Truth and Beauty: Quantum Entanglement Theory and Its Searches at the Colliders

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Outline

Introduction

Quantum Entanglement Theory

The Specific Approach or Decay Approach

The Consistent Searches for Quantum Entanglement at the Colliders

The Cornerstones for Modern Physics

- Quantum Mechanics.
- Special Relativity.

Two Most Genuine Features of Quantum Mechanics

- Quantum entanglement.
- Bell's theorem.

Completeness, Realism, and Locality

- Completeness: Every element of physical reality must have a counterpart in the physical theory in order for the theory to be complete.
- Realism: If the value of a physical quantity can be predicted with certainty, *i.e.*, probability 1, without disturbing the system, then the quantity has physical reality.
- Locality: There is no action at a distance. Measurements on a (sub)system do not affect measurements on (sub)systems that are far away.

Quantum Entanglement

- EPR Paradox: in 1935, Einstein-Podolsky-Rosen demonstrated the conflict between local realism and quantum mechanics by considering quantum entangled states . They claimed that quantum mechanics is incomplete.
- In 1935, Schrödinger called it quantum entanglement, a characteristic of quantum mechanics.
- In 1949, Wu-Shaknov provided the first photon entanglement experiment, and obtained a clearly spatially separated quantum entangled state.
- In 1957, Bohm-Aharonov found that Wu-Shaknov experiment achieved the photon polarization correlation, and showed that the non-entangled state could not give such results. Also, they first presented the spin-spin realization of the EPR Paradox.

Local Hidden Variable Theory

- Einstein attempted to formulate a deterministic counter proposal to quantum mechanics, presenting a paper at a meeting of the Academy of Sciences in Berlin, on 5 May 1927, titled "Bestimmt Schrödinger's Wellenmechanik die Bewegung eines Systems vollständig oder nur im Sinne der Statistik?" ("Does Schrödinger's wave mechanics determine the motion of a system completely or only in the statistical sense?"). However, as the paper was being prepared for publication in the academy's journal, Einstein decided to withdraw it, possibly because he discovered that, contrary to his intention, his use of Schrödinger's field to guide localized particles allowed just the kind of non-local influences he intended to avoid.

Local Hidden Variable Theory

- At the Fifth Solvay Congress, held in Belgium in October 1927 and attended by all the major theoretical physicists of the era, Louis de Broglie presented his own version of a deterministic hidden-variable theory, apparently unaware of Einstein's aborted attempt earlier in the year. In his theory, every particle had an associated, hidden "pilot wave" which served to guide its trajectory through space. The theory was subject to criticism at the Congress, particularly by Wolfgang Pauli, which de Broglie did not adequately answer; de Broglie abandoned the theory shortly thereafter.

Local Hidden Variable Theory

- John von Neumann in his 1932 book Mathematical Foundations of Quantum Mechanics had presented a proof that there could be no "hidden parameters" in quantum mechanics. The validity of von Neumann's proof was questioned by Grete Hermann in 1935, who found a flaw in the proof. The critical issue concerned averages over ensembles.

Local Hidden Variable Theory

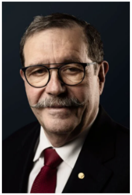
- In 1952, David Bohm proposed a hidden variable theory. Bohm unknowingly rediscovered (and extended) the idea that Louis de Broglie's pilot wave theory had proposed in 1927 (and abandoned) – hence this theory is commonly called "de Broglie-Bohm theory". Assuming the validity of Bell's theorem, any deterministic hidden-variable theory that is consistent with quantum mechanics would have to be non-local, maintaining the existence of instantaneous or faster-than-light relations (correlations) between physically separated entities.

Bell's Theorem

- In 1964, Bell proposed the inequality satisfied by local realism or local hidden-variable theory.
- Quantum entangled states might violate Bell's inequality, but polarization needs to be measured in a direction that is neither parallel nor perpendicular.
- It is not easy to measure the polarizations of high energy photons directly.
- Over decades, quantum entanglement and Bell non-locality have been rigorously confirmed through various experiments violating Bell inequalities and demonstrations of quantum teleportation, primarily in low-energy systems such as photons, atoms, and solid-state qubits.

Nobel Prize in Physics 2022

Nobel Prize in Physics 2022



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Alain Aspect

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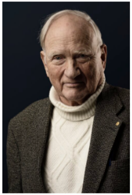

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John F. Clauser

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The Nobel Prize in Physics 2022 was awarded jointly to Alain Aspect, John F. Clauser and Anton Zeilinger "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science"

Quantum Entanglement and Bell's Theorem

- Quantum entanglement and Bell non-locality are closely related. However, there is a subtle difference: quantum entanglement is a necessary condition for Bell non-locality, but not a sufficient condition.
- The exploration of entanglement in high-energy particle physics remains an emerging frontier, where the interplay between quantum correlations and relativistic dynamics opens new avenues to probe fundamental physics.

High Energy Physics

- The foundation of the Stanard Model (SM) is quantum field theory, which is based on quantum mechanics and special relativity.
- We can probe the fundamental properties of quantum mechanics at various high energy physics experiments such as colliders.
- The quantum entanglement in top quark-antiquark ($t\bar{t}$) system has been observed by the ATLAS and CMS Collaborations [1].
- The polarization state of the $t\bar{t}$ system is encoded in its spin density matrix ρ

$$\rho = \frac{I_4 + \sum_i \left(B_i^+ \sigma^i \otimes I_2 + B_i^- I_2 \otimes \sigma^i \right) + \sum_{i,j} C_{ij} \sigma^i \otimes \sigma^j}{4} .$$

[1] G. Aad *et al.* [ATLAS], Nature **633**, no.8030, 542-547 (2024) [arXiv:2311.07288 [hep-ex]]; A. Hayrapetyan *et*

Spin Density Matrix

$$\rho = \frac{1}{4}\begin{bmatrix} 1+B_3^+ + B_3^- + C_{33} & B_1^- + C_{31} - i(B_2^- + C_{32}) & B_1^+ + C_{13} - i(B_2^+ + C_{23}) & C_{11} - C_{22} - i(C_{12} + C_{21}) \\ B_1^- + C_{31} + i(B_2^- + C_{32}) & 1 + B_3^+ - B_3^- - C_{33} & C_{11} + C_{22} + i(C_{12} - C_{21}) & B_1^+ - C_{13} - i(B_2^+ - C_{23}) \\ B_1^+ + C_{13} + i(B_2^+ + C_{23}) & C_{11} + C_{22} + i(C_{21} - C_{12}) & 1 - B_3^+ + B_3^- - C_{33} & B_1^- - C_{31} - i(B_2^- - C_{32}) \\ C_{11} - C_{22} + i(C_{21} + C_{12}) & B_1^+ - C_{13} + i(B_2^+ - C_{23}) & B_1^- - C_{31} + i(B_2^- - C_{32}) & 1 - B_3^+ - B_3^- + C_{33} \end{bmatrix}$$

B_i^+ and B_i^- are respectively the spin polarizations of t and $\bar{t}$, and C_{ij} is the spin correlation matrix. Under CP invariance, we obtain $B_i^+ = B_i^-$, and $C_{ij} = C_{ji}$.

High Energy Physics

- For the decay processes $t \to e^+ + \nu_e + b$ and $\bar{t} \to e^- + \bar{\nu}_e + \bar{b}$, the angular distributions of the e^+ and e^- in their parent particles' rest frames are

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega_+ d\Omega_-} = \frac{1 + \mathbf{B}^+ \cdot \hat{\mathbf{q}}_+ - \mathbf{B}^- \cdot \hat{\mathbf{q}}_- - \hat{\mathbf{q}}_+ \cdot \mathbf{C} \cdot \hat{\mathbf{q}}_-}{(4\pi)^2} \ .$$

- The LHC experiments utilized the observable

$$D = \mathrm{tr}\left[\mathbf{C}\right]/3 = -3\langle \cos\theta_{e^-e^+}\rangle \ ,$$

and $\cos\theta_{e^-e^+} = \hat{\mathbf{q}}_+ \cdot \hat{\mathbf{q}}_-$.
- A value of $D < -\frac{1}{3}$ is a sufficient (though not necessary) condition for quantum entanglement.
- At the e^-e^+ colliders, the leading-order (LO) calculations predict $D \equiv \frac{1}{3}$, independent of beam energy, polarization, or the top quarks' emission angles.

Truth and Beauty: Quantum Entanglement Theory

- The criteria for quantum entanglement and Bell non-localities in the previous studies are all defined via the inequalities [2], and are not universal. Why?
- **Truth**: the quantum entanglement and Bell non-localities have been confirmed in low-energy systems such as photons, atoms, and solid-state qubits.
- Fundamental Principles in Physics: Simplicity, Naturality, Universality, and Beauty.
- **Beauty**: there might exist a quantum entanglement theory which is indeed beautiful?

[2] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880-884 (1969); R. Horodecki, P. Horodecki and M. Horodecki, Phys. Lett. A **200**, no.5, 340-344 (1995); A. Peres, Phys. Rev. Lett. **77**, 1413-1415 (1996) [arXiv:quant-ph/9604005 [quant-ph]]. M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A **223**, 1 (1996) [arXiv:quant-ph/9605038 [quant-ph]].

Mathematical Beauty

- Paul Dirac summarized his philosophy of physics at the Moscow State University in 1956:
  Physical Laws Should Have Mathematical Beauty.
- Jules H. Poincare
  The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful.
  If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living.

Quantum Entanglement Theory and Its Generic Searches

- A great challenge question: can we propose a beautiful Quantum Entanglement Theory (QET) which can define the quantum entanglement exactly?
- We need to provide the solid foundation for quantum entanglement, and probe it via a fundamental approach at the exact level in general.
- Similarly, for any specific approach to probe the quantum entanglement, we need to define the corresponding quantum entanglement criterion exactly as well.

Quantum range, classical range, and quantum entanglement range.

Idea

- Mathematics: Think It Geometrically, Prove It Algebraically.
- The parameter space for a quantum complex system is compact. We define the total spin polarization parameter space as the quantum space, and the spin polarization parameter space for classical theory as the classical space.
- The classical space is a subspace in the quantum space. Thus, we can describe the classical space via the algebraic equations in the quantum space.
- The quantum entanglement space is the difference between the quantum space and classical space, *i.e.*, the quantum space minus classical space.

Outline

Spin Density Matrix ρ

- The trace of ρ is unity: $\mathrm{Tr}\rho = 1$.
- ρ is a Hermitian matrix, and then $\rho^D = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$.
- All the diagonal elements are non-negative.
- ρ satisfies the Schwarz inequality: $|\rho_{ij}|^2 \leq \rho_{ii}\rho_{jj}$.
- $\mathrm{Tr}\rho^2 = \sum_{m=1}^{n} \lambda_m^2 \leq (\sum_{m=1}^{n} \lambda_m)^2 = (\mathrm{Tr}\rho)^2 = 1$.
- For pure and mixed states, we have $n = 1$ and $n > 1$, respectively.

The Classical Correlated State and Quantum Entangled State

Definition of the Classical Correlated State.

The state ρ acting on the Hilbert space $H = H_1 \otimes H_2$ is separable if it can be approxiamted in the trace norm by the states of the form [3]

$$\rho \equiv \sum_{m=1}^{n} p_m \rho_m \ , \quad \text{where} \quad \rho_m \equiv \rho_m^1 \otimes \rho_m^2 \ .$$

where $n \leq N^2$, $N = \dim H$, ρ_m is a pure product state, and ρ_m^1 and ρ_m^2 are the states in H_1 and H_2, respectively.

Definition of the Quantum Entangled State.

The state is not separable.

[3] R. F. Werner, Phys. Rev. A **40**, 4277-4281 (1989); P. Horodecki, Phys. Lett. A **232**, 333 (1997) [arXiv:quant-ph/9703004 [quant-ph]].

Quantum Entanglement Theory

- Based on $SU(N)$ group theory or complex projective space, we can formulate the quantum entanglement theory.
- The quantum entanglement criteria can be derived from its definition.
- The subtle issue: how to define the Bell non-locality in general?

The Two-Fermion $f_1 f_2$ System

The Werner states with a free parameter w for the spin density matrix are

$$\rho_{\mathrm{W}} = \begin{pmatrix} \frac{1-w}{4} & 0 & 0 & 0 \\ 0 & \frac{1+w}{4} & -\frac{w}{2} & 0 \\ 0 & -\frac{w}{2} & \frac{1+w}{4} & 0 \\ 0 & 0 & 0 & \frac{1-w}{4} \end{pmatrix} , \; w \in \left[-\frac{1}{3}, \; 1 \right] \; .$$

The Two-Fermion $f_1 f_2$ System

- Using the Peres-Horodecki criterion, we can prove that the Werner states are classical or separable for $w \leq \frac{1}{3}$, and quantum entangled for $w > \frac{1}{3}$.
- The Bell variable for the CHSH inequality for a Werner state is $2\sqrt{2}\,|w|$, *i.e.*, $\mathcal{B} = 2\sqrt{2}\,|w|$. And thus we can realize Bell non-locality for $w > \frac{1}{\sqrt{2}}$.
- For the Werner states, the quantum entanglement range is $w \in \left(\frac{1}{3},\ 1\right]$, and the Bell non-locality range is $w \in \left(\frac{1}{\sqrt{2}},\ 1\right]$.

The Two-Fermion $f_1 f_2$ System

- The condition $w = \frac{1}{3}$ is the condition that the spin density matrices for f_1 and f_2 satisfy the non-negative condition.
- The upper bound $w = 1$ is the condition that the spin density matrix for the two-fermion $f_1 f_2$ system satisfies the non-negative condition.
- These conditions are determined by the pure states.
- We can define the Bell non-locality for specific cases, but no universal formular. Why?

The Bell Non-Locality via Super-Activation

- Theorem. If ρ for a quantum entangled state is local and satisfy the Bell inequality, $\rho \otimes \rho$ can be non-local and violate the Bell inequality [4].
- Conjecture: The quantum entanglement and Bell non-locality might be equivalent.
- Gisin theorem: They are equivalent for the pure states.
- The quantum entanglement is the fundamental property of the quantum mechanics.

[4] C. Palazuelos, Phys. Rev. Lett. **109**, no.19, 190401 (2012), and references therein.

The Quantum Entanglement Criteria

- We will study the quantum entanglement criteria for pure states.
- Any separable state can be written as a convex combination of finite pure separable states

$$\rho \equiv \sum_{m=1}^{n} p_m \rho_m \ .$$

- With the quantum entanglement criteria for pure states, we can obtain the quantum entanglement criteria for mixed states, which can be given as a set of algebraic equations.
- Application in high energy physics: the quantum entanglement criteria at the lepton colliders can be super simple.

Quantum entanglement criteria for pure states.

Complex Projective Space

- The complex projective space $\mathbb{CP}^n$ is the $(n+1)$-dimensional complex space $\mathbb{C}^{n+1}\backslash\{0\}$ modulo the following equivalent classes

$$z \sim w \ \text{ iff } \ \exists \lambda \in \mathbb{C}\backslash\{0\},\ w = \lambda z \ ,$$

$$(z_0, z_1, z_2, ..., z_n) \sim (\lambda z_0, \lambda z_1, \lambda z_2, ..., \lambda z_n) \ \text{ for all } \ \lambda \in \mathbb{C}\backslash\{0\} \ .$$

- In physics, we assume $|z| = 1$, and then have $|\lambda| = 1$.
- The complex dimension of $\mathbb{CP}^n$ is n, or say the real dimension of $\mathbb{CP}^n$ is $2n$.
- We have $\mathbb{CP}^n \simeq S^{2n+1}/S^1$, where S^n is n-dimensional sphere.
- In particular, we can prove that $\mathbb{CP}^1$ is diffeomorphic to S^2.

Spin Polarization Space

- For a particle P with spin s, the polarization state for its spin (or helicity) space is

$$|P\rangle = \sum_{i=-s}^{s} z_i |i\rangle \ , \quad \sum_{i=-s}^{s} |z_i|^2 = 1 \ .$$

- Two polarization states are equivalent if their coefficients z_i and z_i' satisfy the following equivalent relation

$$(z_{-s}, z_{-s+1}, ..., z_s) \sim (\lambda z'_{-s}, \lambda z'_{-s+1}, ..., \lambda z'_s)$$
$$\text{for all} \ \ \lambda \in \mathbb{C} \ \ \text{and} \ \ |\lambda| = 1 \ .$$

Therefore, we prove that the spin polarization space for a particle P with spin s is $\mathbb{CP}^{2s}$!

