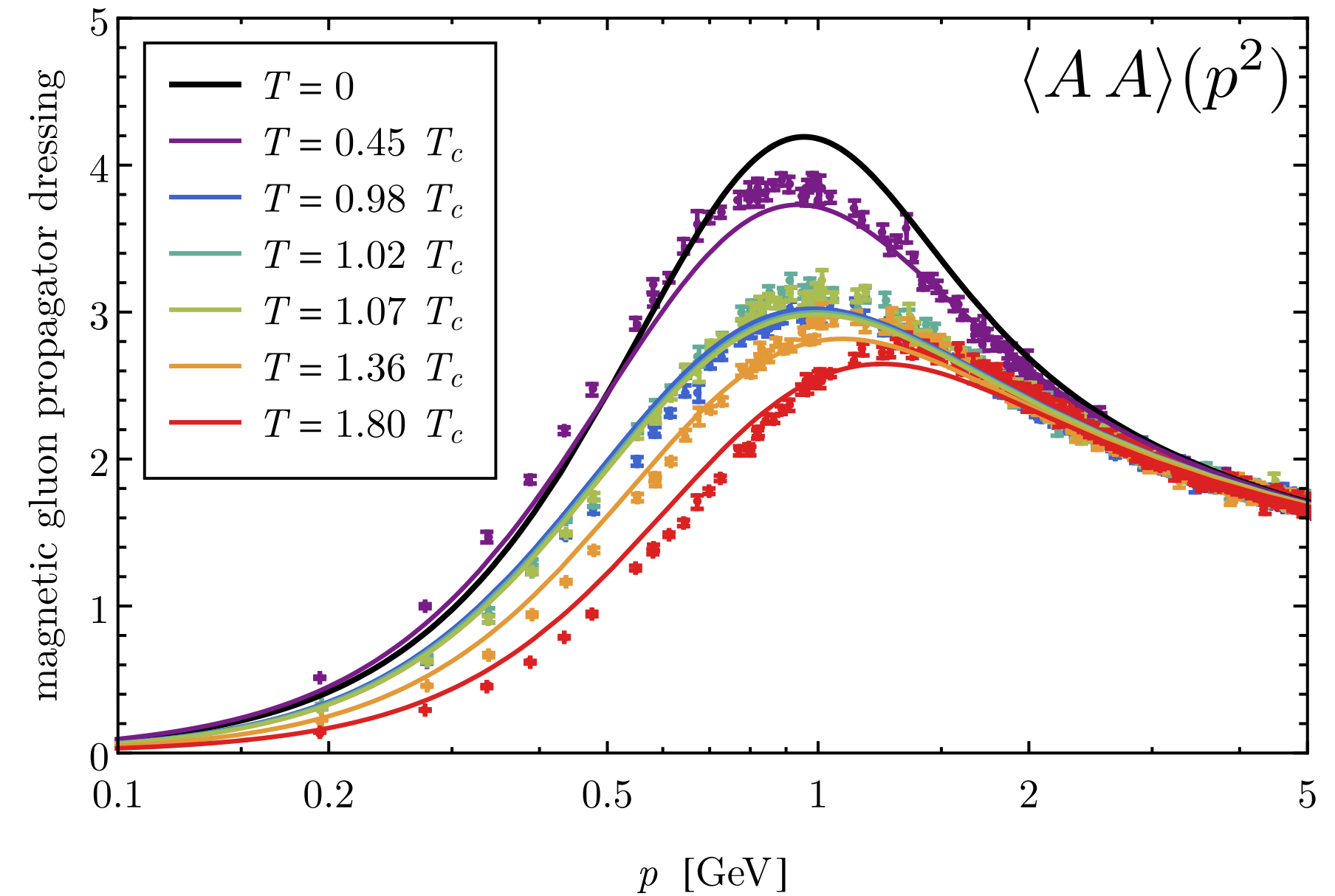
chromo-magnetic propagator



Fister, JMP, arXiv:1112.5440

Lattice: Maas, JMP, Smekal, Spielmann, PRD 85 (2012) 034037

CF model: Reinosa, Serreau, Tissier, Tresmontant, PRD 95 (2017) 045014

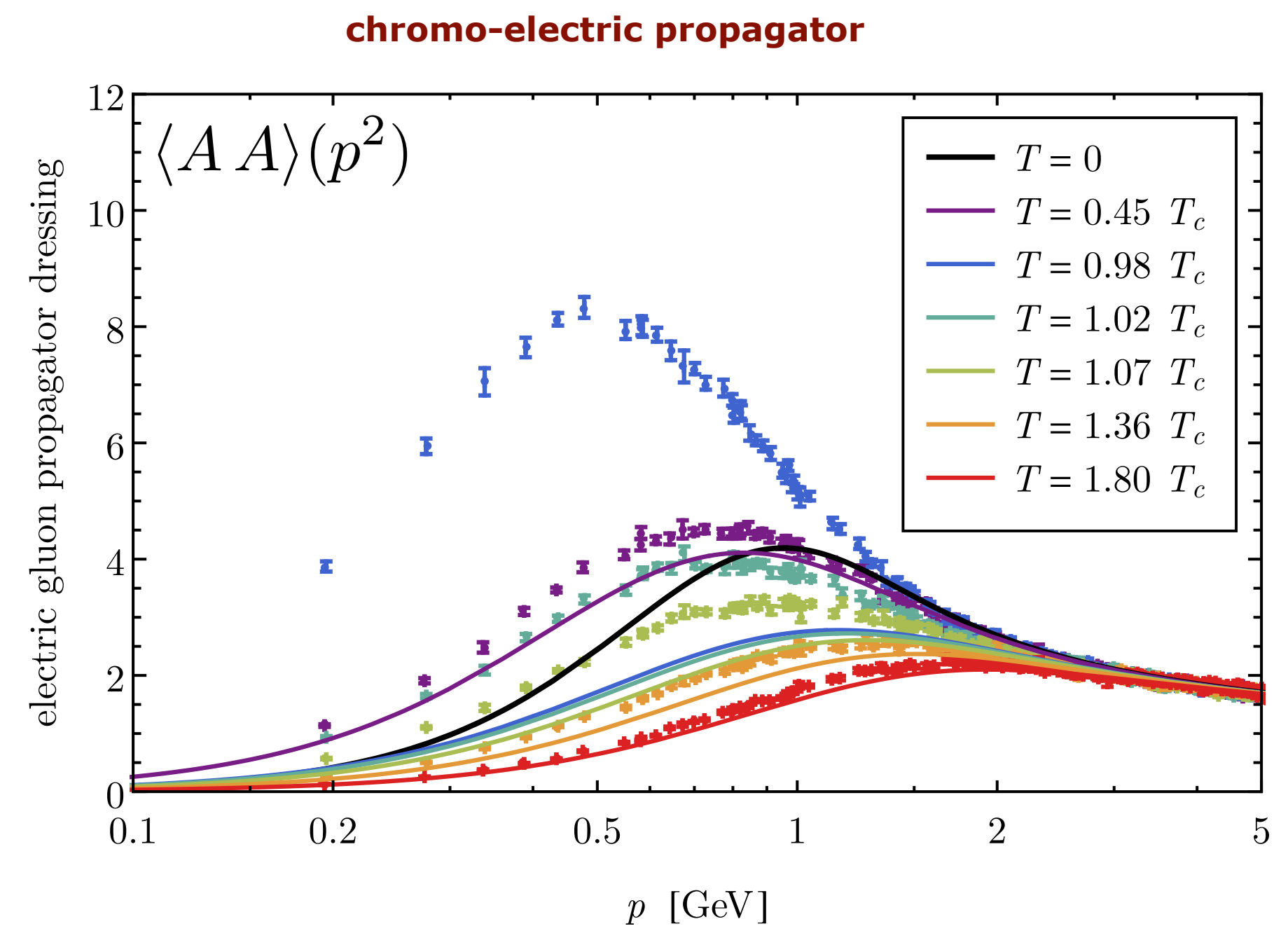
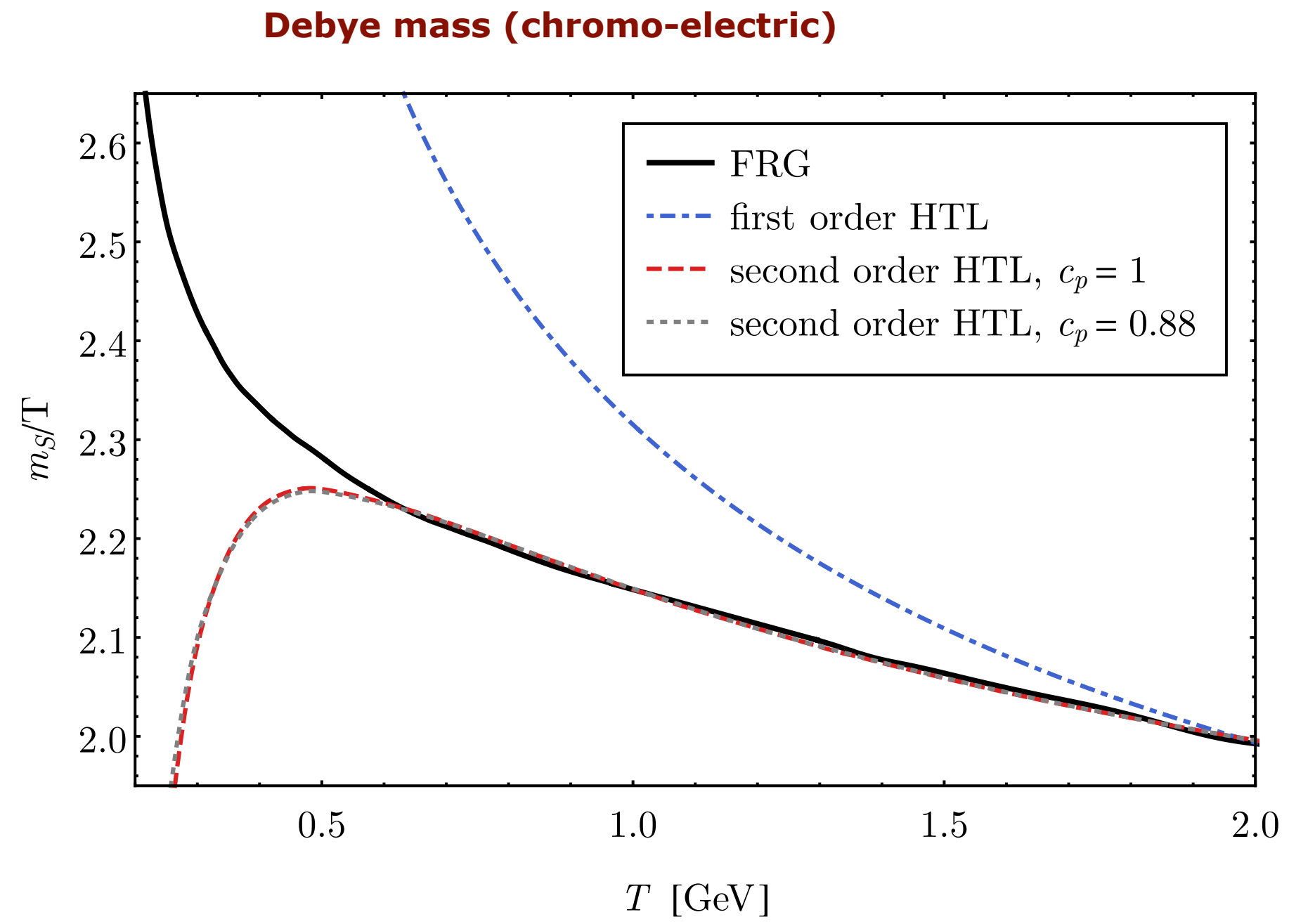


Lattice: Silva, Oliveira, Bicudo, Cardoso, PRD89 (2014) 7, 074503

Aiming at apparent convergence

Cyrol, Fister, Mitter, JMP, Strodthoff, PRD 97 (2018) 5, 054015

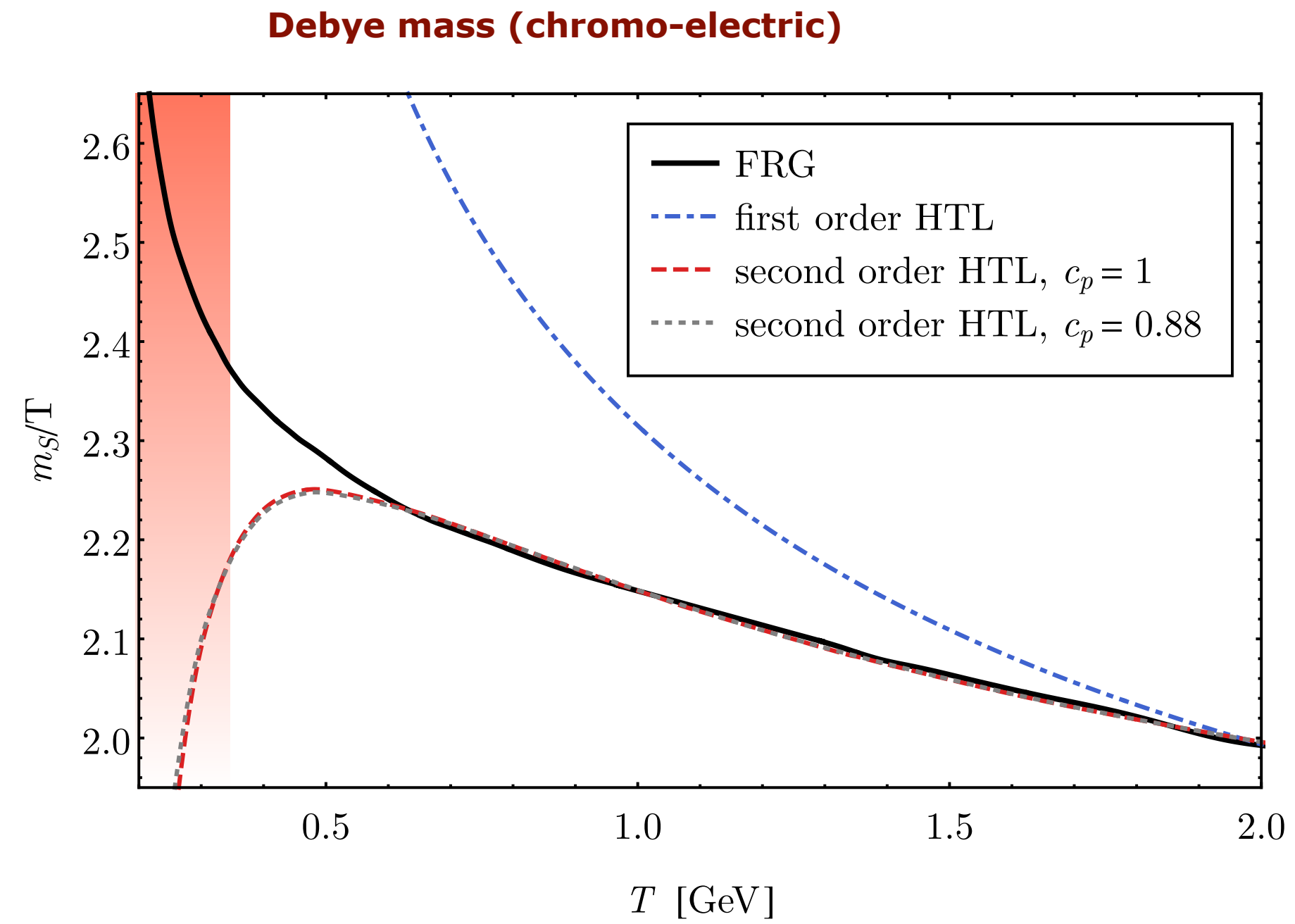
Euclidean gluon propagator at finite T



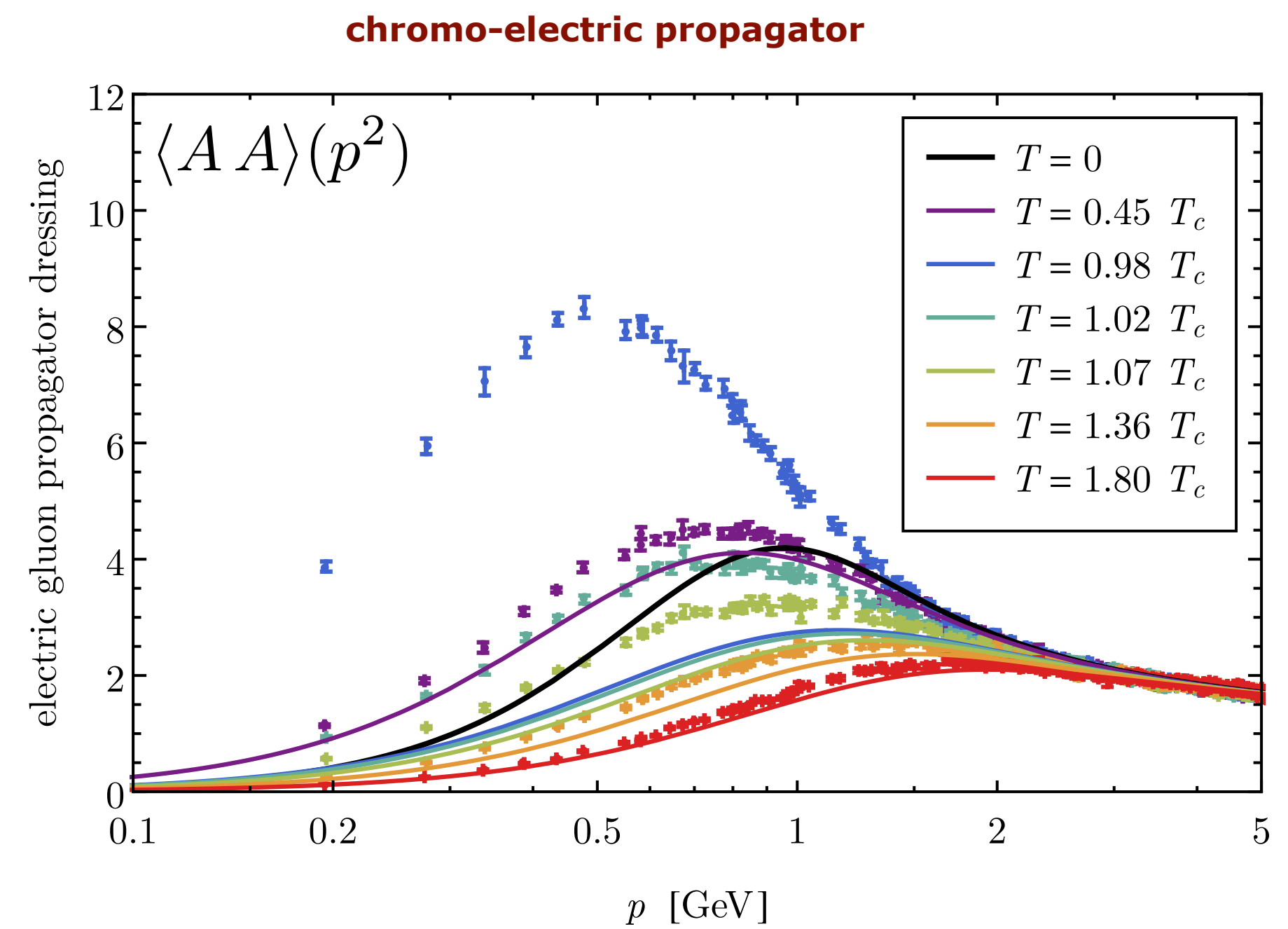
Lattice: Silva, Oliveira, Bicudo, Cardoso, PRD89 (2014) 7, 074503

Cyrol, Fister, Mitter, JMP, Strodthoff, PRD 97 (2018) 5, 054015

Euclidean gluon propagator at finite T



$$\langle A_0 \rangle \neq 0$$



Lattice: Silva, Oliveira, Bicudo, Cardoso, PRD89 (2014) 7, 074503

Cyrol, Fister, Mitter, JMP, Strodthoff, PRD 97 (2018) 5, 054015

Polyakov loop from functional approaches

Confinement

FRG: Braun, Gies, JMP, PLB 684 (2010) 262

FRG, DSE, 2PI: Fister, JMP, PRD 88 (2013) 045010

$$L[A_0] = \frac{1}{N_c} \text{tr} \mathcal{P} \mathbf{e}^{\mathbf{i} \mathbf{g} \int_0^\beta \mathbf{A}_0(\mathbf{x})}$$

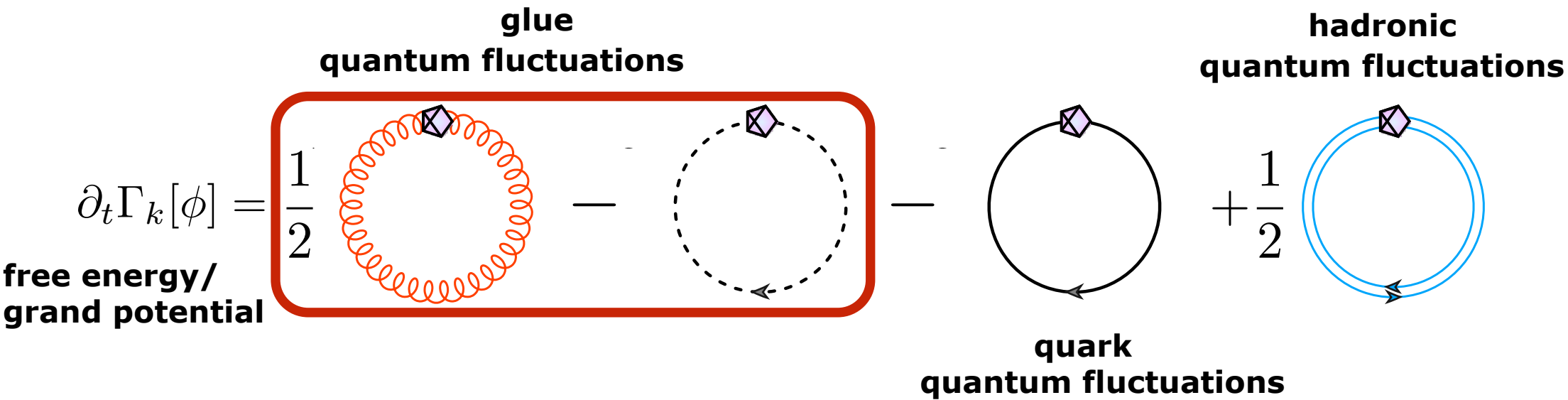
$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \left(\text{glue quantum fluctuations} - \text{quark quantum fluctuations} \right) + \frac{1}{2} \text{hadronic quantum fluctuations}$$

Confinement

FRG: Braun, Gies, JMP, PLB 684 (2010) 262

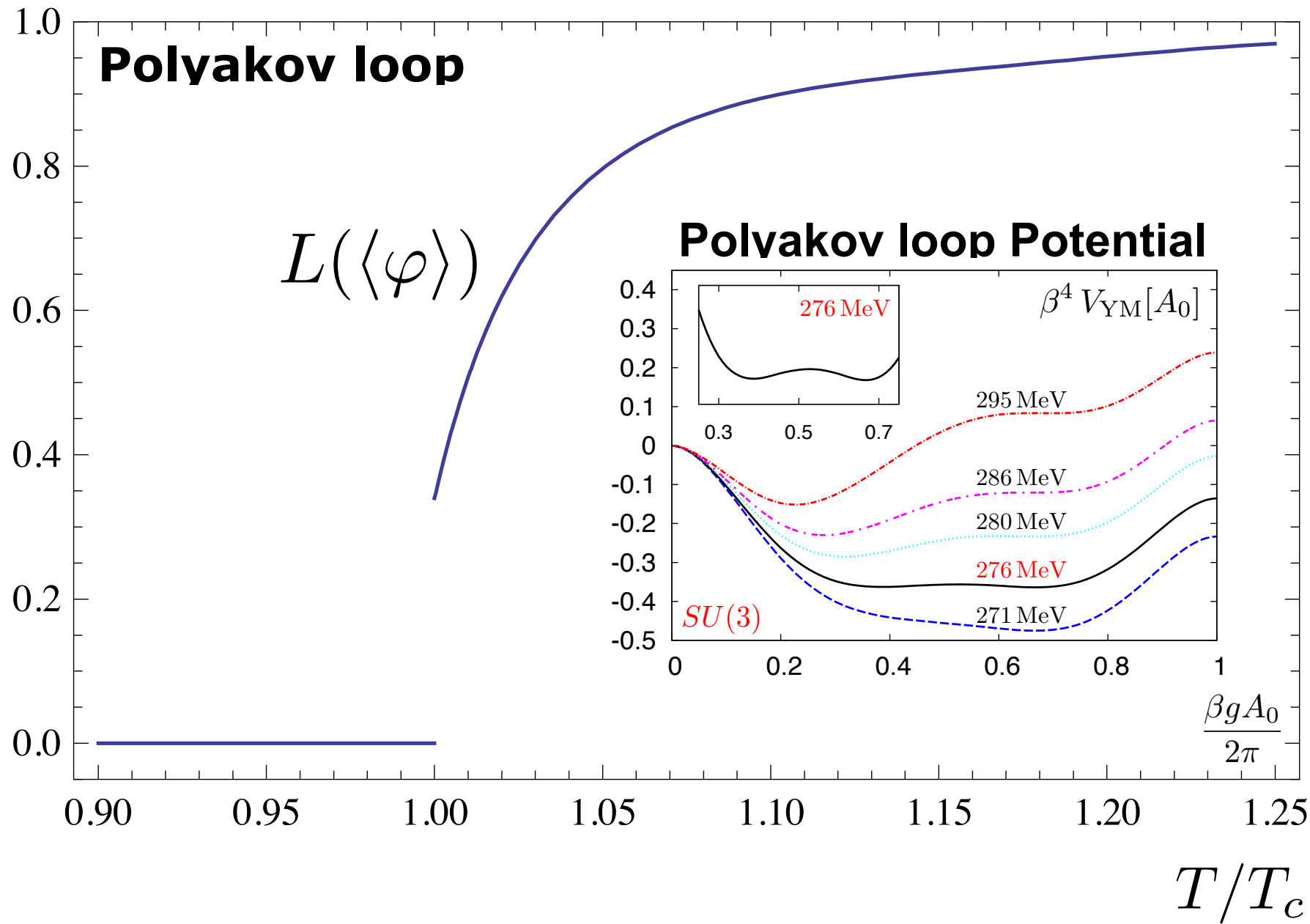
FRG, DSE, 2PI: Fister, JMP, PRD 88 (2013) 045010

$$L[A_0] = \frac{1}{N_c} \text{tr} \mathcal{P} e^{i g \int_0^\beta A_0(x)}$$



Polvakov loop Potential: $V_{\text{YM}}[A_0]$

$$\mathcal{P} e^{i g \int_0^\beta A_0(x)} = e^{i \varphi}$$



Confinement

FRG: Braun, Gies, JMP, PLB 684 (2010) 262

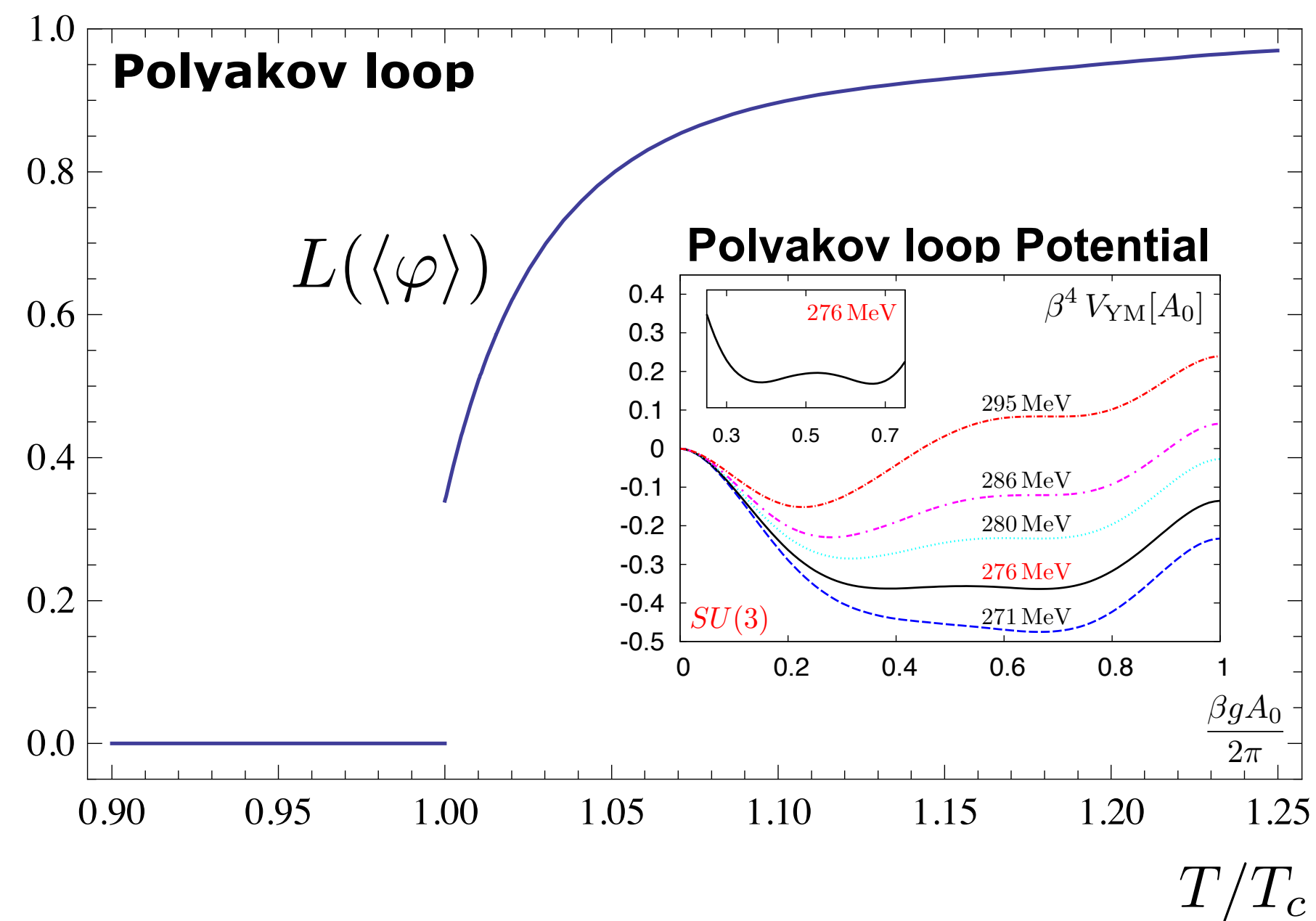
FRG, DSE, 2PI: Fister, JMP, PRD 88 (2013) 045010

$$L[A_0] = \frac{1}{N_c} \text{tr} \mathcal{P} e^{i g \int_0^\beta \mathbf{A}_0(\mathbf{x})}$$

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \left(\text{glue quantum fluctuations} - \text{quark quantum fluctuations} \right) + \frac{1}{2} \text{hadronic quantum fluctuations}$$

