



Nucleon Gravitational Form Factors

—From chiral perturbation theory to dispersion relation—

De-Liang Yao (姚德良)

Hunan University

[X.-H. Cao, F.-K. Guo, Q.-Z. Li and **DLY**, Nature Communications (2025)]

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Probes of nucleon structure

- Electromagnetic, weak, gravitational form factors (GFFs)

$$\text{em: } \partial_\mu J_{\text{em}}^\mu = 0 \quad \langle N' | J_{\text{em}}^\mu | N \rangle \longrightarrow Q = 1.602176487(40) \times 10^{-19} \text{C}$$

$$\mu = 2.792847356(23) \mu_N$$

$$\text{weak: PCAC} \quad \langle N' | J_{\text{weak}}^\mu | N \rangle \longrightarrow g_A = 1.2694(28)$$

$$g_p = 8.06(55)$$

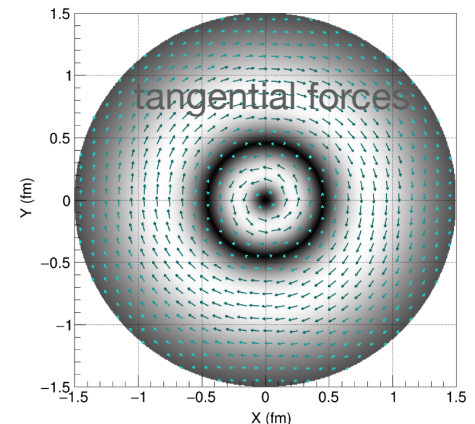
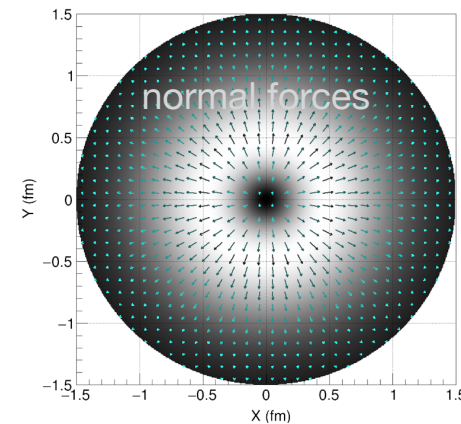
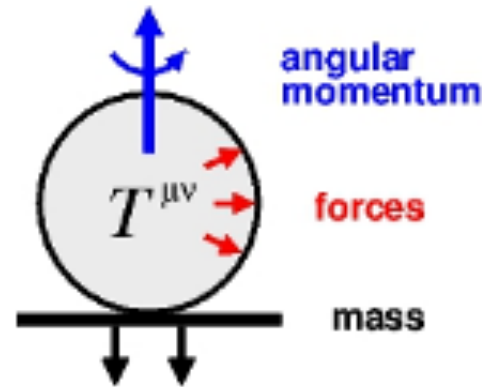
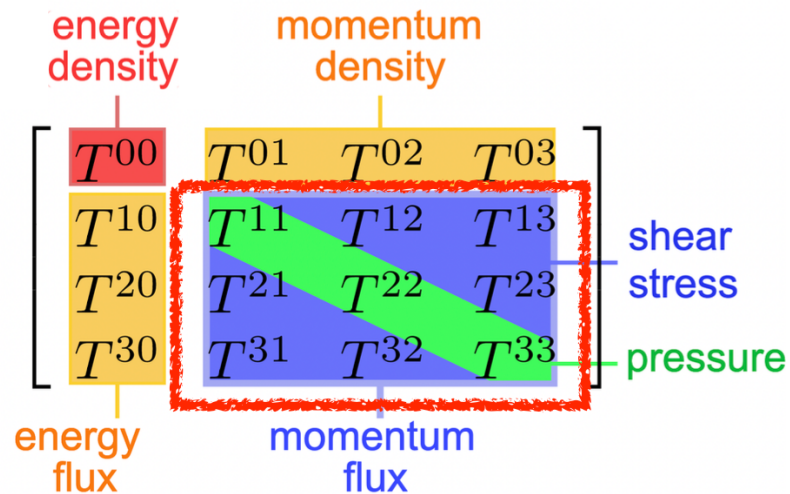
$$\text{gravity: } \partial_\mu T_{\text{grav}}^{\mu\nu} = 0 \quad \langle N' | T_{\text{grav}}^{\mu\nu} | N \rangle \longrightarrow m = 938.272013(23) \text{ MeV}/c^2$$

$$J = \frac{1}{2}$$

$$D = ?$$

Polyakov & Schweitzer, IJMPA (2018) 1830025

D-term: “Last unknown global property”

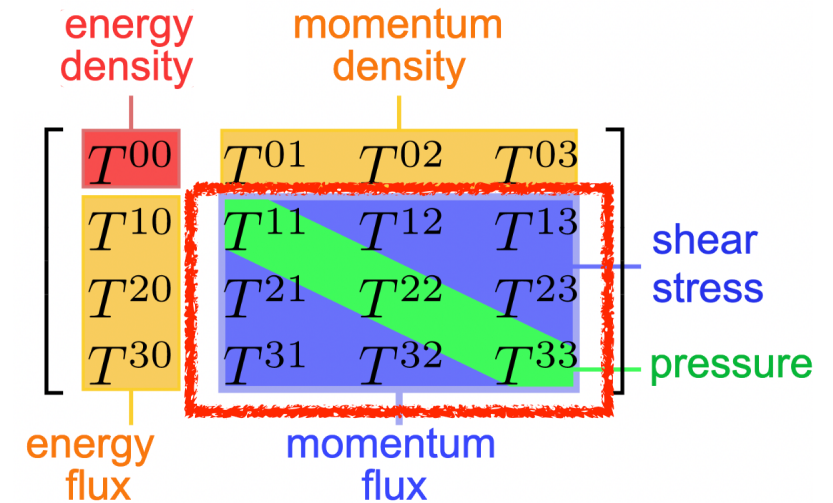


Burkert, et al., Reviews of Modern Physics

The EMT QCD

- A weak classical background gravitational field $g_{\mu\nu}(x)$

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{grav}}}{\delta g^{\mu\nu}} \bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}, \quad S_{\text{grav}} = \int d^4x \sqrt{|g|} \mathcal{L}_{\text{QCD}}$$