Spin Polarization Space

- For a general quantum system with N particles with spin s_1, s_2, ..., and s_n, we obtain that the quantum space (the total spin polarization parameter space) is complex projective space $\mathbb{CP}^{J-1}$ with $J = (2s_1 + 1) \times (2s_2 + 1) \times ... \times (2s_n + 1)$.
- The classical space (the spin polarization parameter space for classical theory) is the cartesian product of the complex projective spaces $\mathbb{CP}^{2s_1} \times \mathbb{CP}^{2s_2} \times ... \times \mathbb{CP}^{2s_n}$.
- In mathematics, the classical space is the (generalized) Segre variety in the quantum space.
- The quantum entanglement space is the difference of these two spaces: quantum space minus classical space.

Spin Polarization Space

- With the property of Cartesian product, we propose the discriminants Δ_i, which are degree 2 homogeneous and holomorphic functions. Thus, the Number of Independent Discriminants (NID) is

$$\mathrm{NID} = J - 1 - \sum_i 2s_i = \Pi_i(2s_i + 1) - 1 - \sum_i 2s_i \ .$$

- We define the corresponding classical spaces as the discriminant locus $\Delta = 0$ for *ff* system, and the intersections of the discriminant loci $\Delta_i = 0$ for all the other systems in the quantum space.
- We define the quantum entanglement spaces as the quantum space with $\Delta \neq 0$ for *ff* system, and the quantum spaces without the intersections of the discriminant loci $\Delta_i = 0$ for all the other systems.

High Energy Physics

- For high energy physics experiments, we can reconstruct the discriminants from various measurements, and probe the quantum entanglement spaces at exact level.
- We can perform such kind of studies in some two-fermion systems, but in general it might be very difficult.
- For classification, this kind of quantum entanglement search can be defined as the fundamental approach, or say kinematic approach.
- To probe the quantum no-locality, we just consider the space-like separated measurements.

High Energy Physics

- We shall study the discriminants in the ff, AA, Af, fff, and ffA systems.
- To be general, we will not distinguish the fermion (f) and anti-fermion ($\bar{f}$).
- We only consider the massive gauge bosons since they need to decay.

The Two-Fermion $f_1 f_2$ System

- The most general polarization state of a two-fermion system $f_1 f_2$ can be written as

$$|f_1 f_2\rangle = \sum_{k,j=\pm\frac{1}{2}} \alpha_{k,j} |k\rangle_{f_1} \otimes |j\rangle_{f_2} \ , \quad \sum_{k,j=\pm\frac{1}{2}} |\alpha_{k,j}|^2 = 1 \ .$$

- In the two-fermion system $f_1 f_2$, the quantum space is $\mathbb{CP}^3$ with complex dimension 3, and the classical space is $\mathbb{CP}^1 \otimes \mathbb{CP}^1$ with complex dimension 2. Thus, there is one discriminant with complex dimension 1

$$\Delta = \alpha_{\frac{1}{2},\frac{1}{2}} \alpha_{-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2}} \alpha_{-\frac{1}{2},\frac{1}{2}} \ .$$

- We can prove that the range of the discriminant Δ is

$$-\frac{1}{2} \leq \Delta \leq \frac{1}{2} \ .$$

The Two-Fermion $f_1 f_2$ System

- We define

$$\alpha_{\frac{1}{2},\frac{1}{2}} \equiv |\alpha_{\frac{1}{2},\frac{1}{2}}| e^{i\eta_{++}} ~,~ \alpha_{\frac{1}{2},-\frac{1}{2}} \equiv |\alpha_{\frac{1}{2},-\frac{1}{2}}| e^{i\eta_{+-}} ~,$$
$$\alpha_{-\frac{1}{2},\frac{1}{2}} \equiv |\alpha_{-\frac{1}{2},\frac{1}{2}}| e^{i\eta_{-+}} ~,~ \alpha_{-\frac{1}{2},-\frac{1}{2}} \equiv |\alpha_{-\frac{1}{2},-\frac{1}{2}}| e^{i\eta_{--}} ~.$$

- The determinant Δ as two real functions Δ_M and Δ_P

$$\Delta_M = |\alpha_{\frac{1}{2},\frac{1}{2}}||\alpha_{-\frac{1}{2},-\frac{1}{2}}| - |\alpha_{\frac{1}{2},-\frac{1}{2}}||\alpha_{-\frac{1}{2},\frac{1}{2}}| ~,$$

$$\Delta_P = \eta_{++} + \eta_{--} - \eta_{+-} - \eta_{-+} ~.$$

- The classical space is the discriminant locus $\Delta = 0$ in quantum space, or say the quantum space with both $\Delta_M = 0$ and $\Delta_P = 0$.

The Two-Fermion $f_1 f_2$ System

- The quantum entanglement space is the quantum space with $\Delta \neq 0$, or $\Delta_M \neq 0$, or $\Delta_P \neq 0$. In particular, even if $\Delta_M = 0$, we might still probe the quantum entanglement space with $\Delta_P \neq 0$.
- We can reconstruct Δ from the collider experiments, for example, the $\Lambda\overline{\Lambda}$ pair productions at the BES experiment. Thus, we can probe the quantum entanglement space via the fundamental approach.
- If the measurements are space-like separated, we can probe the quantum non-locality as well.

The Two-Fermion $f_1 f_2$ System

- We study the relation between our discriminant criterion and the Peres-Horodecki criterion [5].
- In the basis $\left(|\frac{1}{2}\rangle_{f_1} \otimes |\frac{1}{2}\rangle_{f_2}, |\frac{1}{2}\rangle_{f_1} \otimes |-\frac{1}{2}\rangle_{f_2}, |-\frac{1}{2}\rangle_{f_1} \otimes |\frac{1}{2}\rangle_{f_2}, |-\frac{1}{2}\rangle_{f_1} \otimes |-\frac{1}{2}\rangle_{f_2}\right)$, we obtain the spin density matrix.
- Taking partial transpose of ρ, *i.e.*, the transposes of the four 2×2 sub-matrices of ρ, we obtain ρ^{T_2}.

[5] A. Peres, Phys. Rev. Lett. **77**, 1413-1415 (1996) [arXiv:quant-ph/9604005 [quant-ph]]; P. Horodecki, Phys. Lett. A **232**, 333 (1997) [arXiv:quant-ph/9703004 [quant-ph]].

Spin Density Matrix

$$\rho = |f_1 f_2\rangle\langle f_1 f_2| = \begin{pmatrix} |\alpha_{\frac{1}{2},\frac{1}{2}}|^2 & \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} & \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}} & \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} \\ \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} & |\alpha_{\frac{1}{2},-\frac{1}{2}}|^2 & \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}} & \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} \\ \alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} & \alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} & |\alpha_{-\frac{1}{2},\frac{1}{2}}|^2 & \alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} \\ \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} & \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} & \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}} & |\alpha_{-\frac{1}{2},-\frac{1}{2}}|^2 \end{pmatrix}.$$

Spin Density Matrix

$$\rho^{\mathrm{T}_2} = \begin{pmatrix} |\alpha_{\frac{1}{2},\frac{1}{2}}|^2 & \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} & \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}} & \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}} \\ \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} & |\alpha_{\frac{1}{2},-\frac{1}{2}}|^2 & \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} & \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} \\ \alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} & \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} & |\alpha_{-\frac{1}{2},\frac{1}{2}}|^2 & \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}} \\ \alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} & \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} & \alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} & |\alpha_{-\frac{1}{2},-\frac{1}{2}}|^2 \end{pmatrix}.$$

The Two-Fermion $f_1 f_2$ System

- The original Peres-Horodecki criterion provides a sufficient and necessary condition for classical space: ρ^{T_2} is positive semi-definite.
- The four eigenvalues of ρ^{T_2} are

$$-|\Delta| \ , \quad |\Delta| \ , \quad \frac{1}{2}\left(1-\sqrt{1-4|\Delta|^2}\right) \ , \quad \frac{1}{2}\left(1+\sqrt{1-4|\Delta|^2}\right) \ .$$

- The original Peres-Horodecki criterion for classical space is $\Delta = 0$.

We prove that our criterion for classical space is equivalent to the original Peres-Horodecki criterion.

The Two-Fermion $f_1 f_2$ System for Real Case

- We have

$$C_{22} = \rho_{23} + \rho_{32} - \rho_{14} - \rho_{41}$$
$$= \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}} + \alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} - \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} .$$

- If $\alpha_{\frac{1}{2},\frac{1}{2}}$, $\alpha_{\frac{1}{2},-\frac{1}{2}}$, $\alpha_{-\frac{1}{2},\frac{1}{2}}$, and $\alpha_{-\frac{1}{2},-\frac{1}{2}}$ are all real for real case, we obtain

$$\Delta = -\frac{1}{2}C_{22} .$$

- The Bell inequalities and Bell non-localities have been studied in the literatures, and one can easily check that some of their results are somewhat similar to $\Delta = -\frac{1}{2}C_{22} \neq 0$ by defining the Bell variable as $\mathcal{B} \equiv 2\sqrt{1+4\Delta^2}$ or $\mathcal{B} \equiv 2\sqrt{1+C_{22}^2}$, no big difference.

The Two-Fermion $f_1 f_2$ System

- Bell inequality is an equation which distinguishes the Bell local quantum states (or general speaking Bell local states) and the Bell non-local quantum states.
- For a bipartite qubit system, it is the Clauser-Horne-Shimony-Holt (CHSH) inequality [6].
- Thus, we study the relation between our discriminant criterion and the CHSH inequality.
- For simplicity, we consider the equivalent definition of the CHSH inequality [7].

[6] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880-884 (1969)

[7] R. Horodecki, P. Horodecki and M. Horodecki, Phys. Lett. A **200**, no.5, 340-344 (1995).

$B_i^{\pm}$ and C_{ij}

$$
\begin{aligned}
B_1^+ &= \alpha_{\frac12,\frac12}\alpha^*_{-\frac12,\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,\frac12} + \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,-\frac12}\ , \\
B_2^+ &= i(\alpha_{\frac12,\frac12}\alpha^*_{-\frac12,\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,-\frac12} - \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,-\frac12})\ , \\
B_3^+ &= 2\alpha_{\frac12,\frac12}\alpha^*_{\frac12,\frac12} + 2\alpha_{\frac12,-\frac12}\alpha^*_{\frac12,-\frac12} - 1\ , \\
B_1^- &= \alpha_{\frac12,\frac12}\alpha^*_{\frac12,-\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{\frac12,\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{-\frac12,-\frac12}\alpha^*_{-\frac12,\frac12}\ , \\
B_2^- &= i(\alpha_{\frac12,\frac12}\alpha^*_{\frac12,-\frac12} - \alpha_{\frac12,-\frac12}\alpha^*_{\frac12,\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{-\frac12,\frac12})\ , \\
B_3^- &= 2\alpha_{\frac12,\frac12}\alpha^*_{\frac12,\frac12} + 2\alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,\frac12} - 1\ , \\
C_{11} &= \alpha_{\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,-\frac12} + \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,\frac12}\ , \\
C_{12} &= i(\alpha_{\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} - \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,-\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,\frac12})\ , \\
C_{13} &= \alpha_{\frac12,\frac12}\alpha^*_{-\frac12,\frac12} - \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,-\frac12}\ , \\
C_{21} &= i(\alpha_{\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} - \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,-\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,\frac12})\ , \\
C_{22} &= -\alpha_{\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,-\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,\frac12}\ , \\
C_{23} &= i(\alpha_{\frac12,\frac12}\alpha^*_{-\frac12,\frac12} - \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,-\frac12} - \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,\frac12} + \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,-\frac12})\ , \\
C_{31} &= \alpha_{\frac12,\frac12}\alpha^*_{\frac12,-\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{\frac12,\frac12} - \alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{-\frac12,\frac12}\ , \\
C_{32} &= i(\alpha_{\frac12,\frac12}\alpha^*_{\frac12,-\frac12} - \alpha_{\frac12,-\frac12}\alpha^*_{\frac12,\frac12} - \alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{-\frac12,-\frac12}\alpha^*_{-\frac12,\frac12})\ , \\
C_{33} &= 1 - 2\alpha_{\frac12,-\frac12}\alpha^*_{\frac12,-\frac12} - 2\alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,\frac12}\ .
\end{aligned}
$$

The Two-Fermion $f_1 f_2$ System

- If $\alpha_{k,j}$ are all real for real case, we obtain

$$C_{12} = C_{21} = C_{23} = C_{32} = 0 \ .$$

- Defining the matrix C as the matrix with elements C_{ij}, we obtain the eigenvalues of $C^{\mathrm{T}} C$

$$\lambda_1 = 1 \ , \quad \lambda_2 = \lambda_3 = 4 \left|\Delta\right|^2 \ .$$

- Note that $0 \leq |\Delta| \leq \frac{1}{2}$, the ranges of $\lambda_{2,3}$ are given by

$$0 \leq \lambda_{2,3} \leq 1 \ .$$

- Thus, we obtain the Bell variable for the CHSH inequality

$$\mathcal{B} \equiv 2\sqrt{1 + 4|\Delta|^2} \ .$$

The Two-Fermion $f_1 f_2$ System

- Because $0 \leq |\Delta| \leq \frac{1}{2}$, we prove

$$2 \leq \mathcal{B} \leq 2\sqrt{2} \ .$$

- The CHSH inequality is $\mathcal{B} \leq 2$, and thus for the Bell local parameter space the CHSH inequality becomes the CHSH criterion $\mathcal{B} = 2$, *i.e.*, $\Delta = 0$. The CHSH inequality is violated if and only if $\mathcal{B} > 2$, *i.e.*, $\Delta \neq 0$.
- Thus, our discriminant criterion for classical space is the same as the CHSH criterion for the Bell local parameter space, and our discriminant criterion for quantum entanglement space is the same as the CHSH criterion for the Bell non-local parameter space.

The Two-Fermion $f_1 f_2$ System

- Therefore, we prove that our classical space is the same as the Bell local parameter space, and our quantum entanglement space is the same as the Bell non-local parameter space.
- In particular, our quantum entanglement space is Bell non-local in high energy physics.
- To distinguish the classical space and Bell local parameter space, or distinguish the quantum entanglement space and Bell non-local parameter space, we consider the Werner states.
- Because our discriminant criterion is the same as the CHSH criterion, we need to prove that the Werner state, which satisfies the Peres-Horodecki criterion for quantum entanglement space and the CHSH criterion for Bell local parameter space, does not exist in our quantum entanglement space.

The Two-Fermion $f_1 f_2$ System

The Werner states with a free parameter w for the spin density matrix are

$$\rho_{\rm W} = \begin{pmatrix} \frac{1-w}{4} & 0 & 0 & 0 \\ 0 & \frac{1+w}{4} & -\frac{w}{2} & 0 \\ 0 & -\frac{w}{2} & \frac{1+w}{4} & 0 \\ 0 & 0 & 0 & \frac{1-w}{4} \end{pmatrix} ~,~ w \in \left[-\frac{1}{3},~1\right] ~.$$

The Two-Fermion $f_1 f_2$ System

- Using the Peres-Horodecki criterion, we can prove that the Werner states are classical or separable for $w \leq \frac{1}{3}$, and quantum entangled for $w > \frac{1}{3}$.
- The Bell variable for the CHSH inequality for a Werner state is $2\sqrt{2}\,|w|$, *i.e.*, $\mathcal{B} = 2\sqrt{2}\,|w|$. And thus we can realize Bell non-locality for $w > \frac{1}{\sqrt{2}}$.
- For the Werner states, the quantum entanglement range is $w \in \left(\frac{1}{3},\ 1\right]$, and the Bell non-locality range is $w \in \left(\frac{1}{\sqrt{2}},\ 1\right]$.