Polvakov loop Potential: $V_{\text{YM}}[A_0]$

$$\mathcal{P}e^{i g \int_0^\beta A_0(x)} = e^{i\varphi}$$



$$T_c/\sqrt{\sigma} = 0.658 \pm 0.023$$

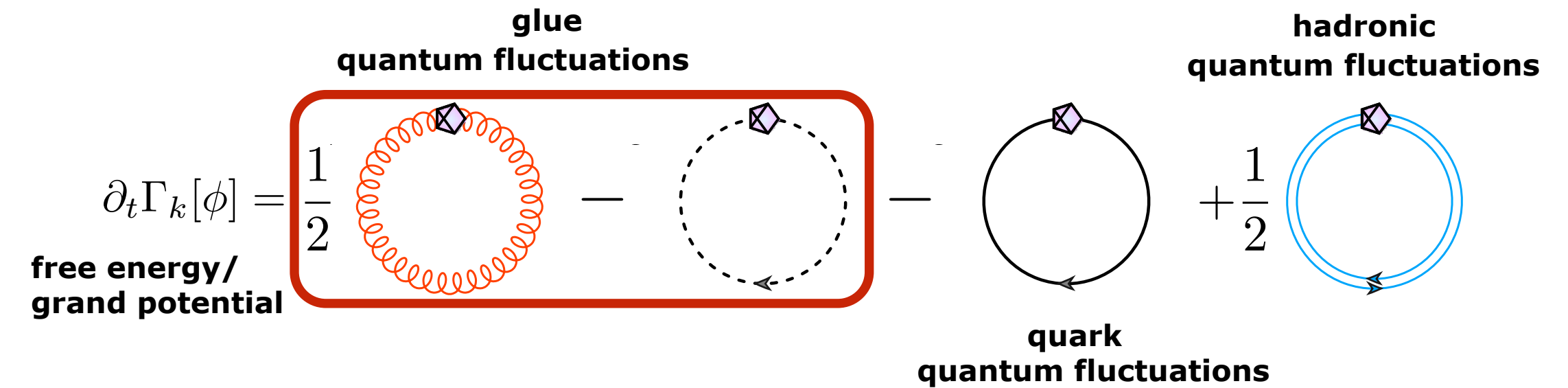
lattice : $T_c/\sqrt{\sigma} = 0.646$

Confinement

FRG: Braun, Gies, JMP, PLB 684 (2010) 262

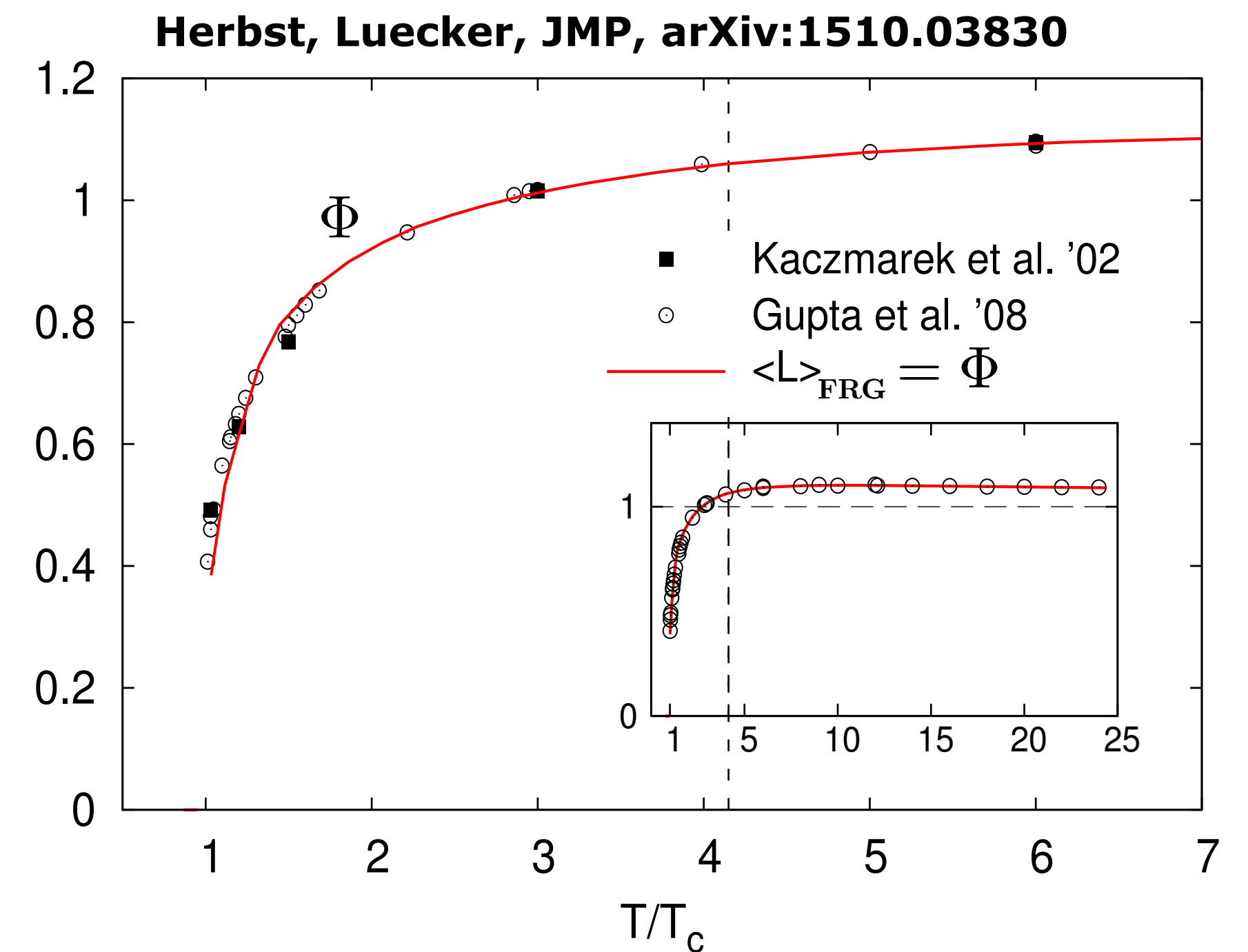
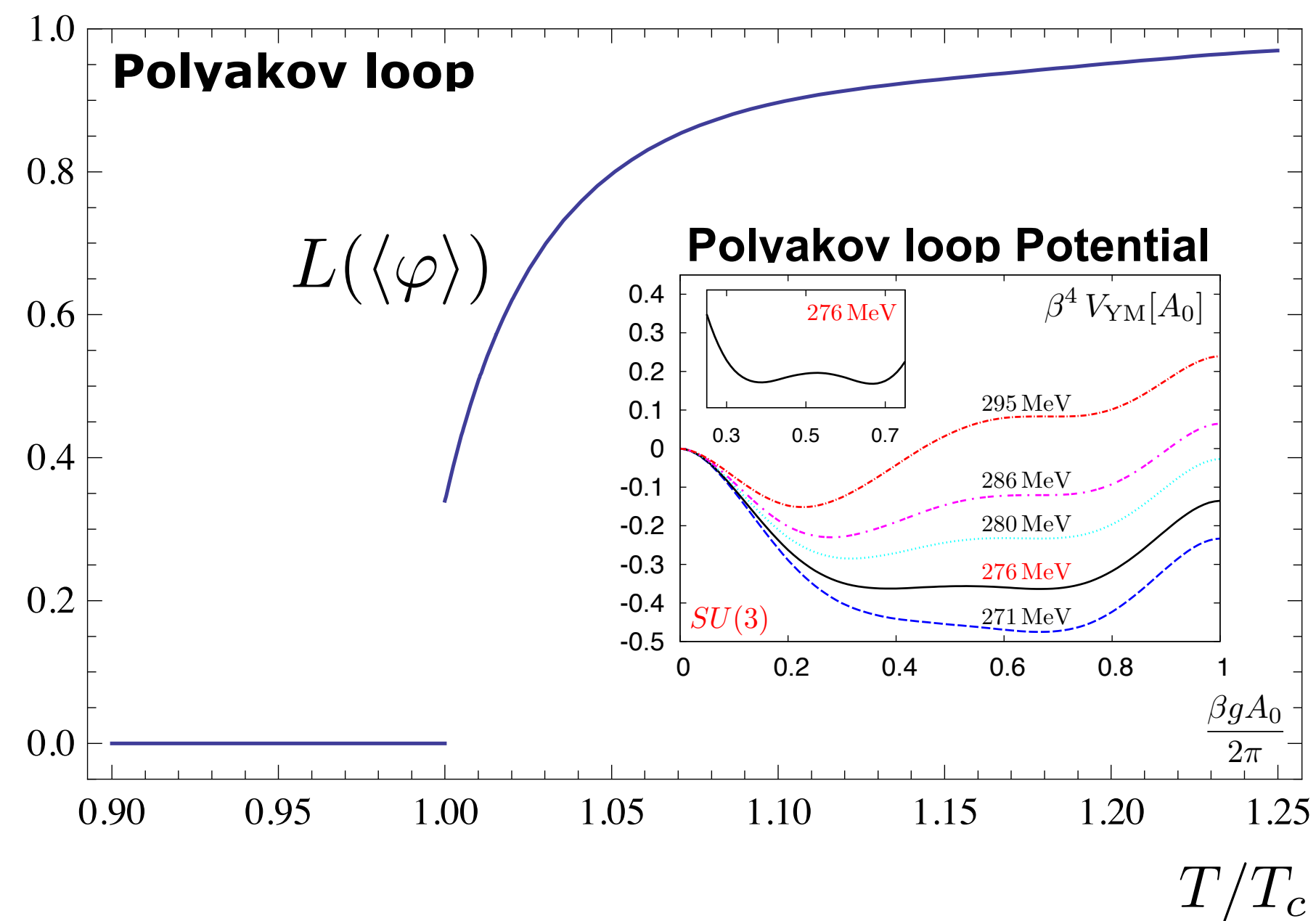
FRG, DSE, 2PI: Fister, JMP, PRD 88 (2013) 045010

$$L[A_0] = \frac{1}{N_c} \text{tr} \mathcal{P} e^{i g \int_0^\beta A_0(x)}$$



Polvakov loop Potential: $V_{\text{YM}}[A_0]$

$$\mathcal{P} e^{i g \int_0^\beta A_0(x)} = e^{i\varphi}$$

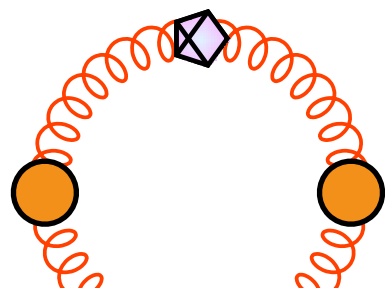


Confinement

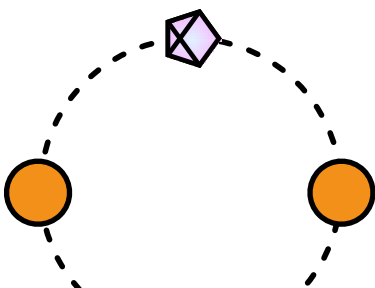
Herbst, Luecker, JMP, arXiv:1510.03830

Flow equation for the Polyakov loop expectation value

$\partial_t \langle L[A_0] \rangle = -\frac{1}{2}$

 $-\frac{\delta^2 \langle L[A_0] \rangle}{\delta A^2}$

$\frac{\delta^2 \langle L[A_0] \rangle}{\delta c \delta \bar{c}}$



quation for composite operators

JMP, AP 322 (2007) 2831

Igarashi, Itoh, Sonoda, PTP Suppl. 181 (2010) 1

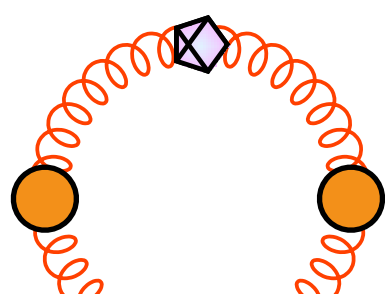
Pagani, PRD 94 (2016) 045001

Confinement

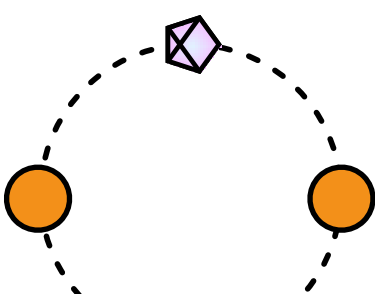
Herbst, Luecker, JMP, arXiv:1510.03830

Flow equation for the Polyakov loop expectation value

$$\partial_t \langle L[A_0] \rangle = -\frac{1}{2}$$


$$\frac{\delta^2 \langle L[A_0] \rangle}{\delta A^2}$$

$$-$$


$$\frac{\delta^2 \langle L[A_0] \rangle}{\delta c \delta \bar{c}}$$

quation for composite operators

JMP, AP 322 (2007) 2831

Igarashi, Itoh, Sonoda, PTP Suppl. 181 (2010) 1

Pagani, PRD 94 (2016) 045001

Parameterisation

$$\langle L[A_0] \rangle = Z_L[\bar{A}, \phi] \cdot L[A_0]$$

with $\phi = (a_\mu, c, \bar{c})$

Confinement

Herbst, Luecker, JMP, arXiv:1510.03830

Flow equation for the Polyakov loop expectation value

$$\partial_t \langle L[A_0] \rangle = -\frac{1}{2} \underbrace{\text{Diagram 1}}_{\frac{\delta^2 \langle L[A_0] \rangle}{\delta A^2}} - \underbrace{\text{Diagram 2}}_{\frac{\delta^2 \langle L[A_0] \rangle}{\delta c \delta \bar{c}}}$$

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Flow for Polyakov loop wave function

$$\partial_t Z_L[\bar{A}, \phi] = \text{Flow}_{Z_L}[\bar{A}; Z_L, G_A, G_c, L[A_0]]$$

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Pagani, PRD 94 (2016) 045001

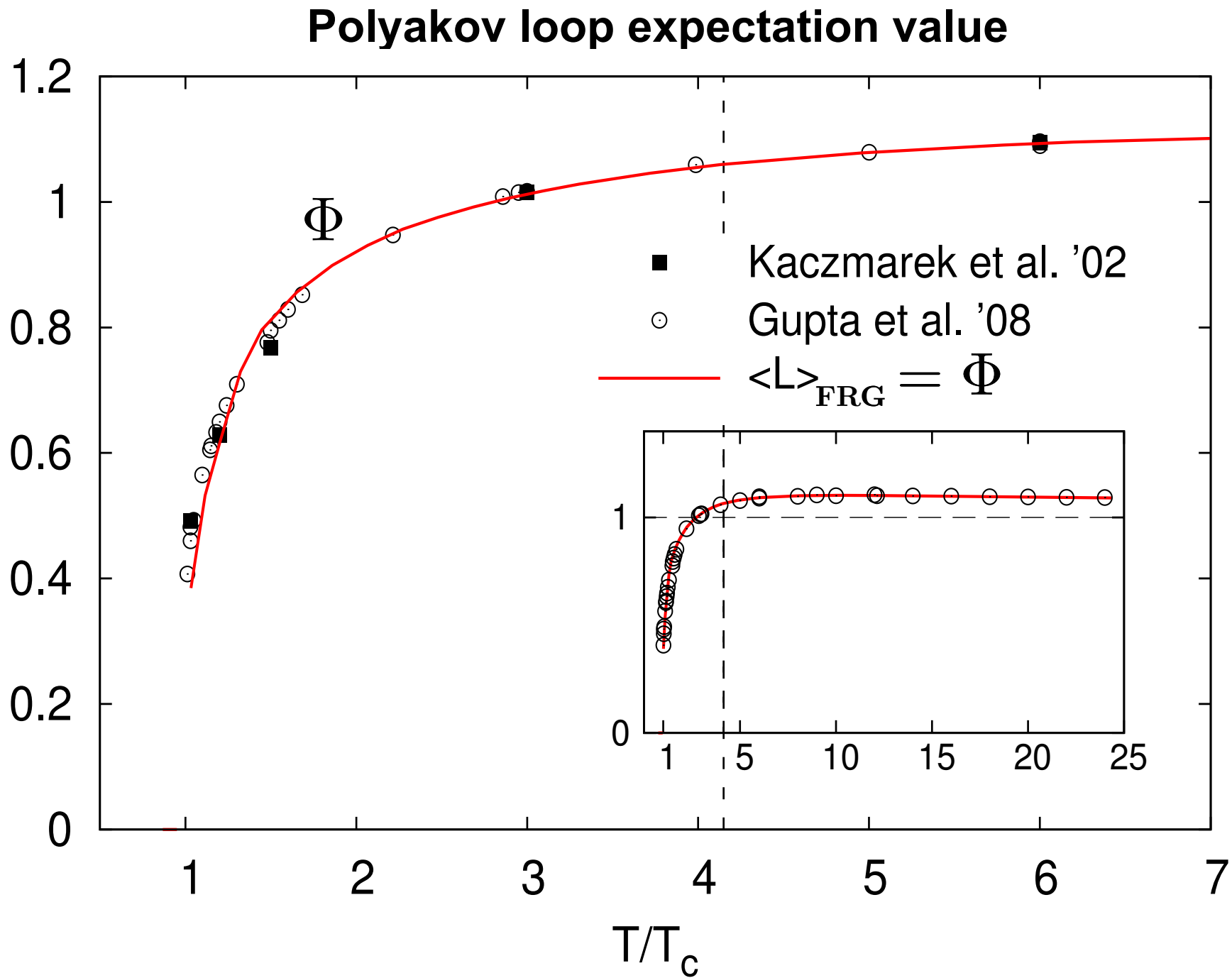
Parameterisation

$$\langle L[A_0] \rangle = Z_L[\bar{A}, \phi] \cdot L[A_0]$$

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Flow for Polyakov loop wave function

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QCD phase structure

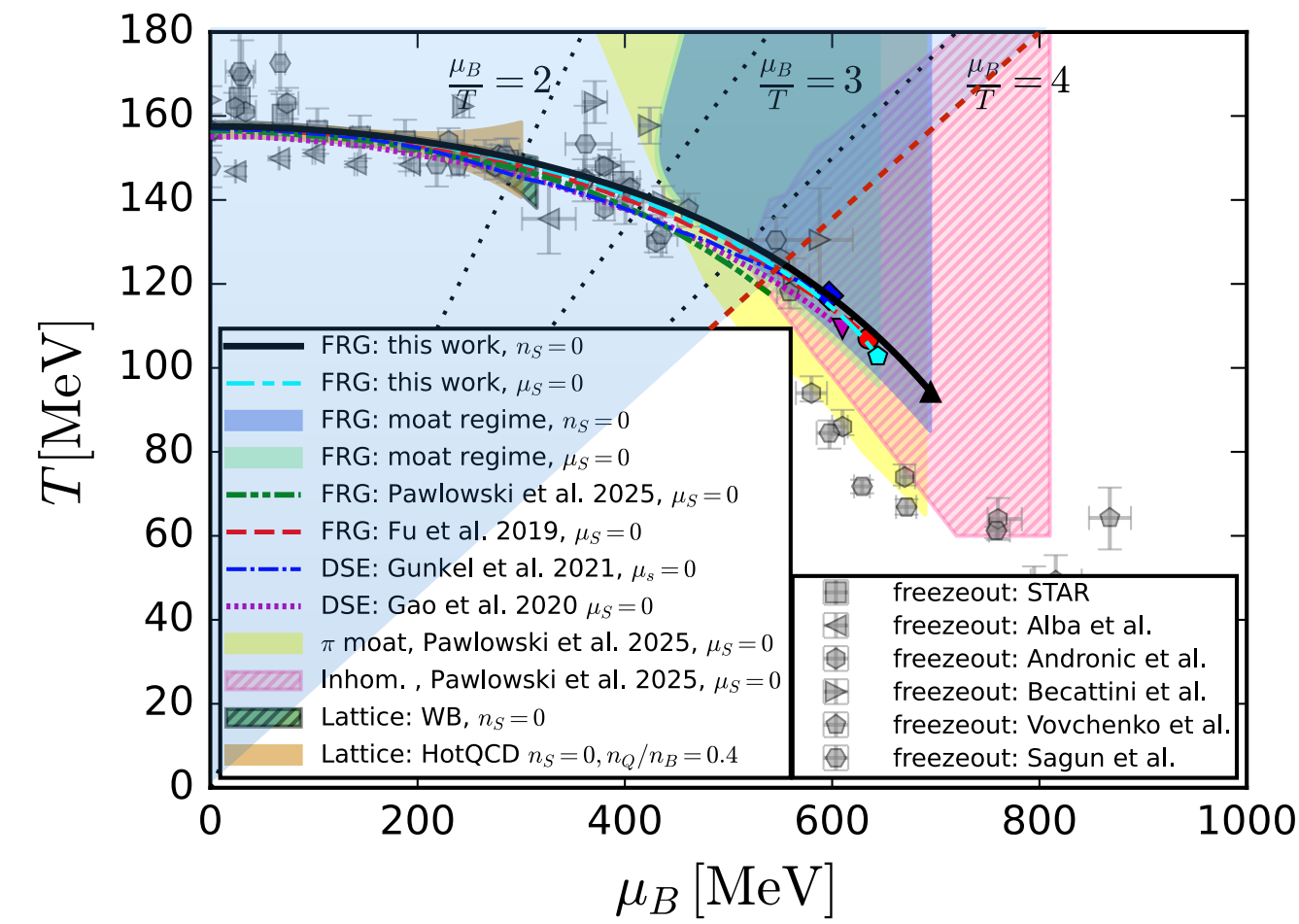
Jump back to: QCD at finite temperature and density

Phase structure from functional QCD: Predictions & estimates

aka

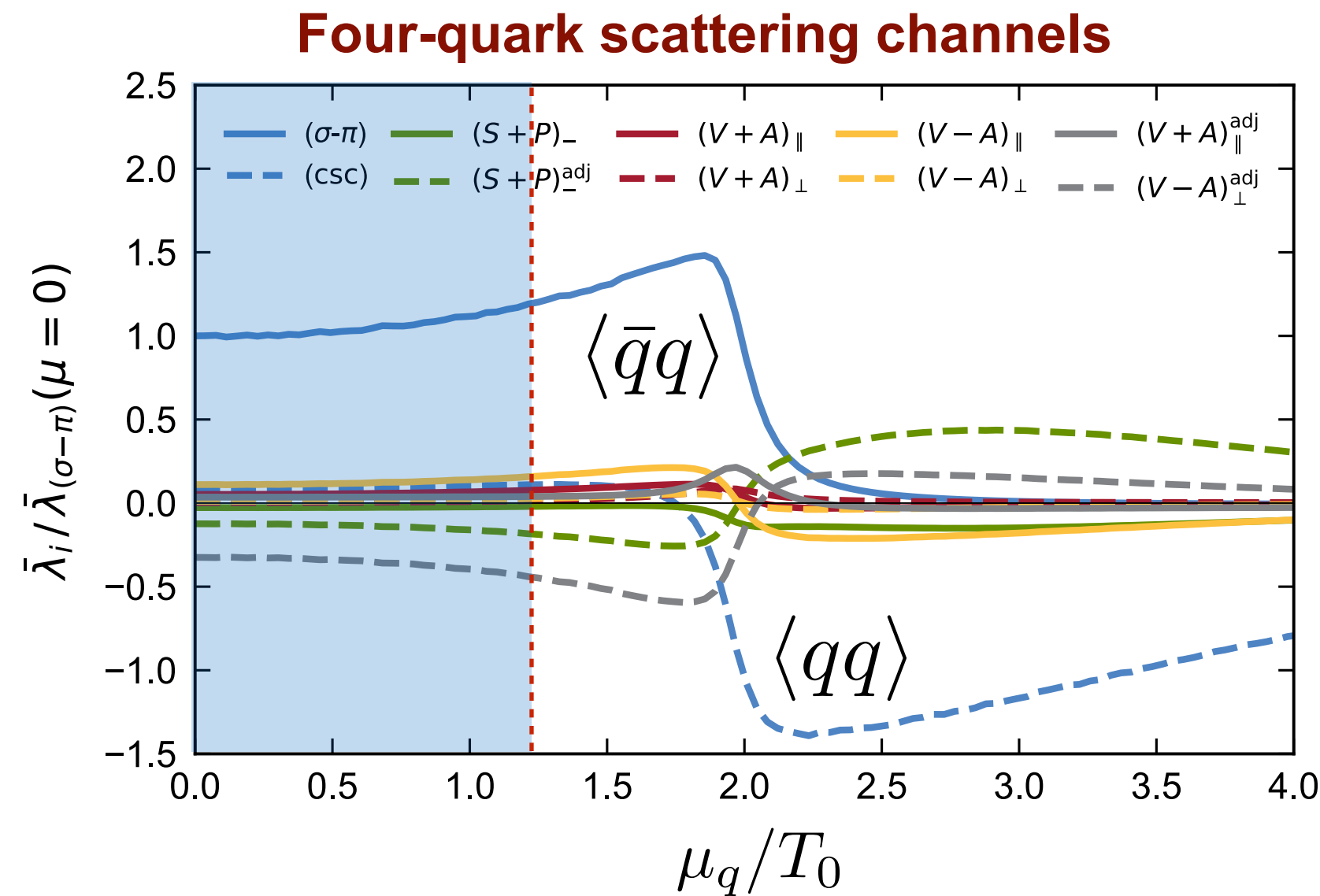
Locating the critical end point

Predictions & estimates



$$\frac{\mu_B}{T} \lesssim 4.5$$

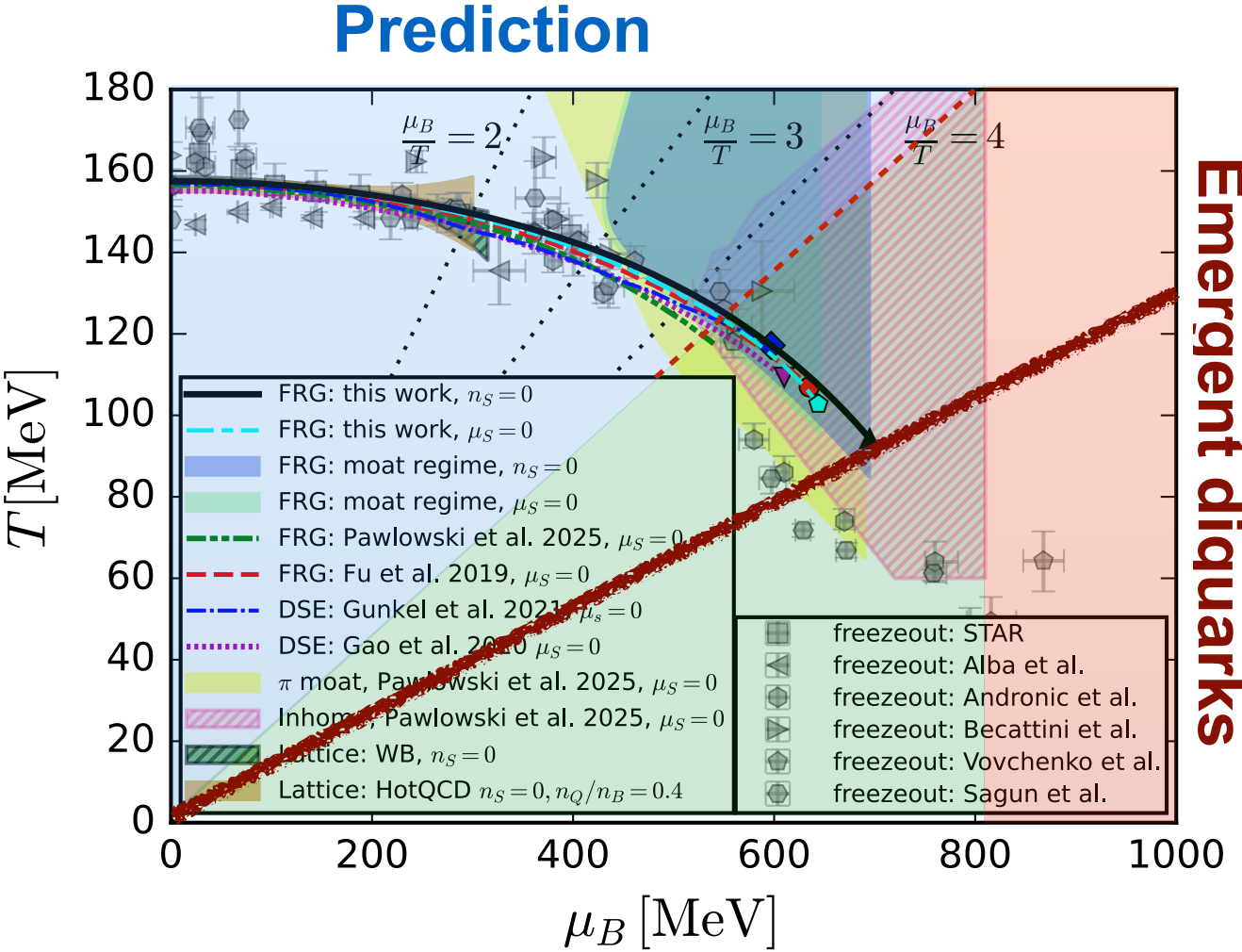
Regime of quantitative reliability
of
current best truncation



Dominance of scalar-pseudoscalar fluctuations
Pions & sigma mode

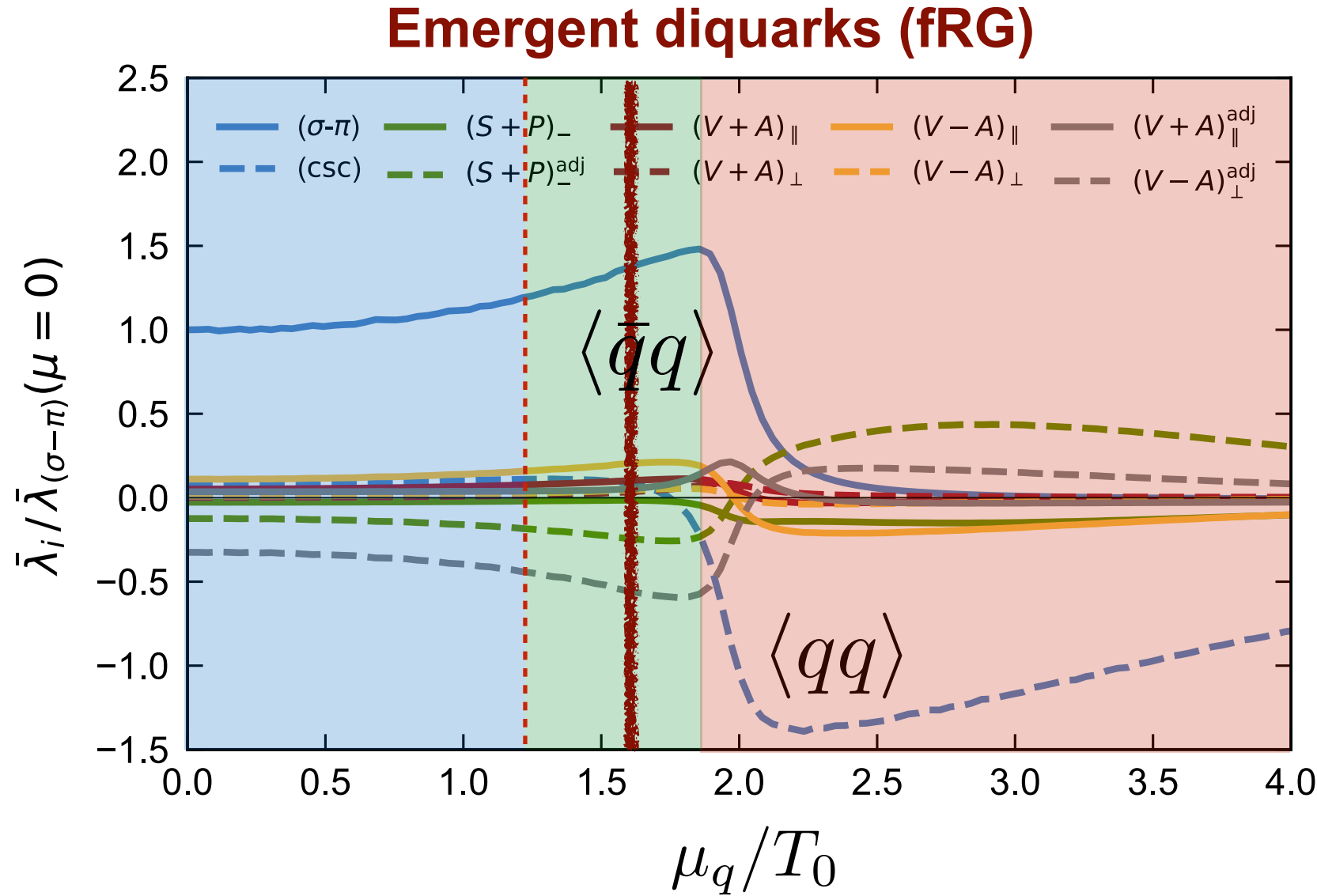
Predictions & estimates

Emergent diquarks



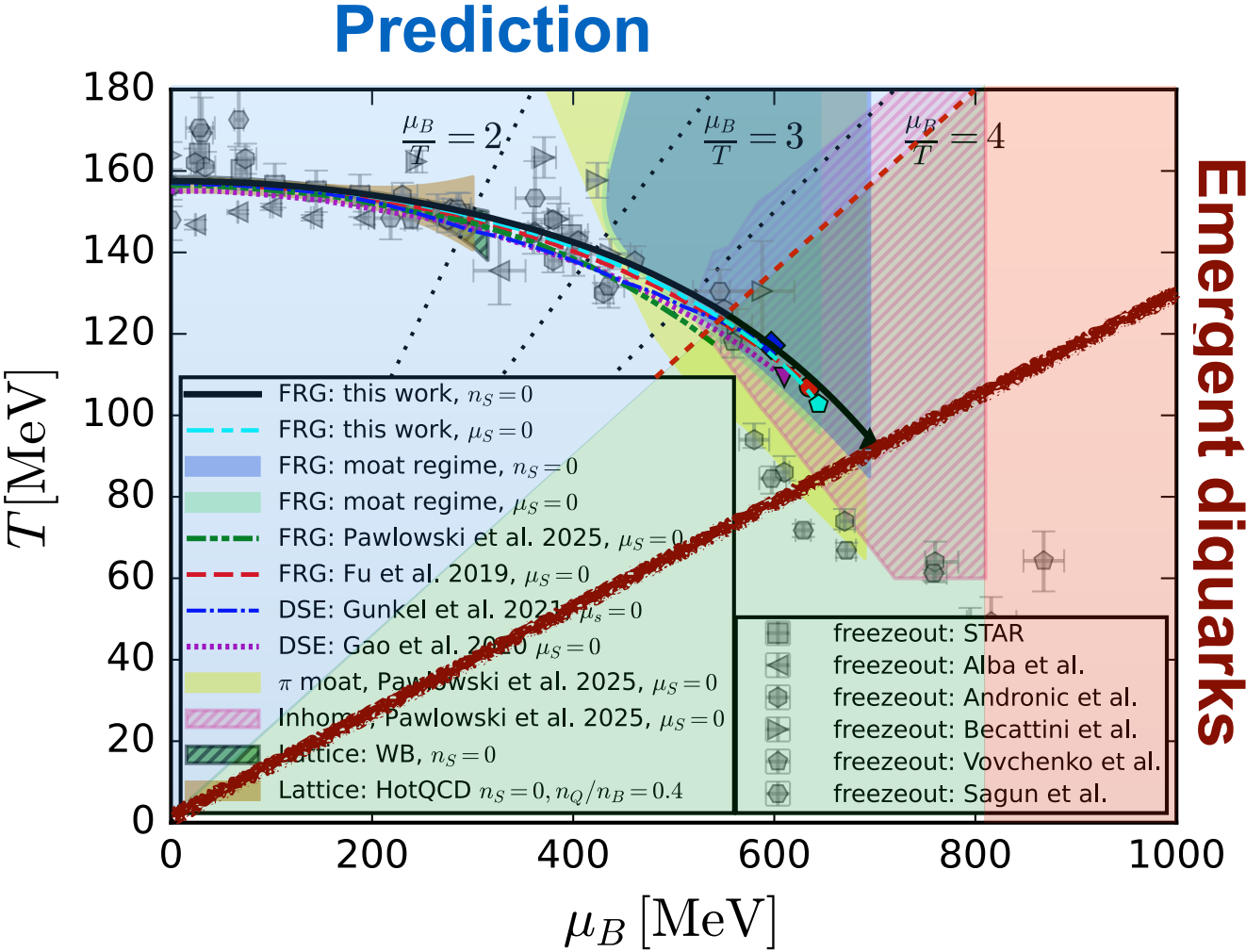
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of
current best truncation