- Symmetric Belinfante-improved total QCD EMT:

► Total EMT = Quark + Gluon

$$\hat{T}^{\mu\nu} = \sum_q T_q^{\mu\nu} + T_g^{\mu\nu}, \quad T_q^{\mu\nu} = \bar{\psi}_q \left(\frac{i}{2} \gamma^{\{\mu} \mathcal{D}^{\nu\}} + \frac{1}{4} g^{\mu\nu} m_q \right) \psi_q, \quad T_g^{\mu\nu} = F^{A,\mu\eta} F^{A,\nu}_{\eta} + \frac{1}{4} g^{\mu\nu} F^{A,\kappa\eta} F^{A}_{\kappa\eta}$$

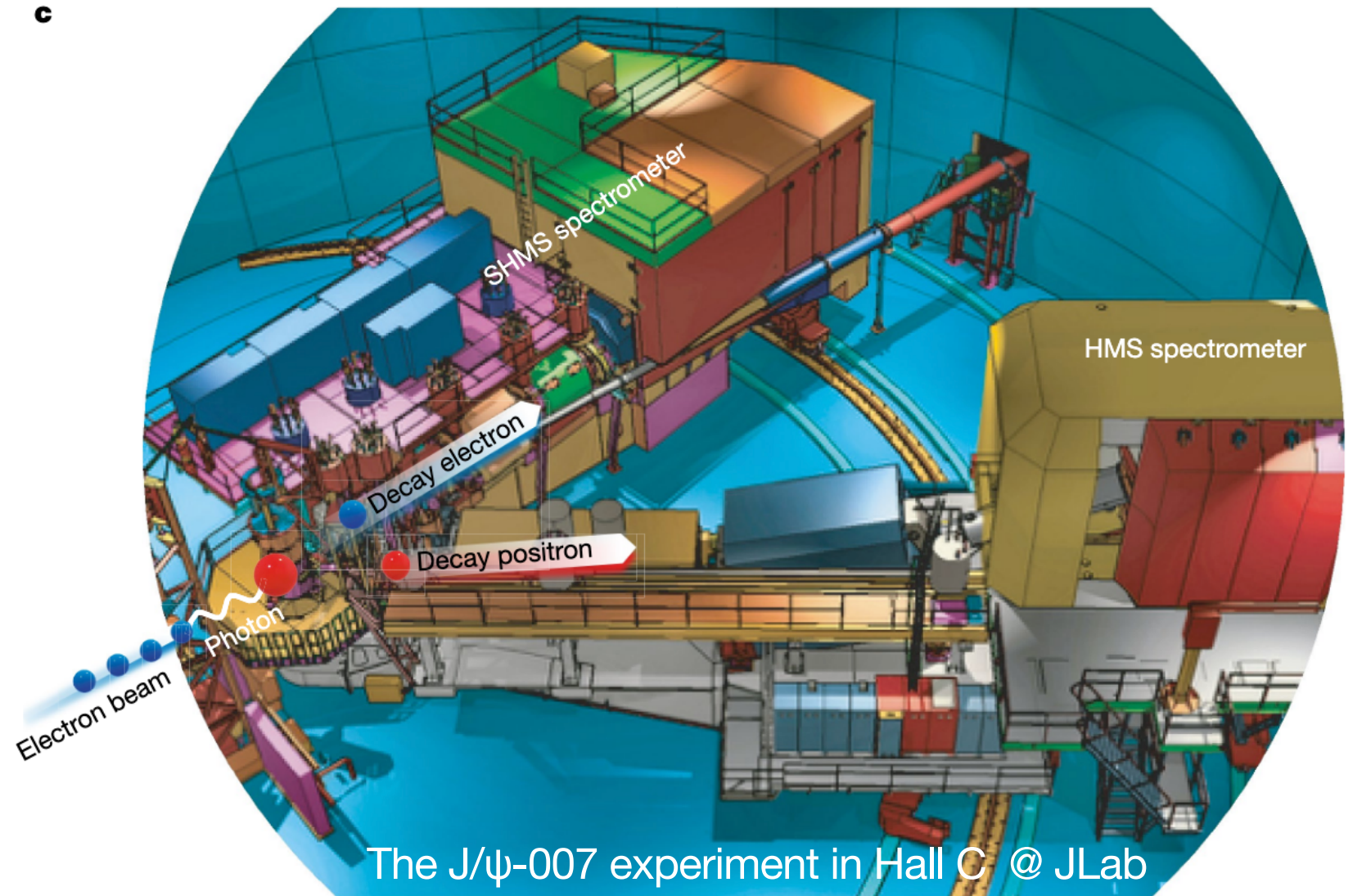
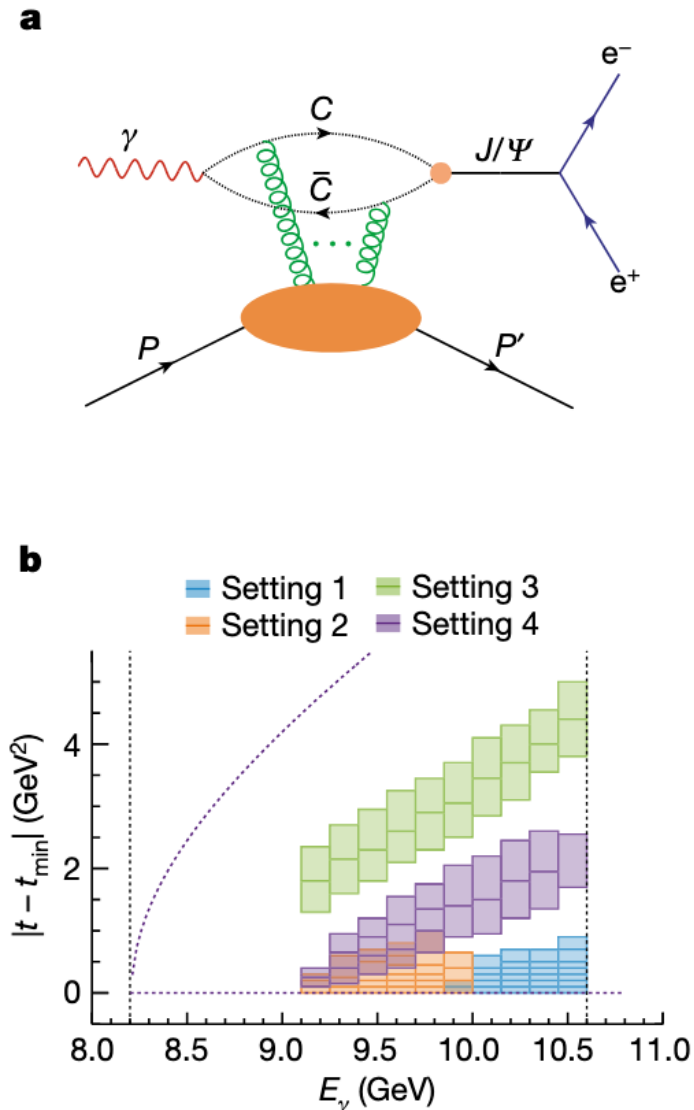
- The total EMT is conserved: $\partial_\mu T^{\mu\nu} = 0$
- Dilation current $j^\mu = x_\nu \hat{T}^{\mu\nu} \longrightarrow \partial_\mu j^\mu = \hat{T}^\mu_\mu = \sum m_q \bar{\psi}_q \psi_q$ (conserved in massless limit)
- Quantum correction due to trace anomaly in non-Abelian gauge theory