The Two-Fermion $f_1 f_2$ System

- Comparing with our spin density matrix, we can easily show that the Werner states with $w \in \left(\frac{1}{3},\ \frac{1}{\sqrt{2}}\right]$ cannot be realized.
- We can only achieve the Werner state with $w = 1$.
- We prove that the Werner state, which satisfies the Peres-Horodecki criterion for quantum entanglement space and the CHSH criterion for Bell local parameter space, does not exist in our quantum entanglement space.
- By the way, if $\alpha_{k,j}$ are all real for real case, we obtain

$$\mathcal{B} \equiv 2\sqrt{1 + C_{22}^2}\ .$$

The Two-Gauge Boson AA System

- The most general polarization state of a physical system with two gauge bosons $A_1 A_2$ can be written as

$$|A_1 A_2\rangle = \sum_{j,k=1,0,-1} \alpha_{j,k} |j\rangle_{A_1} \otimes |k\rangle_{A_2} \ , \quad \sum_{j,k=1,0,-1} |\alpha_{j,k}|^2 = 1 \ .$$

- In the two-gauge boson system $A_1 A_2$, the quantum space is $\mathbb{CP}^8$ with complex dimension 8, and the classical space is $\mathbb{CP}^2 \otimes \mathbb{CP}^2$ with complex dimension 4. Thus, there are four independent discriminants with complex dimension 1.

The Two-Gauge Boson *AA* System

We define the general discriminants as

$$\Delta_1 = \alpha_{1,1}\alpha_{0,0} - \alpha_{1,0}\alpha_{0,1} ,$$

$$\Delta_2 = \alpha_{1,1}\alpha_{0,-1} - \alpha_{1,-1}\alpha_{0,1} ,$$

$$\Delta_3 = \alpha_{1,1}\alpha_{-1,0} - \alpha_{1,0}\alpha_{-1,1} ,$$

$$\Delta_4 = \alpha_{1,1}\alpha_{-1,-1} - \alpha_{1,-1}\alpha_{-1,1} ,$$

$$\Delta_5 = \alpha_{-1,-1}\alpha_{0,0} - \alpha_{-1,0}\alpha_{0,-1} ,$$

$$\Delta_6 = \alpha_{-1,-1}\alpha_{0,1} - \alpha_{-1,1}\alpha_{0,-1} ,$$

$$\Delta_7 = \alpha_{-1,-1}\alpha_{1,0} - \alpha_{-1,0}\alpha_{1,-1} ,$$

$$\Delta_8 = \alpha_{0,0}\alpha_{1,-1} - \alpha_{0,-1}\alpha_{1,0} ,$$

$$\Delta_9 = \alpha_{0,0}\alpha_{-1,1} - \alpha_{0,1}\alpha_{-1,0} .$$

The Two-Gauge Boson *AA* System

- We can prove that the ranges of the discriminants Δ_i are

$$-\frac{1}{2} \leq \Delta_i \leq \frac{1}{2} \ .$$

- We can prove that the four independent discriminants can be chosen as $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$.
- The classical space is the intersection of the discriminant loci $\Delta_i = 0$ in quantum space, and the quantum entanglement space is the quantum space by removing the intersection of the discriminant loci $\Delta_i = 0$, *i.e.*, the quantum space minus the classical space.

The Gauge Boson-Fermion Af System

- The most general polarization state of a physics system with one gauge boson A and one fermion f can be written as

$$|Af\rangle = \sum_{j=1,0,-1} \sum_{k=\pm\frac{1}{2}} \alpha_{j,k} |j\rangle_A \otimes |k\rangle_f \ , \quad \sum_{j=1,0,-1} \sum_{k=\pm\frac{1}{2}} |\alpha_{j,k}|^2 = 1 \ .$$

- In the Af system, the quantum space is $\mathbb{CP}^5$ with complex dimension 5, and the classical space is $\mathbb{CP}^2 \otimes \mathbb{CP}^1$ with complex dimension 3. Thus, there are two independent discriminants with complex dimension 1.
- We define the general discriminants

$$\Delta_1 = \alpha_{1,\frac{1}{2}}\alpha_{0,-\frac{1}{2}} - \alpha_{1,-\frac{1}{2}}\alpha_{0,\frac{1}{2}} \ ,$$

$$\Delta_2 = \alpha_{1,\frac{1}{2}}\alpha_{-1,-\frac{1}{2}} - \alpha_{1,-\frac{1}{2}}\alpha_{-1,\frac{1}{2}} \ ,$$

$$\Delta_3 = \alpha_{0,\frac{1}{2}}\alpha_{-1,-\frac{1}{2}} - \alpha_{0,-\frac{1}{2}}\alpha_{-1,\frac{1}{2}} \ .$$

The Gauge Boson-Fermion *Af* System

- we can prove that the ranges of the discriminants Δ_i are

$$-\frac{1}{2} \leq \Delta_i \leq \frac{1}{2} \ .$$

- The two independent discriminants can be chosen as $\{\Delta_1, \Delta_2\}$.
- The classical space is the intersection of the discriminant loci $\Delta_i = 0$ in quantum space, and the quantum entanglement space is the quantum space without the intersection of the discriminant loci $\Delta_i = 0$, *i.e.*, the quantum space minus the classical space.

The Three Fermion *fff* System

- The most general polarization state of a physics system with three fermions $f_1 f_2 f_3$ can be written as

$$|f_1 f_2 f_3\rangle = \sum_{j,k,l=\pm\frac{1}{2}} \alpha_{j,k,l} |j\rangle_{f_1} \otimes |k\rangle_{f_2} \otimes |l\rangle_{f_2} \ , \quad \sum_{j,k,l=\pm\frac{1}{2}} |\alpha_{j,k,l}|^2 = 1 \ .$$

- In the physics system with three fermions $f_1 f_2 f_3$, the quantum space is $\mathbb{CP}^7$ with complex dimension 7, and the classical space is $\mathbb{CP}^1 \otimes \mathbb{CP}^1 \otimes \mathbb{CP}^1$ with complex dimension 3. Thus, there are four independent discriminants with complex dimension 1.
- The strategy to construct the discriminants of $N+1$ particles is that we fix the spin (helicity) of one particle and construct the corresponding discriminants of N particles, thus, we have $N+1$ kinds. Next, we consider the new discriminants where all the particles have different spins (helicities).

The Three Fermion *fff* System

We define the general discriminants

$$
\begin{aligned}
\Delta_1 &= \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} ~, \\
\Delta_1' &= \alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} ~, \\
\Delta_2 &= \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} ~, \\
\Delta_2' &= \alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} ~, \\
\Delta_3 &= \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} ~, \\
\Delta_3' &= \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} ~, \\
\Delta_4 &= \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} ~, \\
\Delta_4' &= \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} ~, \\
\Delta_4'' &= \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} ~.
\end{aligned}
$$

The Three Fermion *fff* System

- We can prove that the ranges of the discriminants Δ_i are

$$-\frac{1}{2} \leq \Delta_i \leq \frac{1}{2} \ .$$

- The four independent discriminants can be chosen as $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$.
- The classical space is the intersection of the discriminant loci $\Delta_i = 0$ in quantum space, and the quantum entanglement space is the quantum space by removing the intersection of the discriminant loci $\Delta_i = 0$, *i.e.*, the quantum space minus the classical space.

The System with Two Fermions and One Gauge Boson ffA

- The most general polarization state of a physics system with two fermions and one gauge boson $f_1 f_2 A$ can be written as

$$|f_1 f_2 A\rangle = \sum_{j,k=\pm\frac{1}{2}} \sum_{l=1,0,-1} \alpha_{j,k,l} |j\rangle_{f_1} \otimes |k\rangle_{f_2} \otimes |l\rangle_A ~,$$

$$\sum_{j,k=\pm\frac{1}{2}} \sum_{l=1,0,-1} |\alpha_{j,k,l}|^2 = 1 ~.$$

- In the physics system with two fermions and one gauge boson $f_1 f_2 A$, the quantum space is $\mathbb{CP}^{11}$ with complex dimension 11, and the classical space is $\mathbb{CP}^1 \otimes \mathbb{CP}^1 \otimes \mathbb{CP}^2$ with complex dimension 4. Thus, there are seven independent discriminants with complex dimension 1.
- The strategy to construct the discriminants is similar to the three fermion system.

The System with Two Fermions and One Gauge Boson *ffA*

We define the general discriminants

$$\Delta_1 = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},0} - \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{\frac{1}{2},-\frac{1}{2},1} \ ,$$

$$\Delta_1' = \alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{-\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},1} \ ,$$

$$\Delta_2 = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{\frac{1}{2},-\frac{1}{2},1} \ ,$$

$$\Delta_2' = \alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{-\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},1} \ ,$$

$$\Delta_3 = \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{\frac{1}{2},-\frac{1}{2},0} \ ,$$

$$\Delta_3' = \alpha_{-\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{-\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} \ ,$$

$$\Delta_4 = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},0} - \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},\frac{1}{2},1} \ ,$$

$$\Delta_4' = \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{\frac{1}{2},-\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},1} \ ,$$

The System with Two Fermions and One Gauge Boson *ffA*

$$
\begin{aligned}
\Delta_5 &= \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},-1} - \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},\frac{1}{2},1} \ , \\
\Delta_5' &= \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},-\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},1} \ , \\
\Delta_6 &= \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},\frac{1}{2},-1} - \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},\frac{1}{2},0} \ , \\
\Delta_6' &= \alpha_{\frac{1}{2},-\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},-\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} \ , \\
\Delta_7 &= \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},1} - \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},1} \ , \\
\Delta_7' &= \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{\frac{1}{2},-\frac{1}{2},0}\alpha_{-\frac{1}{2},\frac{1}{2},0} \ , \\
\Delta_7'' &= \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},-\frac{1}{2},-1}\alpha_{-\frac{1}{2},\frac{1}{2},-1} \ ,
\end{aligned}
$$

The System with Two Fermions and One Gauge Boson *ffA*

$$\Delta_8 = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},1} ,$$
$$\Delta_8' = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},0} ,$$
$$\Delta_8'' = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},0} ,$$
$$\Delta_9 = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},1} ,$$
$$\Delta_9' = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},-1} ,$$
$$\Delta_9'' = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},-1} .$$

- We can prove that the ranges of the discriminants Δ_i are

$$-\frac{1}{2} \leq \Delta_i \leq \frac{1}{2} \ .$$

- The seven independent discriminants can be chosen as $\{\Delta_1, \Delta_2, \Delta_4, \Delta_5, \Delta_7, \Delta_8, \Delta_9\}$.
- The classical space is the intersection of the discriminant loci $\Delta_i = 0$ in quantum space, and the quantum entanglement space is the quantum space without the intersection of the discriminant loci $\Delta_i = 0$, *i.e.*, the quantum space minus the classical space.

Quantum Entanglement Theory (QET)

- The criterion on the sufficient and necessary condition for classical space might be only found for the simplest two fermion system in 1996 or 1997, *i.e.*, the original Peres-Horodecki criterion [8], 32 or 33 years after Bell's original paper. Thus, it might be a highly non-trivial problem.
- We propose the Quantum Entanglement Theory (QET), and can achieve the criteria on the sufficient and necessary conditions for any generic physics systems.

[8] A. Peres, Phys. Rev. Lett. **77**, 1413-1415 (1996) [arXiv:quant-ph/9604005 [quant-ph]]; P. Horodecki, Phys. Lett. A **232**, 333 (1997) [arXiv:quant-ph/9703004 [quant-ph]].

Outline

Introduction

Quantum Entanglement Theory

The Specific Approach or Decay Approach

The Consistent Searches for Quantum Entanglement at the Colliders

Theoretical framework: two-particle systems

Two-particle systems:

$$|AB\rangle = \sum_{k,j} \alpha_{k,j} |k\rangle_A |j\rangle_B \ .$$

- The existence of quantum entanglement (QE) is independent of the choice of reference frame. So we adopt the center-of-mass (c.m.) frame of the AB system for our analysis without loss of generality.
- The spin projection quantum numbers k for particle A and j for particle B are defined along their respective momentum directions $\hat{e}_A$ and $\hat{e}_B$.

Theoretical framework: two-particle systems

In the decay processes $A \to A_1 + A_2 + \ldots$ and $B \to B_1 + B_2 + \ldots$, the angular distributions of the decay products A_i and B_i are characterized in their respective parent rest frames using spherical coordinates: $(\theta_{A_i}, \phi_{A_i})$ for A_i in the A-rest frame, and $(\theta_{B_i}, \phi_{B_i})$ for B_i in the B-rest frame.

- $\hat{e}_A$ and $\hat{e}_B$ are used as the polar axes for θ_{A_i} and θ_{B_i}, respectively.
- Azimuthal angle reference protocol:
  -**Construct orthogonal bases**: Choose auxiliary axes $\hat{e}'_A$ and $\hat{e}'_B$ orthogonal to $\hat{e}_A$ and $\hat{e}_B$, respectively
  -**Define zero azimuth**: Align $\phi_{A_i} = 0$ with $\hat{e}'_A$ and $\phi_{B_i} = 0$ with $\hat{e}'_B$
  -**Angular measurement**: $\phi_{A_i/B_i} \in [0, 2\pi]$ increases following the right-handed coordinate system about $\hat{e}_{A/B}$

Theoretical framework: two-particle systems

Decay amplitudes:

$$\mathcal{M} = \langle f_A f_B | AB \rangle = \sum_{k,j} \alpha_{k,j} \langle f_A | k \rangle_A \langle f_B | j \rangle_B \ ,$$

$$\langle f_A | k \rangle_A = \sqrt{\frac{2S_A + 1}{4\pi}} e^{i(k-\tilde{\lambda}_{f_A})\phi_{A_1}} d^{S_A}_{k,\tilde{\lambda}_{f_A}}(\theta_{A_1}) H_A(\lambda_{f_A}) \ ,$$

$$\langle f_B | j \rangle_B = \sqrt{\frac{2S_B + 1}{4\pi}} e^{i(j-\tilde{\lambda}_{f_B})\phi_{B_1}} d^{S_B}_{j,\tilde{\lambda}_{f_B}}(\theta_{B_1}) H_B(\lambda_{f_B}) \ .$$

- $S_{A/B}$ denote the spin quantum numbers of particles A/B. k/j represent the spin projection quantum numbers along A/B's momentum direction in the AB c.m. frame.
- We collectively denote the final-state particles as $f_A \equiv (A_1, A_2, \ldots)$ and $f_B \equiv (B_1, B_2, \ldots)$. $\lambda_{f_{A/B}}$ encode polarization configurations: $\lambda_{f_A} = (\lambda_{A_1}, \lambda_{A_2}, \ldots)$, $\lambda_{f_B} = (\lambda_{B_1}, \lambda_{B_2}, \ldots)$, where λ_{A_i} and λ_{B_i} are spin projections defined relative to directions of $(\theta_{A_1}, \phi_{A_1})$ and $(\theta_{B_1}, \phi_{B_1})$, respectively. The helicity summation rules are defined as $\tilde{\lambda}_{f_A} = \sum_i \lambda_{A_i}$ and $\tilde{\lambda}_{f_B} = \sum_i \lambda_{B_i}$.
- $H_A(\lambda_{f_A})/H_B(\lambda_{f_B})$ remains independent of both the angular variables $(\theta_{A_1}, \phi_{A_1})/(\theta_{B_1}, \phi_{B_1})$ and the parent particle spin projections k/j. The Wigner d-functions satisfy the normalization conditions:

$$\int_{-1}^{1} d\cos\theta_{A_1} \left(d^{S_A}_{k,\tilde{\lambda}_{f_A}}(\theta_{A_1}) \right)^2 = \frac{2}{2S_A + 1} \ , \int_{-1}^{1} d\cos\theta_{B_1} \left(d^{S_B}_{j,\tilde{\lambda}_{f_B}}(\theta_{B_1}) \right)^2 = \frac{2}{2S_B + 1} \ .$$

Theoretical framework: two-particle systems

$$\Gamma' = \iint d\pi_{f_A} d\pi_{f_B} \left|\mathcal{M}\right|^2 = \int d\pi'_{f_A} \left|H_A(\lambda_{f_A})\right|^2 \int d\pi'_{f_B} \left|H_B(\lambda_{f_B})\right|^2 \ ,$$

$$\begin{aligned} d\pi_{f_A} &= d\pi'_{f_A} d\phi_{A_1} d\cos\theta_{A_1} \ , \\ d\pi_{f_B} &= d\pi'_{f_B} d\phi_{B_1} d\cos\theta_{B_1} \ . \end{aligned}$$

- $d\pi_{f_A}$ and $d\pi_{f_B}$ correspond to the phase space volume elements for the decay products of particles A and B, respectively.
- Γ' remains independent not only of the polarization coefficients $\alpha_{k,j}$, but also of the invariant mass squared s characterizing the AB system.