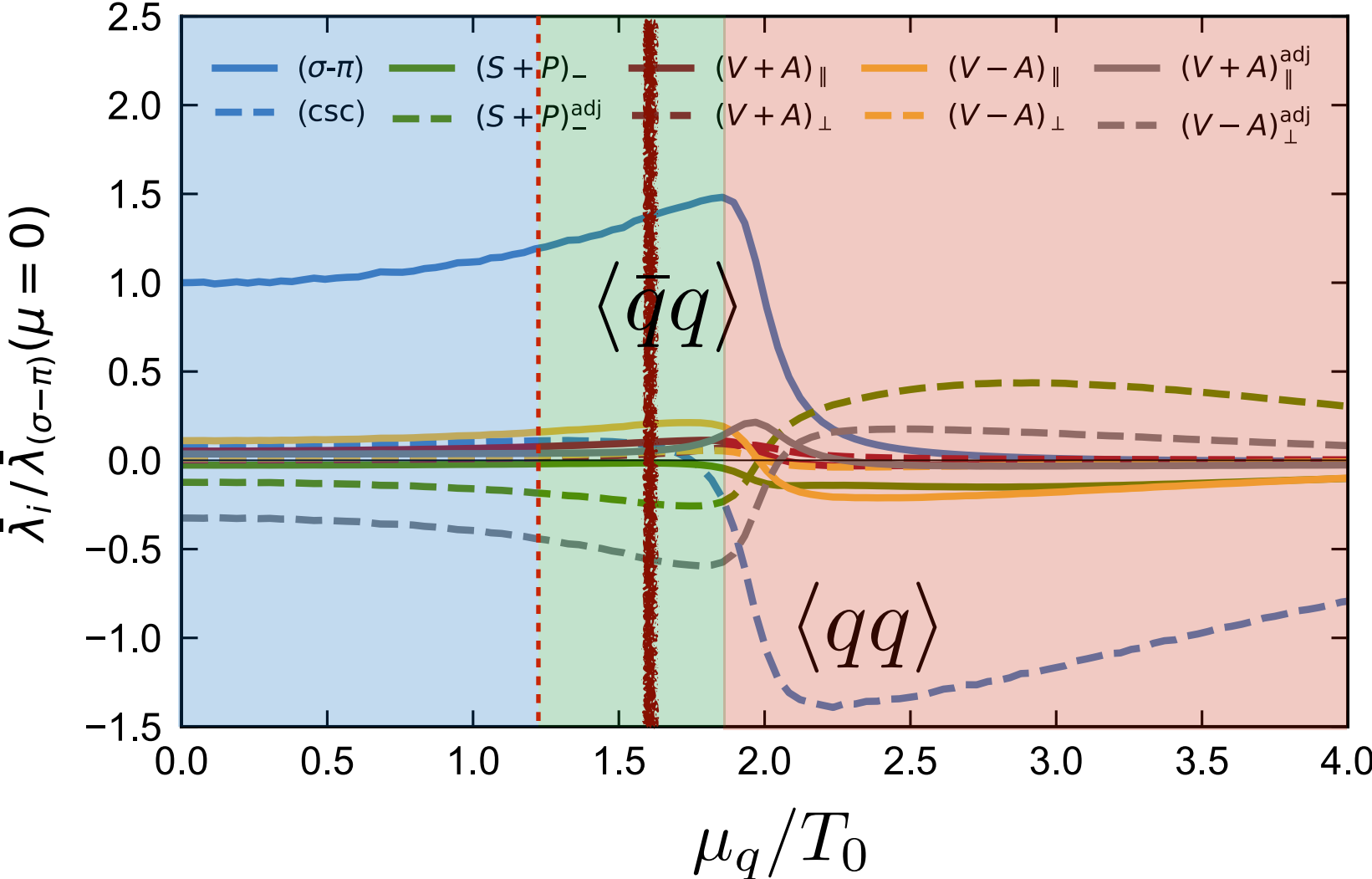


Predictions & estimates

Emergent diquarks



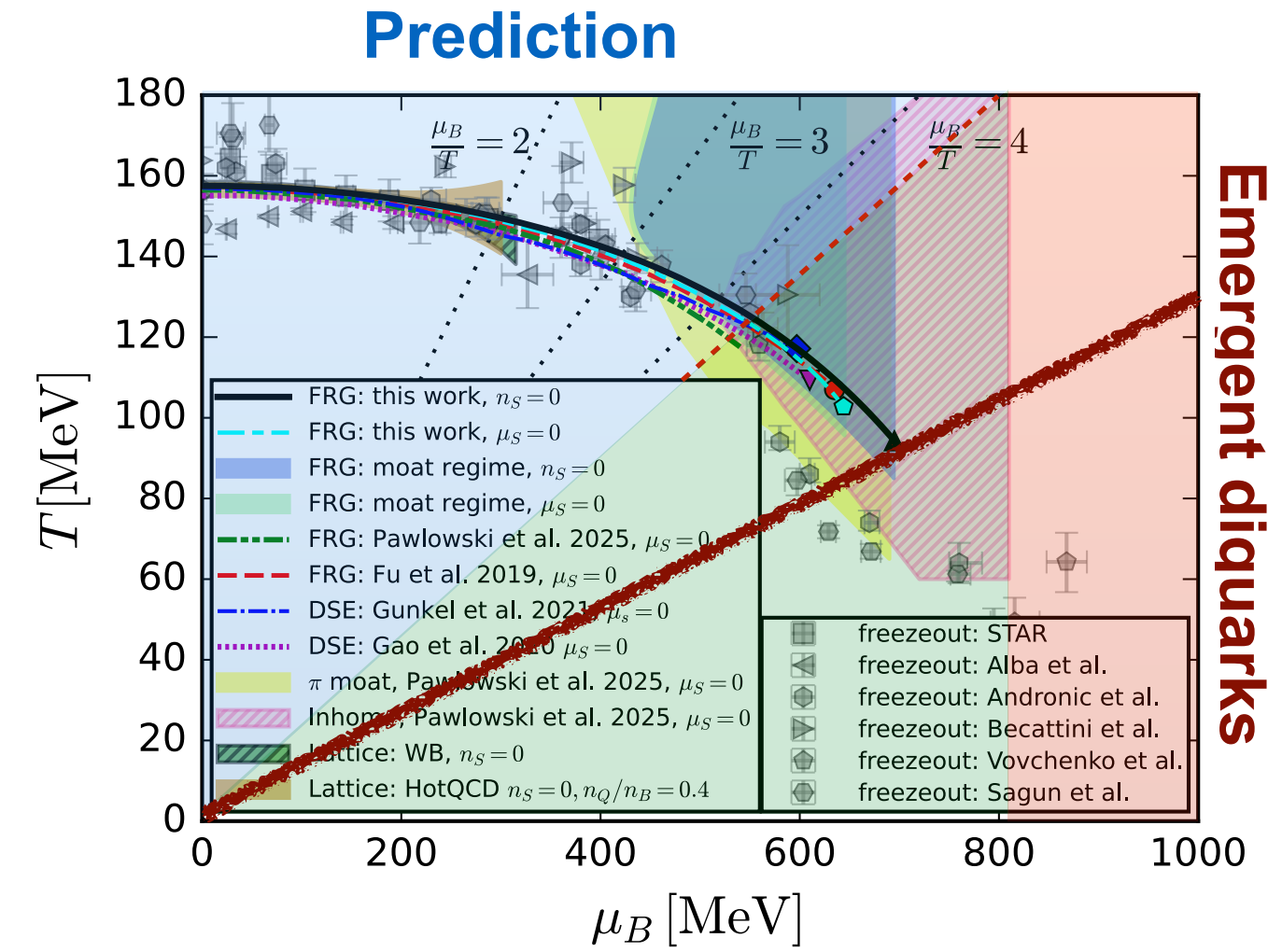
Emergent diquarks (fRG)



Emergent diquarks are not captured by extrapolations

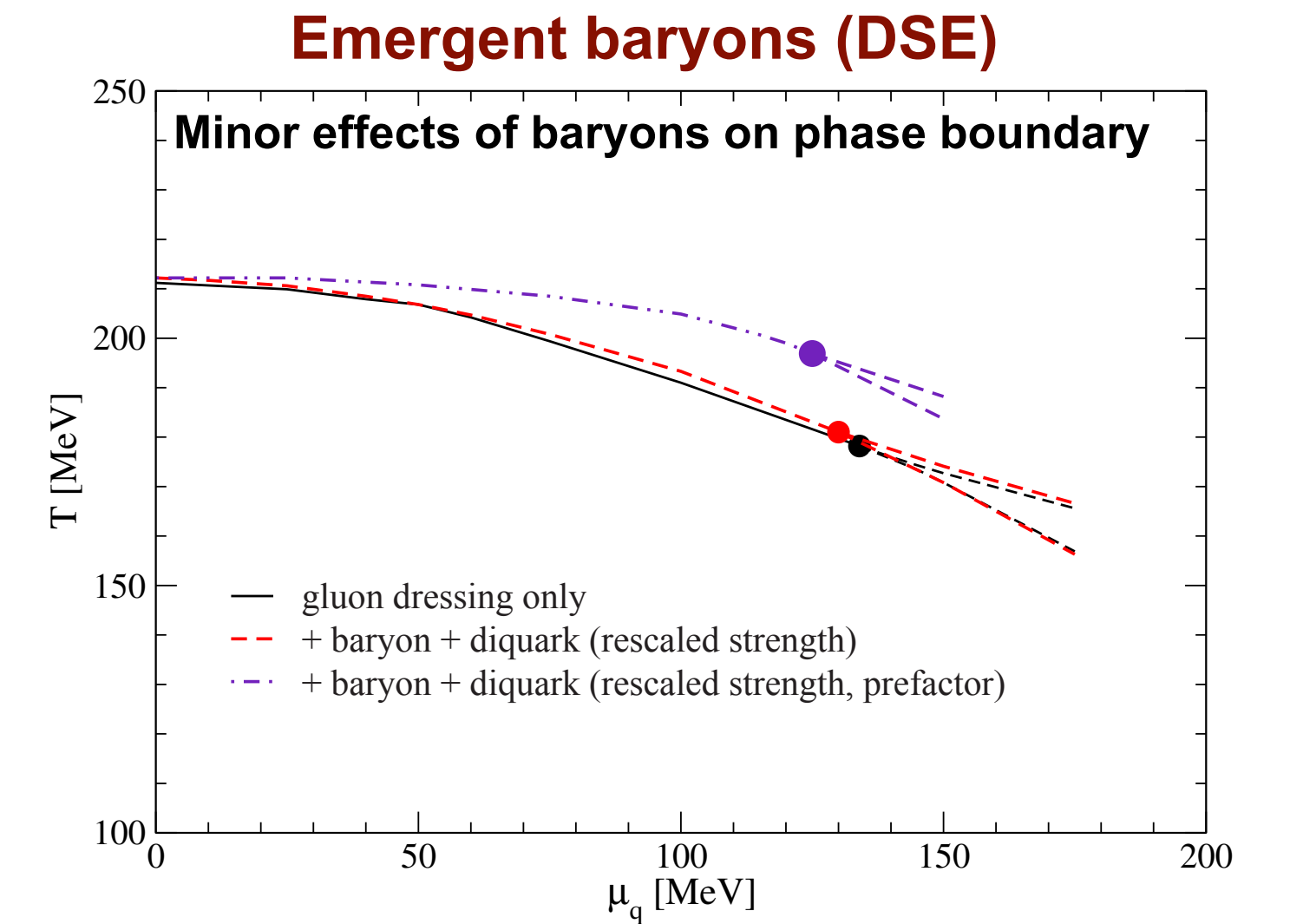
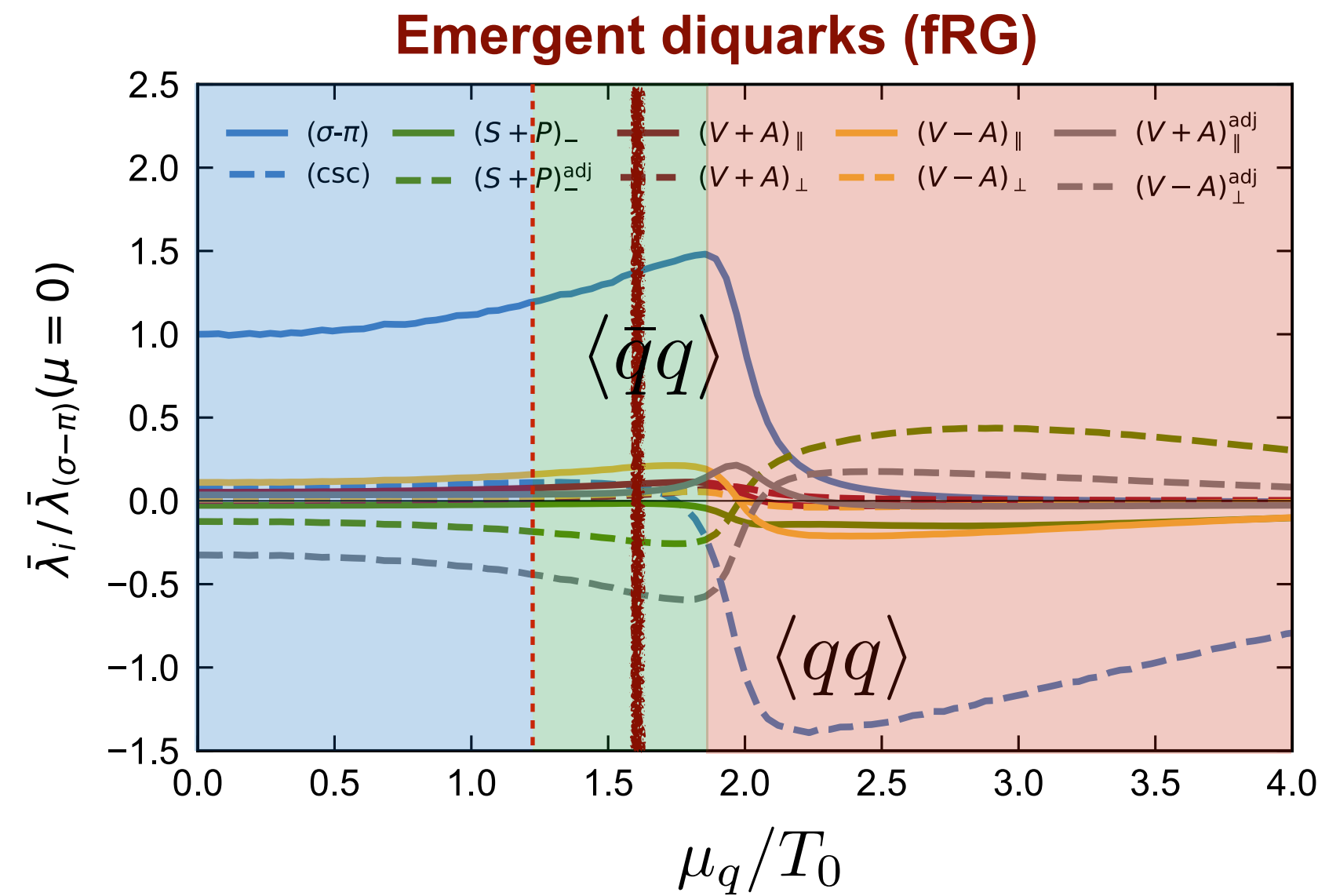
Predictions & estimates

Emergent diquarks



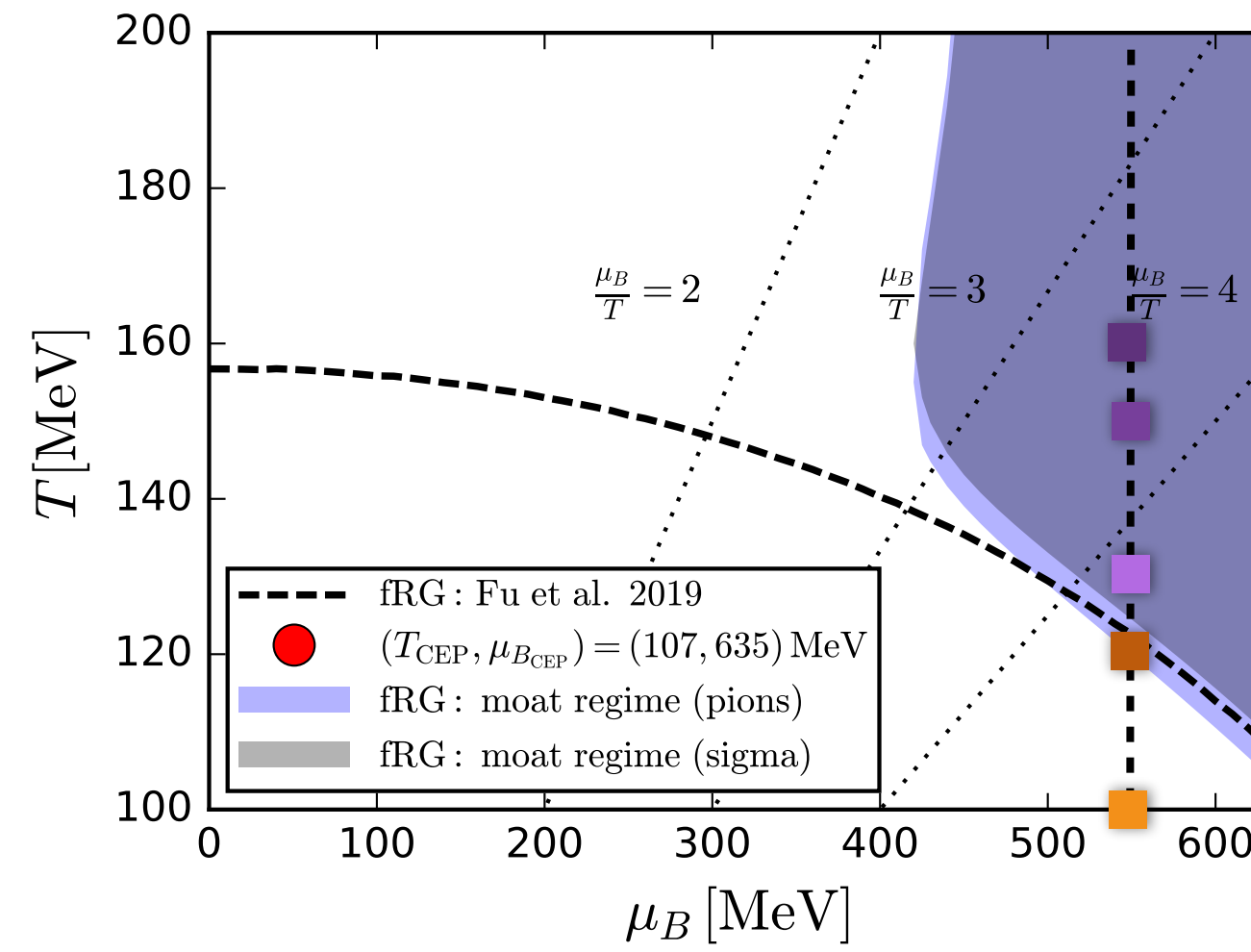
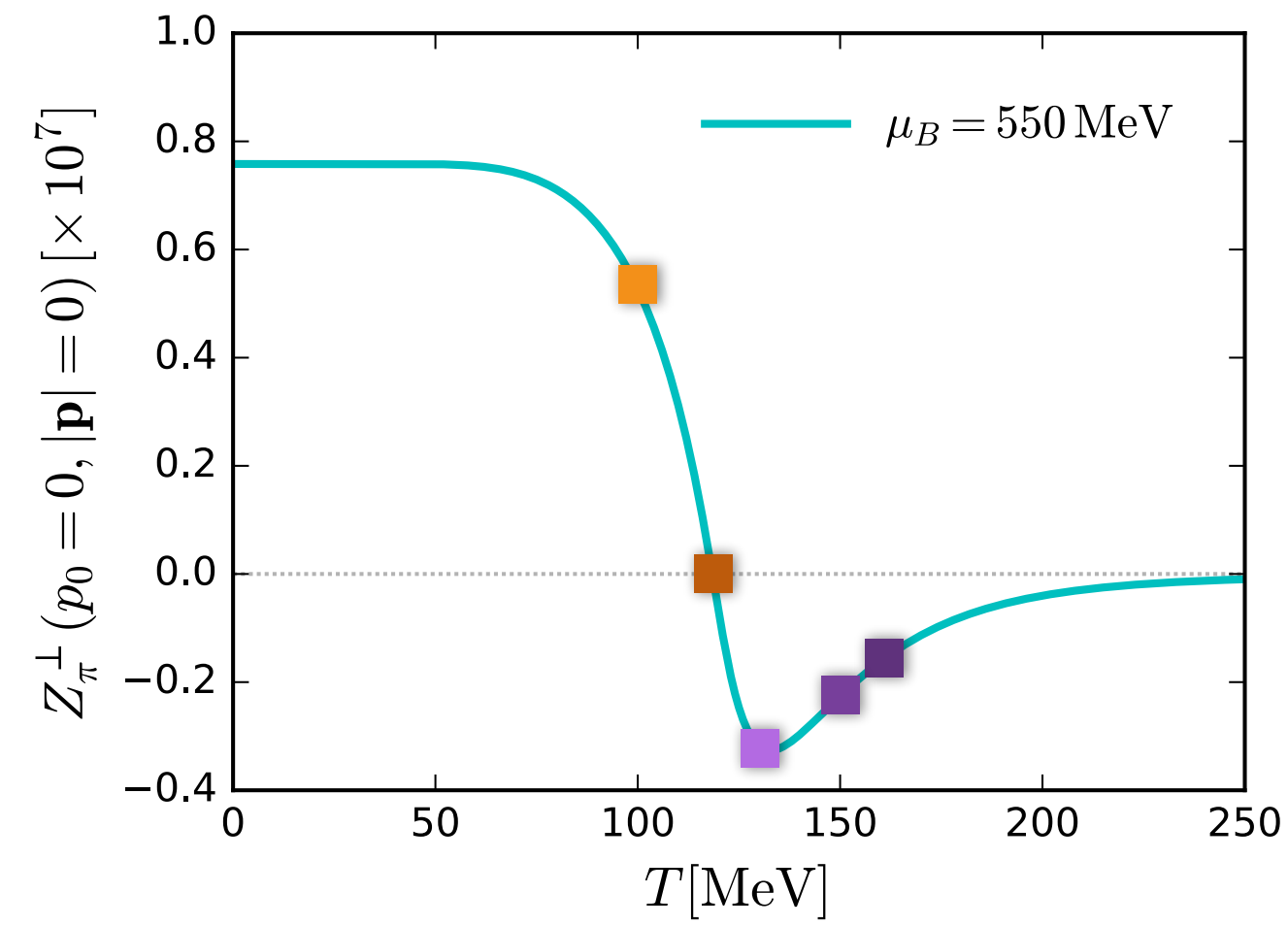
$\frac{\mu_B}{T} \lesssim 4.5$