$$\hat{T}^\mu_\mu \equiv \frac{\beta(g_s)}{2g_s} F^{A,\mu\nu} F^{A}_{\mu\nu} + \left(1 + \gamma_m \right) \sum_q m_q \bar{\psi}_q \psi_q \quad \text{Nucleon sigma term} \sim 60 \text{ MeV from Higgs}$$

Gluonic GFFs from Experiment

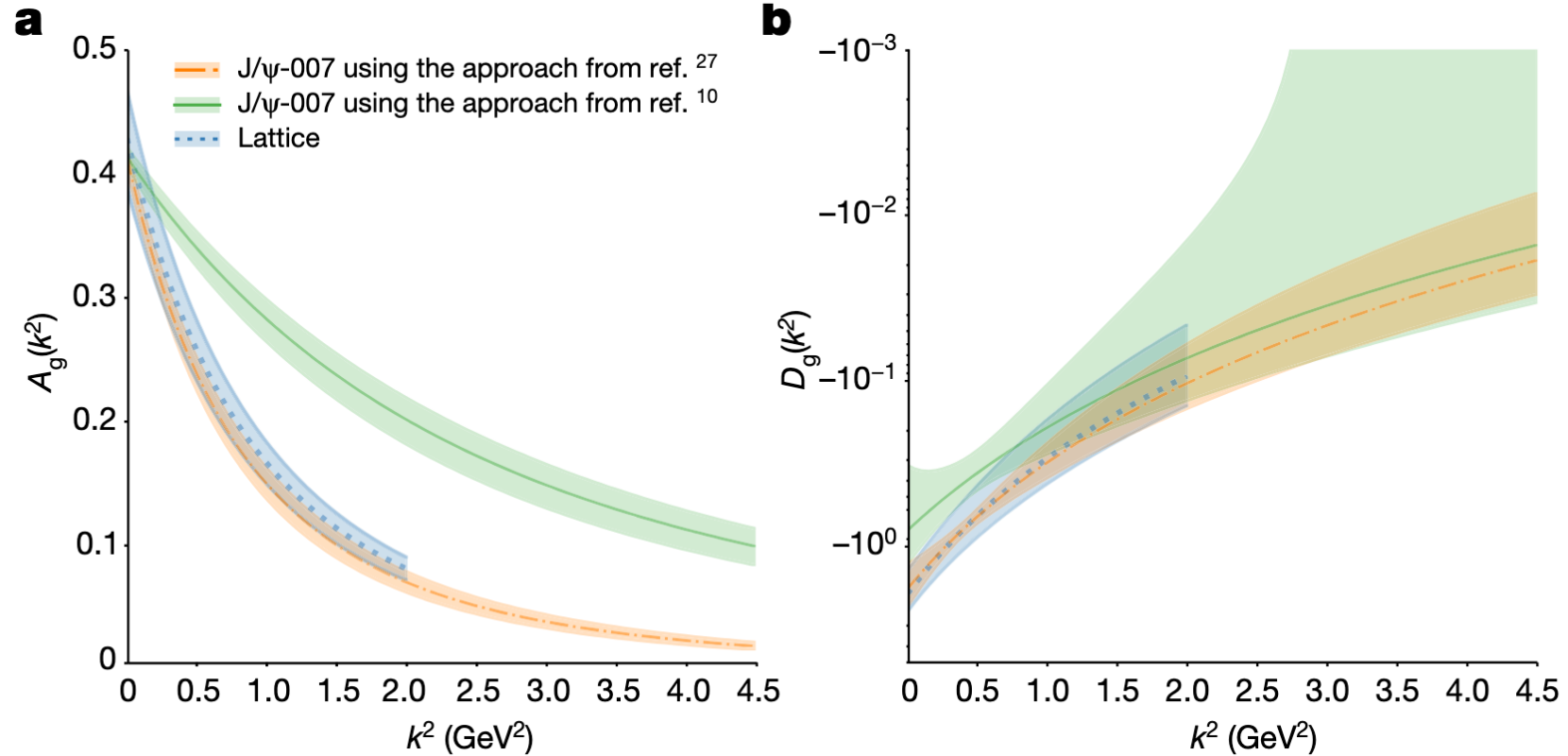
- Threshold J/ψ photo-production @ JLab

Duran, et al., Nature 615 (2023) 7954



Gluonic GFFs from Experiment

- Extracted mass radius ~ 0.5 fm

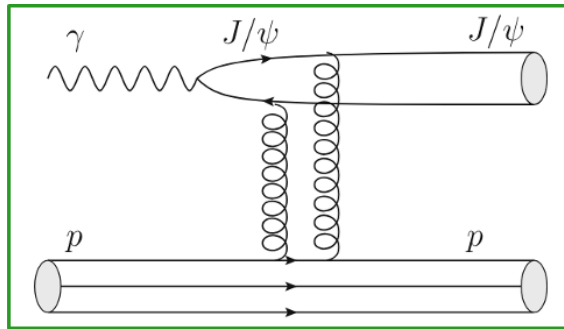


$$\left\langle N(p') \bar{N}(p) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle = \frac{1}{4m_N} \bar{u}(p') \left[A(t) \Delta^\mu \Delta^\nu + J(t) \left(i \Delta^{\{\mu} \sigma^{\nu\} \rho} P_\rho \right) + D(t) (P^\mu P^\nu - t g^{\mu\nu}) \right] u(p)$$