Theoretical framework: two-particle systems

Given that θ_{A_1}, θ_{B_1}, ϕ_{A_1}, and ϕ_{B_1} represent measurable quantities, they naturally serve as building blocks for constructing composite observables $\mathcal{O}(\theta_{A_1}, \theta_{B_1}, \phi_{A_1}, \phi_{B_1})$.

$$\langle \mathcal{O}(\theta_{A_1}, \theta_{B_1}, \phi_{A_1}, \phi_{B_1}) \rangle = \sum_{k,j,m,n} \mathcal{O}_{k,j;m,n} \alpha_{k,j} \alpha^*_{m,n} \ .$$

$$\mathcal{O}_{k,j;m,n} = \frac{(2S_A+1)(2S_B+1)}{16\pi^2} \sum_{\lambda_{f_A}, \lambda_{f_B}} w_{\lambda_{f_A}, \lambda_{f_B}} \left(\int_0^{2\pi} d\phi_{A_1} \int_{-1}^{1} d\cos\theta_{A_1} \int_0^{2\pi} d\phi_{B_1} \int_{-1}^{1} d\cos\theta_{B_1} \right.$$
$$\left. \mathcal{O}(\theta_{A_1}, \theta_{B_1}, \phi_{A_1}, \phi_{B_1}) e^{i(k-m)\phi_{A_1}} e^{i(j-n)\phi_{B_1}} d^{S_A}_{k,\tilde{\lambda}_{f_A}}(\theta_{A_1}) d^{S_A}_{m,\tilde{\lambda}_{f_A}}(\theta_{A_1}) d^{S_B}_{j,\tilde{\lambda}_{f_B}}(\theta_{B_1}) d^{S_B}_{n,\tilde{\lambda}_{f_B}}(\theta_{B_1}) \right) \ ,$$

$$w_{\lambda_{f_A}, \lambda_{f_B}} = \frac{\int d\pi'_{f_A} \left| H_A(\lambda_{f_A}) \right|^2 \int d\pi'_{f_B} \left| H_B(\lambda_{f_B}) \right|^2}{\sum_{\lambda'_{f_A}, \lambda'_{f_B}} \int d\pi'_{f_A} \left| H_A(\lambda'_{f_A}) \right|^2 \int d\pi'_{f_B} \left| H_B(\lambda'_{f_B}) \right|^2} \ ,$$

$$\sum_{\lambda_{f_A}, \lambda_{f_B}} w_{\lambda_{f_A}, \lambda_{f_B}} = 1 \ .$$

Theoretical framework: two-particle systems

Some specific results:

$$\langle f(\theta_{A_1},\theta_{B_1})\cos(d_A\phi_{A_1}+d_B\phi_{B_1})\rangle$$
$$=\frac{(2S_A+1)(2S_B+1)}{8}\sum_{k,j}\left(\alpha_{k,j}\alpha^*_{k+d_A,j+d_B}+\alpha_{k+d_A,j+d_B}\alpha^*_{k,j}\right)\sum_{\lambda_{f_A},\lambda_{f_B}}w_{\lambda_{f_A},\lambda_{f_B}}$$
$$\left(\int_{-1}^{1}d\cos\theta_{A_1}\int_{-1}^{1}d\cos\theta_{B_1}\ f(\theta_{A_1},\theta_{B_1})d^{S_A}_{k,\tilde\lambda_{f_A}}(\theta_{A_1})d^{S_A}_{k+d_A,\tilde\lambda_{f_A}}(\theta_{A_1})d^{S_B}_{j,\tilde\lambda_{f_B}}(\theta_{B_1})d^{S_B}_{j+d_B,\tilde\lambda_{f_B}}(\theta_{B_1})\right)\ ,$$

$$\langle f(\theta_{A_1},\theta_{B_1})\sin(d_A\phi_{A_1}+d_B\phi_{B_1})\rangle$$
$$=\frac{(2S_A+1)(2S_B+1)}{8i}\sum_{k,j}\left(\alpha_{k,j}\alpha^*_{k+d_A,j+d_B}-\alpha_{k+d_A,j+d_B}\alpha^*_{k,j}\right)\sum_{\lambda_{f_A},\lambda_{f_B}}w_{\lambda_{f_A},\lambda_{f_B}}$$
$$\left(\int_{-1}^{1}d\cos\theta_{A_1}\int_{-1}^{1}d\cos\theta_{B_1}\ f(\theta_{A_1},\theta_{B_1})d^{S_A}_{k,\tilde\lambda_{f_A}}(\theta_{A_1})d^{S_A}_{k+d_A,\tilde\lambda_{f_A}}(\theta_{A_1})d^{S_B}_{j,\tilde\lambda_{f_B}}(\theta_{B_1})d^{S_B}_{j+d_B,\tilde\lambda_{f_B}}(\theta_{B_1})\right)\ .$$

Theoretical framework: multi-particle systems

Multi-particle systems:

$$|P_1 P_2 \ldots P_N\rangle = \sum_{k_1,k_2,\ldots,k_N} \alpha_{k_1,k_2,\ldots,k_N} |k_1\rangle_{P_1} |k_2\rangle_{P_2} \ldots |k_N\rangle_{P_N} \ ,$$

$$\sum_{k_1,k_2,\ldots,k_N} \left|\alpha_{k_1,k_2,\ldots,k_N}\right|^2 = 1 \ .$$

Theoretical framework: multi-particle systems

Multi-particle systems:

- k_i ($i = 1, 2, \ldots, N$) denote spin projection quantum numbers for respective particles P_i, with quantization axes $\hat{e}_i$ aligned to each particle's momentum direction in the system's c.m. frame.
- For each parent particle P_i, we identify a corresponding daughter particle D_i within its decay final-state products. Within the rest frame of P_i, the momentum orientation of daughter particle D_i is parameterized by polar angle θ_i and azimuthal angle ϕ_i.
- The polar angle θ_i is defined with respect to the quantization axis $\hat{e}_i$. The azimuthal angle ϕ_i is established through the following coordinate convention:
  - Select an arbitrary fixed auxiliary axis $\hat{e}'_i$ orthogonal to $\hat{e}_i$
  - Define the reference direction $\phi_i = 0$ via $\hat{e}'_i$
  - The angular parameter $\phi_i \in [0, 2\pi]$ increases following the right-hand rule about $\hat{e}_i$

Theoretical framework: multi-particle systems

Any physical observable $\mathcal{O}(\theta_1, \theta_2, \ldots, \theta_N, \phi_1, \phi_2, \ldots, \phi_N)$ constructed from angular parameters:

$$\langle \mathcal{O}(\theta_1, \theta_2, \ldots, \theta_N, \phi_1, \phi_2, \ldots, \phi_N) \rangle = \sum_{k_1, k_2, \ldots, k_N, k'_1, k'_2, \ldots, k'_N} \mathcal{O}_{k_1, k_2, \ldots, k_N; k'_1, k'_2, \ldots, k'_N} \alpha_{k_1, k_2, \ldots, k_N} \alpha^*_{k'_1, k'_2, \ldots, k'_N} ,$$

$$\mathcal{O}_{k_1, k_2, \ldots, k_N; k'_1, k'_2, \ldots, k'_N} = \frac{\prod_{i=1}^N \left(2 S_{P_i} + 1\right)}{(4\pi)^N} \sum_{\lambda_{f_{P_1}}, \lambda_{f_{P_2}}, \ldots, \lambda_{f_{P_N}}} w_{\lambda_{f_{P_1}}, \lambda_{f_{P_2}}, \ldots, \lambda_{f_{P_N}}} \times$$
$$\left(\int_0^{2\pi} d\phi_1 \int_{-1}^{1} d\cos\theta_1 \int_0^{2\pi} d\phi_2 \int_{-1}^{1} d\cos\theta_2 \ldots \int_0^{2\pi} d\phi_N \int_{-1}^{1} d\cos\theta_N \right.$$
$$\mathcal{O}(\theta_1, \theta_2, \ldots, \theta_N, \phi_1, \phi_2, \ldots, \phi_N) e^{i\left(k_1 - k'_1\right)\phi_1} e^{i\left(k_2 - k'_2\right)\phi_2} \ldots e^{i\left(k_N - k'_N\right)\phi_N} \times$$
$$\left. d^{S_{P_1}}_{k_1, \tilde{\lambda}_{f_{P_1}}}(\theta_1) d^{S_{P_1}}_{k'_1, \tilde{\lambda}_{f_{P_1}}}(\theta_1) d^{S_{P_2}}_{k_2, \tilde{\lambda}_{f_{P_2}}}(\theta_2) d^{S_{P_2}}_{k'_2, \tilde{\lambda}_{f_{P_2}}}(\theta_2) \ldots d^{S_{P_N}}_{k_N, \tilde{\lambda}_{f_{P_N}}}(\theta_N) d^{S_{P_N}}_{k'_N, \tilde{\lambda}_{f_{P_N}}}(\theta_N) \right) ,$$

Theoretical framework: multi-particle systems

$$w_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}} = \frac{\prod_{i=1}^{N}\int d\pi'_{f_{P_i}}\left|H_{P_i}(\lambda_{f_{P_i}})\right|^2}{\sum_{\lambda'_{f_{P_1}},\lambda'_{f_{P_2}},\ldots,\lambda'_{f_{P_N}}}\left(\prod_{i=1}^{N}\int d\pi'_{f_{P_i}}\left|H_{P_i}(\lambda'_{f_{P_i}})\right|^2\right)} ,$$

$$\sum_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}} w_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}} = 1 .$$

Theoretical framework: multi-particle systems

Some epecific results:

$$\left\langle f(\theta_1,\theta_2,\ldots,\theta_N)\cos\left(\sum_{i=1}^{N} d_i\phi_i\right)\right\rangle = \frac{\prod_{i=1}^{N}\left(2S_{P_i}+1\right)}{2^{N+1}}\times$$

$$\sum_{k_1,k_2,\ldots,k_N}\left(\alpha_{k_1,k_2,\ldots,k_N}\alpha^*_{k_1+d_1,k_2+d_2,\ldots,k_N+d_N}+\alpha_{k_1+d_1,k_2+d_2,\ldots,k_N+d_N}\alpha^*_{k_1,k_2,\ldots,k_N}\right)\times$$

$$\sum_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}} w_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}}\left(\int_{-1}^{1} d\cos\theta_1\int_{-1}^{1} d\cos\theta_2\ldots\int_{-1}^{1} d\cos\theta_N f(\theta_1,\theta_2,\ldots,\theta_N)\right.$$

$$\left. d^{S_{P_1}}_{k_1,\tilde\lambda_{f_{P_1}}}(\theta_1)d^{S_{P_1}}_{k_1+d_1,\tilde\lambda_{f_{P_1}}}(\theta_1)d^{S_{P_2}}_{k_2,\tilde\lambda_{f_{P_2}}}(\theta_2)d^{S_{P_2}}_{k_2+d_2,\tilde\lambda_{f_{P_2}}}(\theta_2)\ldots d^{S_{P_N}}_{k_N,\tilde\lambda_{f_{P_N}}}(\theta_N)d^{S_{P_N}}_{k_N+d_N,\tilde\lambda_{f_{P_N}}}(\theta_N)\right),$$

Theoretical framework: multi-particle systems

Some epecific results:

$$\left\langle f(\theta_1,\theta_2,\ldots,\theta_N)\sin\left(\sum_{i=1}^{N} d_i\phi_i\right)\right\rangle = \frac{\prod_{i=1}^{N}\left(2S_{P_i}+1\right)}{2^{N+1}i}\times$$

$$\sum_{k_1,k_2,\ldots,k_N}\left(\alpha_{k_1,k_2,\ldots,k_N}\alpha^*_{k_1+d_1,k_2+d_2,\ldots,k_N+d_N} - \alpha_{k_1+d_1,k_2+d_2,\ldots,k_N+d_N}\alpha^*_{k_1,k_2,\ldots,k_N}\right)\times$$

$$\sum_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}} w_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}}\left(\int_{-1}^{1} d\cos\theta_1\int_{-1}^{1} d\cos\theta_2\ldots\int_{-1}^{1} d\cos\theta_N f(\theta_1,\theta_2,\ldots,\theta_N)\right.$$

$$\left. d^{S_{P_1}}_{k_1,\tilde{\lambda}_{f_{P_1}}}(\theta_1) d^{S_{P_1}}_{k_1+d_1,\tilde{\lambda}_{f_{P_1}}}(\theta_1) d^{S_{P_2}}_{k_2,\tilde{\lambda}_{f_{P_2}}}(\theta_2) d^{S_{P_2}}_{k_2+d_2,\tilde{\lambda}_{f_{P_2}}}(\theta_2)\ldots d^{S_{P_N}}_{k_N,\tilde{\lambda}_{f_{P_N}}}(\theta_N) d^{S_{P_N}}_{k_N+d_N,\tilde{\lambda}_{f_{P_N}}}(\theta_N)\right)\ .$$

$t\bar{t}$

The polarization state of $t\bar{t}$:

$$|t\bar{t}\rangle = \sum_{k,j=\pm\frac{1}{2}} \alpha_{k,j} |k\rangle_t |j\rangle_{\bar{t}} \ .$$

- For a given production channel, described as *initial states* $\rightarrow t\bar{t}$, we define the amplitude of producing the $t\bar{t}$ pair in the state $|k\rangle_t|j\rangle_{\bar{t}}$ as $\tilde{\mathcal{M}}_{k,j}$.
- The coefficients $\alpha_{k,j}$ can be calculated using the expression

$$\alpha_{k,j} = \tilde{\mathcal{M}}_{k,j} / \sqrt{\sum_{k,j=\pm\frac{1}{2}} |\tilde{\mathcal{M}}_{k,j}|^2} \ .$$

- The Sufficient and Necessary Condition for indicating QE in the $t\bar{t}$ system:

$$\alpha_{\frac{1}{2},\frac{1}{2}} \alpha_{-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2}} \alpha_{-\frac{1}{2},\frac{1}{2}} \neq 0 \ .$$

$t\bar{t}$: observable I

The decay processes of $t \to e^+ + \nu_e + b$ and $\bar{t} \to e^- + \bar{\nu}_e + \bar{b}$ after the production of on-shell t and $\bar{t}$:

$$e^+ : (\theta_{e^+}, \phi_{e^+}) \, , \quad e^- : (\theta_{e^-}, \phi_{e^-}) \, .$$

- Observable I: $D = -3 \cdot \langle \cos\theta_{e^+e^-} \rangle$

$$\cos\theta_{e^+e^-} = -\cos\theta_{e^+} \cos\theta_{e^-} + \sin\theta_{e^+} \sin\theta_{e^-} \cos\left(\phi_{e^+} + \phi_{e^-}\right) ,$$

$$\langle \cos\theta_{e^+e^-} \rangle = \frac{1}{9}\left(|\alpha_{\frac{1}{2},\frac{1}{2}}|^2 + |\alpha_{-\frac{1}{2},-\frac{1}{2}}|^2 - |\alpha_{\frac{1}{2},-\frac{1}{2}}|^2 - |\alpha_{-\frac{1}{2},\frac{1}{2}}|^2 \right) + \frac{1}{9}\left(2\alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} + 2\alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} \right) .$$

Observables	Quantum Range	Classical Range	criteria for entanglement
D	$[-1, 1/3]$	$[-1/3, 1/3]$	$[-1, -1/3)$

$t\bar{t}$: observable I

The LO results at the e^+e^- collider satisfy the relation

$$\alpha_{\frac{1}{2},\frac{1}{2}} = -\alpha_{-\frac{1}{2},-\frac{1}{2}}\ , \quad \left|\alpha_{\frac{1}{2},\frac{1}{2}}\right|^2 = \left|\alpha_{-\frac{1}{2},-\frac{1}{2}}\right|^2 \leq \frac{1}{4}\ .$$

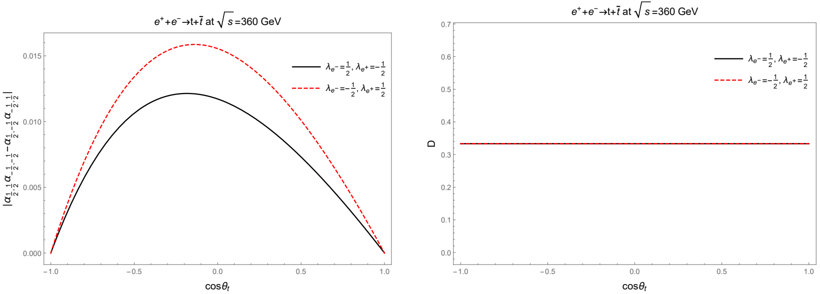

Figure: The LO predictions of $|\alpha_{\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2}}|$ and $D = -3 \cdot \langle \cos\theta_{e^+e^-} \rangle$ for $t\bar{t}$ pairs produced at an e^+e^- collider operating at a c.m. energy of $\sqrt{s} = 360$ GeV. Here, θ_t denotes the polar angle of the top quark t in the laboratory frame. The symbols $\lambda_{e^\pm}$ represent the helicities of the e^+ and e^- beams, defined along their respective momentum directions in the laboratory frame.