Regime of quantitative reliability
of
current best truncation

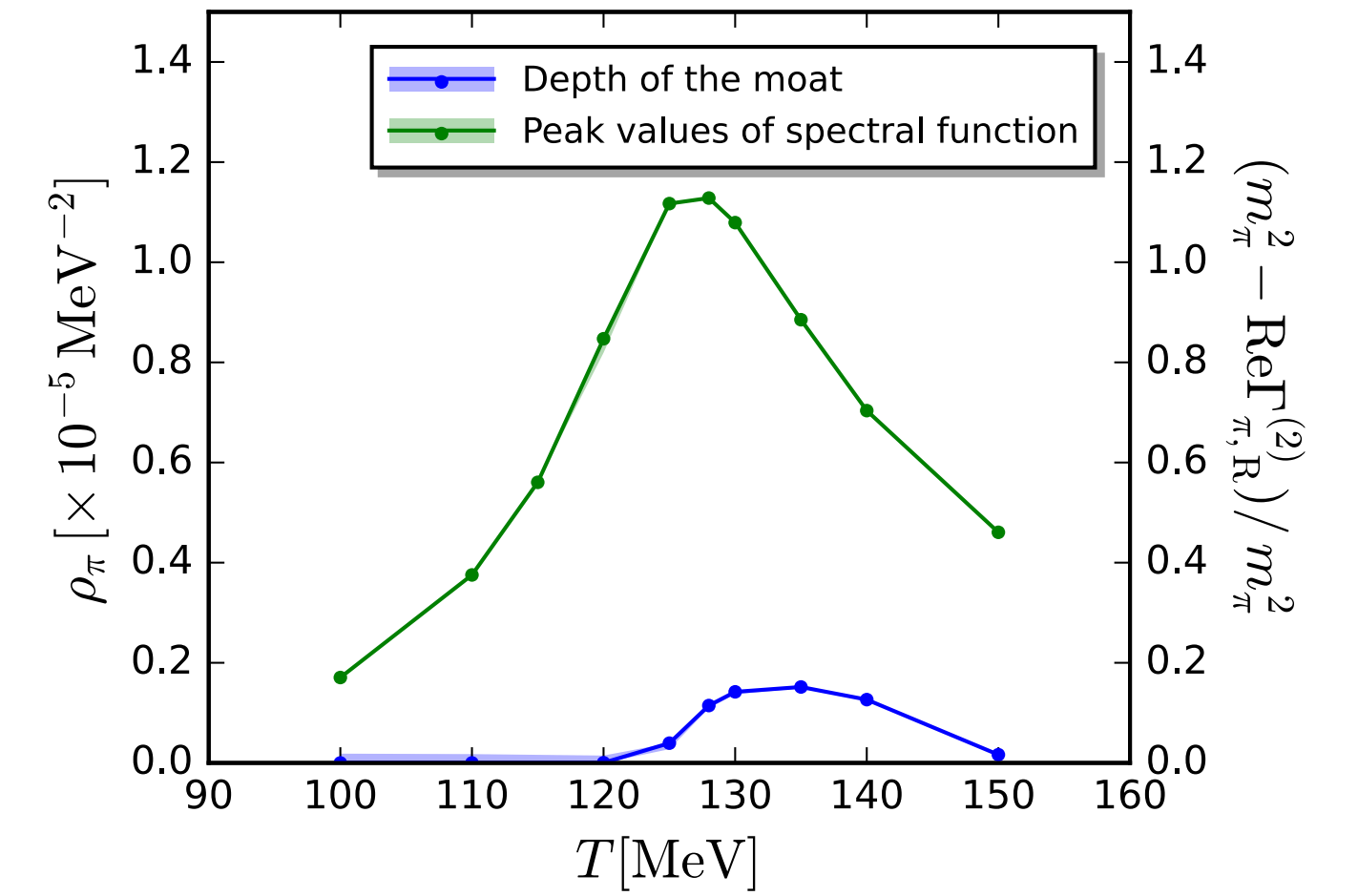


Predictions & estimates

Spatial wave function of the pion at p=0



Depth of the moat & spectral peak of the pion



Dissecting the moat & the moaton

$\mu_B = 650$ MeV

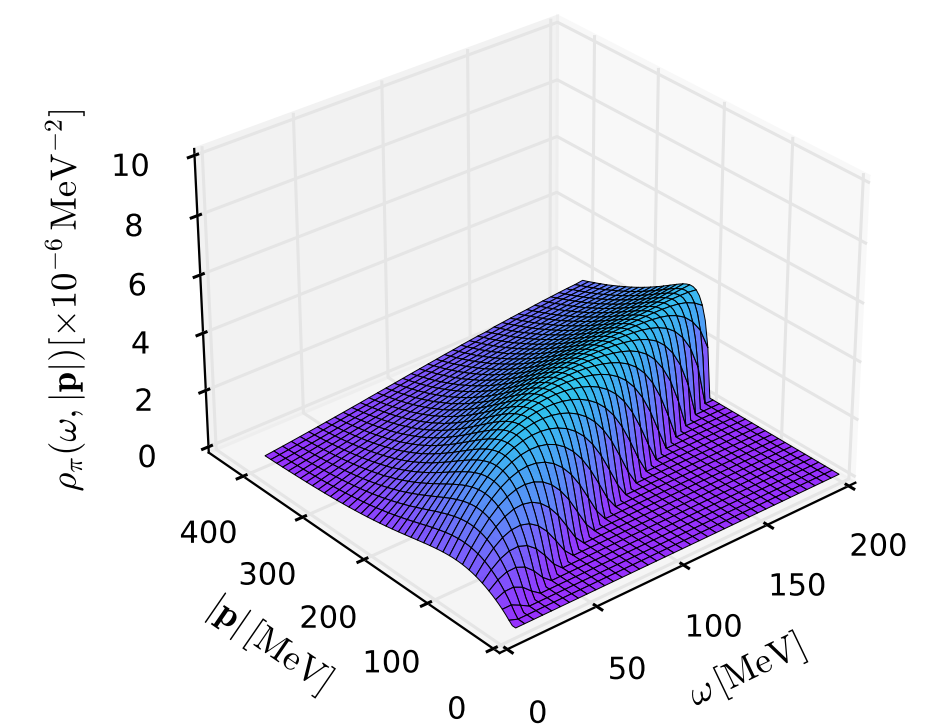
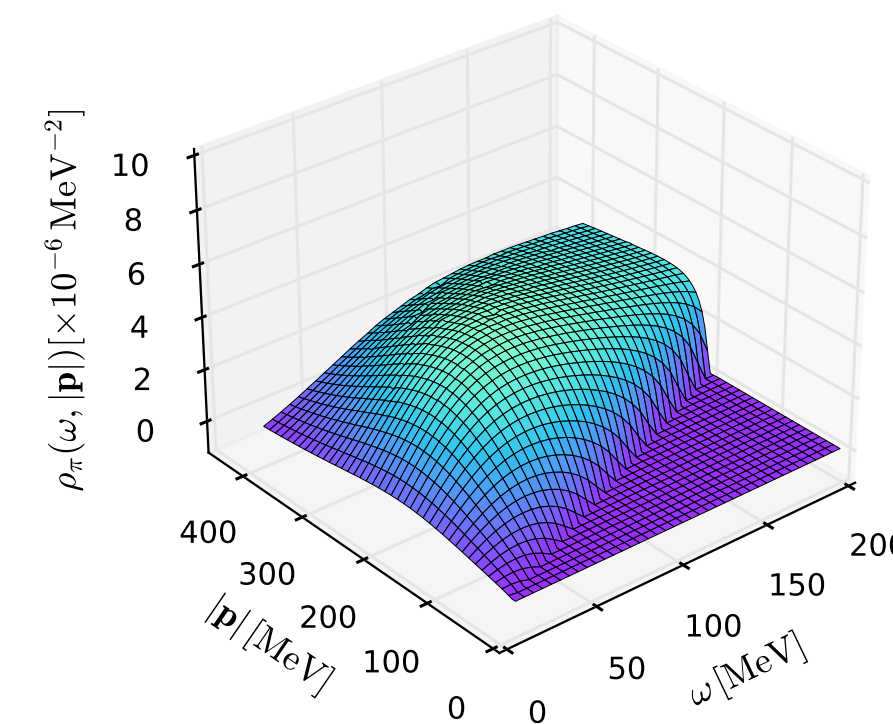
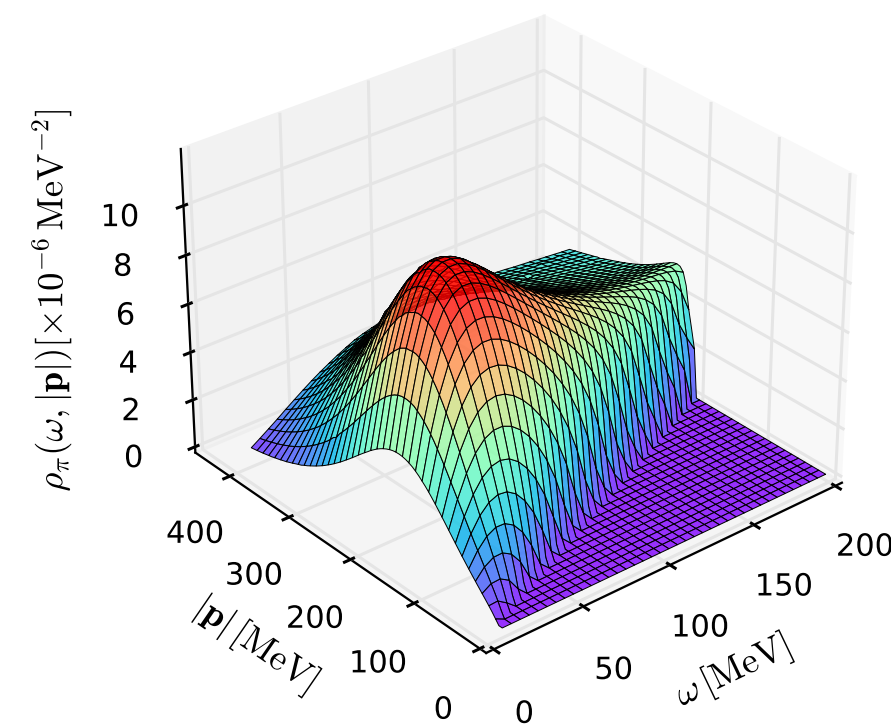
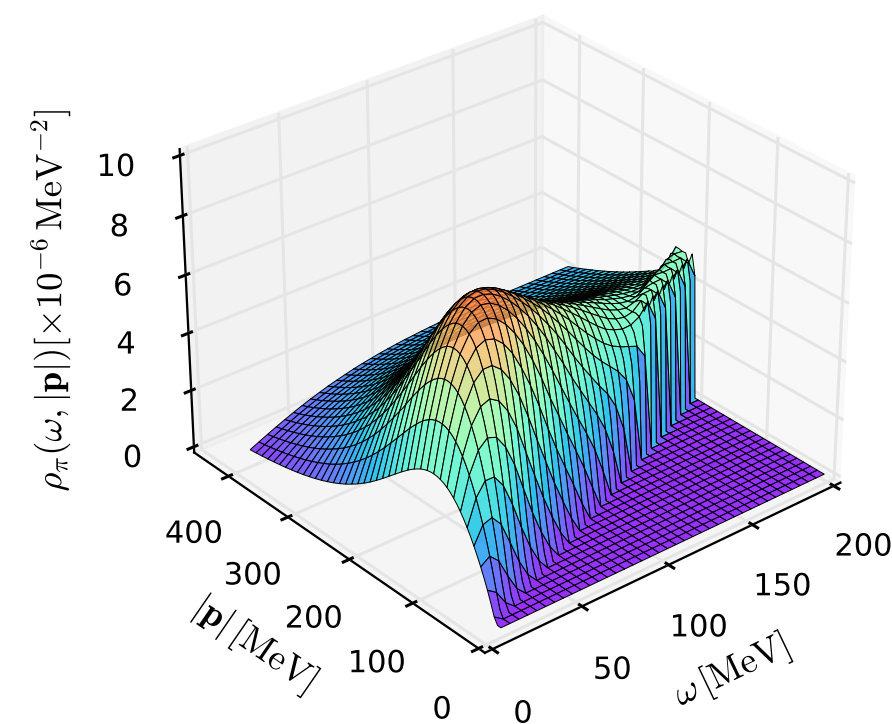
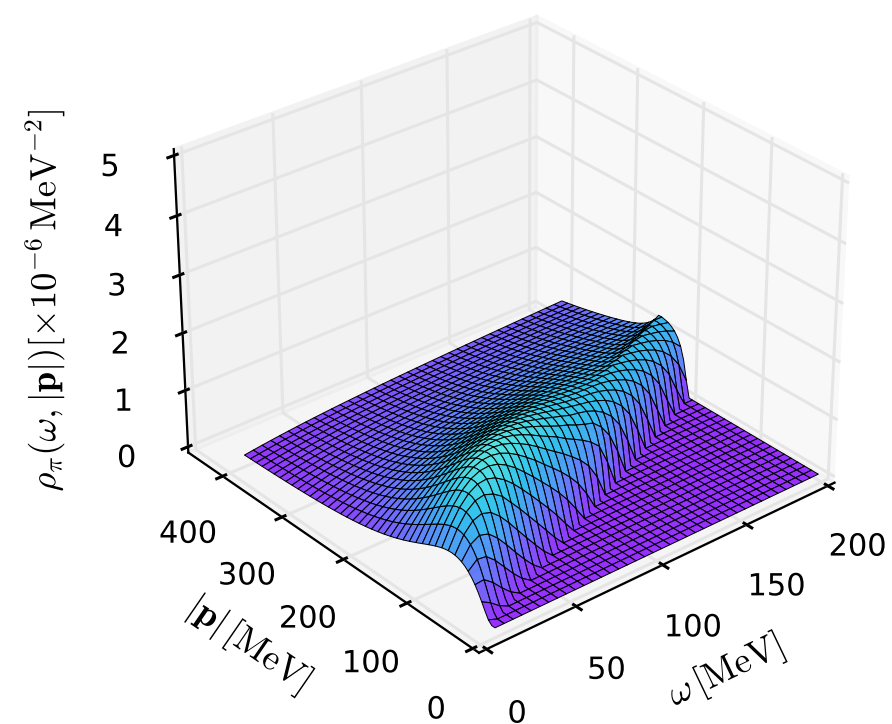
■ T=100 MeV

■ T=120 MeV

■ T=130 MeV

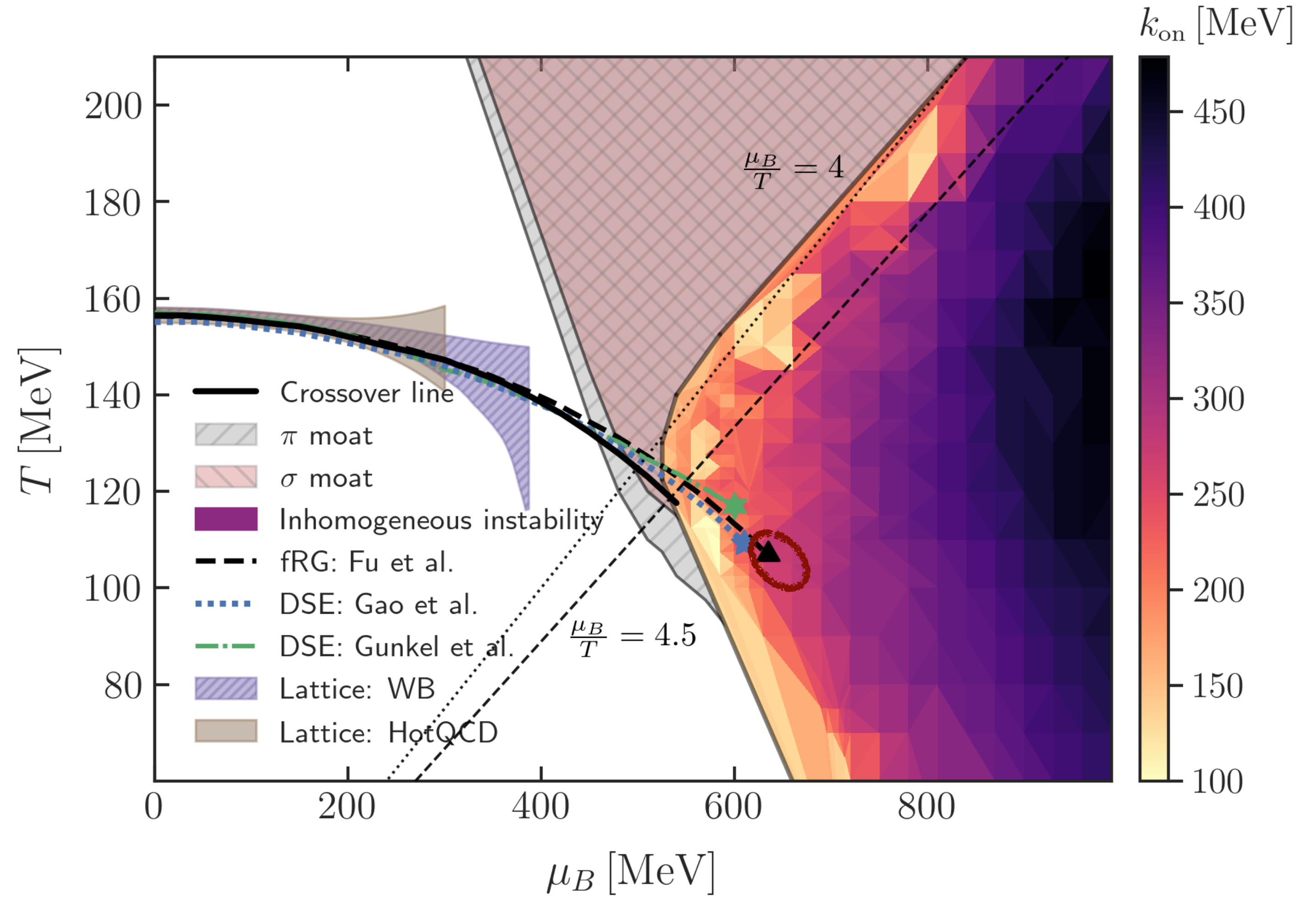
■ T=150 MeV

■ T=160 MeV



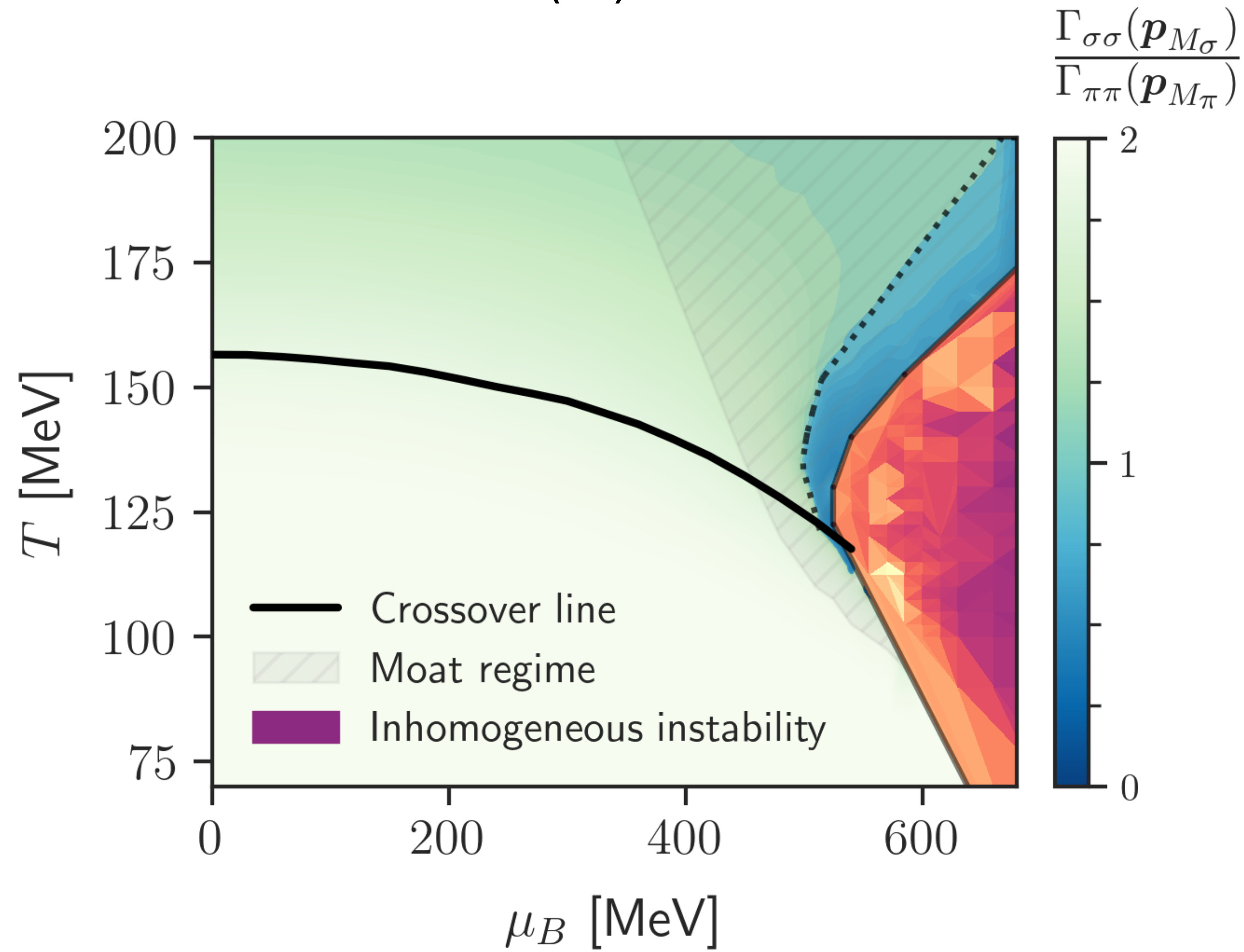
The QCD Moat reloaded

 **Estimated location of CEP**



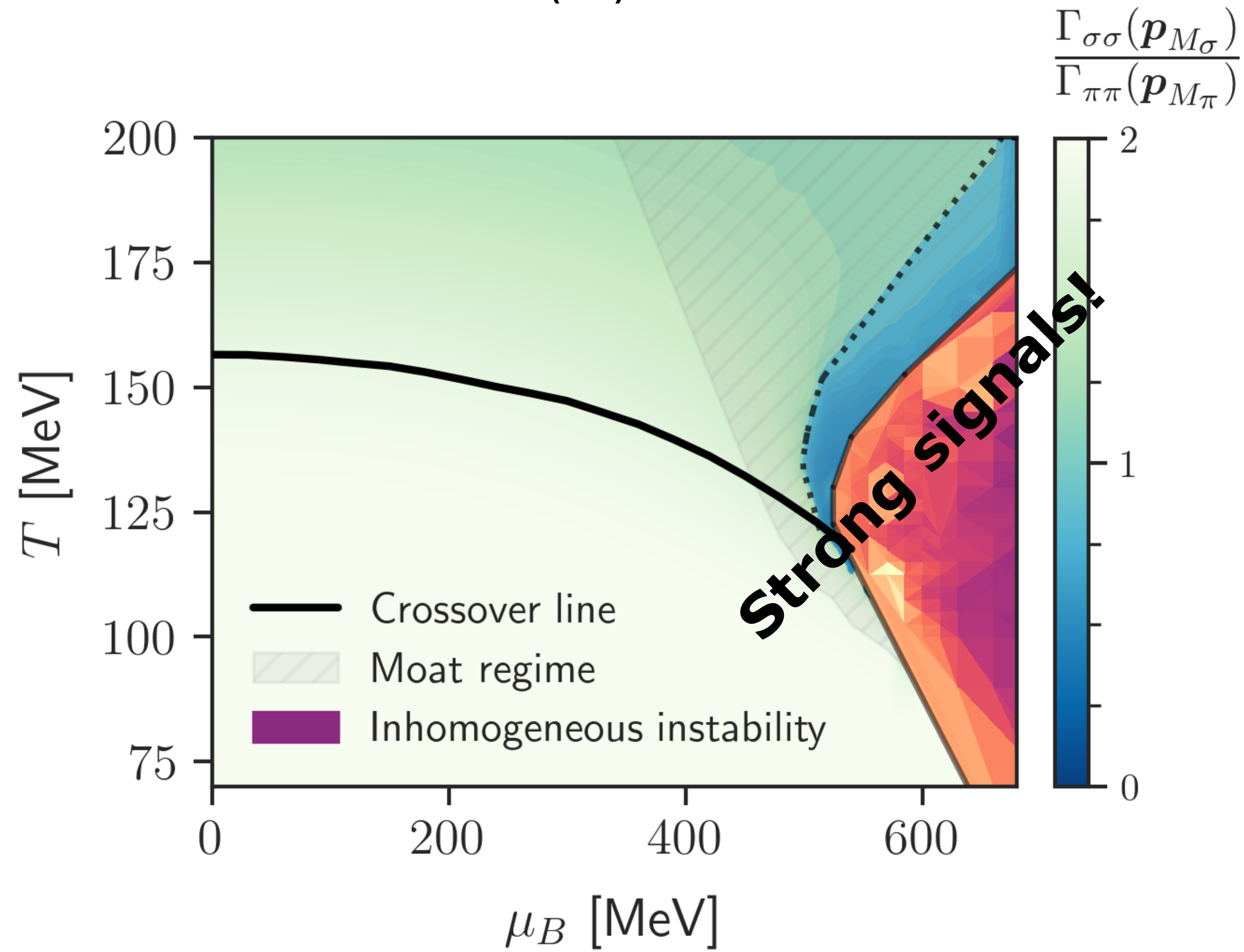
The QCD Moat

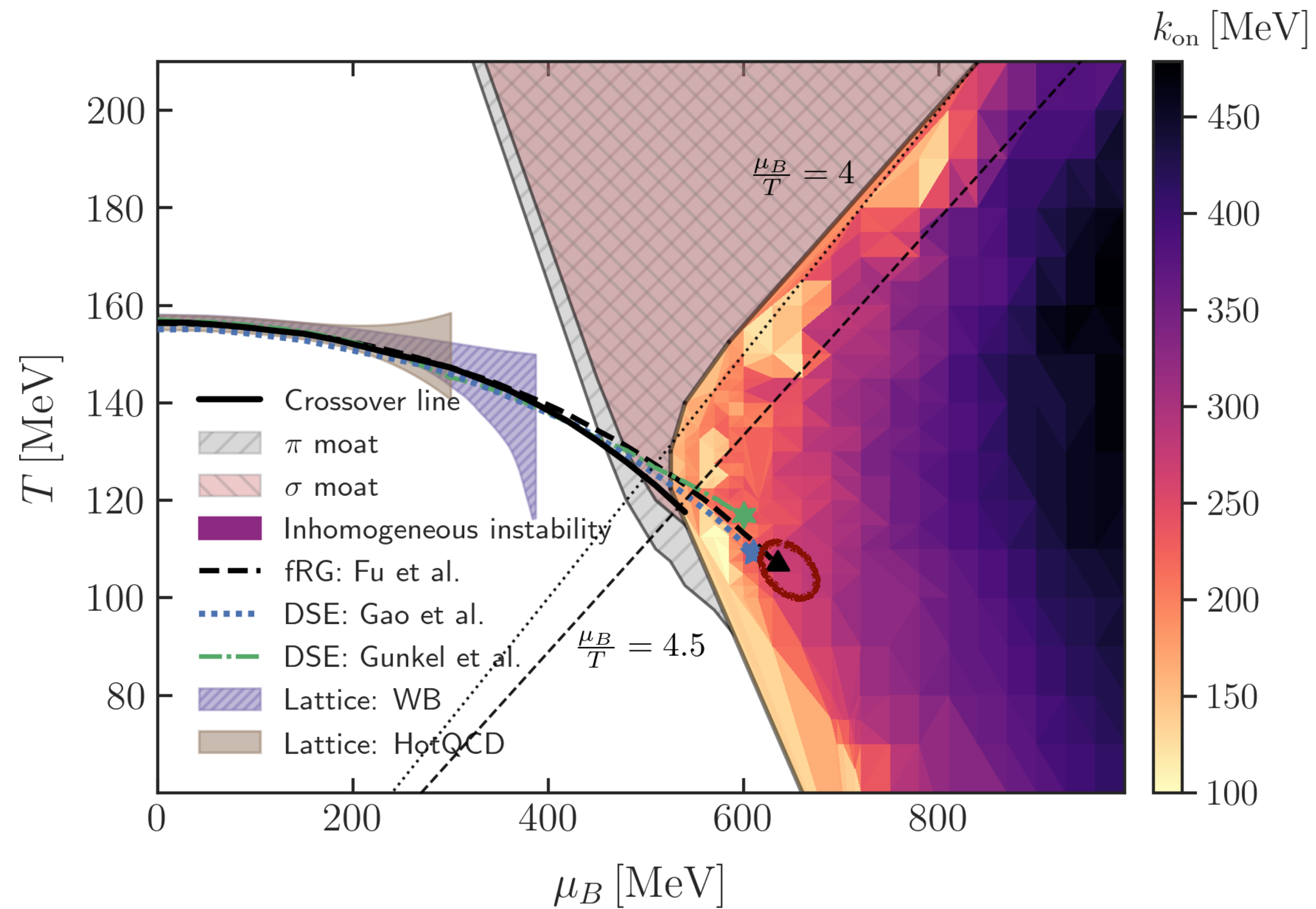
Soft(est) mode



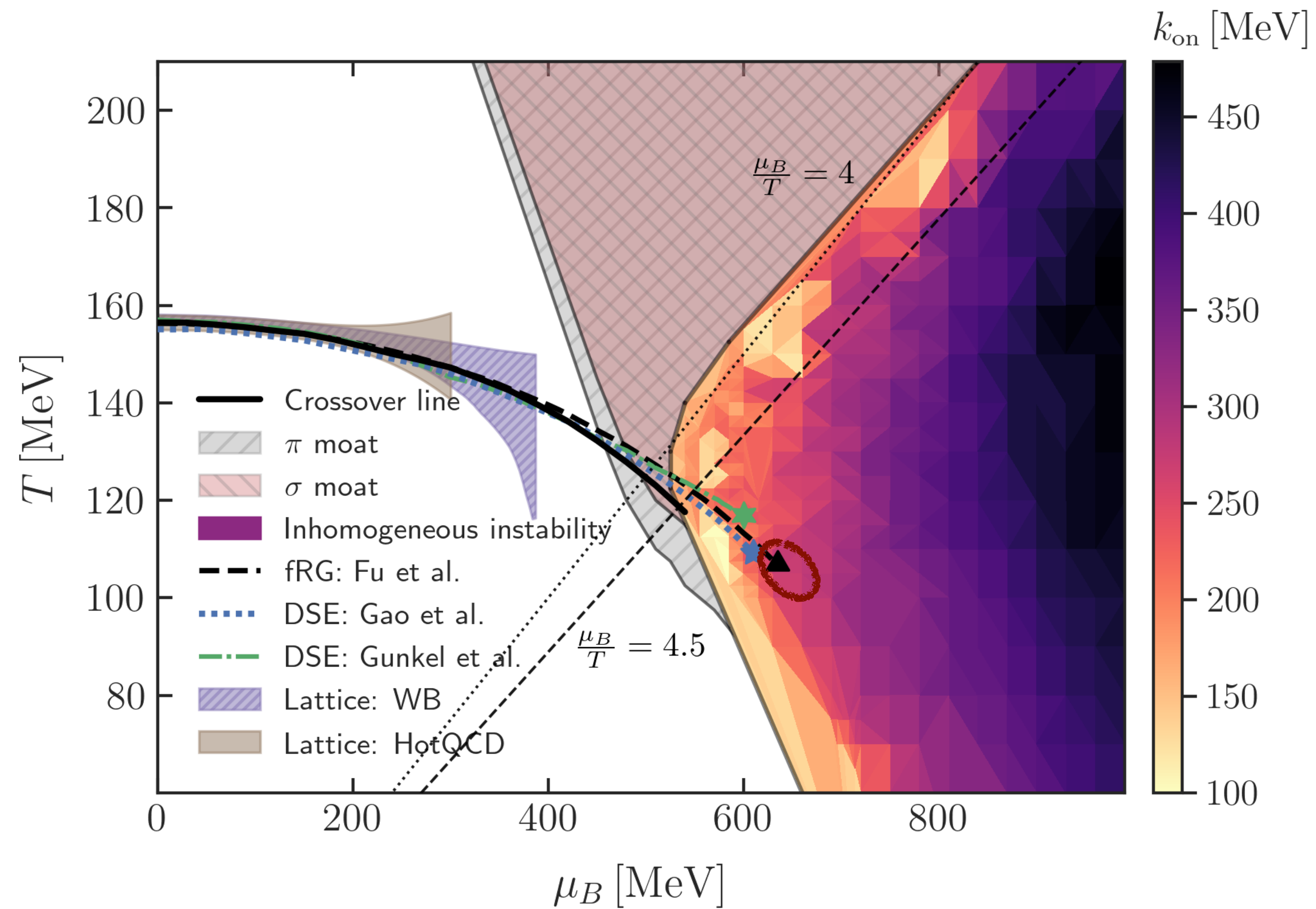
The QCD Moat

Soft(est) mode



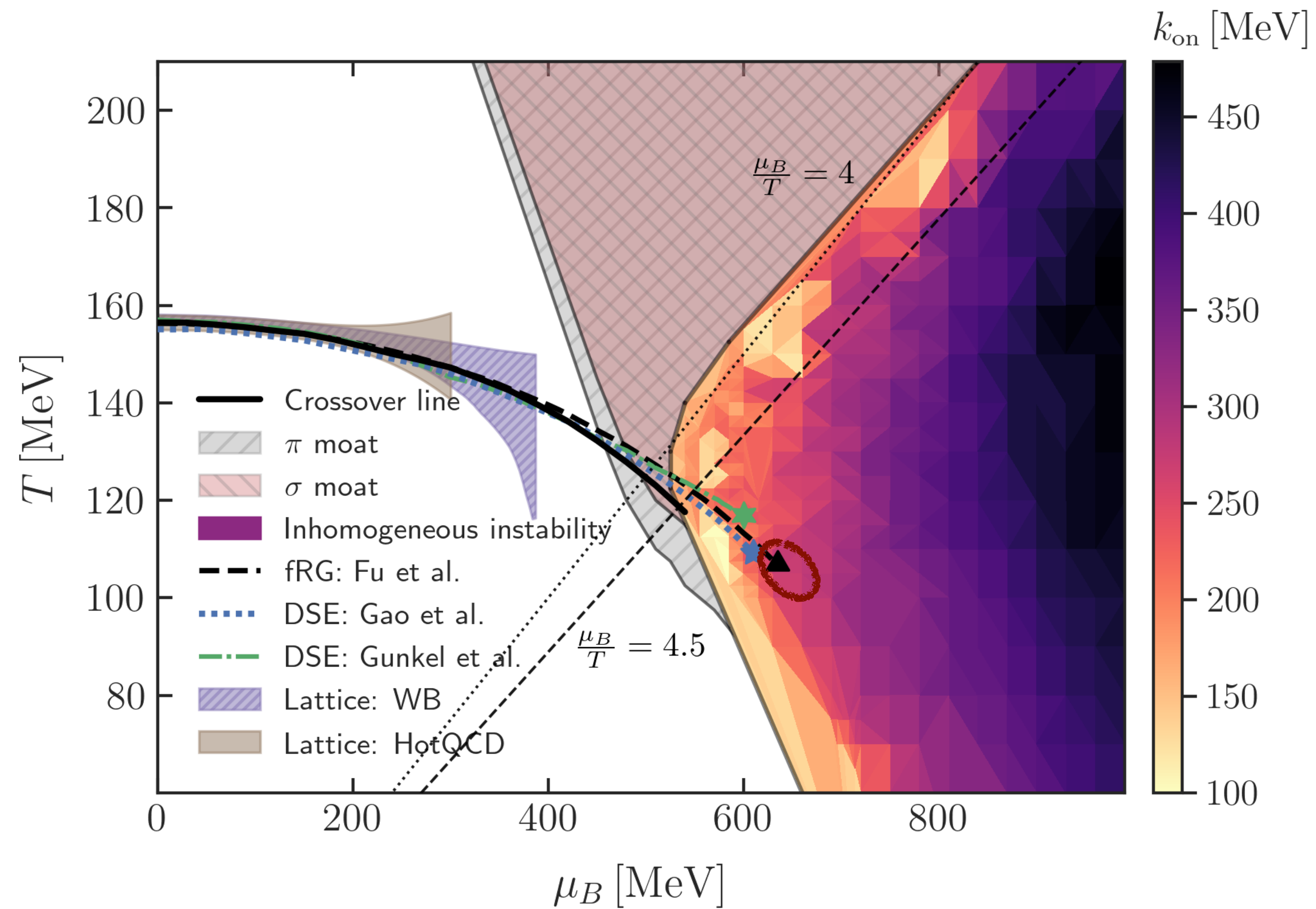


The moat
 &
 the Onset regime of New Phases



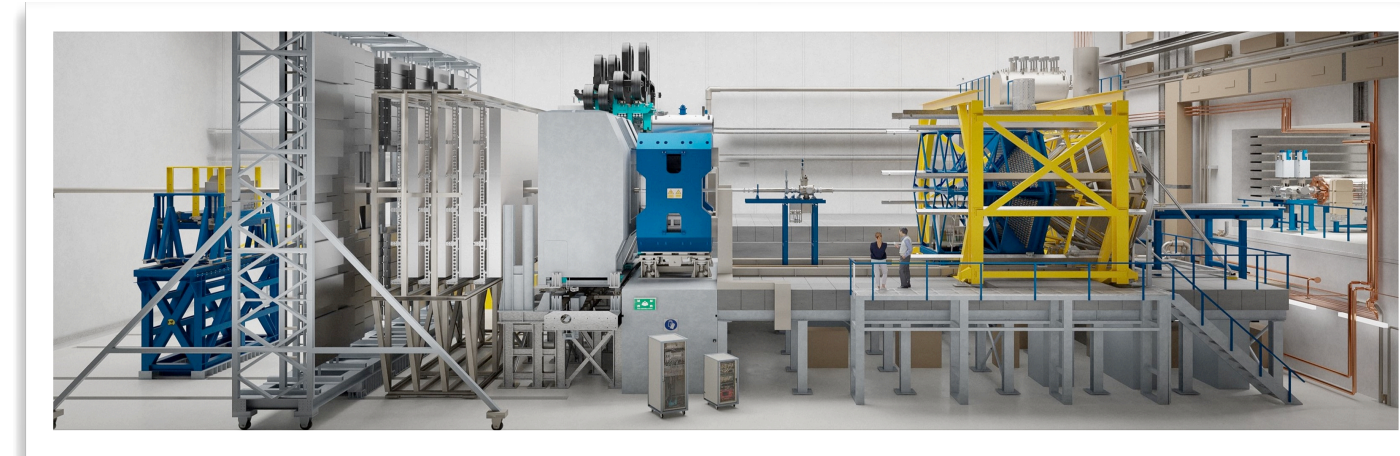
The moat
&
the Onset regime of New Phases

are in the CBM regime
and
require high statistics



The moat
&
the Onset regime of New Phases

are in the CBM regime
and
require high statistics



!Go for it!