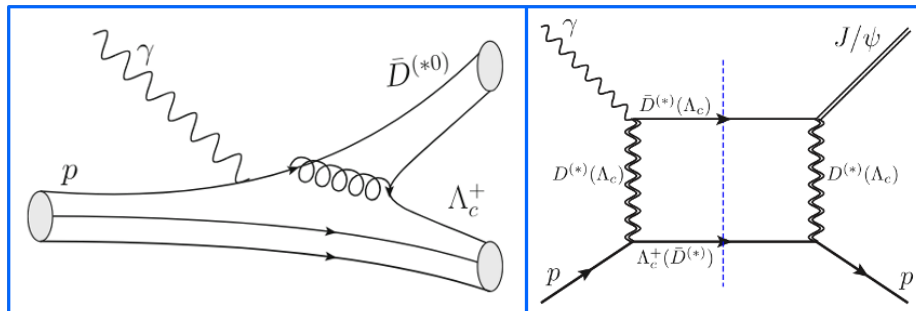
Gluonic GFFs from Experiment

- J/ψ photo-production @ GlueX

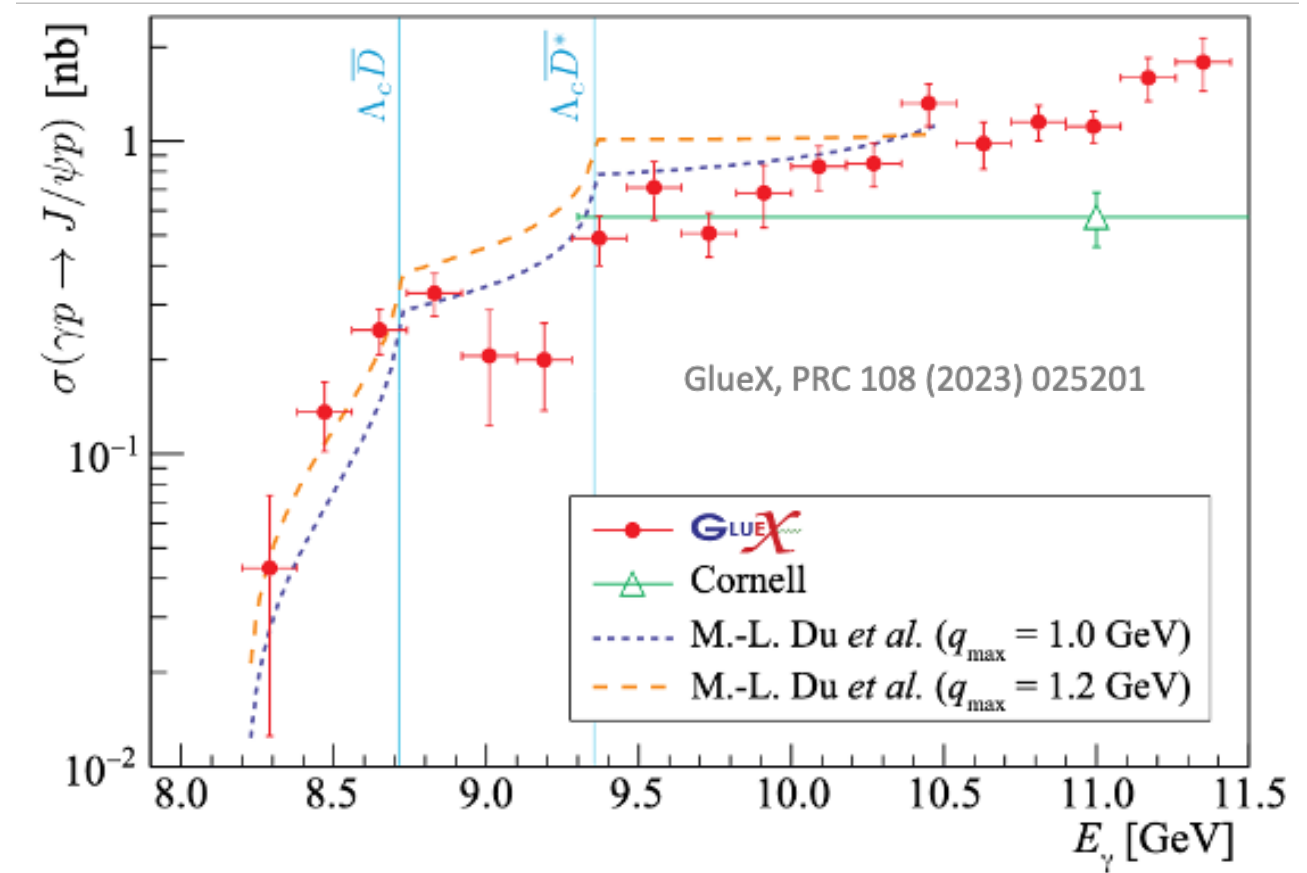
- ▶ Gluon exchange mechanism



- ▶ Coupled channel mechanism



M.-L. Du et al., EPJC 80(2023)1053

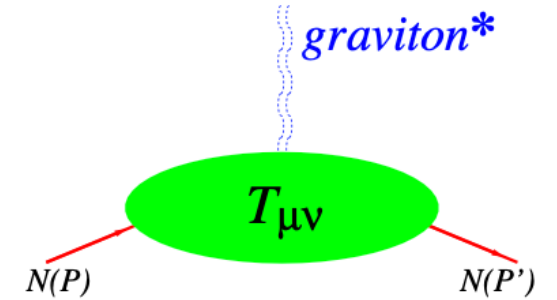


Cusps at open-charm thresholds

Deeply virtual Compton scattering

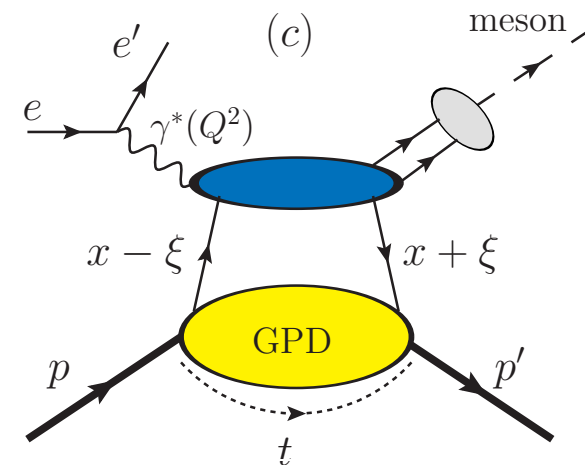
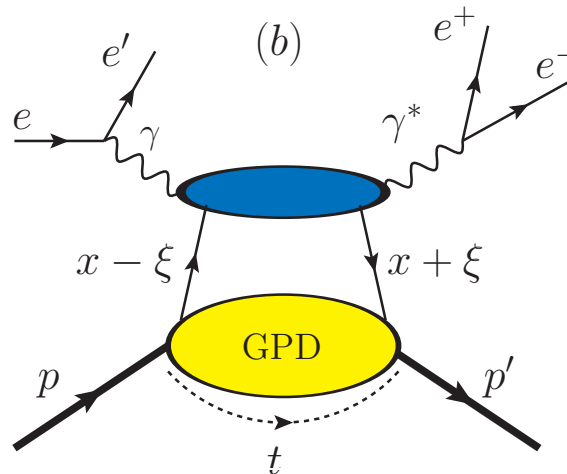
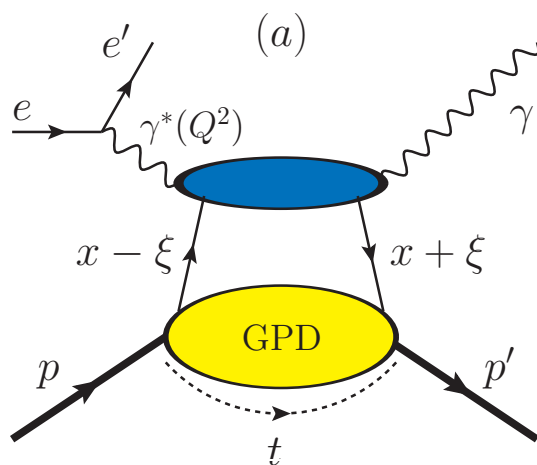
- Deeply virtual Compton scattering (DVCS) $eN \rightarrow e'N'\gamma$: H^a, E^a (GPD)