$t\bar{t}$: observable II

$$\langle \cos\left(\phi_{e+} - \phi_{e-}\right)\rangle = \frac{\pi^2}{32}\left(\alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} + \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}}\right) \ , \quad D' = \frac{32}{\pi^2}\langle \cos\left(\phi_{e+} - \phi_{e-}\right)\rangle \ . \qquad (1)$$

Observables	Quantum Range	Classical Range	criteria for entanglement
D'	$[-1,1]$	$[-1/2,1/2]$	$[-1,-\frac{1}{2}) \cup (\frac{1}{2},1]$

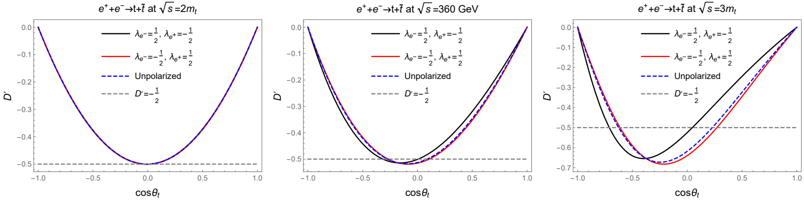

Figure: The LO predictions of $D' = \frac{32}{\pi^2}\langle \cos\left(\phi_{e+} - \phi_{e-}\right)\rangle$ for $t\bar{t}$ pairs produced at e^+e^- collider with $\sqrt{s} = 2m_t$, 360 GeV, and $3m_t$, respectively. The angle θ_t represents the polar angle of the top quark t in the laboratory frame. The symbols $\lambda_{e\pm}$ indicate the helicities of the e^+ and e^- beams, defined along their respective momentum directions in the laboratory frame.

$t\bar{t}$: observable III

$$D'' = \frac{9}{2}\langle \sin\theta_{e^+} \sin\theta_{e^-} \cos\left(\phi_{e^+} + \phi_{e^-}\right)\rangle = \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} + \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} .$$

Observables	Quantum Range	Classical Range	criteria for entanglement
D''	$[-1, 1]$	$[-1/2, 1/2]$	$[-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$

- Within the SM at e^+e^- colliders, we obtain

$$\alpha_{\frac{1}{2},\frac{1}{2}} = -\alpha_{-\frac{1}{2},-\frac{1}{2}} , \quad \left|\alpha_{\frac{1}{2},\frac{1}{2}}\right|^2 = \left|\alpha_{-\frac{1}{2},-\frac{1}{2}}\right|^2 \leq \frac{1}{4} \Longrightarrow D'' \in [-\frac{1}{2}, 0) .$$

- Considering beyond-Standard-Model scenarios with a Higgs-like particle exhibiting Yukawa coupling:

$$\propto h' t\bar{t} .$$

Direct calculation reveals that $t\bar{t}$ pairs from $h' \to t + \bar{t}$ decays satisfy

$$\alpha_{\frac{1}{2},\frac{1}{2}} = -\alpha_{-\frac{1}{2},-\frac{1}{2}} , \quad \left|\alpha_{\frac{1}{2},\frac{1}{2}}\right|^2 = \left|\alpha_{-\frac{1}{2},-\frac{1}{2}}\right|^2 = \frac{1}{2} .$$

For such h'-mediated $t\bar{t}$ production, we obtain $D'' = -1$.

$\tau^+\tau^-$: observable I

The decay processes of $\tau^+ \to e^+ + \nu_e + \bar{\nu}_\tau$ and $\tau^- \to e^- + \bar{\nu}_e + \nu_\tau$ after the production of on-shell τ^+ and τ^-:

$$e^+ : (\theta_{e^+}, \phi_{e^+}) \; . \; e^- : (\theta_{e^-}, \phi_{e^-}) \; .$$

- Observable I:

$$\begin{aligned}
\cos\theta_{e^+e^-} =& -\cos\theta_{e^+}\cos\theta_{e^-} + \sin\theta_{e^+}\sin\theta_{e^-}\cos(\phi_{e^+} + \phi_{e^-}) \; , \\
\langle\cos\theta_{e^+e^-}\rangle =& 0.01254 \Big(|\alpha_{\frac{1}{2},\frac{1}{2}}|^2 + |\alpha_{-\frac{1}{2},-\frac{1}{2}}|^2 - |\alpha_{\frac{1}{2},-\frac{1}{2}}|^2 - |\alpha_{-\frac{1}{2},\frac{1}{2}}|^2 \\
& + 2\alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} + 2\alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} \Big) \; , \\
=& 0.01254 \times (-1) \; .
\end{aligned}$$

$\tau^+\tau^-$: observable II

$$\langle \cos\left(\phi_{e^+} - \phi_{e^-}\right)\rangle = 0.03426\left(\alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} + \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}}\right)\ ,\quad D'_{\tau^+\tau^-} = \frac{1}{0.03426}\langle \cos\left(\phi_{e^+} - \phi_{e^-}\right)\rangle\ .$$

Observables	Quantum Range	Classical Range	criteria for entanglement
$D'_{\tau^+\tau^-}$	$[-1,1]$	$[-1/2,1/2]$	$[-1,-\frac{1}{2})\cup(\frac{1}{2},1]$

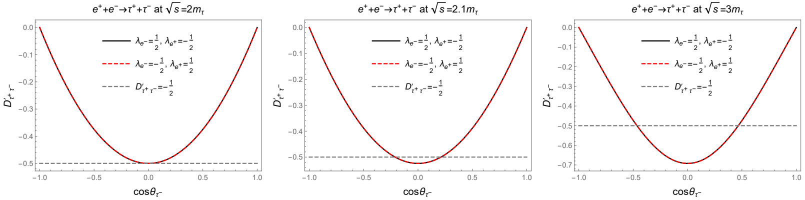

Figure: The LO predictions of $D'_{\tau^+\tau^-} = \langle\cos\left(\phi_{e^+} - \phi_{e^-}\right)\rangle/0.03426$ for $\tau^+\tau^-$ pairs produced at e^+e^- collider with $\sqrt{s} = 2m_\tau$, $2.1m_\tau$, and $3m_\tau$, respectively. The angle θ_{τ^-} represents the polar angle of τ^- in the laboratory frame. The symbols $\lambda_{e^\pm}$ indicate the helicities of the e^+ and e^- beams, defined along their respective momentum directions in the laboratory frame.

W^-W^+: observable I

The decay processes of $W^- \to e^- + \bar{\nu}_e$ and $W^+ \to e^+ + \nu_e$ after the production of on-shell W^- and W^+:

$$e^- : (\theta_{e^-}, \phi_{e^-}) \ , \quad e^+ : (\theta_{e^+}, \phi_{e^+}) \ .$$

- Observable I:

$$\cos\theta_{e^+e^-} = -\cos\theta_{e^+}\cos\theta_{e^-} + \sin\theta_{e^+}\sin\theta_{e^-}\cos\left(\phi_{e^+} + \phi_{e^-}\right) \ ,$$

$$\langle\cos\theta_{e^-e^+}\rangle = \frac{1}{4} \times \Big(|\alpha_{-1,-1}|^2 + |\alpha_{1,1}|^2 - |\alpha_{-1,1}|^2 - |\alpha_{1,-1}|^2 - \alpha_{-1,-1}\alpha^*_{0,0} - \alpha_{0,0}\alpha^*_{-1,-1}$$
$$-\alpha_{1,1}\alpha^*_{0,0} - \alpha_{0,0}\alpha^*_{1,1} - \alpha_{-1,0}\alpha^*_{0,1} - \alpha_{0,1}\alpha^*_{-1,0} - \alpha_{0,-1}\alpha^*_{1,0} - \alpha_{1,0}\alpha^*_{0,-1}\Big) \ .$$

Observables	Quantum Range	Classical Range	criteria for entanglement
$\langle\cos\theta_{e^-e^+}\rangle$	$[-1/4, 1/2]$	$[-1/4, 1/4]$	$(1/4, 1/2]$

W^-W^+: observable I

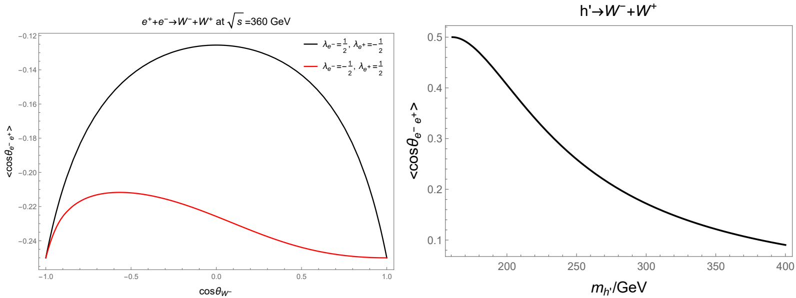

Figure: Left: The LO predictions of $\langle \cos\theta_{e^-e^+} \rangle$ for the W^-W^+ pair produced at e^+e^- collider with $\sqrt{s} = 360$ GeV. The angle θ_{W^-} represents the polar angle of W^- in the laboratory frame. The symbols $\lambda_{e^\pm}$ indicate the helicities of the e^+ and e^- beams, defined along their respective momentum directions in the laboratory frame. Right: The LO predictions of $\langle \cos\theta_{e^-e^+} \rangle$ for the W^-W^+ pair produced from h' decay ($\propto h' g^{\mu\nu} W^-_\mu W^+_\nu$).

W^-W^+: observable II

$$\langle \cos(2\phi_{e^+} - 2\phi_{e^-}) \rangle = \frac{1}{8}\left(\alpha_{-1,1}\alpha^*_{1,-1} + \alpha_{1,-1}\alpha^*_{-1,1}\right) ,$$

$$D'_{W^-W^+} = 8\langle \cos(2\phi_{e^+} - 2\phi_{e^-}) \rangle .$$

Observables	Quantum Range	Classical Range	criteria for entanglement
$D'_{W^-W^+}$	$[-1,1]$	$[-1/2,1/2]$	$[-1,-\frac{1}{2}) \cup (\frac{1}{2},1]$

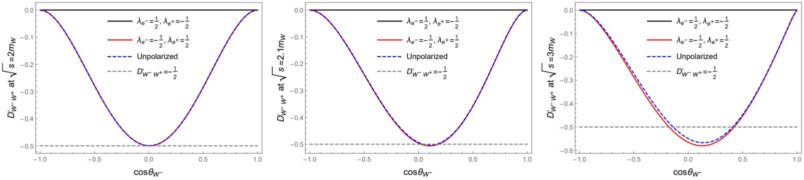

Figure: The LO predictions of $D'_{W^-W^+} = 8\langle\cos\left(2\phi_{e^+} - 2\phi_{e^-}\right)\rangle$ for W^+W^- pairs produced at e^+e^- collider with $\sqrt{s} = 2m_W$, $2.1m_W$ GeV, and $3m_W$, respectively. The angle θ_{W^-} represents the polar angle of the top quark W^- in the laboratory frame. The symbols $\lambda_{e^\pm}$ indicate the helicities of the e^+ and e^- beams, defined

$W^- t$

The decay processes of $W^- \to e^- + \bar{\nu}_e$ and $t \to W^+ + b$ after the production of on-shell W^- and t:

$$e^- : (\theta_{e-}, \phi_{e-}) \ , \quad W^+ : (\theta_{W^+}, \phi_{W^+}) \ .$$

- Observable:

$$\begin{aligned} \cos\theta_{e-W+} =& -\cos\theta_{W+}\cos\theta_{e-} + \sin\theta_{W+}\sin\theta_{e-}\cos\left(\phi_{W+}+\phi_{e-}\right) \ , \\ \langle\cos\theta_{e-W+}\rangle =& 0.0658 \times \left(|\alpha_{-1,-\frac{1}{2}}|^2 + |\alpha_{1,\frac{1}{2}}|^2 - |\alpha_{-1,\frac{1}{2}}|^2 - |\alpha_{1,-\frac{1}{2}}|^2 \right. \\ & \left. + \sqrt{2}\left(\alpha_{-1,-\frac{1}{2}}\alpha^*_{0,\frac{1}{2}} + \alpha_{0,\frac{1}{2}}\alpha^*_{-1,-\frac{1}{2}} + \alpha_{0,-\frac{1}{2}}\alpha^*_{1,\frac{1}{2}} + \alpha_{1,\frac{1}{2}}\alpha^*_{0,-\frac{1}{2}} \right) \right) \ , \\ D'_{W-t} =& \langle\cos\theta_{e-W+}\rangle / 0.0658 \ . \end{aligned}$$

Observables	Quantum Range	Classical Range	criteria for entanglement
D'_{W-t}	$[-1, 2]$	$[-1, 1]$	$(1, 2]$

$W^- t$

Supposing a BSM bottom-like quark, denoted as b', the interaction of b' with the W boson and the top quark t is analogous to that of the SM bottom quark b: $\propto W^+_\mu \bar{t}_L \gamma^\mu b'_L + h.c.$

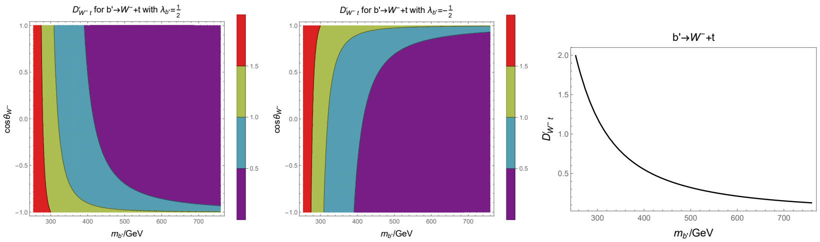

Figure: Left and middle: The LO predictions of $D'_{W^-t} = \langle \cos\theta_{e^-W^+} \rangle / 0.0658$ for the W^-t pair produced by b' decay. Here, θ_{W^-} represents the angle between the momentum direction of the W^- in the b' rest frame and the direction of b' motion in the laboratory frame. The symbols $\lambda_{b'}$ denote the helicities of b', defined along the momentum direction of b' in the laboratory frame. Right: The LO predictions of $D'_{W^-t} = \langle \cos\theta_{e^-W^+} \rangle / 0.0658$ for the W^-t produced by b' decay, averaged over $\cos\theta_{W^-}$.