Jump forward to: Summary

Ripples of the CEP

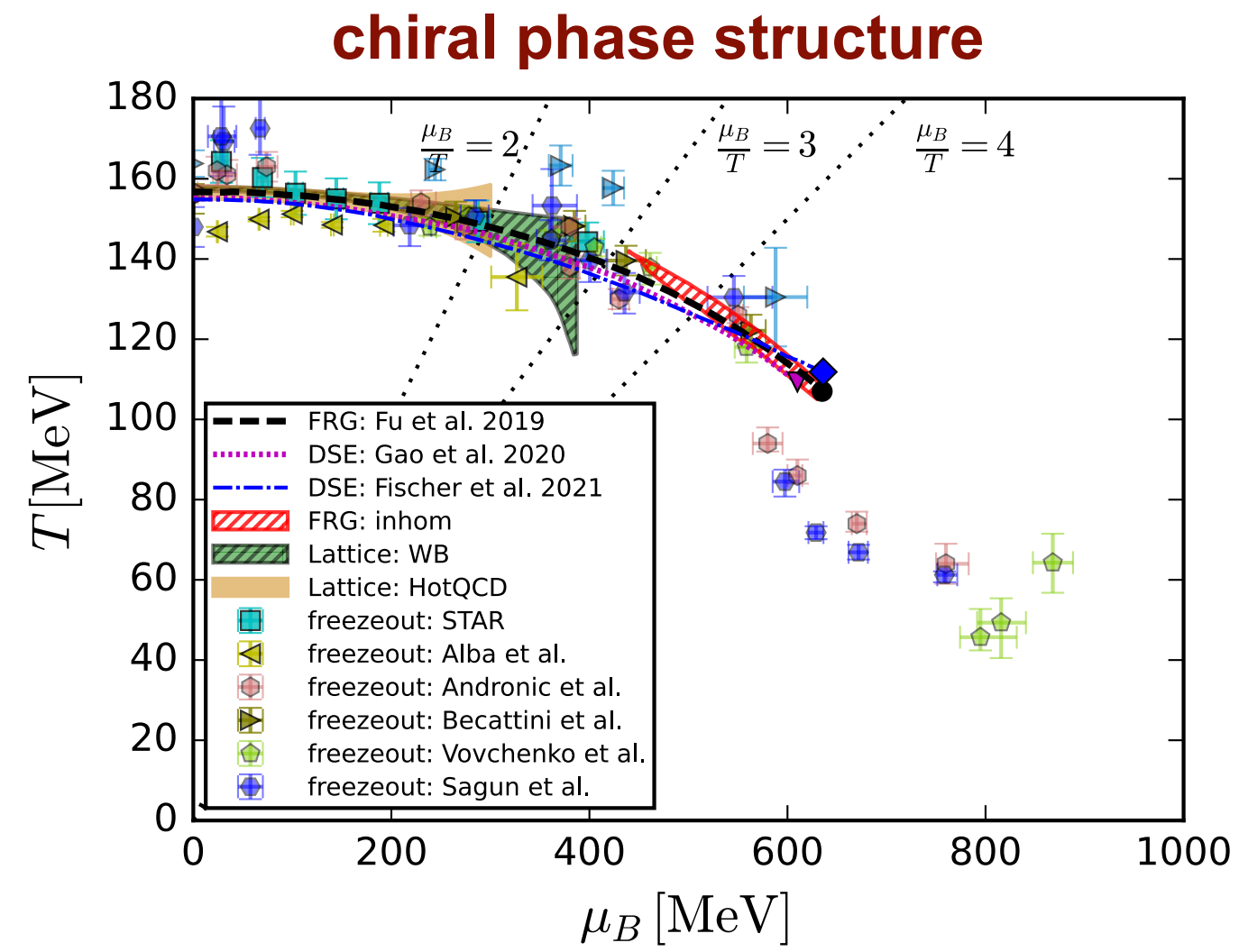


aka

(Strong) signals of the ONP

QCD-assisted low energy effective theory

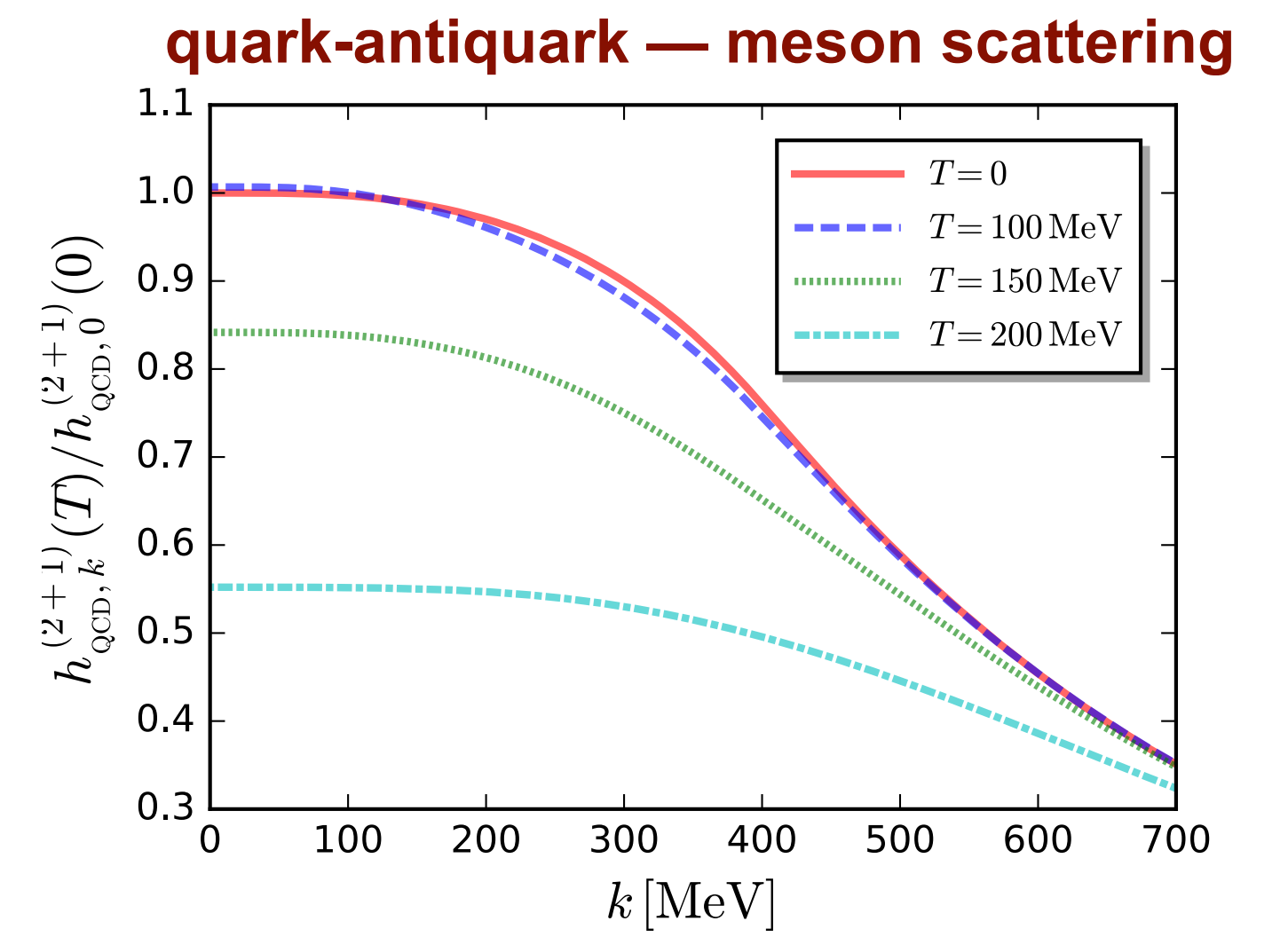
Direct QCD input



+

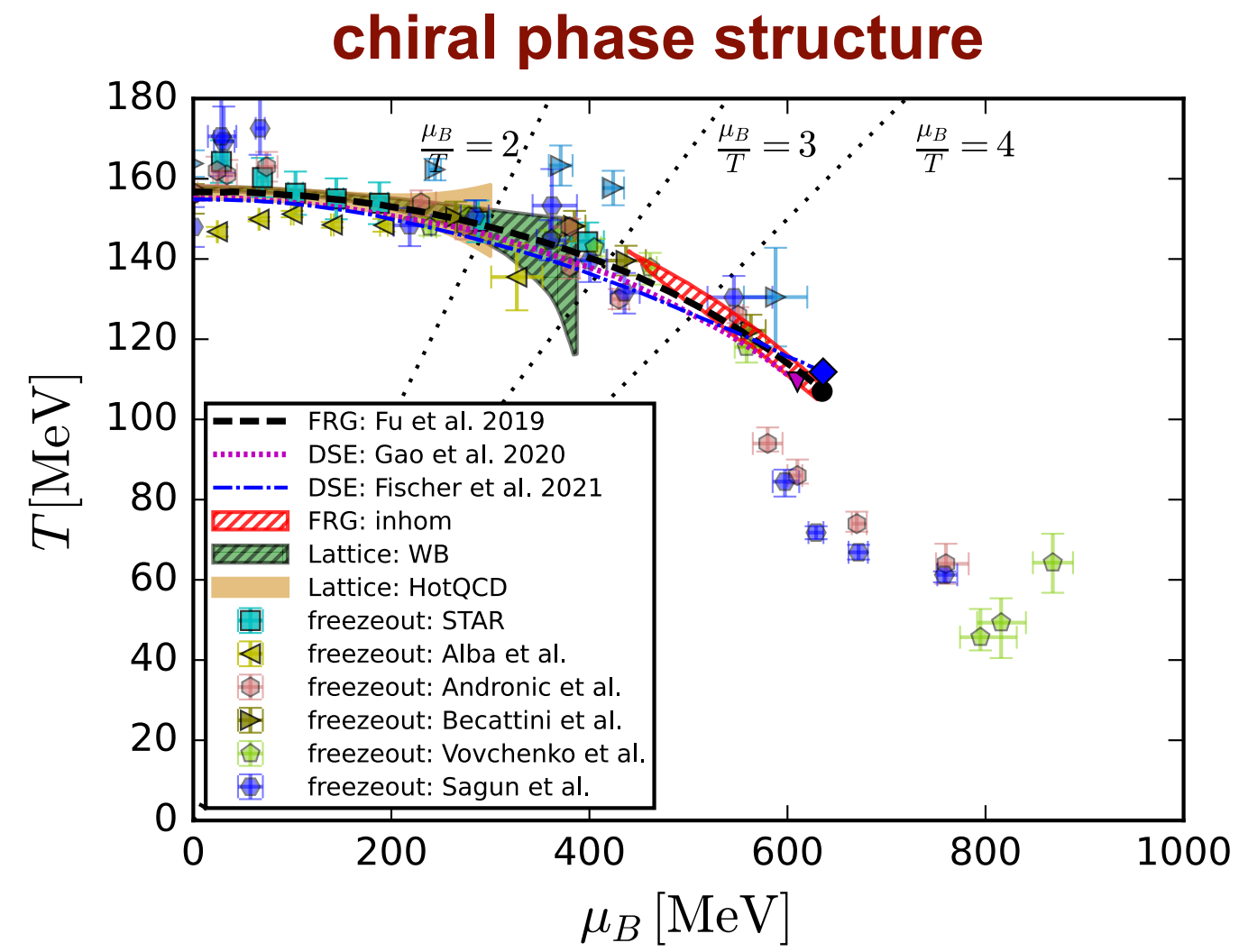


Fu, JMP, Rennecke, PRD 101 (2020) 054032



QCD-assisted low energy effective theory

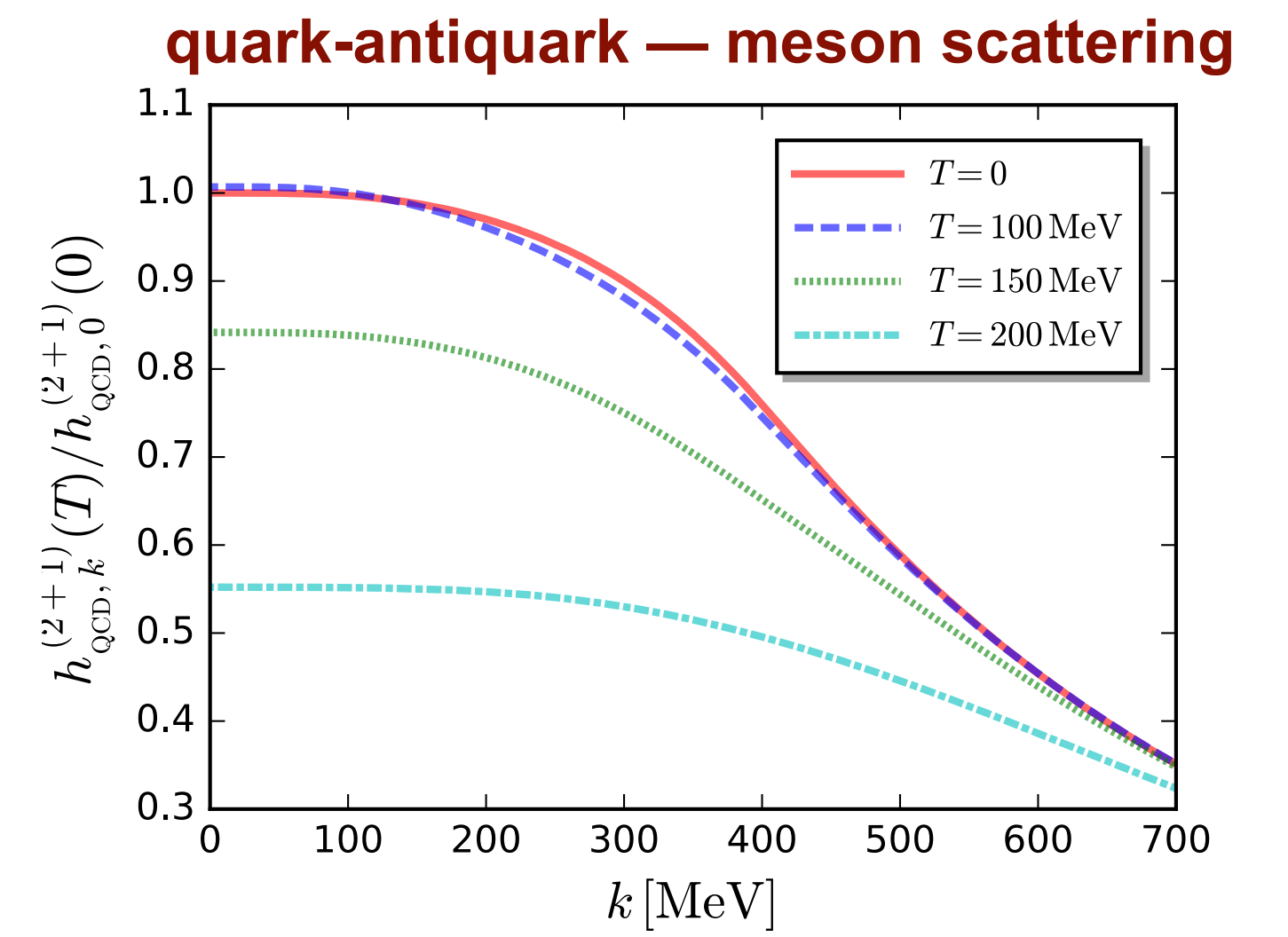
Direct QCD input



+



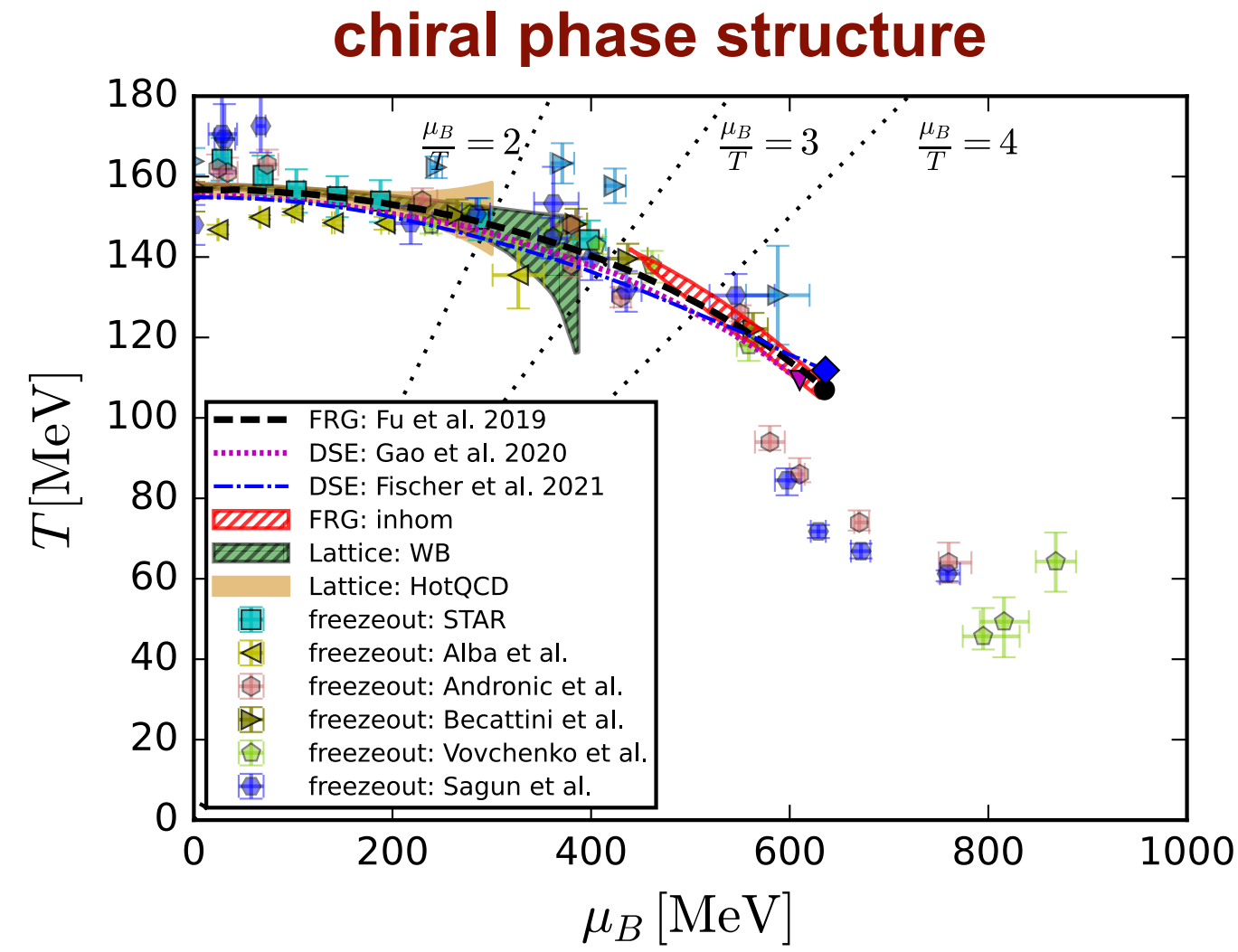
Fu, JMP, Rennecke, PRD 101 (2020) 054032



low energy quantum, thermal & density fluctuations via fRG (QCD-assisted PQM model)

QCD-assisted low energy effective theory

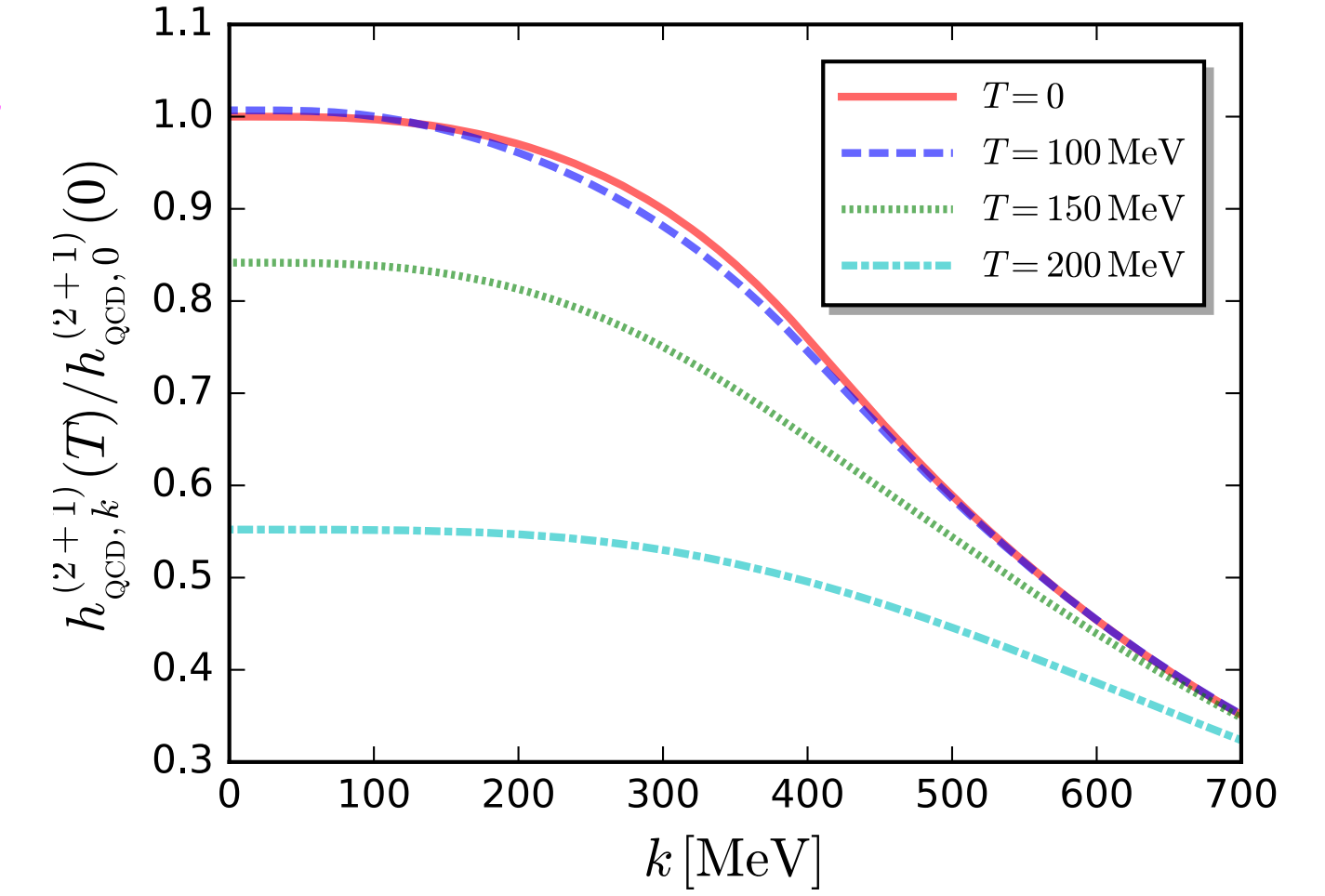
Direct QCD input



Fu, JMP, Rennecke, PRD 101 (2020) 054032

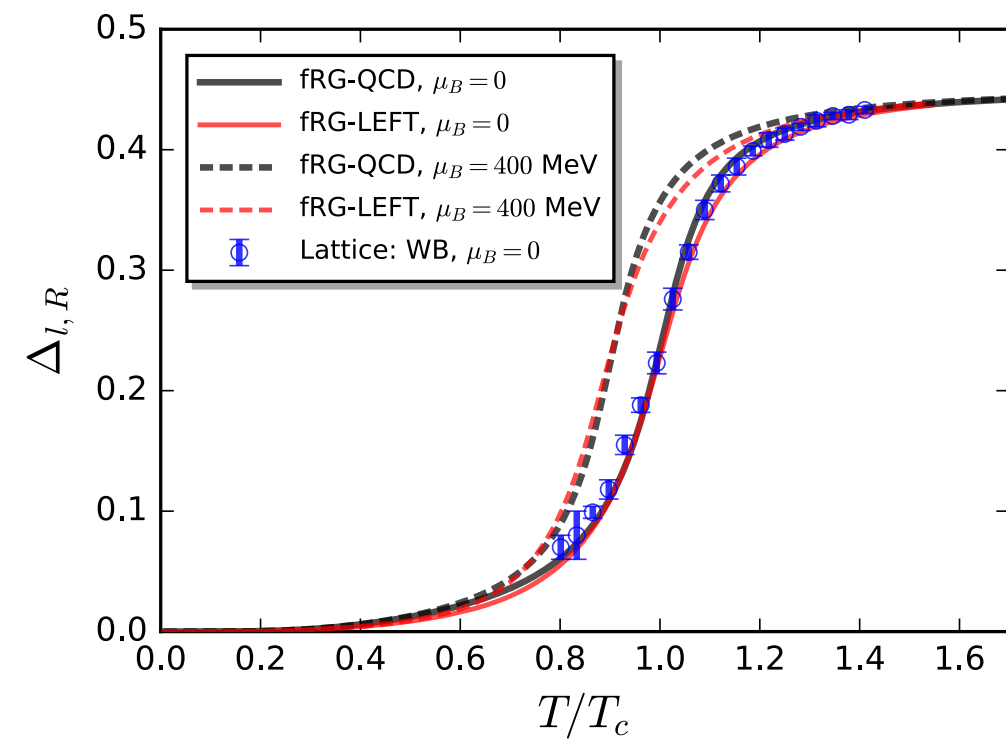


quark-antiquark — meson scattering



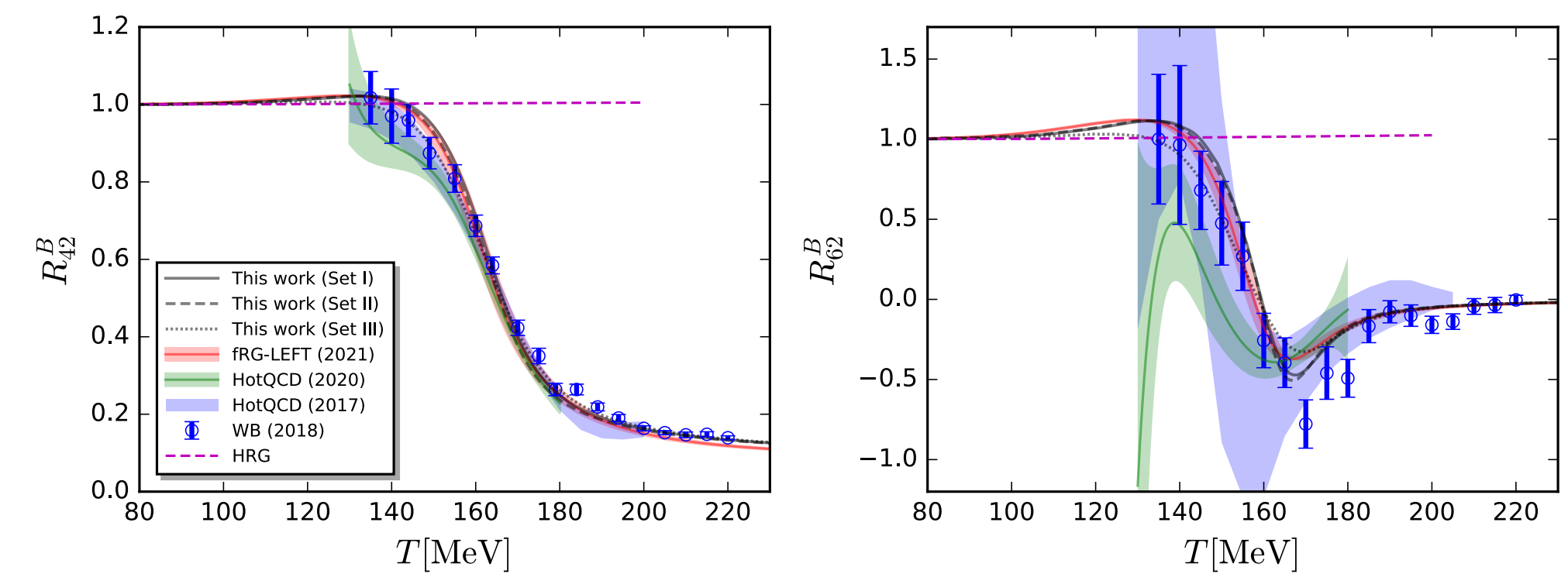
low energy quantum, thermal & density fluctuations via fRG (QCD-assisted PQM model)

renormalised chiral condensate



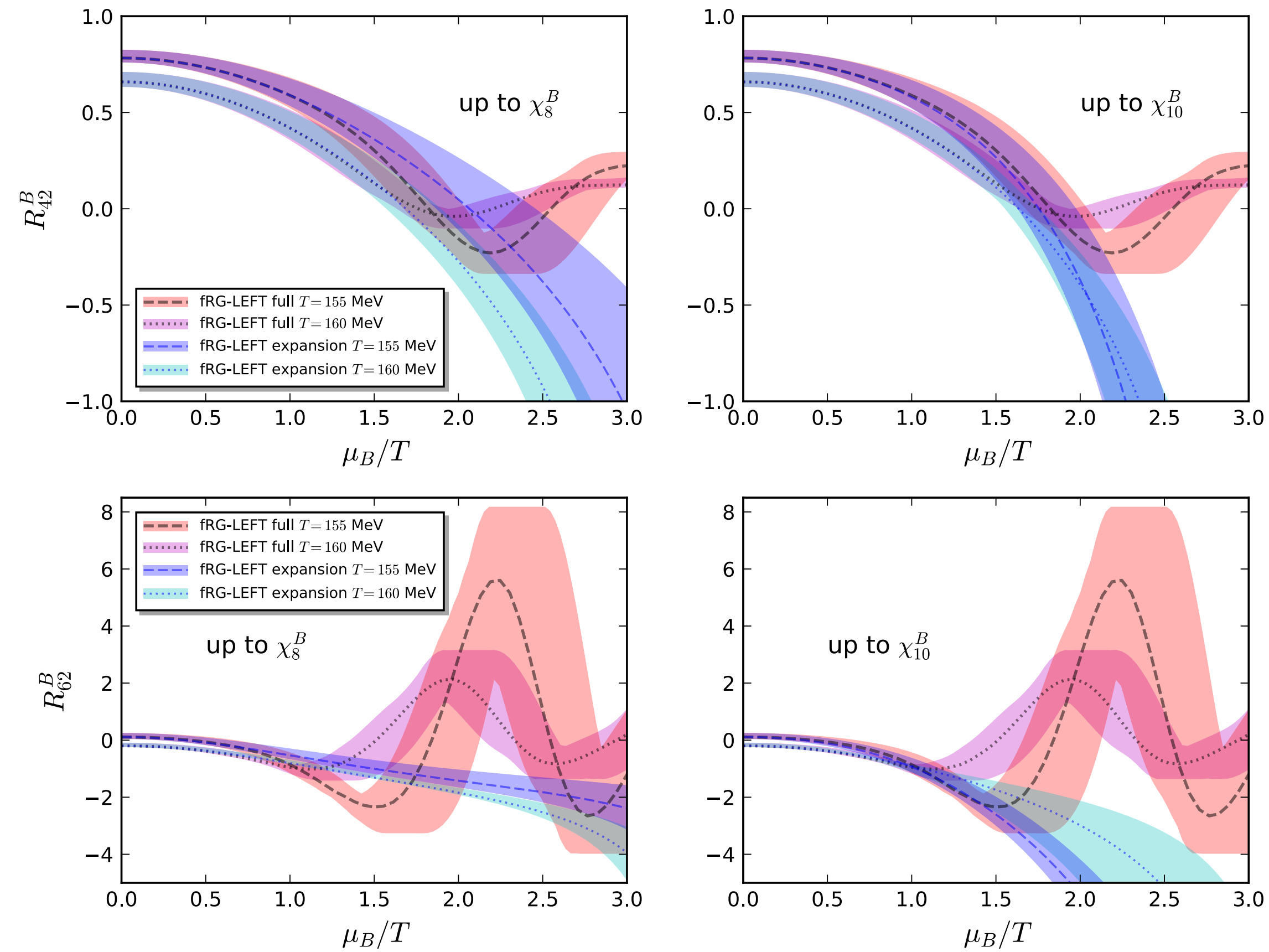
Benchmarks with lattice and fQCD
at
vanishing density and fQCD at finite density

baryon number fluctuations



Fluctuations of conserved charges

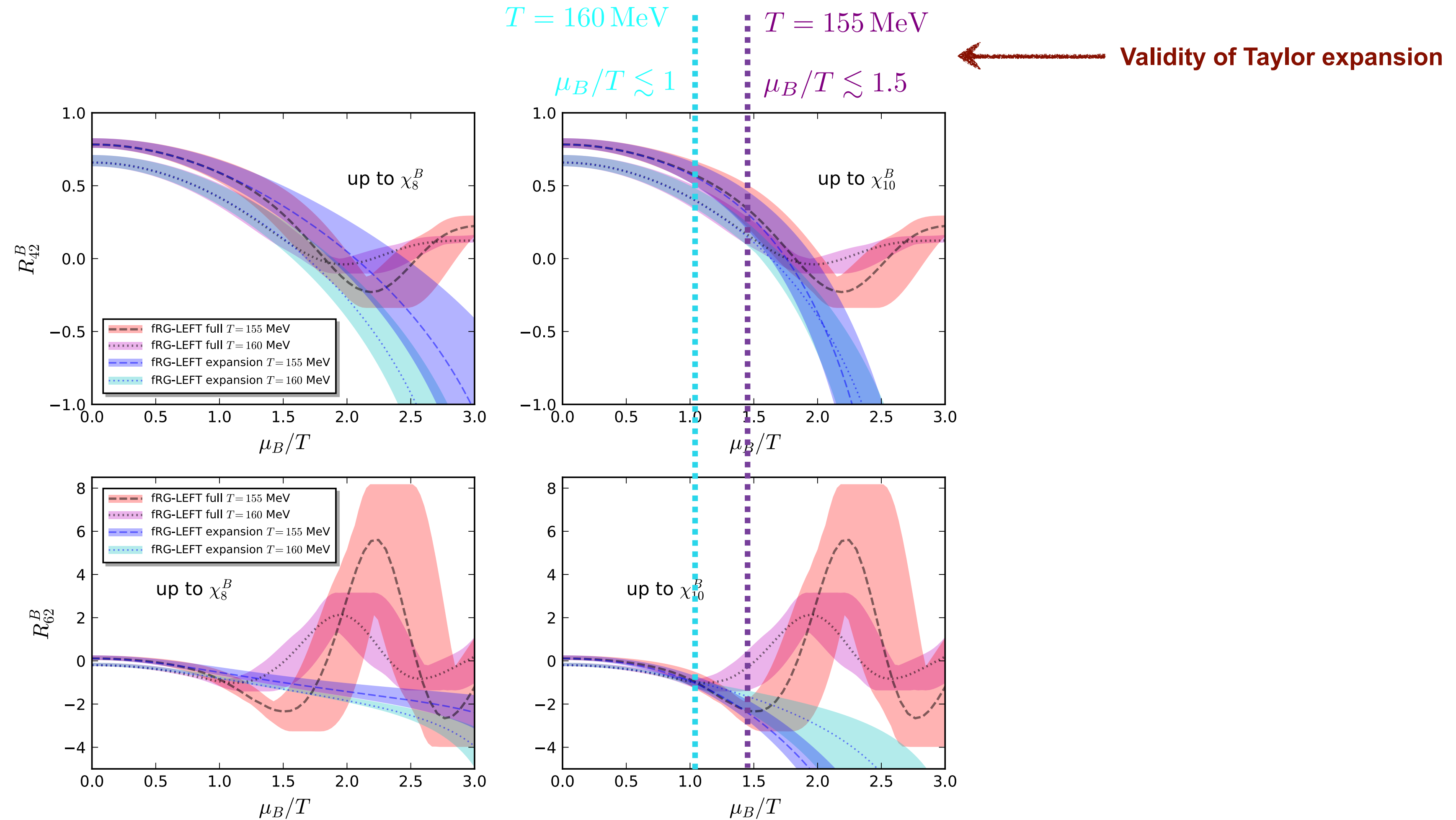
Fluctuations of conserved charges



Great opportunity for a combined analysis of high density QCD (Exp. data + lattice QCD + functional QCD)