- The n -th order Mellin moments:

$$\int_{-1}^1 dx x^{n-1} H^a(x, \xi, t), \quad \int_{-1}^1 dx x^{n-1} E^a(x, \xi, t),$$



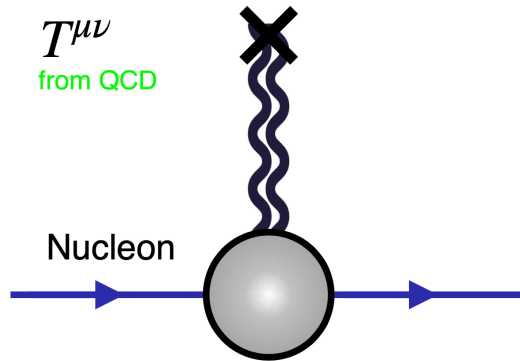
- The first Mellin moments $\rightarrow$ Pauli and Dirac FFs
- The second Mellin moments $\rightarrow$ GFFs

$$\int_{-1}^1 dx x H^a(x, \xi, t) = A^a(t) + \xi^2 D^a(t), \quad \int_{-1}^1 dx x E^a(x, \xi, t) = B^a(t) - \xi^2 D^a(t).$$



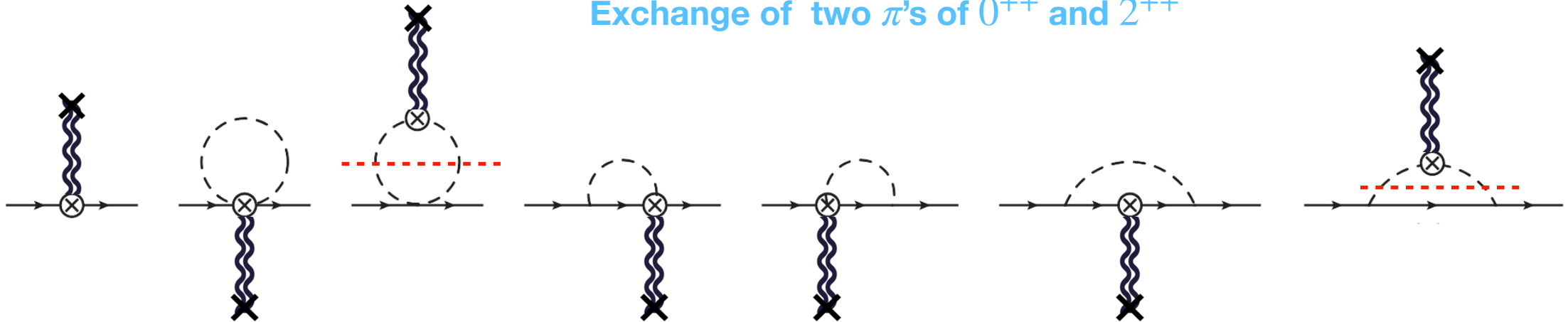
GFFs in ChPT

- Systematical treatment in chiral perturbation theory (ChPT)