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For the decay process $t_i \to W_i^+ + b_i$ for each top quark $(i = 1, 2, 3)$, we define the spherical coordinates (θ_i, ϕ_i) for the W_i^+ momentum direction in respective t_i rest frames. The momentum unit vectors are expressed as

$$\hat{e}_{W_i^+} = (\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i) \ .$$

The triple product correlation observable:

$$\begin{aligned}
&\left(\hat{e}_{W_1^+} \times \hat{e}_{W_2^+}\right) \cdot \hat{e}_{W_3^+} \\
=& -\sin\theta_1 \sin\theta_2 \cos\theta_3 \sin(\phi_1 - \phi_2) - \sin\theta_2 \sin\theta_3 \cos\theta_1 \sin(\phi_2 - \phi_3) - \sin\theta_3 \sin\theta_1 \cos\theta_2 \sin(\phi_3 - \phi_1) \\
=& 0.004557 \ i \ \times (\\
& -\alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha^*_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} + \alpha^*_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} + \alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha^*_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha^*_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} \\
& -\alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} \alpha^*_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} + \alpha^*_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} \alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} + \alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} \alpha^*_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} - \alpha^*_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} \alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \\
& -\alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} \alpha^*_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} + \alpha^*_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} + \alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha^*_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} - \alpha^*_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} \Big) \ , \\
& D'_{3t} = \left\langle \left(\hat{e}_{W_1^+} \times \hat{e}_{W_2^+}\right) \cdot \hat{e}_{W_3^+} \right\rangle / 0.004557 \ .
\end{aligned}$$

ttt

Observables	Quantum Range	Classical Range	criteria for entanglement
D'_{3t}	$[-\sqrt{3}, \sqrt{3}]$	$[-1/2, 1/2]$	$\left[-\sqrt{3}, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \sqrt{3}\right]$

$t\bar{t}W^-$

Considering the decay channels $t \to W^+ b$, $\bar{t} \to W^- \bar{b}$, and $W^- \to e^- \bar{\nu}_e$, we analyze the angular correlations of final-state particles (W^+, W^-, and e^-) in their respective parent particle rest frames.

$$\hat{e}_{W+} = (\sin\theta_1 \cos\phi_1, \sin\theta_1 \sin\phi_1, \cos\theta_1) \ , \quad \hat{e}_{W-} = (\sin\theta_2 \cos\phi_2, \sin\theta_2 \sin\phi_2, \cos\theta_2) \ ,$$
$$\hat{e}_{e-} = (\sin\theta_3 \cos\phi_3, \sin\theta_3 \sin\phi_3, \cos\theta_3) \ .$$

The triple product correlation observable:

$$
\begin{aligned}
&\left(\hat{e}_{W+} \times \hat{e}_{W-}\right) \cdot \hat{e}_{e-} \\
=& -\sin\theta_1 \sin\theta_2 \cos\theta_3 \sin(\phi_1 - \phi_2) - \sin\theta_2 \sin\theta_3 \cos\theta_1 \sin(\phi_2 - \phi_3) - \sin\theta_3 \sin\theta_1 \cos\theta_2 \sin(\phi_3 - \phi_1) \\
=& 0.0122433\ i \times (\\
& - \alpha_{-\frac{1}{2},-\frac{1}{2},0}\alpha^*_{-\frac{1}{2},\frac{1}{2},-1} + \alpha^*_{-\frac{1}{2},-\frac{1}{2},0}\alpha_{-\frac{1}{2},\frac{1}{2},-1} - \alpha_{-\frac{1}{2},-\frac{1}{2},1}\alpha^*_{-\frac{1}{2},\frac{1}{2},0} + \alpha^*_{-\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},0} \\
& + \alpha_{-\frac{1}{2},-\frac{1}{2},0}\alpha^*_{\frac{1}{2},-\frac{1}{2},-1} - \alpha^*_{-\frac{1}{2},-\frac{1}{2},0}\alpha_{\frac{1}{2},-\frac{1}{2},-1} - \sqrt{2}\alpha_{-\frac{1}{2},\frac{1}{2},-1}\alpha^*_{\frac{1}{2},-\frac{1}{2},-1} + \sqrt{2}\alpha^*_{-\frac{1}{2},\frac{1}{2},-1}\alpha_{\frac{1}{2},-\frac{1}{2},-1} \\
& + \alpha_{-\frac{1}{2},-\frac{1}{2},1}\alpha^*_{\frac{1}{2},-\frac{1}{2},0} - \alpha^*_{-\frac{1}{2},-\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},0} + \sqrt{2}\alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha^*_{\frac{1}{2},-\frac{1}{2},1} - \sqrt{2}\alpha^*_{-\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},1} \\
& - \alpha_{-\frac{1}{2},\frac{1}{2},0}\alpha^*_{\frac{1}{2},\frac{1}{2},-1} + \alpha^*_{-\frac{1}{2},\frac{1}{2},0}\alpha_{\frac{1}{2},\frac{1}{2},-1} + \alpha_{\frac{1}{2},-\frac{1}{2},0}\alpha^*_{\frac{1}{2},\frac{1}{2},-1} - \alpha^*_{\frac{1}{2},-\frac{1}{2},0}\alpha_{\frac{1}{2},\frac{1}{2},-1} \\
& -\alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha^*_{\frac{1}{2},\frac{1}{2},0} + \alpha^*_{-\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},\frac{1}{2},0} + \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha^*_{\frac{1}{2},\frac{1}{2},0} - \alpha^*_{\frac{1}{2},-\frac{1}{2},1}\alpha_{\frac{1}{2},\frac{1}{2},0}) \\
& D'_{t\bar{t}W-} = \left\langle \left(\hat{e}_{W+} \times \hat{e}_{W-}\right) \cdot \hat{e}_{e-} \right\rangle / 0.0122433 \ .
\end{aligned}
$$

$t\bar{t}W^-$

Observables	Quantum Range	Classical Range	criteria for entanglement
D'_{3t}	$[-2, 2]$	$\left[-\sqrt{2}/2, \sqrt{2}/2\right]$	$\left[-2, -\frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, 2\right]$

$\Lambda\bar{\Lambda}$

arXiv:2505.09931

- The weak decay processes:

$$\Lambda \to p + \pi^- \ , \quad \bar{\Lambda} \to \bar{p} + \pi^+ \ .$$

- The relevant couplings:

$$W_\mu^+ \bar{p} \left(g_V \gamma^\mu + g_A \gamma^\mu \gamma_5\right) \Lambda + W_\mu^- \bar{\Lambda} \left(g_V' \gamma^\mu + g_A' \gamma^\mu \gamma_5\right) p \ ,$$
$$\propto W_\mu^- \partial^\mu \pi^+ + \text{H.C.} \ .$$

The independent coupling parameters g_V, g_A and g_V', g_A' allow for potential CP-violating effects.

- In the rest frames of Λ and $\bar{\Lambda}$, the polar angle distributions of final-state p and $\bar{p}$ satisfy

$$\frac{1}{\Gamma_{\Lambda \to p+\pi^-}} \frac{d\Gamma_{\Lambda \to p+\pi^-}}{d\cos\theta_p} = \frac{1}{2}\left(1 + \alpha_\Lambda \cos\theta_p\right) \ ,$$
$$\frac{1}{\Gamma_{\bar{\Lambda} \to \bar{p}+\pi^+}} \frac{d\Gamma_{\bar{\Lambda} \to \bar{p}+\pi^+}}{d\cos\theta_{\bar{p}}} = \frac{1}{2}\left(1 + \alpha_{\bar{\Lambda}} \cos\theta_{\bar{p}}\right) \ .$$

$\Lambda\bar{\Lambda}$: decay processes

We parameterize the momentum directions of p in the Λ rest frame and $\bar{p}$ in the $\bar{\Lambda}$ rest frame using spherical coordinates:

$$p : (\theta_1, \phi_1) \to \hat{e}_p = (\sin\theta_1 \cos\phi_1, \sin\theta_1 \sin\phi_1, \cos\theta_1) \ ,$$
$$\bar{p} : (\theta_2, \phi_2) \to \hat{e}_{\bar{p}} = (\sin\theta_2 \cos\phi_2, \sin\theta_2 \sin\phi_2, \cos\theta_2) \ .$$

- The polar angles θ_1 and θ_2 are defined with respect to the Λ momentum direction $\hat{e}_\Lambda$ in the $\Lambda\bar{\Lambda}$ center-of-mass (c.m.) frame.
- The azimuthal angles ϕ_1 and ϕ_2 are measured from an arbitrary reference axis orthogonal to $\hat{e}_\Lambda$, increasing in the right-handed screw direction about $\hat{e}_\Lambda$ with $\phi \in [0, 2\pi]$.
- We define the opening angle $\theta_{p\bar{p}}$ between the proton momenta as:

$$\cos\theta_{p\bar{p}} = \hat{e}_p \cdot \hat{e}_{\bar{p}} = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) \ .$$

$\Lambda\bar{\Lambda}$: decay processes

Helicity amplitudes:

$$\langle p, \pi^- | k \rangle_\Lambda = \frac{1}{\sqrt{2\pi}} e^{i(k-\lambda_p)\phi_1} d^{1/2}_{k,\lambda_p}(\theta_1) H_\Lambda(\lambda_p) \ , \ ,$$

$$\langle \bar{p}, \pi^+ | j \rangle_{\bar{\Lambda}} = \frac{1}{\sqrt{2\pi}} e^{-i(j+\lambda_{\bar{p}})\phi_2} d^{1/2}_{\lambda_{\bar{p}},j}(\pi - \theta_2) H_{\bar{\Lambda}}(\lambda_{\bar{p}}) \ .$$

- k/j represent the spin projection quantum numbers along the momentum direction of $\Lambda/\bar{\Lambda}$ in the $\Lambda\bar{\Lambda}$ c.m. frame.
- λ_p and $\lambda_{\bar{p}}$ ($\lambda_p, \lambda_{\bar{p}} = \pm\frac{1}{2}$) are spin projections of p and $\bar{p}$ defined relative to directions of $\hat{e}_p$ and $\hat{e}_{\bar{p}}$, respectively.
- $H_\Lambda(\lambda_p)/H_{\bar{\Lambda}}(\lambda_{\bar{p}})$ remain independent of both the angular variables $(\theta_1, \phi_1)/(\theta_2, \phi_2)$ and the parent particle spin projections k/j.
- The Wigner d-functions are

$$d^{1/2}_{\frac{1}{2},\frac{1}{2}}(\theta) = d^{1/2}_{-\frac{1}{2},-\frac{1}{2}}(\theta) = \cos\frac{\theta}{2} \ , \quad d^{1/2}_{-\frac{1}{2},\frac{1}{2}}(\theta) = -d^{1/2}_{\frac{1}{2},-\frac{1}{2}}(\theta) = \sin\frac{\theta}{2} \ .$$

$\Lambda\bar{\Lambda}$: observables

For any physical observable $\mathcal{O}(\theta_1, \theta_2, \phi_1, \phi_2)$ constructed from angular variables, its statistical average can be expressed as:

$$\langle \mathcal{O}(\theta_1, \theta_2, \phi_1, \phi_2) \rangle = \sum_{k,j,m,n} \mathcal{O}_{k,j;m,n} \alpha_{k,j} \alpha^*_{m,n} \ . \tag{2}$$

$$\mathcal{O}_{k,j;m,n} = \frac{1}{4\pi^2} \sum_{\lambda_p, \lambda_{\bar{p}}} \Bigg(w_{\lambda_p, \lambda_{\bar{p}}} \int_0^{2\pi} d\phi_1 \int_{-1}^{1} d\cos\theta_1 \int_0^{2\pi} d\phi_2 \int_{-1}^{1} d\cos\theta_2$$
$$\mathcal{O}(\theta_1, \theta_2, \phi_1, \phi_2) e^{i(k-m)\phi_1} e^{i(n-j)\phi_2} d^{1/2}_{k,\lambda_p}(\theta_1) d^{1/2}_{m,\lambda_p}(\theta_1) d^{1/2}_{\lambda_{\bar{p}},j}(\pi - \theta_2) d^{1/2}_{\lambda_{\bar{p}},n}(\pi - \theta_2) \Bigg) \ .$$

$$w_{\lambda_p, \lambda_{\bar{p}}} = \frac{\left| H_\Lambda(\lambda_p) \right|^2 \left| H_{\bar{\Lambda}}(\lambda_{\bar{p}}) \right|^2}{\sum_{\lambda'_p, \lambda'_{\bar{p}}} \left| H_\Lambda(\lambda'_p) \right|^2 \left| H_{\bar{\Lambda}}(\lambda'_{\bar{p}}) \right|^2} \ .$$

$\Lambda\bar{\Lambda}$: observables

- The hyperon decay parameters:

$$\alpha_{\Lambda/\bar{\Lambda}} = \frac{\left|H_{\Lambda/\bar{\Lambda}}(-\frac{1}{2})\right|^2 - \left|H_{\Lambda/\bar{\Lambda}}(\frac{1}{2})\right|^2}{\left|H_{\Lambda/\bar{\Lambda}}(\frac{1}{2})\right|^2 + \left|H_{\Lambda/\bar{\Lambda}}(-\frac{1}{2})\right|^2} \; .$$

- The weight factors:

$$w_{\frac{u_1}{2},\frac{u_2}{2}} = \frac{1}{4}(1 - u_1\alpha_\Lambda)(1 - u_2\alpha_{\bar{\Lambda}}) \; , \quad u_1, u_2 = \pm 1 \; .$$

$\Lambda\bar{\Lambda}$: observables

$$\mathcal{O}_0 = \langle \cos\theta_{p\bar{p}} \rangle / \left(-\frac{1}{9}\alpha_\Lambda \alpha_{\bar{\Lambda}} \right) = \left| \alpha_{\frac{1}{2},\frac{1}{2}} \right|^2 + \left| \alpha_{-\frac{1}{2},-\frac{1}{2}} \right|^2 - \left| \alpha_{\frac{1}{2},-\frac{1}{2}} \right|^2 - \left| \alpha_{-\frac{1}{2},\frac{1}{2}} \right|^2 \ ,$$

$$+ 2\alpha_{-\frac{1}{2},-\frac{1}{2}} \alpha^*_{\frac{1}{2},\frac{1}{2}} + 2\alpha_{\frac{1}{2},\frac{1}{2}} \alpha^*_{-\frac{1}{2},-\frac{1}{2}} \ ,$$

$$\mathcal{O}_1 = \langle \cos(\phi_1 + \phi_2) \rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda \alpha_{\bar{\Lambda}} \right) = \alpha_{-\frac{1}{2},\frac{1}{2}} \alpha^*_{\frac{1}{2},-\frac{1}{2}} + \alpha_{\frac{1}{2},-\frac{1}{2}} \alpha^*_{-\frac{1}{2},\frac{1}{2}} \ ,$$

$$\mathcal{O}_2 = \langle \cos(\phi_1 - \phi_2) \rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda \alpha_{\bar{\Lambda}} \right) = \alpha_{-\frac{1}{2},-\frac{1}{2}} \alpha^*_{\frac{1}{2},\frac{1}{2}} + \alpha_{\frac{1}{2},\frac{1}{2}} \alpha^*_{-\frac{1}{2},-\frac{1}{2}} \ ,$$

$$\mathcal{O}_3 = \langle \sin(\phi_1 + \phi_2) \rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda \alpha_{\bar{\Lambda}} \right) = \frac{1}{i} \left(\alpha_{-\frac{1}{2},\frac{1}{2}} \alpha^*_{\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2}} \alpha^*_{-\frac{1}{2},\frac{1}{2}} \right) \ ,$$

$$\mathcal{O}_4 = \langle \sin(\phi_1 - \phi_2) \rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda \alpha_{\bar{\Lambda}} \right) = \frac{1}{i} \left(\alpha_{-\frac{1}{2},-\frac{1}{2}} \alpha^*_{\frac{1}{2},\frac{1}{2}} - \alpha_{\frac{1}{2},\frac{1}{2}} \alpha^*_{-\frac{1}{2},-\frac{1}{2}} \right) \ .$$

$\Lambda\bar{\Lambda}$: observables

Observables	Quantum Ranges	Classical Ranges	criteria for entanglement
$\mathcal{O}_0$	$[-1, 3]$	$[-1, 1]$	$(1, 3]$
$\mathcal{O}_1$	$[-1, 1]$	$[-1/2, 1/2]$	$[-1, -1/2) \cup (1/2, 1]$
$\mathcal{O}_2$	$[-1, 1]$	$[-1/2, 1/2]$	$[-1, -1/2) \cup (1/2, 1]$
$\mathcal{O}_3$	$[-1, 1]$	$[-1/2, 1/2]$	$[-1, -1/2) \cup (1/2, 1]$
$\mathcal{O}_4$	$[-1, 1]$	$[-1/2, 1/2]$	$[-1, -1/2) \cup (1/2, 1]$

$\Lambda\bar{\Lambda}$: production

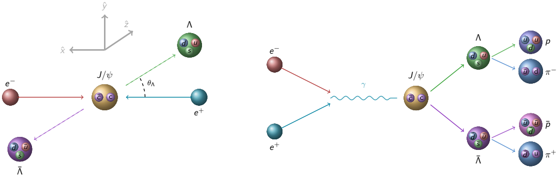

Figure: The processes of
$e^+ + e^- \to J/\psi \to \Lambda(\to p + \pi^-) + \bar{\Lambda}(\to \bar{p} + \pi^+)$.