Fluctuations of conserved charges

Fluctuations of conserved charges

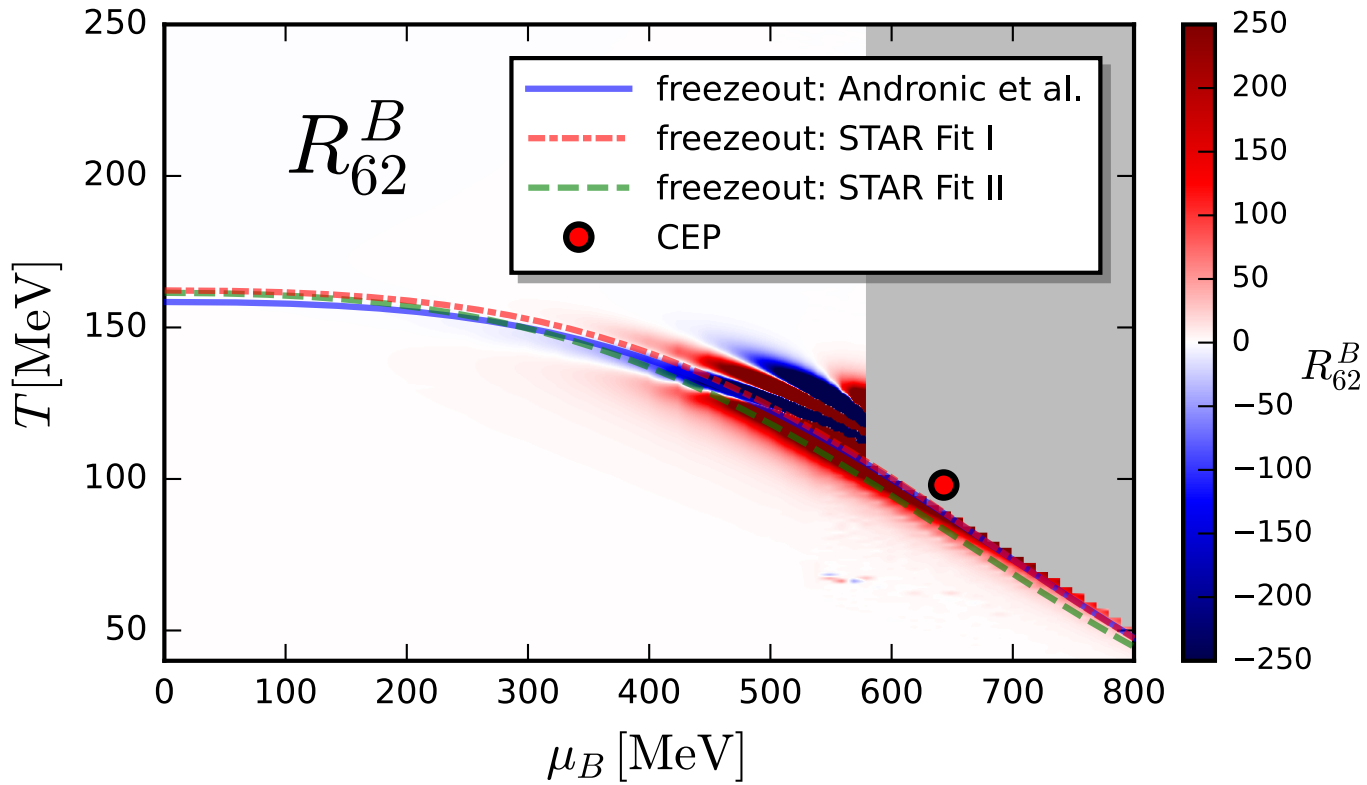
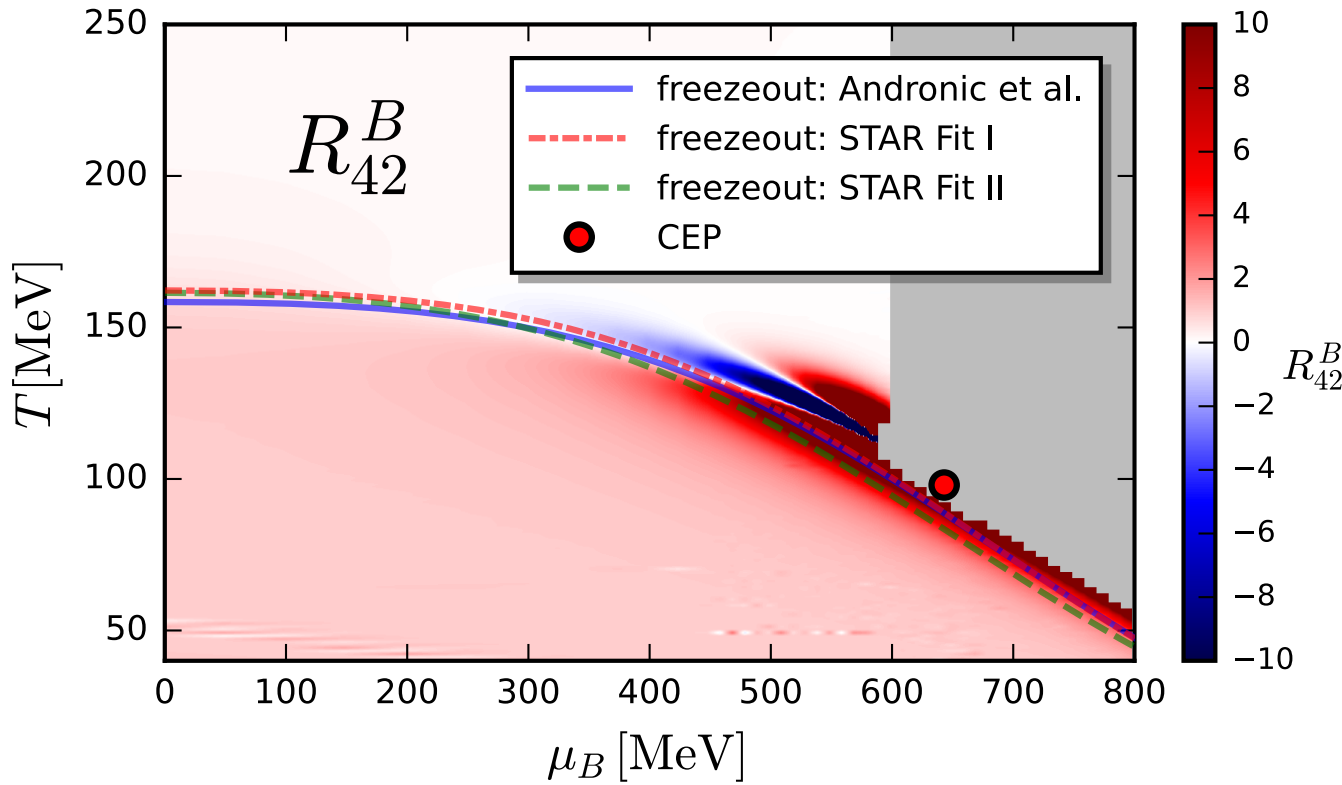


Great opportunity for a combined analysis of high density QCD (Exp. data + lattice QCD + functional QCD)

Ripples of the critical point

baryon number fluctuations in the phase structure

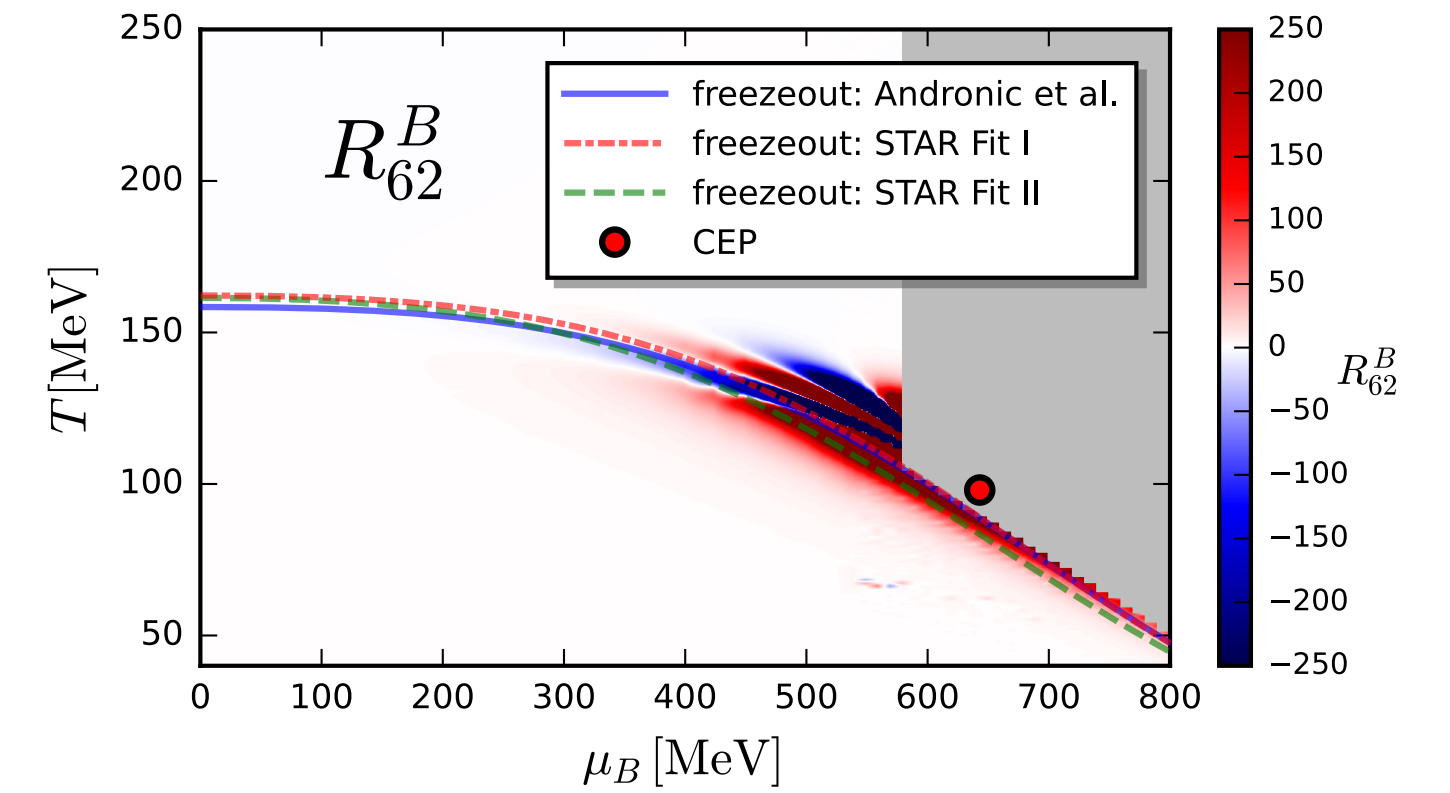
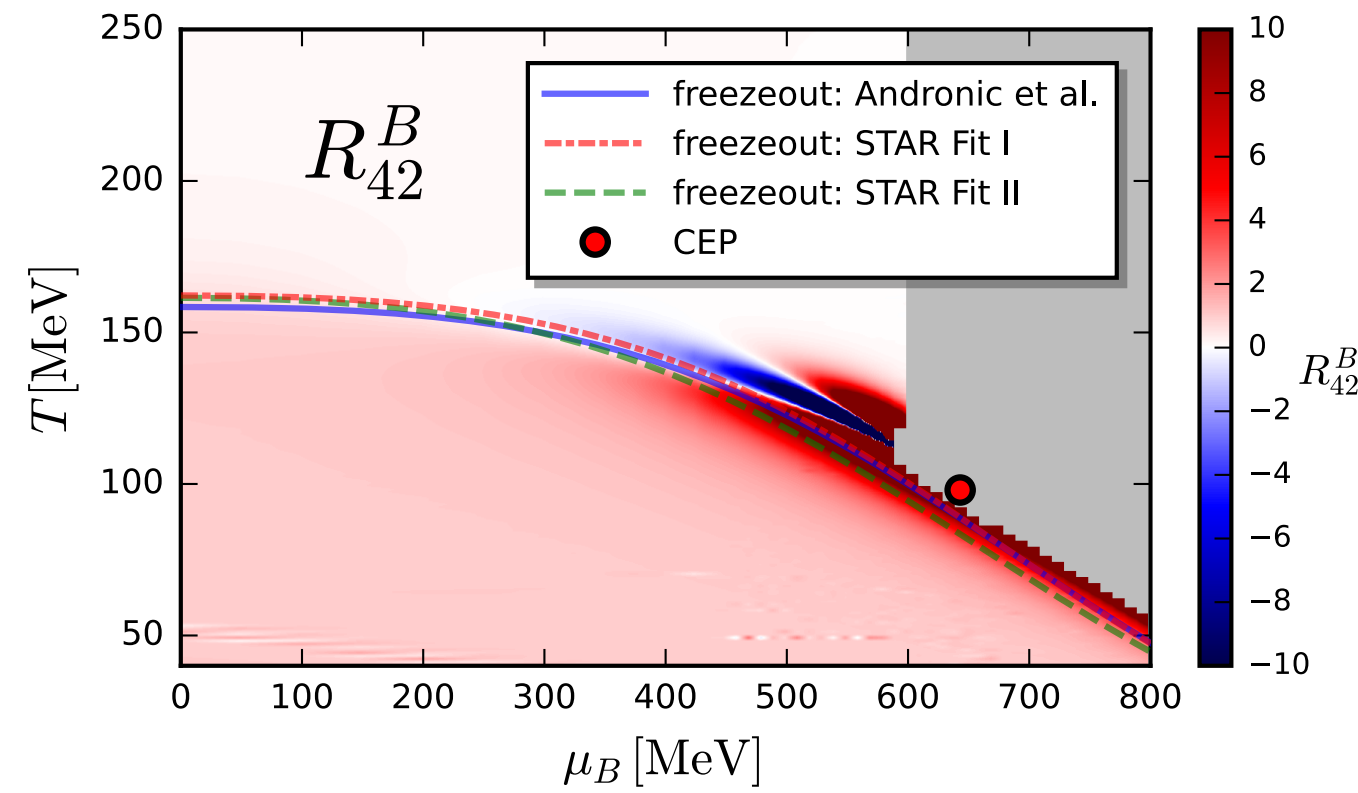
$$(T_{\text{CEP}}, \mu_{B_{\text{CEP}}}) = (98, 643) \text{ MeV}$$



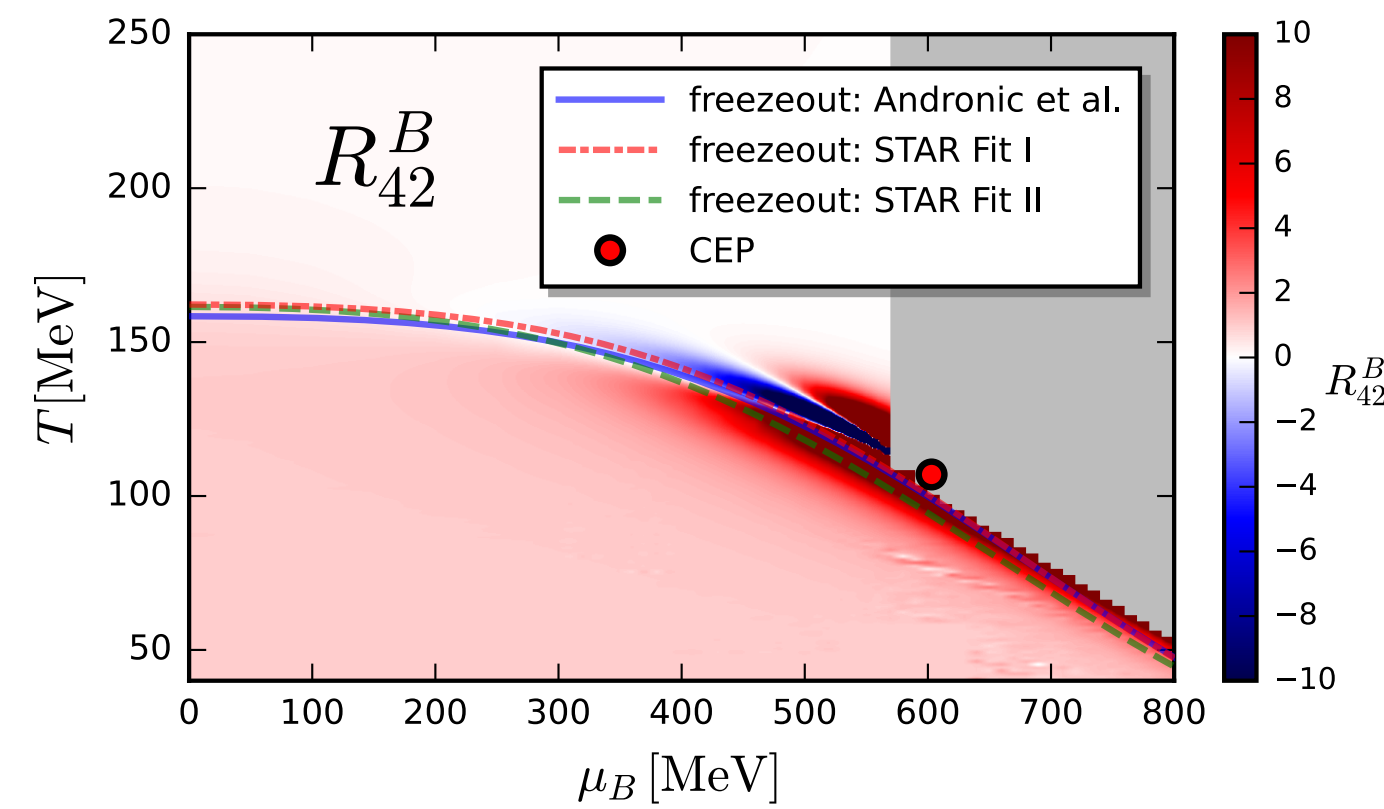
Ripples of the critical point

baryon number fluctuations in the phase structure

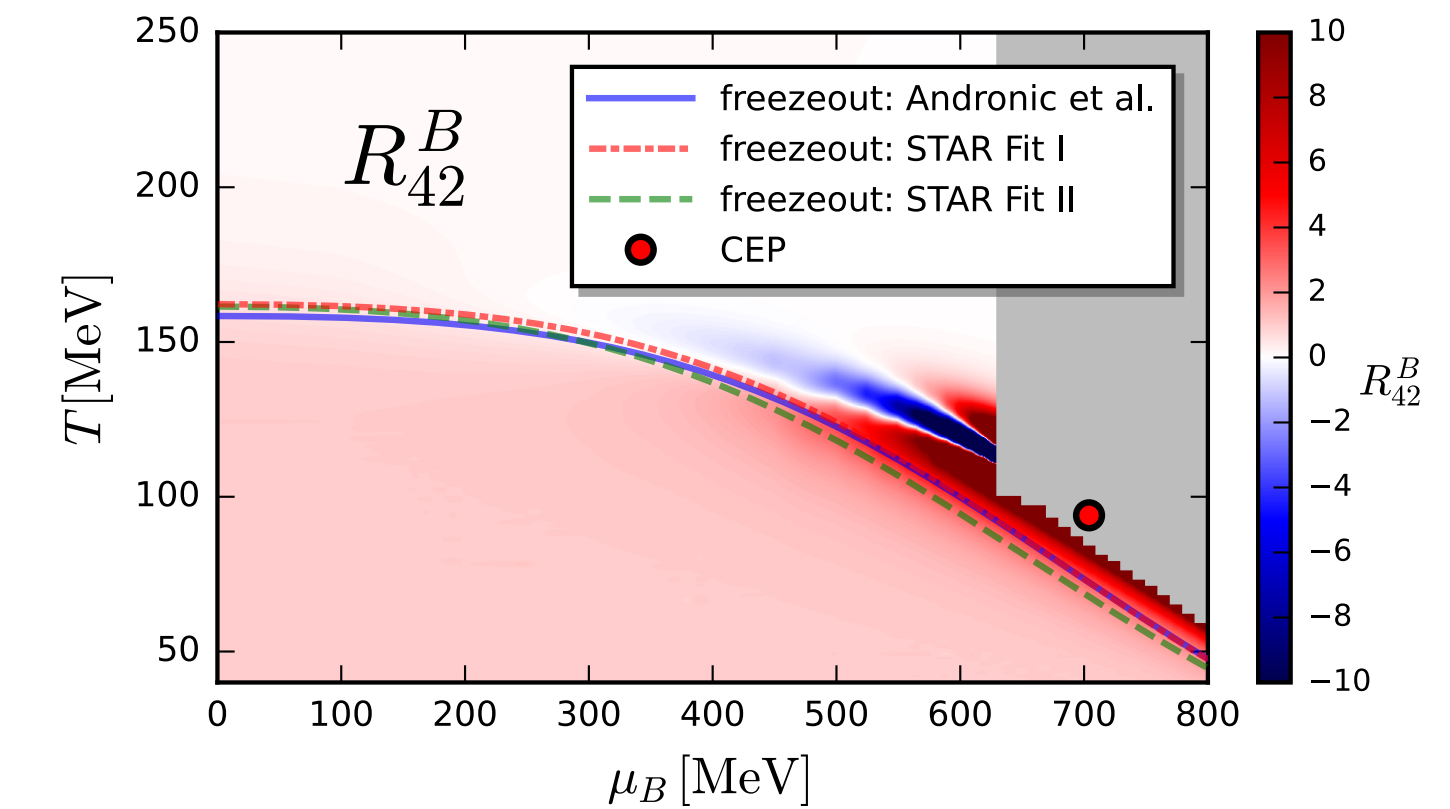
$$(T_{\text{CEP}}, \mu_{B_{\text{CEP}}}) = (98, 643) \text{ MeV}$$



Variations of the CEP in the allowed estimate regime



$$(T_{\text{CEP}}, \mu_{B_{\text{CEP}}}) = (108, 604) \text{ MeV}$$



$$(T_{\text{CEP}}, \mu_{B_{\text{CEP}}}) = (94, 704) \text{ MeV}$$

Ripples of the critical point

Canonical corrections via subensemble acceptance

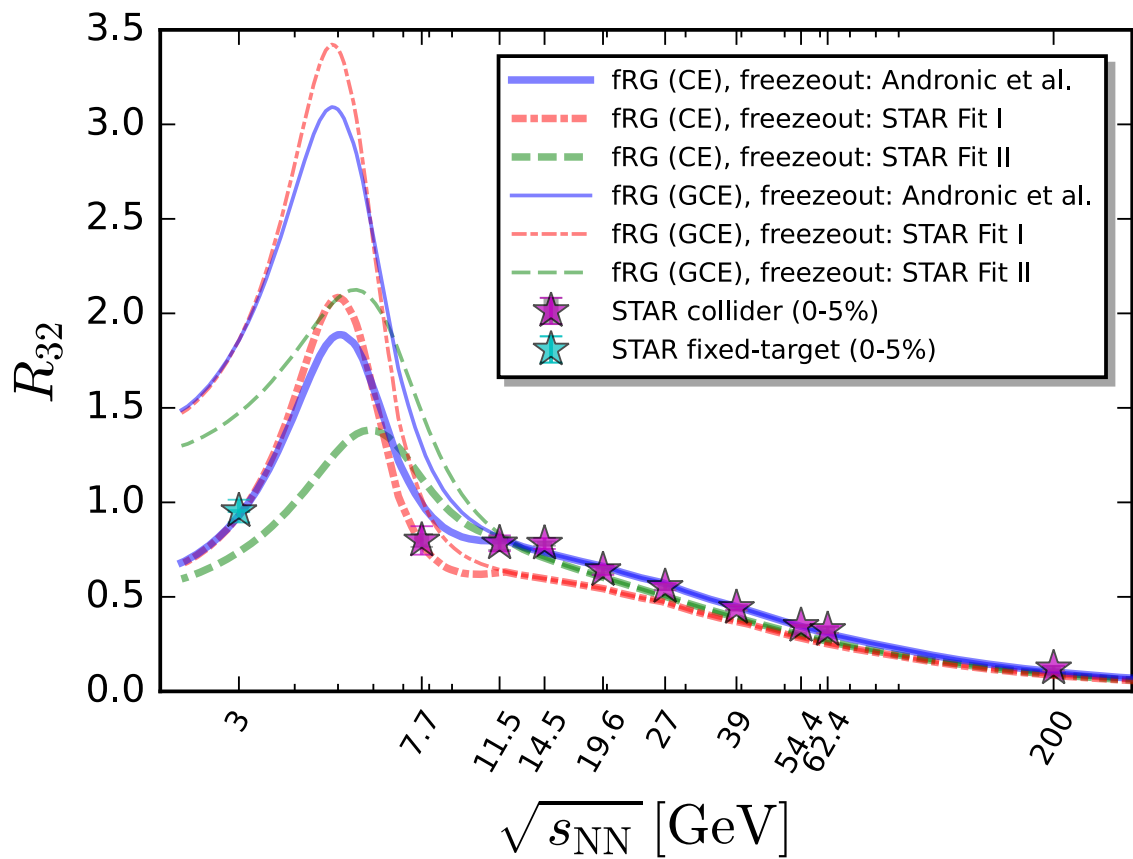
Vovchenko, Savchuk, Poberezhnyuk, Gorenstein, Koch, **PLB 811, 135868 (2020)**

fixing the subensemble volume

subensemble volume system volume

$$V_1 = \alpha V$$

$$\bar{R}_{32}^B = (1 - 2\alpha) R_{32}^B$$



Ripples of the critical point

Canonical corrections via subensemble acceptance

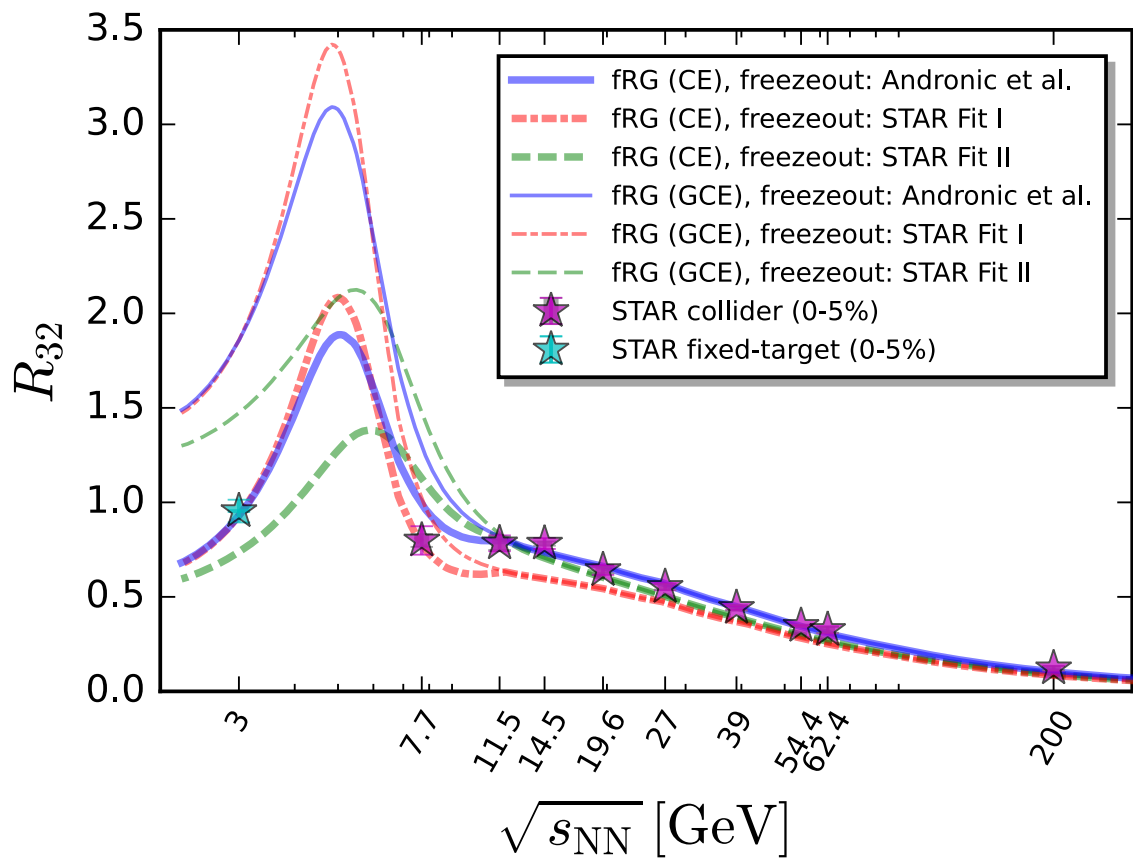
Vovchenko, Savchuk, Poberezhnyuk, Gorenstein, Koch, **PLB 811, 135868 (2020)**