- ▶ At low energies: chiral correction is important
- ▶ Hadron exchanges take over gluon and quark exchanges

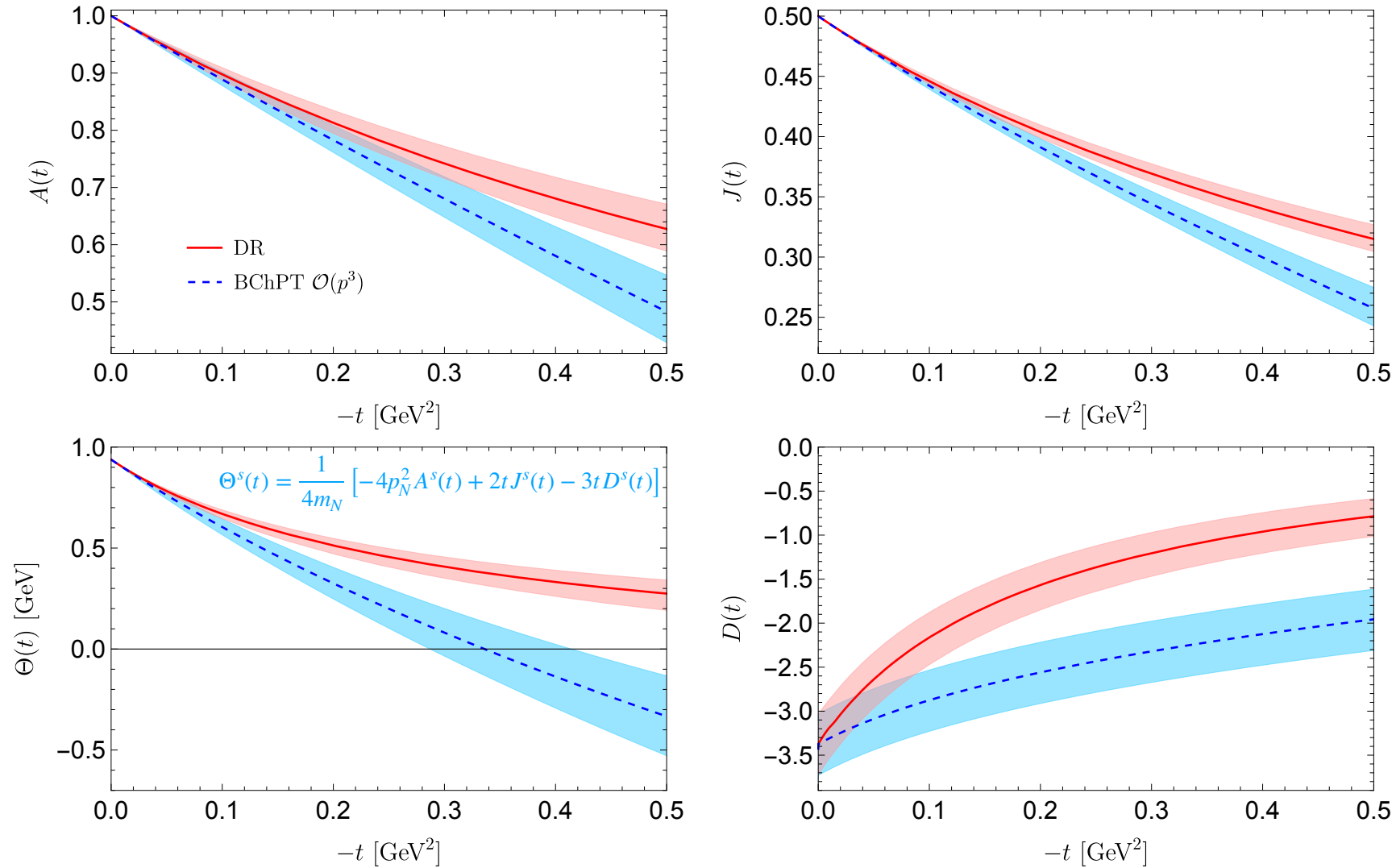
Exchange of two π 's of 0^{++} and 2^{++}



Alharazin, et al., PRD (2020)