$\Lambda\bar{\Lambda}$: production

- The $J/\psi - \Lambda\bar{\Lambda}$ interaction is conventionally parameterized as:

$$\psi_\mu \bar{\Lambda} \left(G_M \gamma^\mu + \frac{2m_\Lambda}{m_\psi^2 - 4m_\Lambda^2} \left(G_M - G_E \right) \left(p_\Lambda^\mu - p_{\bar{\Lambda}}^\mu \right) \right) \Lambda \ ,$$
$$G_E / G_M = R e^{i\Delta\phi} \ .$$

- In the J/ψ rest frame, the Λ polar angle distribution follows:

$$\frac{1}{\Gamma_{\psi\to\Lambda+\bar{\Lambda}}} \frac{d\Gamma_{\psi\to\Lambda+\bar{\Lambda}}}{d\cos\theta_\Lambda} = \frac{1}{2 + \frac{2}{3}\alpha_\psi} \left(1 + \alpha_\psi \cos^2\theta_\Lambda \right) \ .$$

$\Lambda\bar{\Lambda}$: production

- For the $J/\psi \to \Lambda + \bar{\Lambda}$ process, we derive:

$$\mathcal{P}(\Delta s^2 < 0) = \sqrt{1 - 4m_\Lambda^2/m_\psi^2} \ .$$

This result implies that 69.3% of the decay events involving $\Lambda \to p + \pi^-$ and $\bar{\Lambda} \to \bar{p} + \pi^+$ are spacelike-separated.

- For $\Lambda\bar{\Lambda}$ pairs produced through $e^+ + e^- \to J/\psi \to \Lambda + \bar{\Lambda}$, theoretical calculations under decoherence-free conditions yield:

$$\alpha_{\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2}} = \frac{1-\alpha_\psi}{4}\frac{1-\cos^2\theta_\Lambda}{1+\alpha_\psi\cos^2\theta_\Lambda}\left(\frac{1+\alpha_\psi}{1-\alpha_\psi} - e^{2i\Delta\phi}\right) \ .$$

$\Lambda\bar{\Lambda}$: measurements

The complete angular distribution for the cascade decay of $e^+ + e^- \to J/\psi \to \Lambda(\to p + \pi^-) + \bar{\Lambda}(\to \bar{p} + \pi^+)$:

$$\begin{aligned}\mathcal{W} =& \mathcal{T}_0 + \alpha_\psi \mathcal{T}_5 \\ & + \alpha_\Lambda \alpha_{\bar{\Lambda}} \left(\mathcal{T}_1 + \sqrt{1-\alpha_\psi^2} \cos(\Delta\phi) \mathcal{T}_2 + \alpha_\psi \mathcal{T}_6 \right) \\ & + \sqrt{1-\alpha_\psi^2} \sin(\Delta\phi) \left(\alpha_\Lambda \mathcal{T}_3 + \alpha_{\bar{\Lambda}} \mathcal{T}_4 \right) \ ,\end{aligned}$$

where

$$\begin{aligned}\mathcal{T}_0 &= 1 \ , \\ \mathcal{T}_1 &= \sin\theta_\Lambda^2 \sin\theta_1 \sin\theta_2 \cos\phi_1 \cos\phi_2 + \cos^2\theta_\Lambda \cos\theta_1 \cos\theta_2 \ , \\ \mathcal{T}_2 &= \sin\theta_\Lambda \cos\theta_\Lambda \left(\sin\theta_1 \cos\theta_2 \cos\phi_1 + \cos\theta_1 \sin\theta_2 \cos\phi_2 \right) \ , \\ \mathcal{T}_3 &= \sin\theta_\Lambda \cos\theta_\Lambda \sin\theta_1 \sin\phi_1 \ , \\ \mathcal{T}_4 &= \sin\theta_\Lambda \cos\theta_\Lambda \sin\theta_2 \sin\phi_2 \ , \\ \mathcal{T}_5 &= \cos^2\theta_\Lambda \ , \\ \mathcal{T}_6 &= \cos\theta_1 \cos\theta_2 - \sin^2\theta_\Lambda \sin\theta_1 \sin\theta_2 \sin\phi_1 \sin\phi_2 .\end{aligned}$$

$\Lambda\bar{\Lambda}$: measurements

$$\langle \mathcal{O}(\theta_1,\theta_2,\phi_1,\phi_2)\rangle = \frac{1}{N}\int \mathcal{O}(\theta_1,\theta_2,\phi_1,\phi_2)\ \mathcal{W}\ d\Omega_p\ d\Omega_{\bar{p}}\ ,$$

$$N = \int \mathcal{W}\ d\Omega_p\ d\Omega_{\bar{p}}\ , \quad d\Omega_{p/\bar{p}} = d\cos\theta_{1/2} d\phi_{1/2}\ .$$

- This leads to the observables:

$$\mathcal{O}_0 = -1\ ,$$
$$\mathcal{O}_1 = -\frac{1}{2}(1+\alpha_\psi)\frac{1-\cos^2\theta_\Lambda}{1+\alpha_\psi\cos^2\theta_\Lambda}\ ,$$
$$\mathcal{O}_2 = -\frac{1}{2}(1-\alpha_\psi)\frac{1-\cos^2\theta_\Lambda}{1+\alpha_\psi\cos^2\theta_\Lambda}\ ,$$
$$\mathcal{O}_3 = 0\ ,$$
$$\mathcal{O}_4 = 0\ .$$

- $\mathcal{O}_0$, $\mathcal{O}_3$, and $\mathcal{O}_4$ all do not violate their respective separable state boundaries, thus can not demonstrate quantum entanglement in $\Lambda\bar{\Lambda}$

$\Lambda\bar{\Lambda}$: measurements

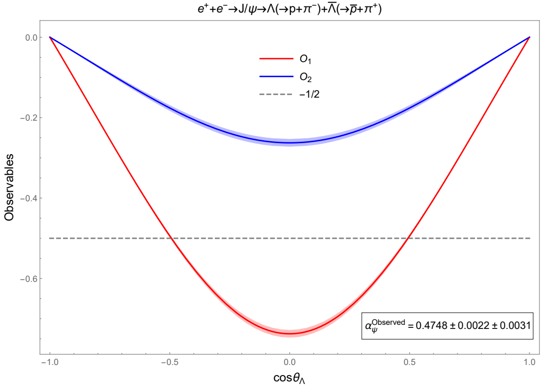

Figure: The $\mathcal{O}_1 = \langle\cos(\phi_1+\phi_2)\rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda\alpha_{\bar{\Lambda}}\right)$ and $\mathcal{O}_2 = \langle\cos(\phi_1-\phi_2)\rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda\alpha_{\bar{\Lambda}}\right)$ for $\Lambda\bar{\Lambda}$ pairs produced at e^+e^- collider by using $\alpha_\psi^{\text{Observed}} = 0.4748 \pm 0.0022 \pm 0.0031$ from the BESIII experiment. The solid lines are given by the central value of $\alpha_\psi^{\text{Observed}}$, while the shaded bands correspond to the 5σ confidence

Outline

Munchhausen Trilemma in Philosophy

- A circular argument.
- An infinite regression.
- Dogmatism

No-Go Theorem for Collider Physics

- No-Go Theorem: we cannot probe the quantum entanglement and Bell non-localtiy at the colliders [9].
- Rule out the Local Hidden Variable Theories (LHVTs).
- Do not use quantum mechanics (QM) and QFT since we test QM against the LHVTs.
- Two approaches: Fundamental Approach, and Decay Approach.

It is very difficult in decay approach since we have done the calculations in the QFT.

[9] S. A. Abel, M. Dittmar and H. K. Dreiner, Phys. Lett. B **280**, 304-312 (1992); H. K. Dreiner, [arXiv:hep-ph/9211203 [hep-ph]]; S. Li, W. Shen and J. M. Yang, Eur. Phys. J. C **84**, no.11, 1195 (2024); M. Fabbrichesi, R. Floreanini and L. Marzola, [arXiv:2503.18535 [quant-ph]]; P. Bechtle, C. Breuning, H. K. Dreiner and C. Duhr, [arXiv:2507.15947 [hep-ph]]; S. A. Abel, H. K. Dreiner, R. Sengupta and L. Ubaldi, [arXiv:2507.15949 [hep-ph]]; M. Low, [arXiv:2508.10979 [hep-ph]].

No-Go Theorem for Collider Physics

- In the current collider experiments, we cannot conduct the direct spin measurements, and thus the spin correlations cannot be directly measured.
- The main idea for the collider studies is that the spin correlations can be measured indirectly from the appropriate angular correlations between the final states from the intermediate particle decays.
- The bridges between the spin correlations and angular correlations are the spin-analyzing powers of the intermediate particles.
- The great challenge is that we must measure the spin-analyzing powers without the assumptions of the QM and QFT.

New Physics Searches and SM Precision Measurements

- Quantum entanglement and Bell non-locality have been studied traditionally in the Quantum Information Theory (QIT).
- We can employ the QIT concepts, and propose the new physics observables to probe the new physics beyond the SM and study the SM precision measurements.
- The important question is what kind of new physics observables can we propose?

The General Framework

- At the e^+e^- collider, we consider the production of a pair of fermions F_aF_b, *i.e.*, $e^+e^- \to F_aF_b$, which subsequently decay

$$F_{a/b} \to f_{a/b,1} + f_{a/b,2} + \ldots + f_{a/b,N} \ .$$

- We choose the helicity rest frames for F_a and F_b, respectively, and define the same coordinate system for them

$$\hat{\mathbf{y}} \equiv \frac{\hat{\mathbf{p}}_{e^-} \times \hat{\mathbf{p}}_{F_a}}{|\hat{\mathbf{p}}_{e^-} \times \hat{\mathbf{p}}_{F_a}|}, \ \hat{\mathbf{z}} \equiv \hat{\mathbf{p}}_{F_a}, \ \hat{\mathbf{x}} \equiv \hat{\mathbf{y}} \times \hat{\mathbf{z}} \ .$$

- This coordinate system is convenient because the difference between the rest frames of F_a and F_b is a pure boost along their momentum directions.

Coordinate System

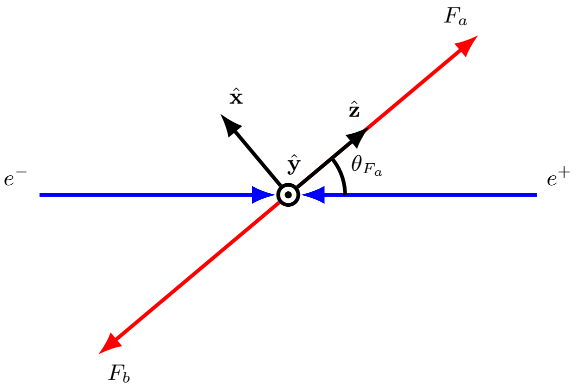

Figure: The coordinate system $(\hat{x}, \hat{y}, \hat{z})$ for the production process $e^+e^- \to F_aF_b$.

The General Framework

- Selecting two fermions $f_{a,1}$ and $f_{b,1}$ respectively from the decay products of F_a and F_b, we assume that the unit momentum directions for $f_{a,1}$ and $f_{b,1}$ in the rest frames of F_a and F_b are $\vec{q}_{a1}$ and $\vec{q}_{b1}$, respectively.
- We consider the angular correlations between $\vec{q}_{a1}$ and $\vec{q}_{b1}$, *i.e.*, the expectation values $\langle q^i_{a1} q^j_{b1} \rangle$, which can be measured in the collider experiments.
- We study the relations between such angular correlations and the spin correlations for F_a and F_b.

Spin Density Matrix

- The spin polarization state for the composite quantum $F_a F_b$ system is described by a spin density matrix

$$\rho = \frac{I_4 + \sum_i \left(B_i^+ \sigma^i \otimes I_2 + B_i^- I_2 \otimes \sigma^i \right) + \sum_{i,j} C_{ij} \sigma^i \otimes \sigma^j}{4} .$$

- Choosing the basis $(|\frac{1}{2}\rangle_{F_a} \otimes |\frac{1}{2}\rangle_{F_b}, |\frac{1}{2}\rangle_{F_a} \otimes |-\frac{1}{2}\rangle_{F_b}, |-\frac{1}{2}\rangle_{F_a} \otimes |\frac{1}{2}\rangle_{F_b}, |-\frac{1}{2}\rangle_{F_a} \otimes |-\frac{1}{2}\rangle_{F_b})$, we can write the spin density matrix as follows

$$\rho \equiv \sum_{i=1}^{n} p_i |\Psi_i\rangle\langle\Psi_i| \ , \ \sum_{i=1}^{n} p_i = 1 \ .$$

Spin Density Matrix

- $|\Psi_i\rangle\langle\Psi_i|$ is a pure state, and its quantum state (polarization state) $|\Psi_i\rangle$ is defined as

$$|\Psi_i\rangle = \sum_{k,j=\pm\frac{1}{2}} \alpha^i_{k,j} |k\rangle_{F_a} \otimes |j\rangle_{F_b} \ , \quad \sum_{k,j=\pm\frac{1}{2}} |\alpha^i_{k,j}|^2 = 1 \ .$$

- We obtain

$$\mathrm{Tr}[C] = 2\left(\rho_{11} + \rho_{23} + \rho_{32} + \rho_{44}\right) - 1 \ .$$

$$\mathrm{Tr}[C] = 1 - 2\sum_{i=1}^{n} p_i |\alpha^i_{1/2,-1/2} - \alpha^i_{-1/2,1/2}|^2 \ .$$

The Correlation Relations in the QFT

- The relations between the spin correlations C_{ij} and angular correlations $\langle q^i_{a1} q^j_{b1} \rangle$ in the QFT are [10]

$$C_{ij} = \frac{9}{\alpha_{F_a}\alpha_{F_b}} \langle q^i_{a1} q^j_{b1} \rangle \ .$$

- α_{F_a} and α_{F_a} are spin-analyzing powers for F_a and F_b, respectively. In particular, we have $-1 \leq \alpha_{F_{a/b}} \leq 1$.

[10] T. Han, M. Low and Y. Su, JHEP **10**, 217 (2025) [arXiv:2501.04801 [hep-ph]]; P. Bechtle, C. Breuning, H. K. Dreiner and C. Duhr, [arXiv:2507.15947 [hep-ph]].

The Correlation Relations in the LHVTs

In the LHVTs, to derive the relations between the spin correlations and angular correlations, we make the following assumptions [11]

1. A LHVT with a set of local hidden variables.
2. Special relativity and Poincaré-invariance hold.
3. The decays of F_a and F_b are independent of each other.
4. The spin for each particle is an element of reality in the EPR sense, *i.e.*, a vector with a definite orientation. And thus the spins play the role of hidden variables.
5. If the particles F_a and F_b have spins $\hat{s}_a$ and $\hat{s}_b$, the probability distributions (in their rest frames) of the momenta of the daughter particles $f_{a,1}$ and $f_{b,1}$ are always the same and depend only on $\vec{s}_a$ and $\vec{s}_b$, respectively.

[11] P. Bechtle, C. Breuning, H. K. Dreiner and C. Duhr, [arXiv:2507.15947 [hep-ph]]; J. A. Aguilar-Saavedra, J. A. Casas and J. M. Moreno, [arXiv:2603.19389 [hep-ph]].

The Correlation Relations in the LHVTs

- With these assumptions, we can obtain the same relations in the LHVTs as these in QFT.
- The key difference is that we now have $-3 \leq \alpha_{F_{a/b}} \leq 3$.
- With such ranges for the spin-analyzing powers, we can show that we can not probe the Bell non-locality at the collider.
- Therefore, the great challenge is how to measure the product of the spin-analyzing powers $\alpha_{F_a}\alpha_{F_b}$ without the QM and QFT assumptions.