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$$V_1 = \alpha V$$

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qualitative adjustment

$$\alpha(\bar{s}) = a \left(1 - \sqrt{\bar{s}}\right) \theta(1 - \bar{s})$$

$$a = 0.33 \qquad \sqrt{\bar{s}} = \frac{\sqrt{s_{NN}}}{11.9 \text{ GeV}}$$

Ripples of the critical point

Canonical corrections via subensemble acceptance

Vovchenko, Savchuk, Poberezhnyuk, Gorenstein, Koch, PLB 811, 135868 (2020)

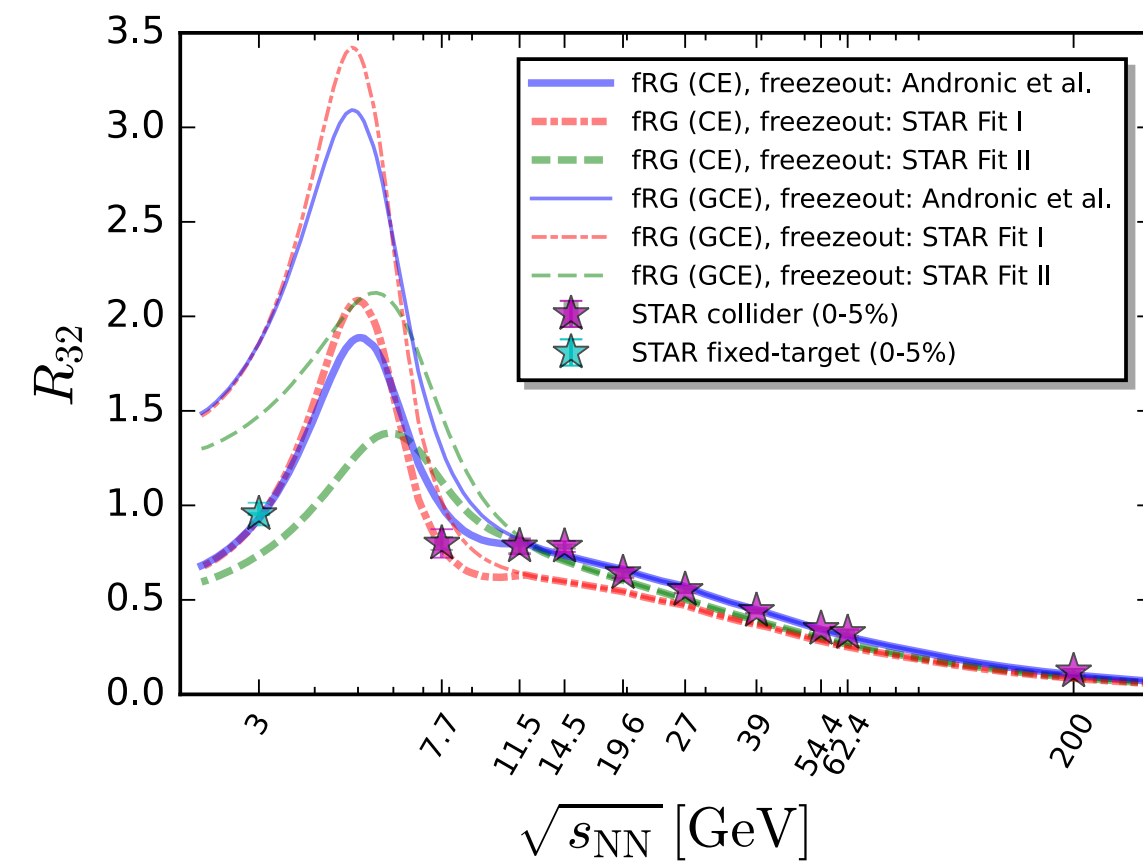
fixing the subensemble volume

baryon & proton number fluctuations

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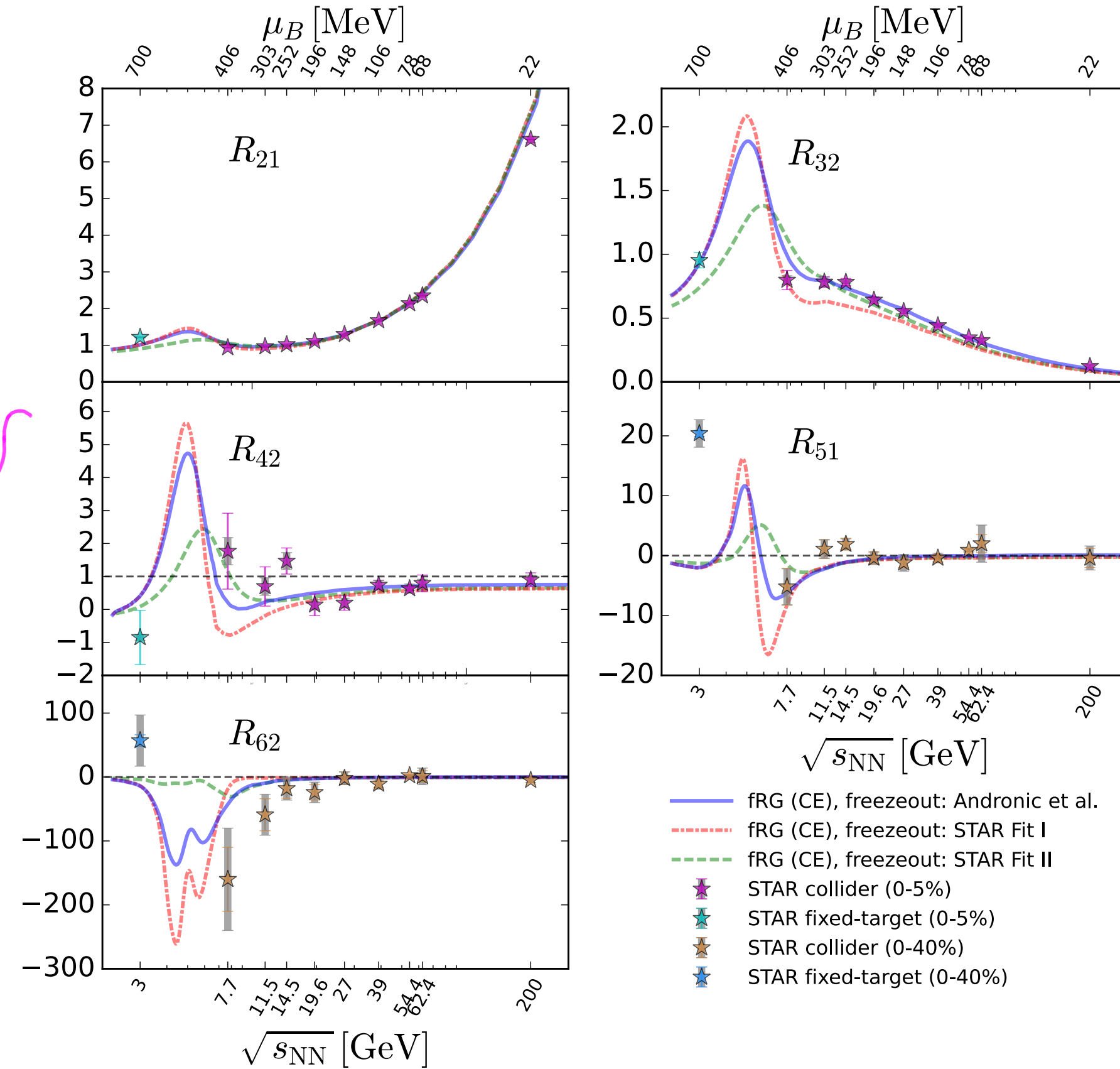


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Handwritten note: α



Ripples of the critical point

Canonical corrections via subensemble acceptance

Vovchenko, Savchuk, Poberezhnyuk, Gorenstein, Koch, PLB 811, 135868 (2020)

fixing the subensemble volume

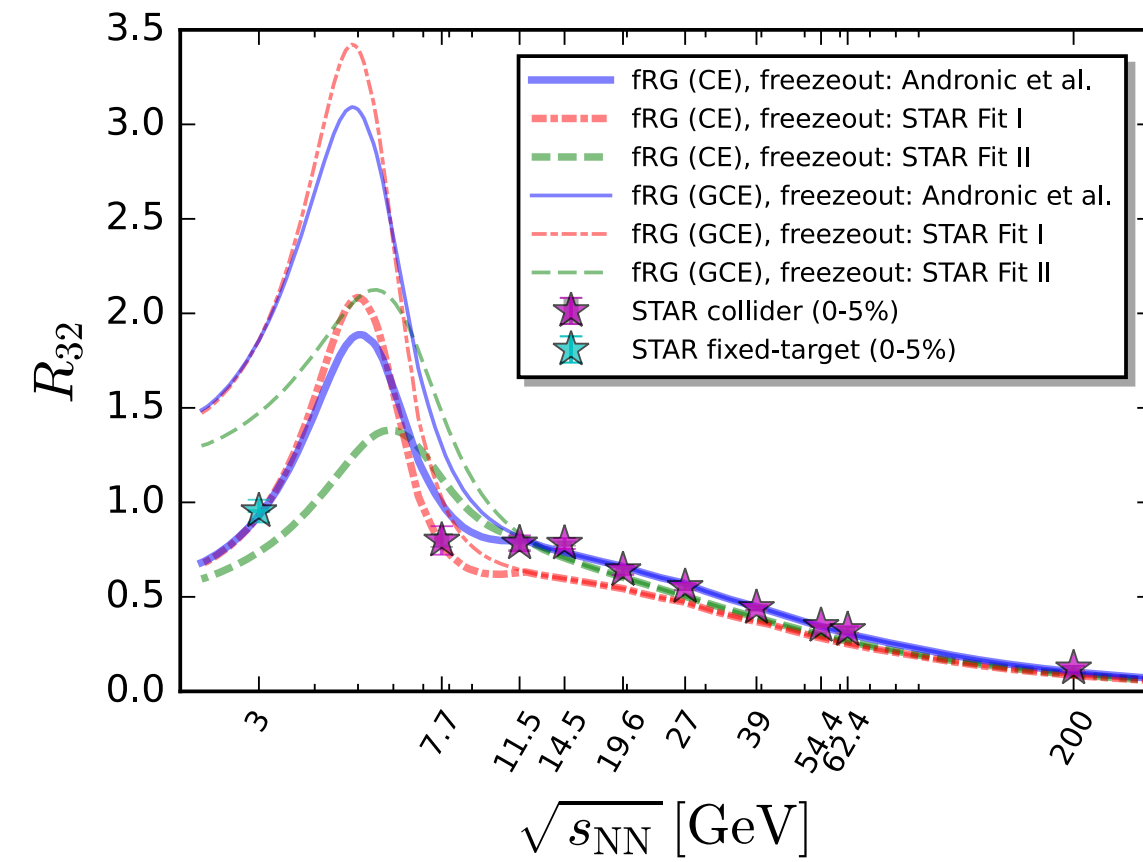
baryon & proton number fluctuations

baryon fluctuations with functional QCD

subensemble volume system volume

$$V_1 = \alpha V$$

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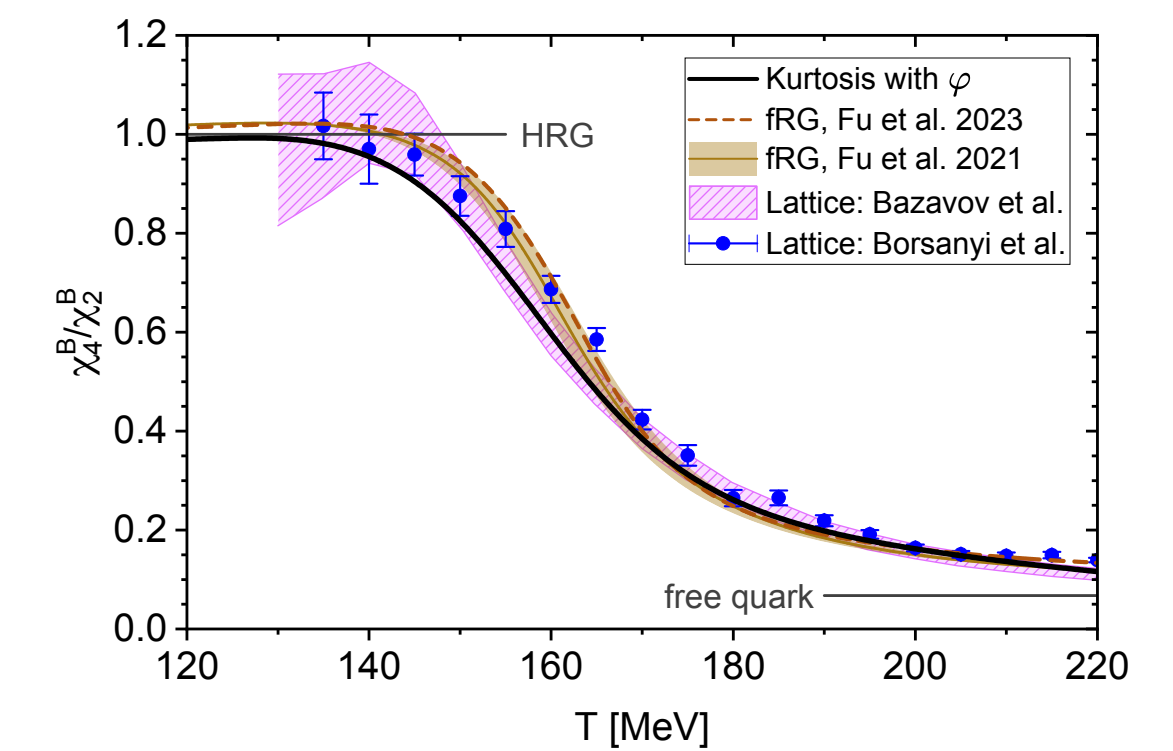
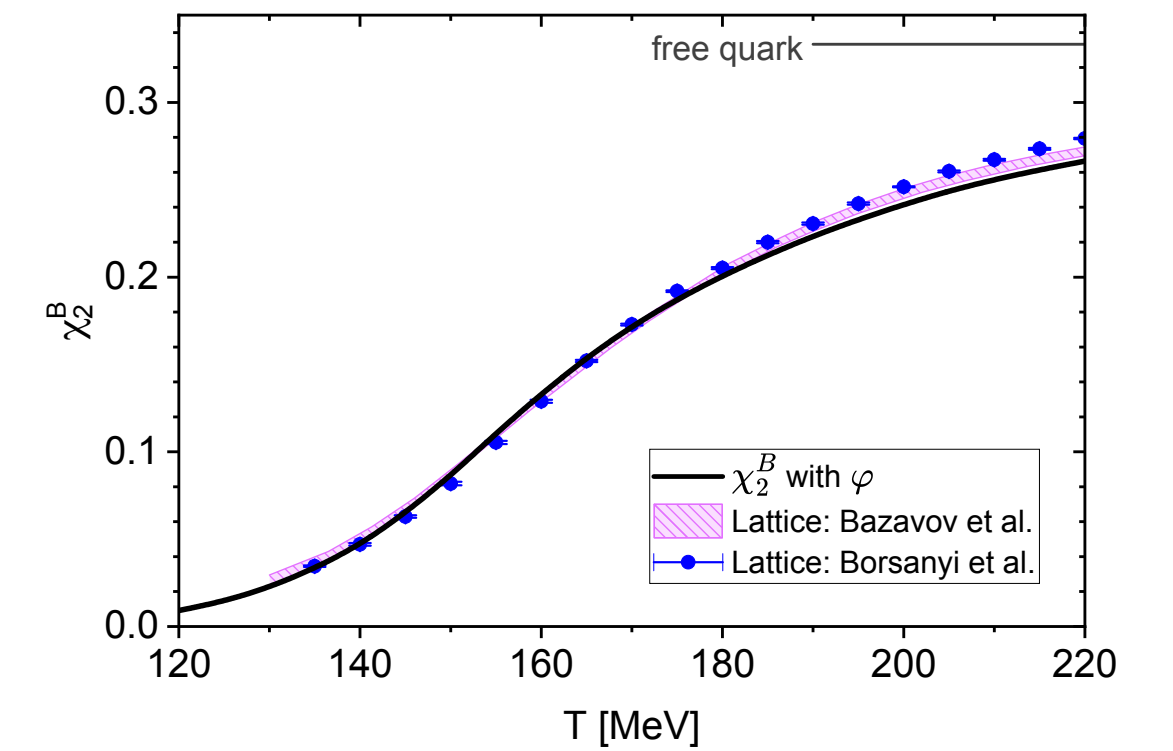
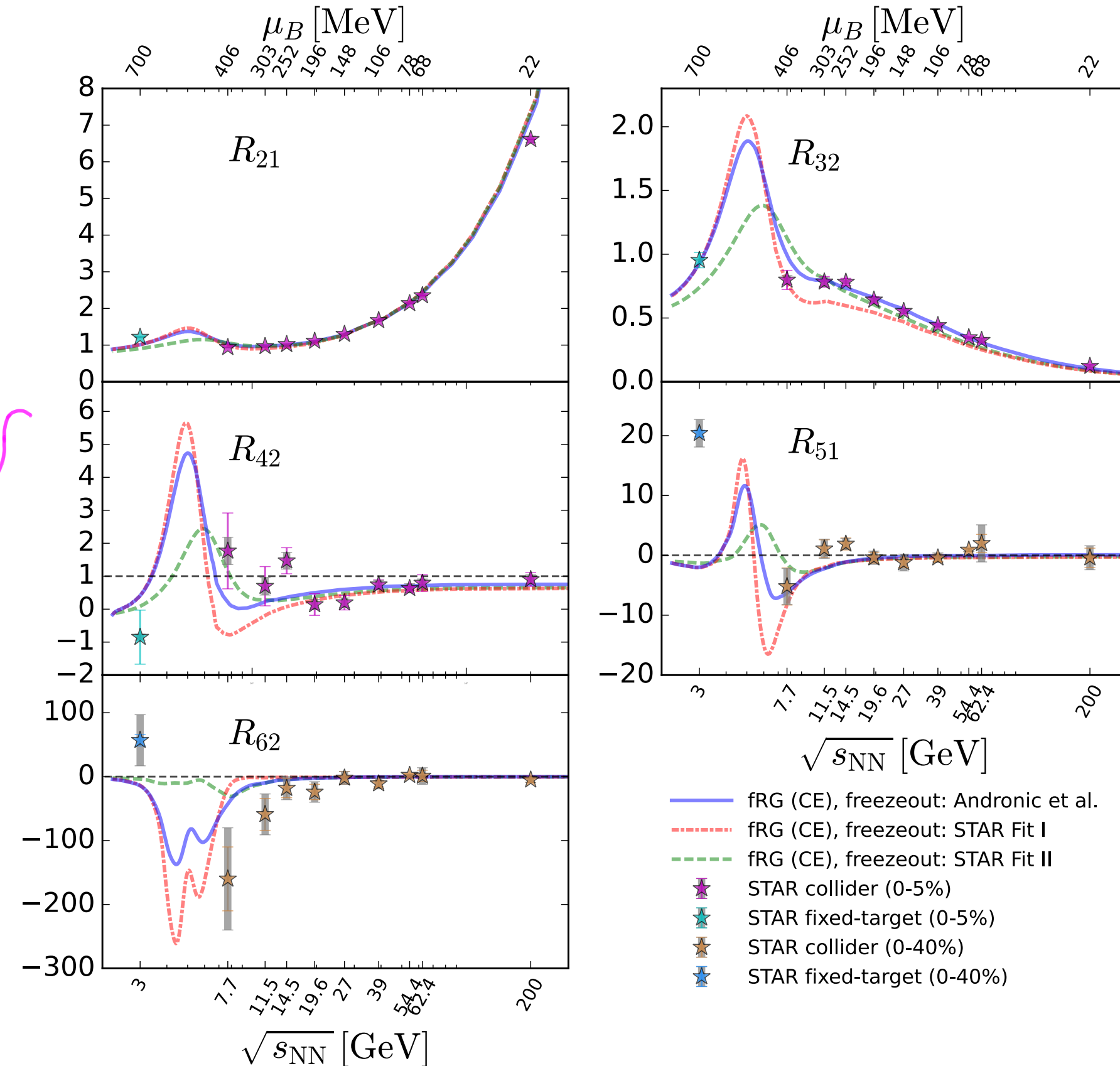


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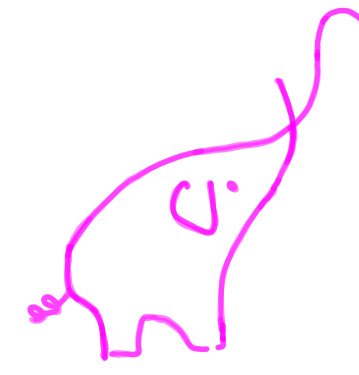
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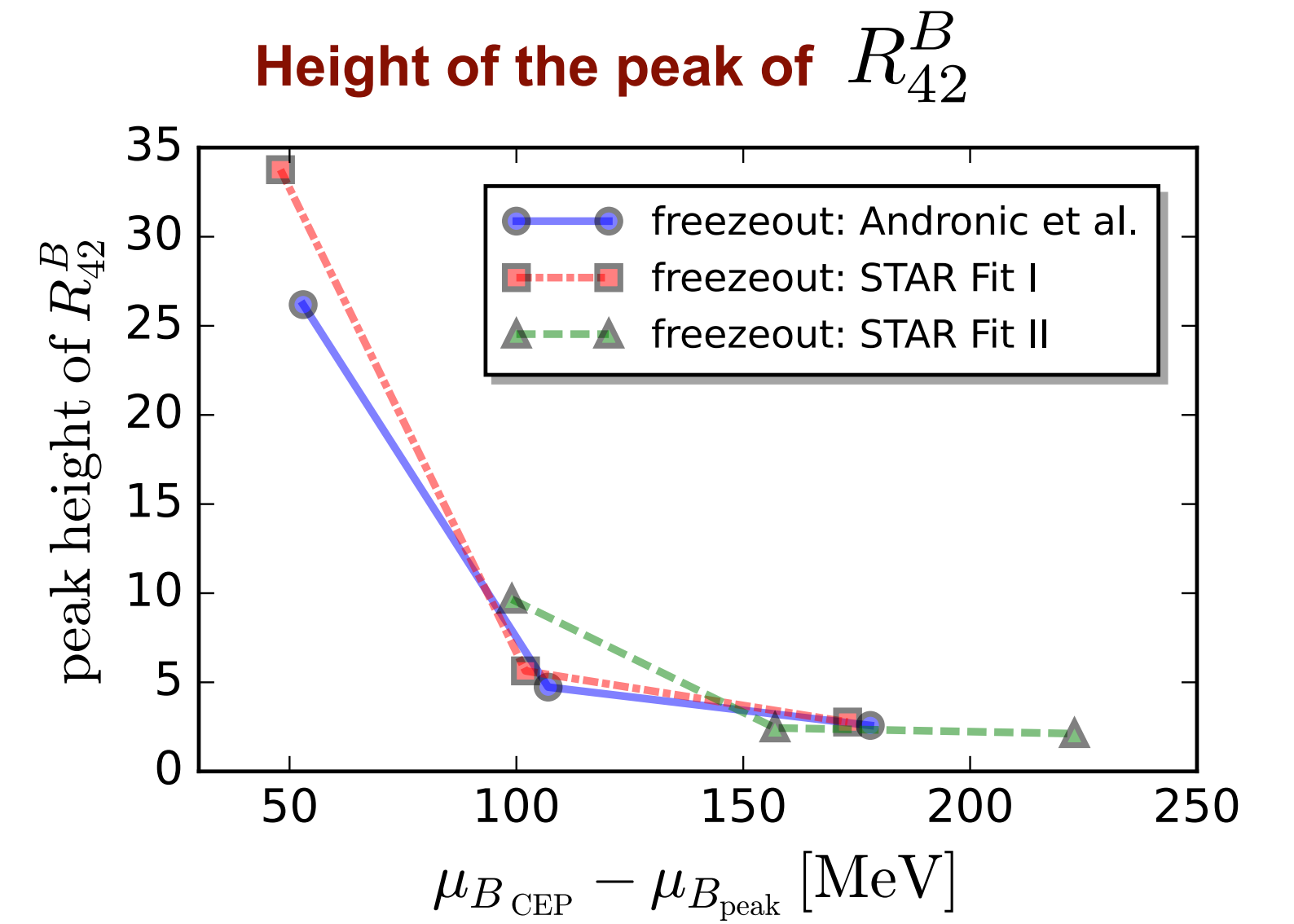
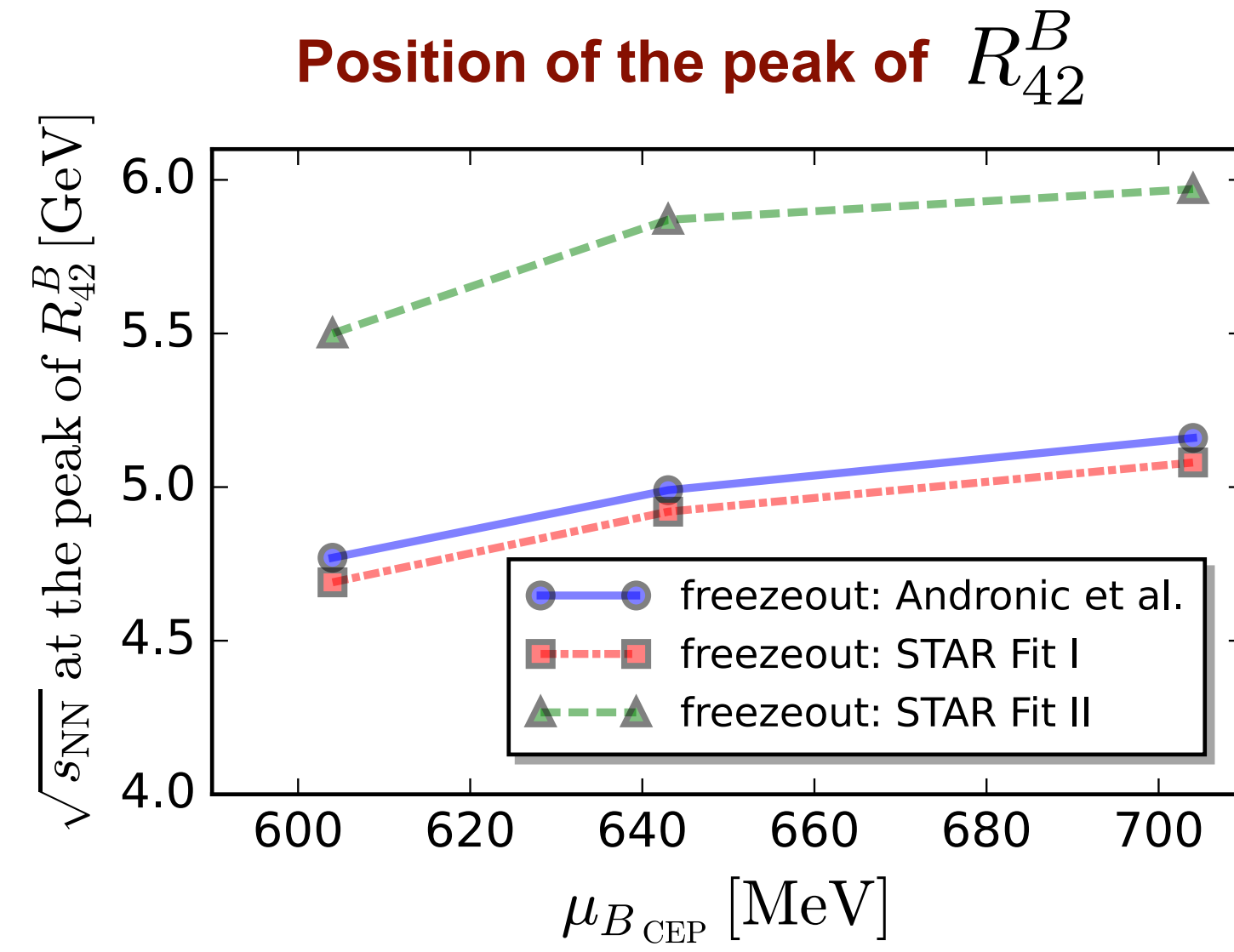


Gao, Liu, Lu, JMP, PRD 113 (2026) 054019

Ripples of the critical point



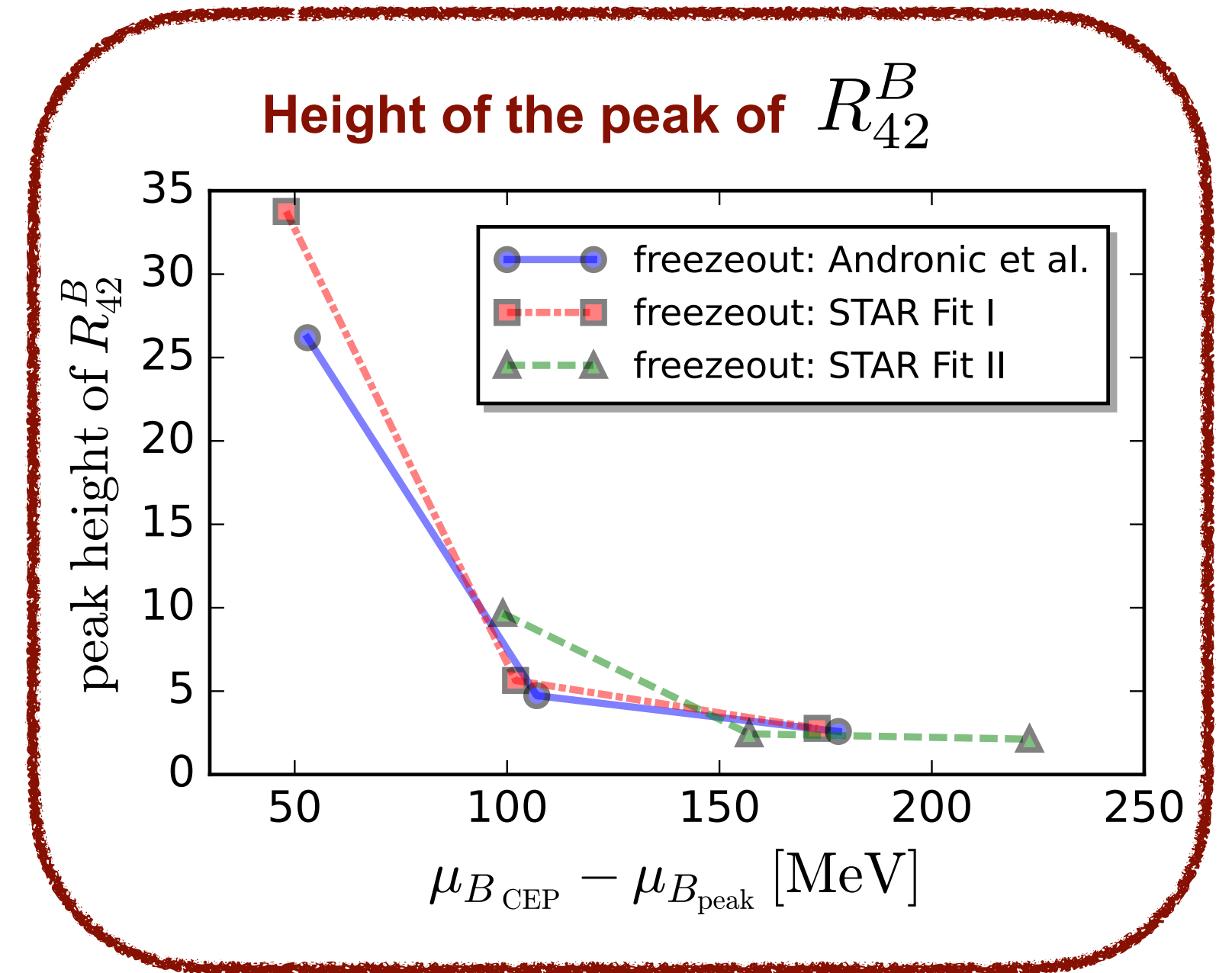
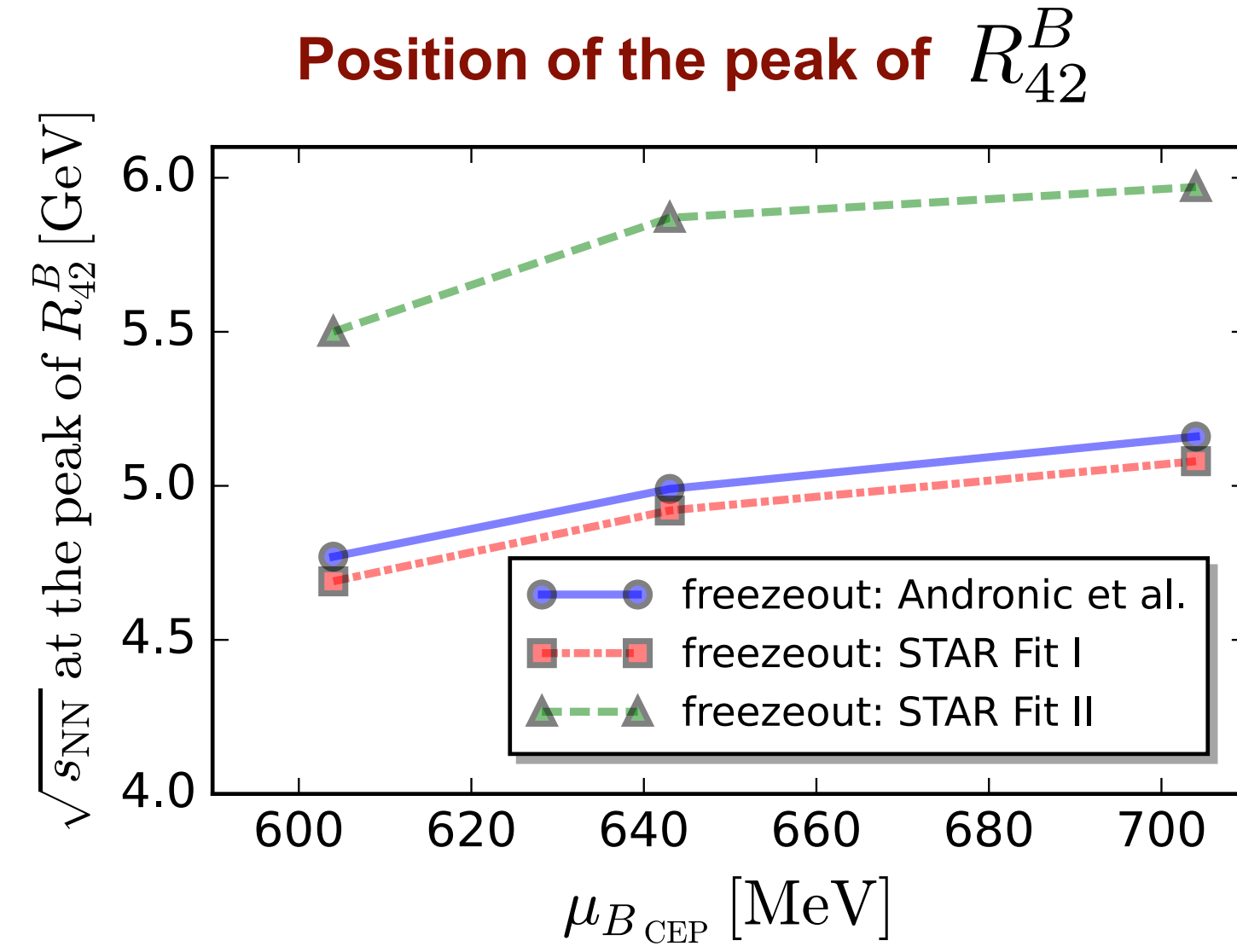
Reconstructing the CEP