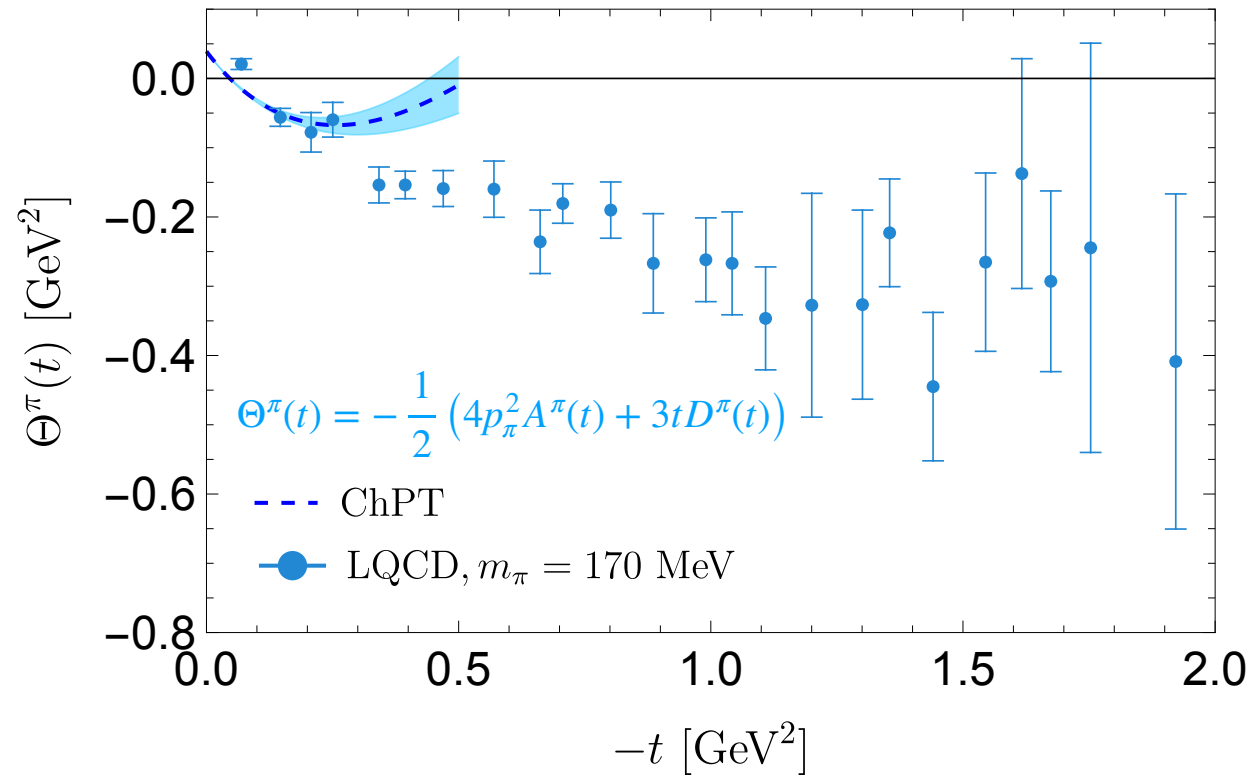
ChPT results

- Nucleon GFFs up to $\mathcal{O}(p^3)$ in BChPT (blue curves)

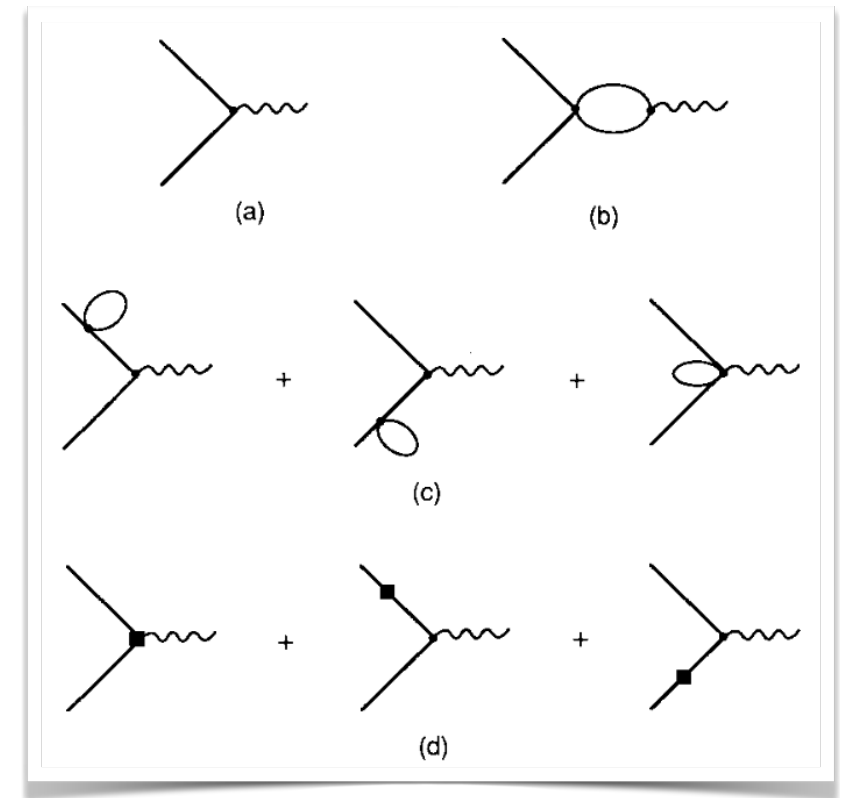


ChPT results

- Pion GFFs up to $\mathcal{O}(p^4)$ in mesonic ChPT



$$\left\langle \pi^a(p') \left| \hat{T}^{\mu\nu}(0) \right| \pi^b(p) \right\rangle = \frac{\delta^{ab}}{2} \left[\textcolor{green}{A}^\pi(t) P^\mu P^\nu + \textcolor{blue}{D}^\pi(t) (\Delta^\mu \Delta^\nu - t g^{\mu\nu}) \right]$$