Three Proposals to Evade the No-Go Theorem

- The absolute values of the spin-analyzing powers for $F_a F_b$ are equal to 1, *i.e.*, the maximal value for fermion in QFT, for example, $t\bar{t}$.
- Invariants of the spin correlation matrices.
- Invariants of the spin density matrices.

Invariants of the Spin Correlation Matrices

With the assumption that the spin is defined via the Lorentz symmetry or considering the implicit symmetry in the spin density matrix, we can show that we can determine the product of the spin-analyzing powers $\alpha_{F_a}\alpha_{F_b}$ by the invariants of the spin correlation matrices, and probe the Bell non-locality at the lepton collider.

Definition of Spin via the Lorentz Symmetry

- Poincaré symmetry is the basic symmetry for the special relativity, and contains the Lorentz symmetry and translation symmetry.
- There are two Casmir operators for Poincaré algebra: $P^\mu P_\mu$ and $W^\mu W_\mu$, where P_μ and W_μ are the four-momentum and Pauli–Lubanski pseudo-vector, respectively. These two Casmir operators define the rest mass and spin of a particle, respectively.
- We consider the proper orthochronous group $SO(3,1)$ for Lorentz symmetry. The complexified Lorentz algebra for $SO(3,1)$ is locally isomorphic to the direct sum of two $SU(2)$ algebra, and thus we can formally write it as $SO(3,1) \simeq SU(2)_1 \times SU(2)_2$.

Definition of Spin via the Lorentz Symmetry

- Every finite-dimensional representation of the Lorentz group can be written as two spins (j_1, j_2), where j_i is an integer or half-integer.
- We obtain the representations for the scalar, left-handed Weyl fermion, right-handed Weyl fermion, gauge boson as $(0, 0)$, $(1/2, 0)$, $(0, 1/2)$, $(1/2, 1/2)$, respectively.
- The spin $SU(2)_S$ group can be considered as the diagonal subgroup of $SU(2)_1 \times SU(2)_2$.

Assumptions: Strong to Weak

- The definition assumption is the solid and fundamental assumption to probe quantum entanglement and Bell non-locality at the colliders. However, it can be relaxed.
- The key point is the following. The Lorentz symmetry is not violated, and the spin density matrix describes this composite quantum system. And thus, Lorentz symmetry is an implicit symmetry in the spin density matrix.
- In fact, there exists a $SU(2)_a \times SU(2)_b$ symmetry in the spin density matrix. The generators for $SU(2)_a$ and $SU(2)_b$ are $\sigma^i \otimes I_2/4$ and $I_2 \otimes \sigma^i/4$, respectively.

Assumptions: Strong to Weak

- $(|\frac{1}{2}\rangle_{F_a}, |-\frac{1}{2}\rangle_{F_a})^T$ forms the fundamental representation of $SU(2)_a$, and $(|\frac{1}{2}\rangle_{F_b}, |-\frac{1}{2}\rangle_{F_b})^T$ forms the fundamental representation of $SU(2)_b$.
- The spin $SU(2)_S$ group can be considered as the diagonal subgroup of $SU(2)_a \times SU(2)_b$.

Invariants of the Spin Correlation Matrices

- Because both $(|\frac{1}{2}\rangle_{F_a}, |-\frac{1}{2}\rangle_{F_a})^T$ and $(|\frac{1}{2}\rangle_{F_b}, |-\frac{1}{2}\rangle_{F_b})^T$ belong to the fundamental representation of $SU(2)_S$, we can decompose the product of these two fundamental representations into the irreducible representations of $SU(2)_S$, *i.e.*, decompose the basis of spin density matrix into the irreducible representations of $SU(2)_S$. Such decomposition is

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \ .$$

- The singlet $\mathbf{1}$ and triplet $\mathbf{3}$ belong to the anti-symmetric and symmetric representations of $SU(2)_S$. In the collider experiments, $\mathbf{1}$ and $\mathbf{3}$ correspond to the CP-odd scalar and gauge boson, respectively, which can couple to both e^+e^- and F_aF_b.

Quantum States

By definition, the quantum state for the singlet in the anti-symmetric representation is

$$|\Psi_S\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}\rangle_{F_a} \otimes |-\frac{1}{2}\rangle_{F_b} - |-\frac{1}{2}\rangle_{F_a} \otimes |\frac{1}{2}\rangle_{F_b}\right) \ .$$

And the quantum states for the triplet in the symmetric representation are

$$|\Psi_T^{(1,1)}\rangle = |\frac{1}{2}\rangle_{F_a} \otimes |\frac{1}{2}\rangle_{F_b} \ ,$$

$$|\Psi_T^{(1,0)}\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}\rangle_{F_a} \otimes |-\frac{1}{2}\rangle_{F_b} + |-\frac{1}{2}\rangle_{F_a} \otimes |\frac{1}{2}\rangle_{F_b}\right) \ ,$$

$$|\Psi_T^{(1,-1)}\rangle = |-\frac{1}{2}\rangle_{F_a} \otimes |-\frac{1}{2}\rangle_{F_b} \ .$$

Quantum States

For the spin density matrix, we can obtain the non-zero components for the singlet and triplet

$$
\begin{aligned}
|\Psi_S\rangle : &\quad C_{11} = C_{22} = C_{33} = -1; \mathrm{Tr}[C] = -3 \ , \\
|\Psi_T^{(1,1)}\rangle : &\quad B_3^{\pm} = 1, C_{33} = 1; \mathrm{Tr}[C] = 1 \ , \\
|\Psi_T^{(1,0)}\rangle : &\quad C_{11} = C_{22} = -C_{33} = 1; \mathrm{Tr}[C] = 1 \ , \\
|\Psi_T^{(1,-1)}\rangle : &\quad B_3^{\pm} = -1, C_{33} = 1; \mathrm{Tr}[C] = 1 \ .
\end{aligned}
$$

Invariants of the Spin Correlation Matrices

- The quantum state for the CP-odd scalar exchange is $|\Psi_S\rangle$, and then we have $\mathrm{Tr}[C] = -3$.
- The quantum states for the gauge boson exchanges are the linear combinations of $|\Psi_T^{(1,1)}\rangle$, $|\Psi_T^{(1,0)}\rangle$, and $|\Psi_T^{(1,-1)}\rangle$, whose coefficients are functions of θ_{F_a}. And thus we have $\mathrm{Tr}[C] = 1$.
- The quantum state for the CP-even scalar exchange is $|\Psi_T^{(1,0)}\rangle$, so we have $\mathrm{Tr}[C] = 1$.

Prove $\mathrm{Tr}[C] = 1$ for Gauge Boson Exchanges

- Because $|\Psi_i\rangle$ and $|\Psi_S\rangle$ are orthogonal to each other, we obtain

$$\alpha^i_{1/2,-1/2} = \alpha^i_{-1/2,1/2} \ .$$

- And thus, we prove that

$$\mathrm{Tr}[C] = 1 - 2\sum_{i=1}^{n} p_i |\alpha^i_{1/2,-1/2} - \alpha^i_{-1/2,1/2}|^2 = 1 \ .$$

Invariants under Basis Rotations

- $\mathrm{Tr}[C]$ is invariant under the basis rotation of quantization axes.
- Assuming that $\vec{e}_i$ and $\vec{e}_i'$ with $i = 1, 2, 3$ are two different quantization axis choices, which relate to each other via a $SO(3)$ rotation R_{ij}, *i.e.*, $\vec{e}_i' = \vec{e}_j R_{ji}$, we obtain [12]

$$C'_{ij} = R^T_{ik} C_{kl} R_{lj} = \left(R^T C R\right)_{ij} \ .$$

- Thus, we prove that

$$\mathrm{Tr}[C'] = \mathrm{Tr}[R^T C R] = \mathrm{Tr}[C] \ .$$

[12] K. Cheng, T. Han and M. Low, Phys. Rev. D **109**, no.11, 11 (2024) [arXiv:2311.09166 [hep-ph]]; K. Cheng, T. Han and M. Low, Phys. Rev. D **111**, no.3, 033004 (2025) [arXiv:2407.01672 [hep-ph]].

Summary

- For the two-fermion $F_a F_b$ productions and decays via one mediator exchange at the e^+e^- collider, we show that the trace $\mathrm{Tr}[C]$ of the spin correlation matrix is an invariant quantity, which is independent on the scattering angle.
- $\mathrm{Tr}[C]$ is equal to 1, 1, and -3 for the exchanges of gauge boson, CP-even scalar, and CP-odd scalar, respectively.
- Thus, for one mediator exchange, for example, photon, we can determine the product of the spin-analyzing powers for $F_a F_b$ via $\mathrm{Tr}[C]$, and reconstruct the spin correlation matrix.

Summary

- With the CHSH-Horodecki criterion [13], we can probe the Bell non-locality, and evade the no-go theorem.
- In particular, the invariant $\mathrm{Tr}[C]$ is a new physics observable to probe the new physics beyond the SM and study the SM precision measurements.
- For the CP-even scalar and CP-odd scalar changes, C_{11}, C_{22}, and C_{33} are all invariants as well, which are independent on the scattering angle. In particular, $C_{33} = -1$ for both CP-even and CP-odd scalars, and then we do not need to know the CP property of the scalar.

[13] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880-884 (1969); R. Horodecki, P. Horodecki and M. Horodecki, Phys. Lett. A **200**, no.5, 340-344 (1995).

Summary

- We can employ the observables C_{ii} and $\mathrm{Tr}[C]$ to determine the product of the spin-analyzing powers and study the Bell non-locality, and to probe the new physics beyond the SM and study the SM precision measurements.
- Similarly, we can study the Bell non-locality for the Higgs to $\tau^+\tau^-$ at the LHC.
- We can perform the calculations in the traditional helicity formalism [14]. And then the decomposition is $\mathbf{2} \otimes \overline{\mathbf{2}} = \mathbf{1} \oplus \mathbf{3}$, where the singlet $\mathbf{1}$ and triplet $\mathbf{3}$ belong to the trivial and adjoint representations of $SU(2)_S$. Although the expressions for the quantum states are different, the physics results are the same.

[14] H. Murayama, I. Watanabe and K. Hagiwara, KEK-91-11.

Bell Non-Locality at the BESIII Experiment

- We study the Bell non-locality for the $\Lambda\bar{\Lambda}$ pair production via the $e^+e^- \to J/\psi \to \Lambda\bar{\Lambda}$ process at the BESIII experiment.
- The subsequent decays are: $\Lambda \to p + \pi^-$, and $\bar{\Lambda} \to \bar{p} + \pi^+$.
- So we have $F_a \equiv \Lambda$ and $F_b \equiv \bar{\Lambda}$, and choose $f_{a,1} \equiv p$ and $f_{b,1} \equiv \bar{p}$.

Bell Non-Locality at the BESIII Experiment

- With $\mathrm{Tr}[C] = 1$, we obtain

$$\langle q_p^k q_{\bar{p}}^k \rangle = \frac{1}{9} \alpha_\Lambda \alpha_{\bar{\Lambda}} \ .$$

- And thus we have the spin correlation matrix with

$$C_{ij} = \frac{1}{\langle q_p^k q_{\bar{p}}^k \rangle} \langle q_p^i q_{\bar{p}}^j \rangle \ .$$

- Thus, the spin correlation matrix can be determined by the angular correlation measurements.

Bell Non-Locality at the BESIII Experiment

- We denote the eigenvalues of $C^T C$ as M_1, M_2, M_3, which satisfy $M_1 \geq M_2 \geq M_3$.
- The Bell variable is defined as [15]

$$\mathcal{B} = 2\sqrt{M_1 + M_2}\ .$$

- If $2 < \mathcal{B} \leq 2\sqrt{2}$, we realize the Bell non-locality via the CHSH-Horodecki criterion.

[15] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880-884 (1969); R. Horodecki, P. Horodecki and M. Horodecki, Phys. Lett. A **200**, no.5, 340-344 (1995).

Bell Non-Locality at the BESIII Experiment

- Because we do not have the BESIII experimental data, we consider the spin correlation matrix from the QFT calculations [16].
- With $\alpha_\Lambda = 0.7519 \pm 0.0036 \pm 0.0024$, $\alpha_{\bar{\Lambda}} = -0.7559 \pm 0.0036 \pm 0.0030$, $\alpha_\psi = 0.4748 \pm 0.0022 \pm 0.0031$, and $\Delta\Phi = 0.7521 \pm 0.0042 \pm 0.0066$ [17], we present the numerical results in the following figure with solid line.
- Within 5σ uncertainty in $(\alpha_\psi, \Delta\Phi)$ parameter space, the Bell non-locality is realized in range $|\cos\theta_\Lambda| < 0.44150$

[16] E. Perotti, G. Fäldt, A. Kupsc, S. Leupold and J. J. Song, Phys. Rev. D **99**, no.5, 056008 (2019) [arXiv:1809.04038 [hep-ph]]; V. Batozskaya, A. Kupsc, N. Salone and J. Wiechnik, Phys. Rev. D **108**, no.1, 016011 (2023) [arXiv:2302.07665 [hep-ph]]; S. Wu, C. Qian, Q. Wang and X. R. Zhou, Phys. Rev. D **110**, no.5, 054012 (2024) [arXiv:2406.16298 [hep-ph]].

[17] M. Ablikim *et al.* [BESIII], Phys. Rev. Lett. **129**, no.13, 131801 (2022) [arXiv:2204.11058 [hep-ex]].

Spin Correlation Matrix from the QFT

$$C_{ij} = \frac{1}{1+\alpha_\psi \cos^2\theta_\Lambda} \times \begin{pmatrix} \sin^2\theta_\Lambda & 0 & \gamma_\psi \sin\theta_\Lambda \cos\theta_\Lambda \\ 0 & -\alpha_\psi \sin^2\theta_\Lambda & 0 \\ \gamma_\psi \sin\theta_\Lambda \cos\theta_\Lambda & 0 & \alpha_\psi + \cos^2\theta_\Lambda \end{pmatrix} .$$

Figure: Here, $\alpha_\psi \in [-1, +1]$ is the decay parameter of the vector charmonium ψ, and $\gamma_\psi \equiv \sqrt{1-\alpha_\psi^2}\cos(\Delta\Phi)$ with $\Delta\Phi \in (-\pi, +\pi]$ the relative form factor phase.

Bell Non-Locality at the BESIII Experiment

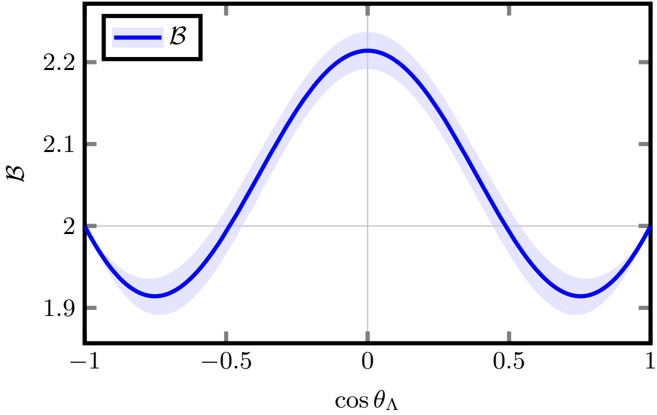

Figure: Bell variable vs $\cos\theta_\Lambda$. The shaded region represents the 5σ uncertainty in $(\alpha_\psi, \Delta\Phi)$ parameter space.

Summary

- We propose a new formalism for quantum entanglement, and study its generic searches at the colliders.
- We show that the quantum space is complex projective space, and the classical space is the cartesian product of the complex projective spaces, which can be defined in the quantum spacce via the discriminant loci. Thus, the quantum entanglement space is the difference of these two spaces and can be define exactly.
- We can reconstruct the discriminants from various measurements at high energy physics experiments, and probe the quantum entanglement spaces via a fundamental approach at exact level.

Summary

- For the specific approach, we propose a generic method to calculate the quantum range and classical range for the expectation value of any physics observable, and can probe the quantum entanglement ranges which the previous ways cannot.
- We study the invariants of the spin correlation matrices. For one mediator exchanges such as scalar and gauge boson, we can determine the product of the spin-analyzing powers, and reconstruct the spin correlation matrix. With the CHSH-Horodecki criterion, we can probe the Bell non-locality, and evade the no-go theorem.
- These invariants provide new physics observables to probe the new physics beyond the SM and study the SM precision measurements.

Thank You Very Much for Your Attention!