Ripples of the critical point



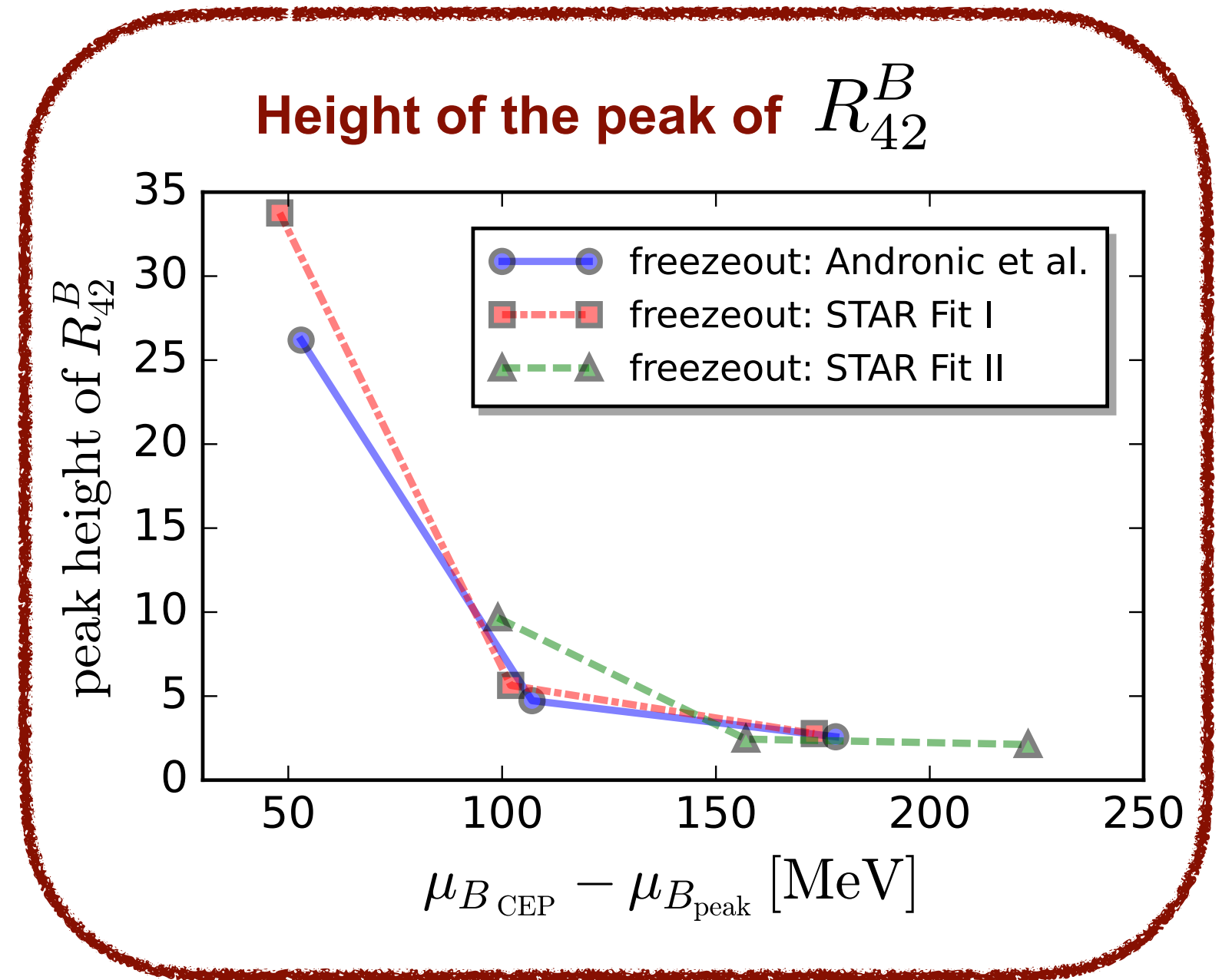
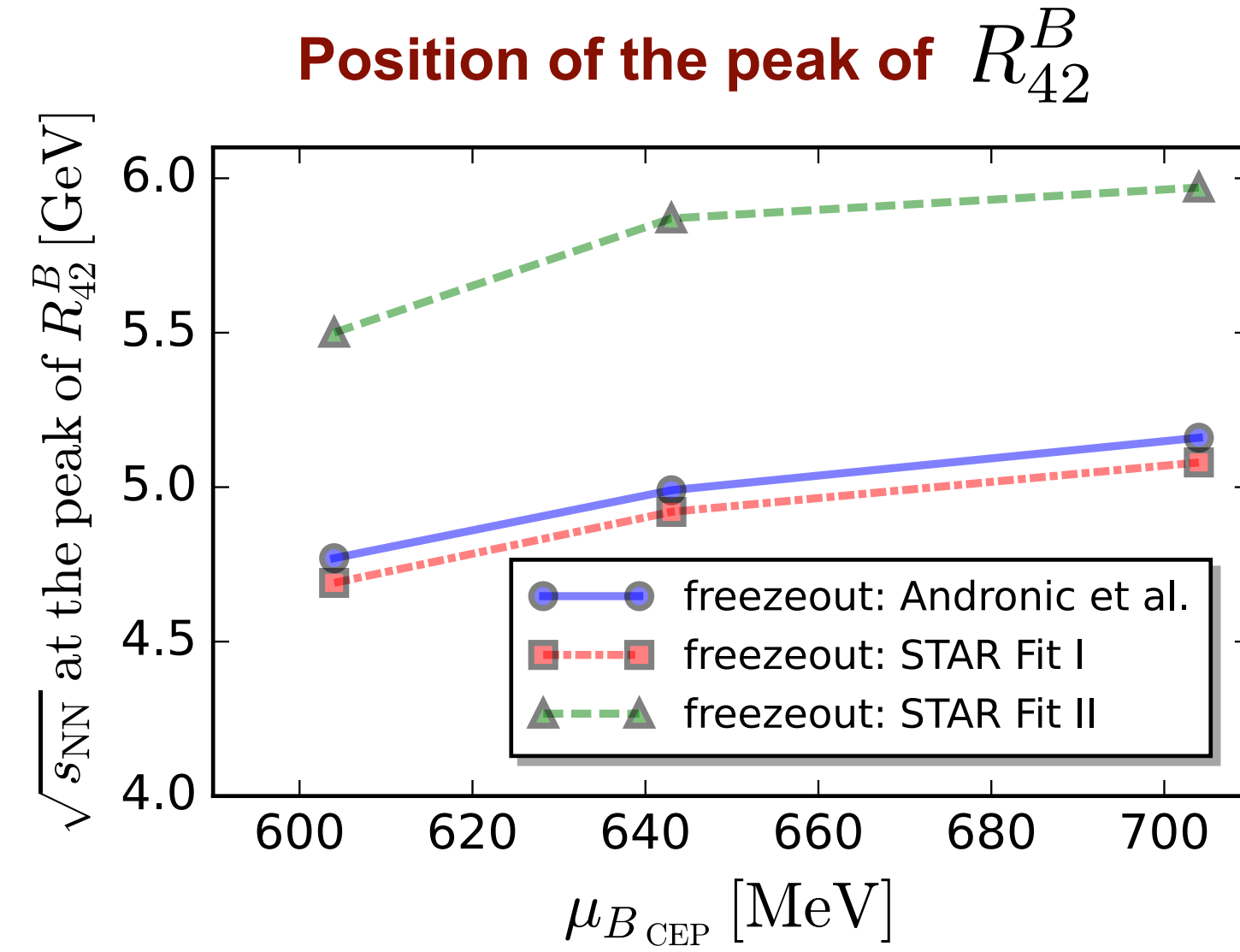
Reconstructing the CEP



Ripples of the critical point



Reconstructing the CEP

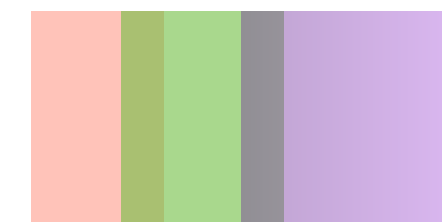
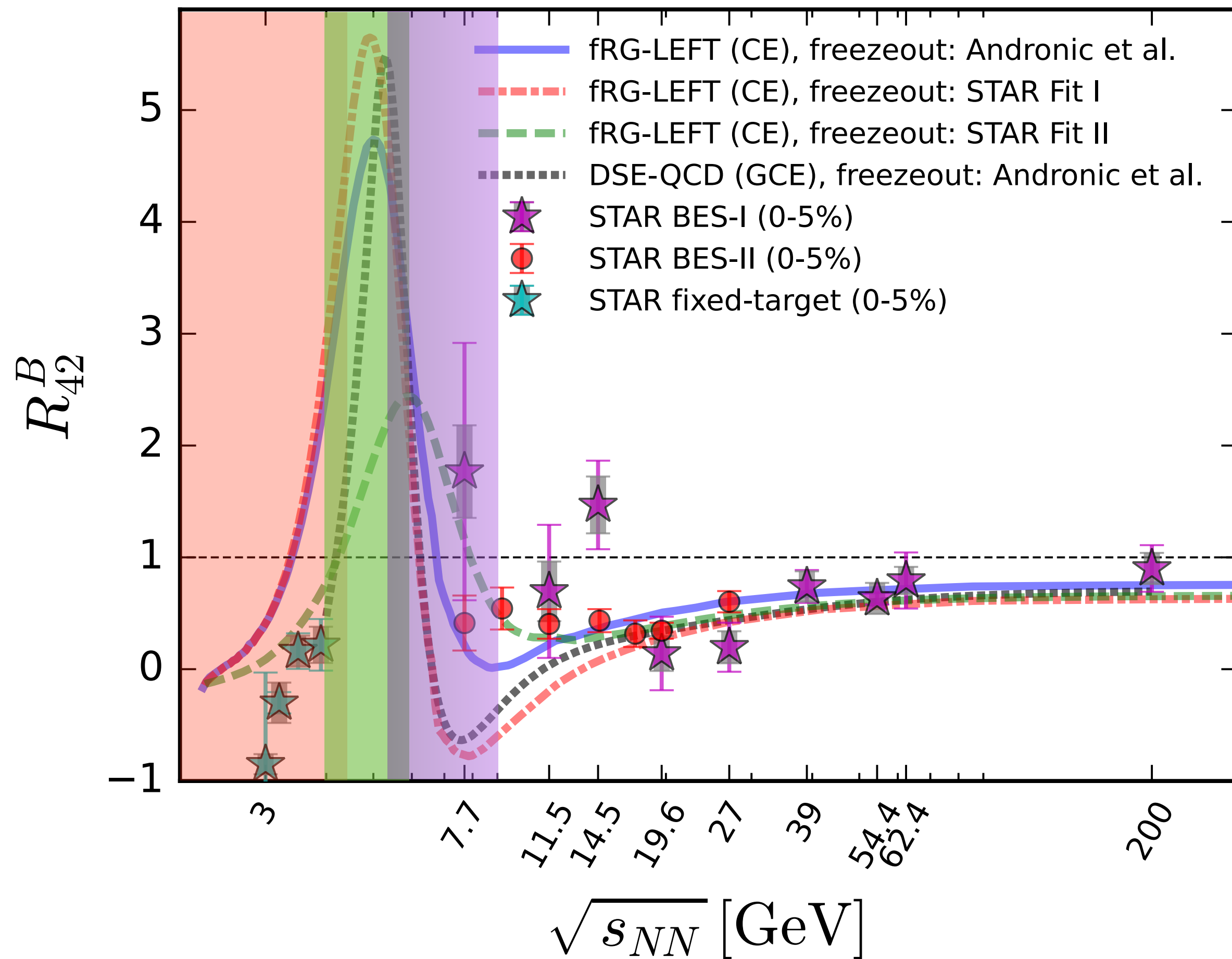


Unfolding the high density regime with new phases & physics

Great opportunity for a combined high precision analysis of high density QCD (Exp. data + lattice QCD + functional QCD)

Ripples of the critical point

net-baryon fluctuations in QCD vs net-proton fluctuations at STAR



Dominated by non-critical soft modes

Results:
1st principles functional QCD computations
& low energy effective theories/extrapolations

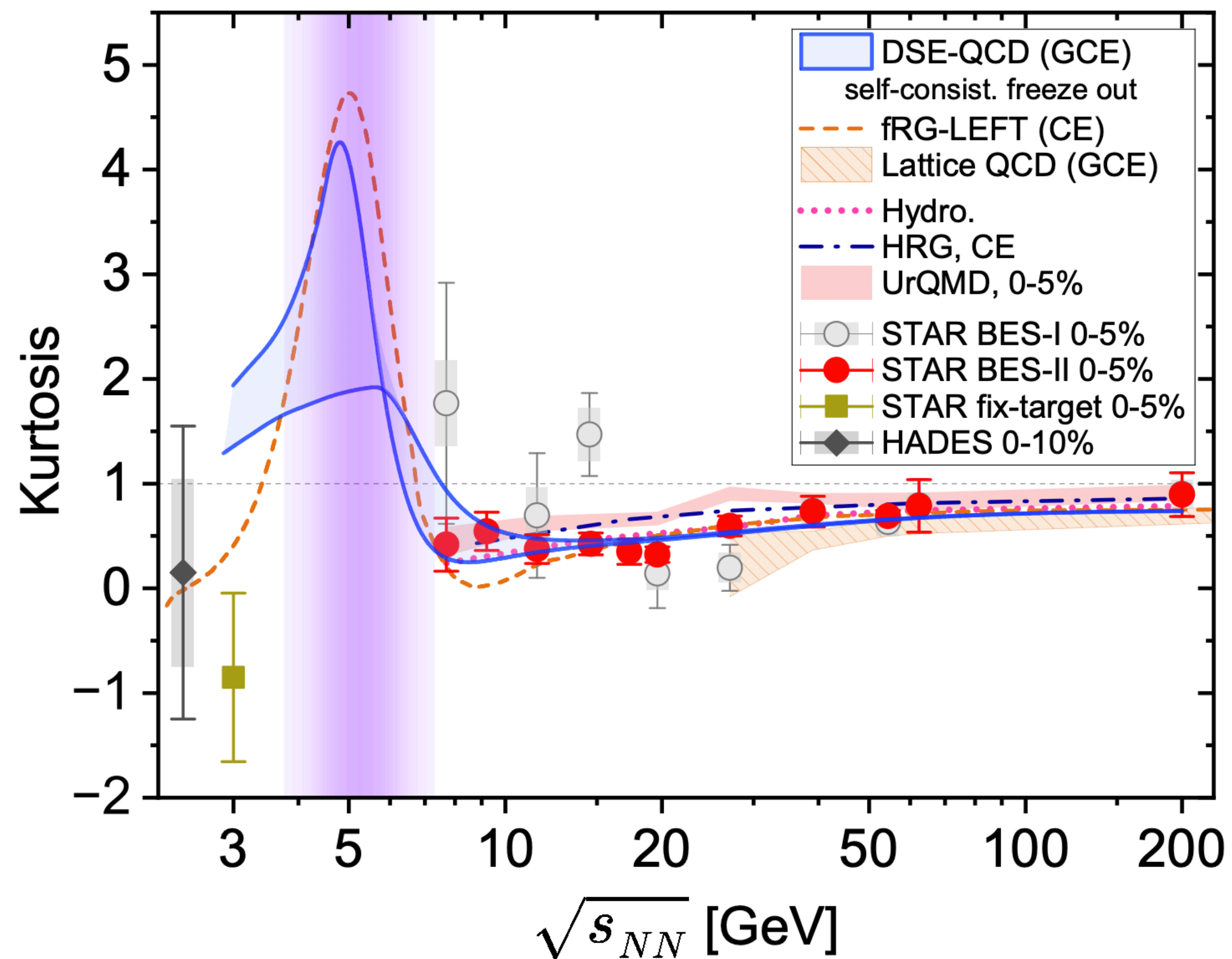
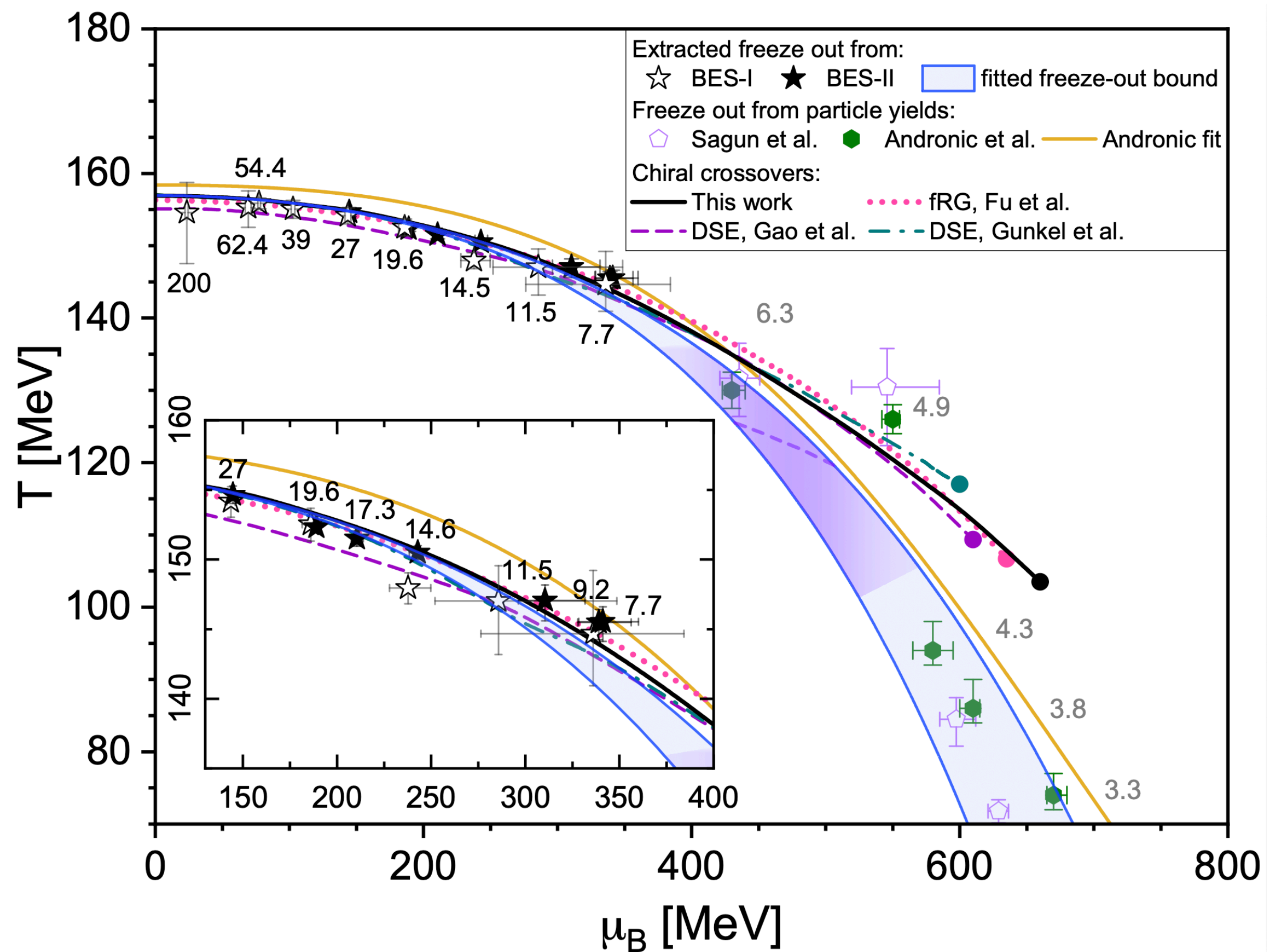
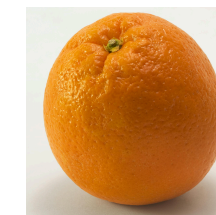
Fu, Luo, JMP, Rennecke, Wen, Yin, PRD 104 (2021) 9

Fu, Luo, JMP, Rennecke, Yin, PRD 111, L031502

Lu, Gao, Liu, JMP, PRD 113 (2026) 054019

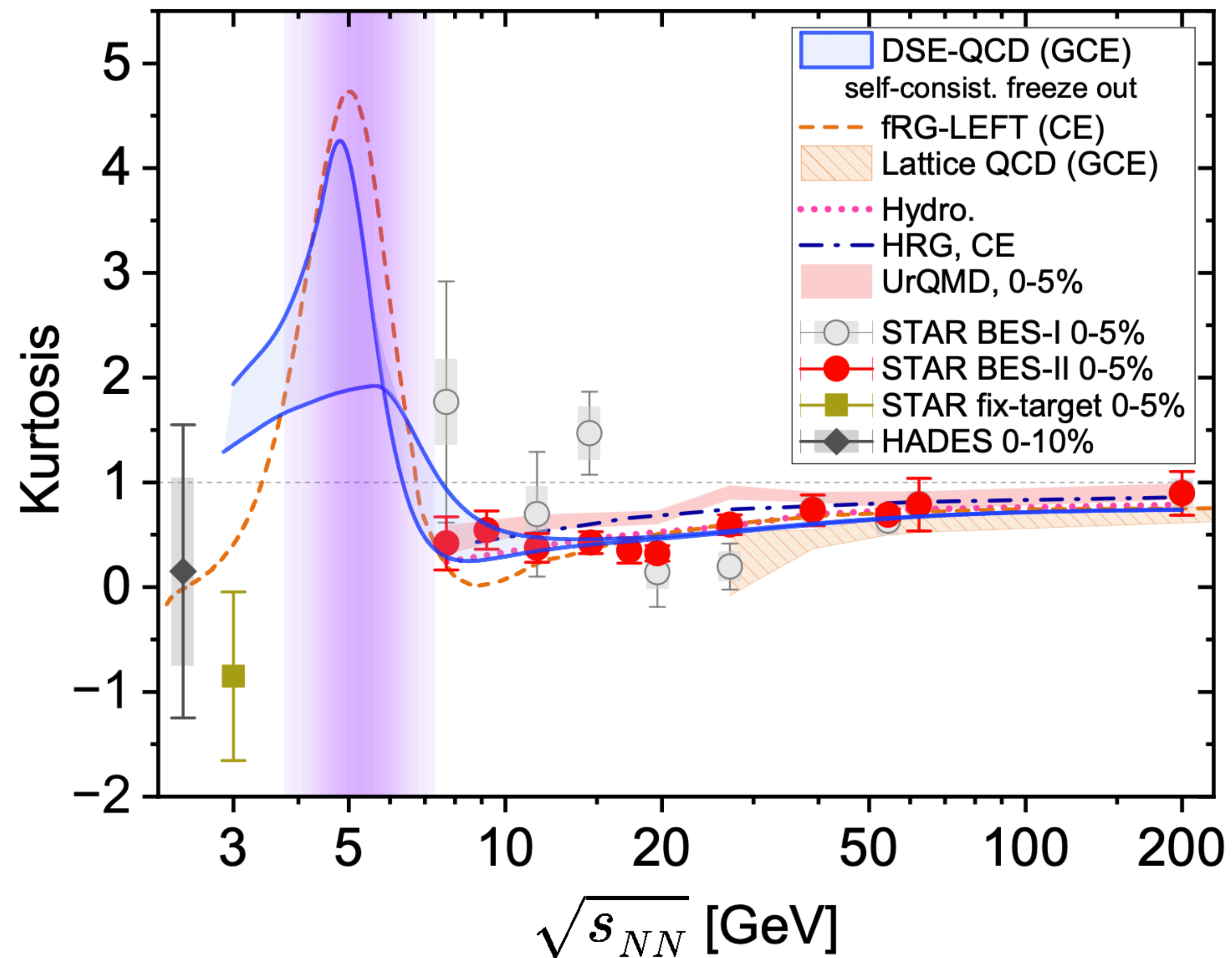
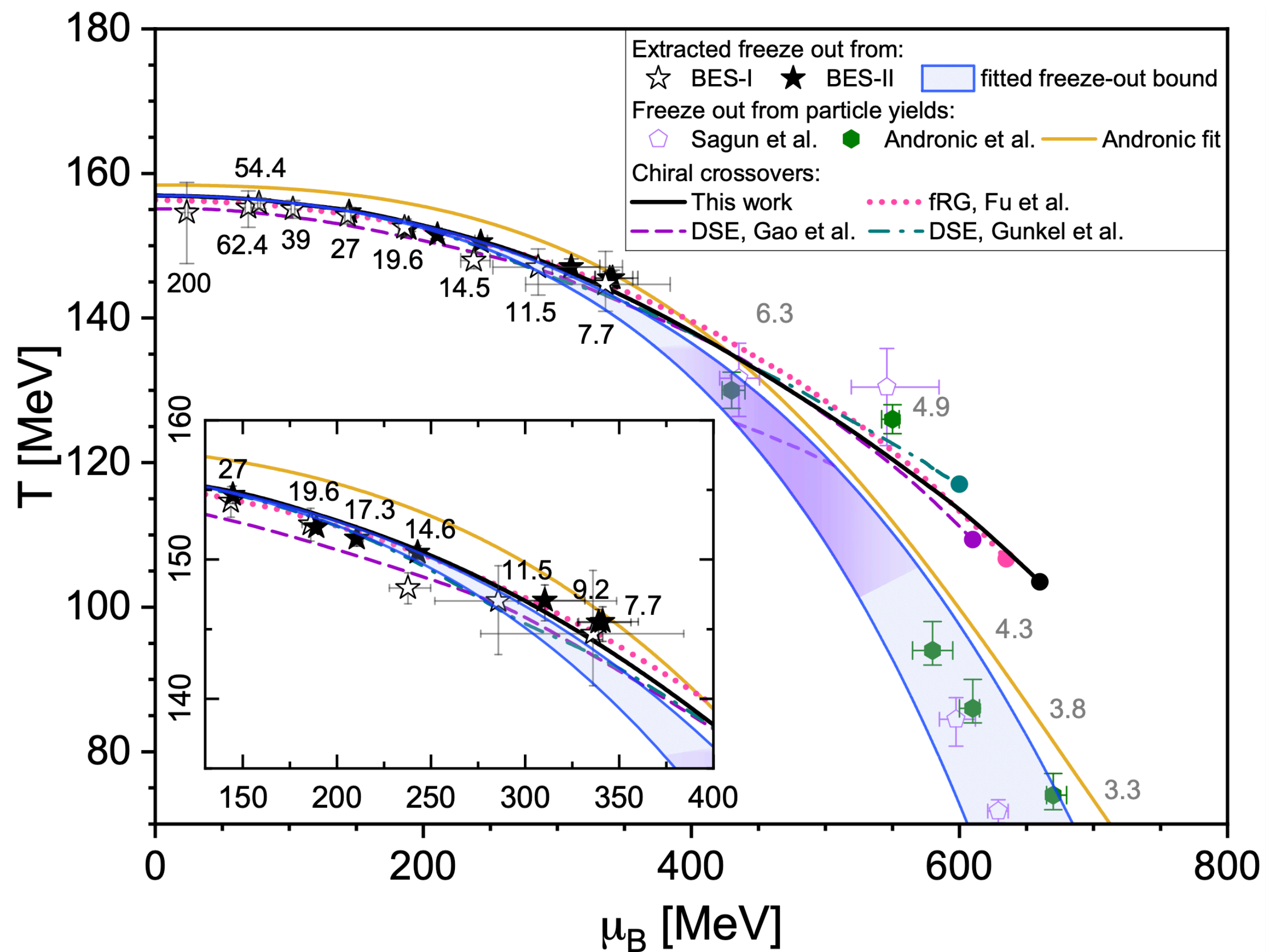
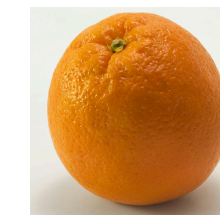


Ripples of the critical point at chemical freeze-out





Ripples of the critical point at chemical freeze-out



Demand: build in dynamics

EMMI workshop: Collective phenomena and the EoS of dense baryonic matter

Summary

Jump back to: Ripples of the CEP or the onset of new phases (ONP)

(I) Functional Renormalisation Group for QCD

- The renormalisation group is a one-loop exact functional approach
 - consistent RG-scaling ✓
 - systematic expansion schemes & error control ✓
 - compatibility with other functional approaches ✓
- Resonances via dynamical hadronisation
 - hadrons as exchange fields of quarks scattering vertices
 - BSE wave function  quark-hadron vertex
 - Stable dynamical emergence of low energy effective theories
- Quark-gluon-meson correlation functions
 - Self-consistent results: all correlation functions computed are fed back
 - Dynamical chiral symmetry breaking & confinement
 - Quantitative agreement with lattice results

(II) Functional QCD and the QCD phase structure

- QCD at finite temperature and density
 - all available benchmarks in the vacuum passed
 - confinement-deconfinement phase transition
 - compatibility with other functional approaches
- Locating the QCD phase structure and the critical end point
 - Quantitative predictions for $\mu_B/T \lesssim 4.5$, estimates for $\mu_B/T \lesssim 9$
 - Estimate for the location of the **CEP**: $(\mu_B, T)_{\text{CEP}} \sim (600 - 650, 105 - 115) \text{ MeV}$
 - Diquark domination for $\mu_B/T \gtrsim 8$
- Fluctuations of conserved charges: Ripples of the critical end point
 - QCD-assisted low energy effective theory with the phase structure of QCD & functional QCD
 - Quantitative agreement of the fluctuations of conserved charges with lattice results
 - Qualitative accounting for canonical effects with the sub-ensemble method
 - Remarkable compatibility with the new STAR data (baryon vs proton fluctuations, volume effects, ...)