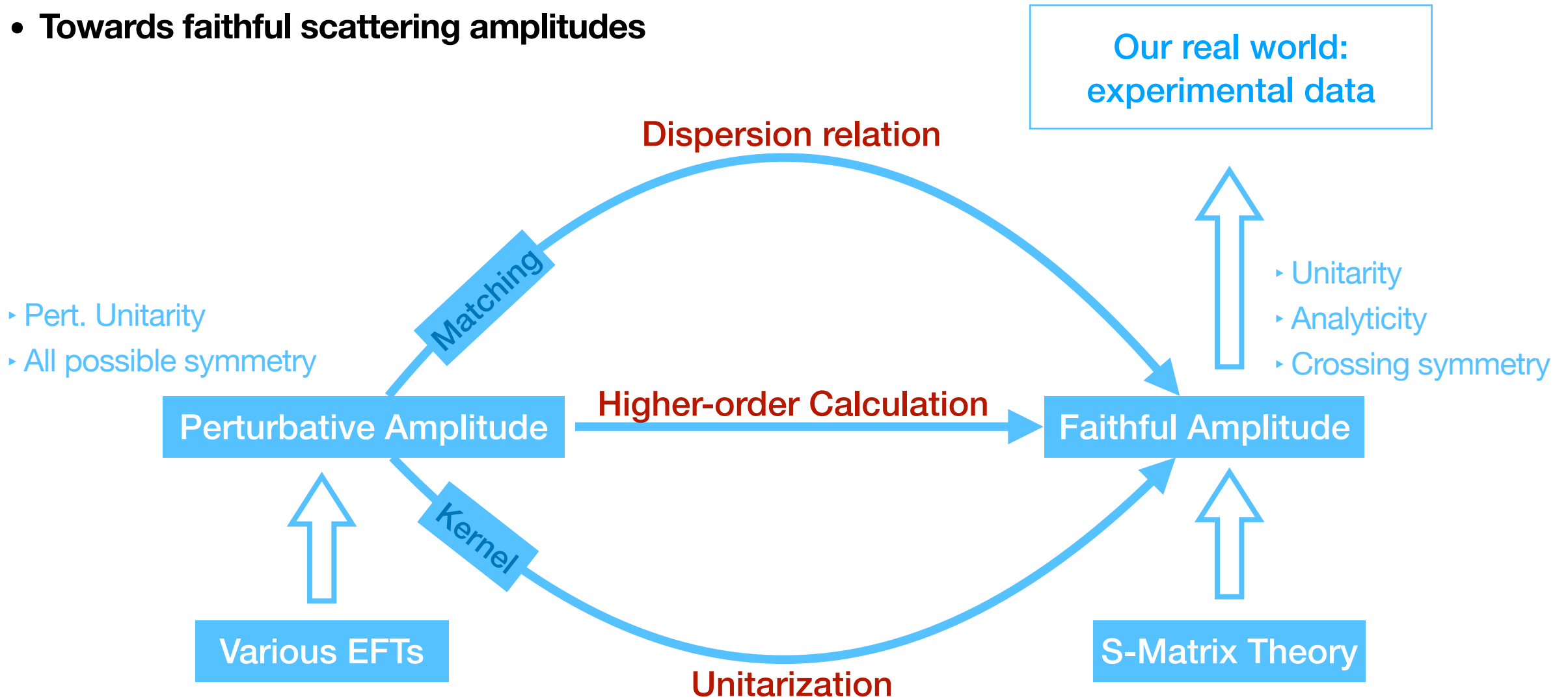


Donoghue & Leutwyler, Z. Phys. C (1991)



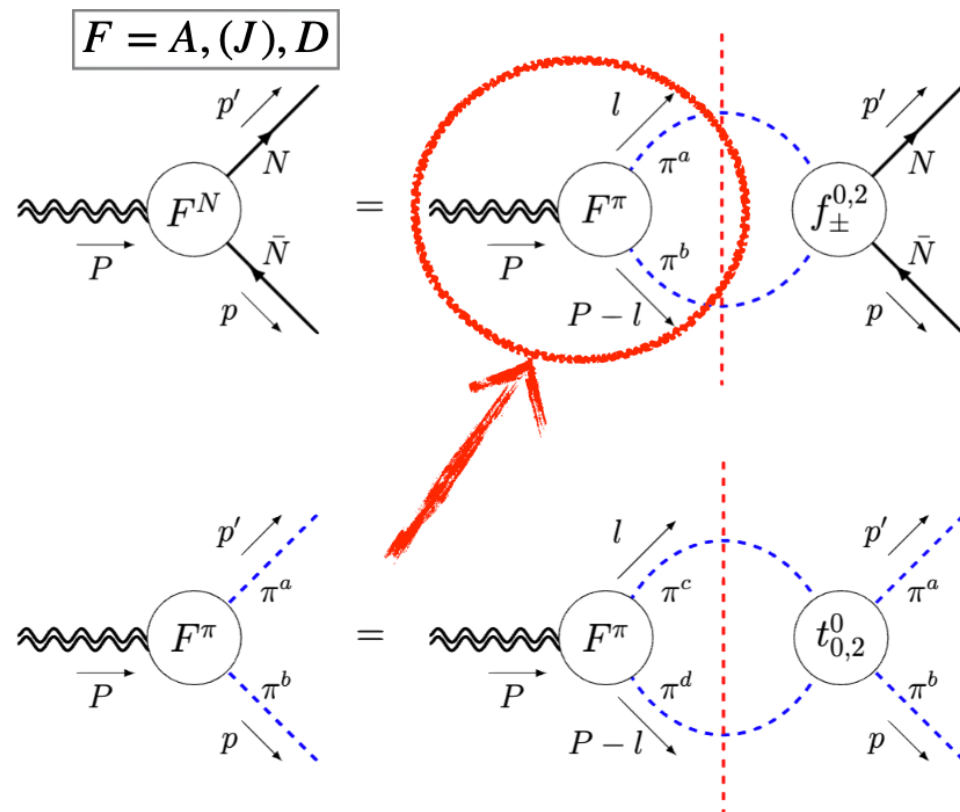
Extension of valid region

- Towards faithful scattering amplitudes



Dispersive Representation

- Crossing: space-like $\langle N(p') | T^{\mu\nu} | N(p) \rangle \Leftrightarrow$ time-like $\langle N(p') \bar{N}(p) | T^{\mu\nu} | 0 \rangle$



Take $A(t)$ as an example

- Nucleon sector:

$$\text{Im } A(t) = \frac{3p_{\pi}^5}{\sqrt{6}t} \left[f_{-}^2(t) + \sqrt{\frac{3}{2}} \frac{m_N}{p_N^2} \Gamma^2(t) \right]^* A^{\pi}(t),$$

...

$$\Gamma^2(t) = m_N \sqrt{\frac{2}{3}} f_{-}^2(t) - f_{+}^2(t)$$

- Meson sector:

$$\text{Im } A^{\pi}(t) = \rho_{\pi}(t) (t_2^0(t))^* A^{\pi}(t),$$

...

πN amplitudes $f_{\pm}^{0,2}$ from modern Roy-Steiner eq. analyses

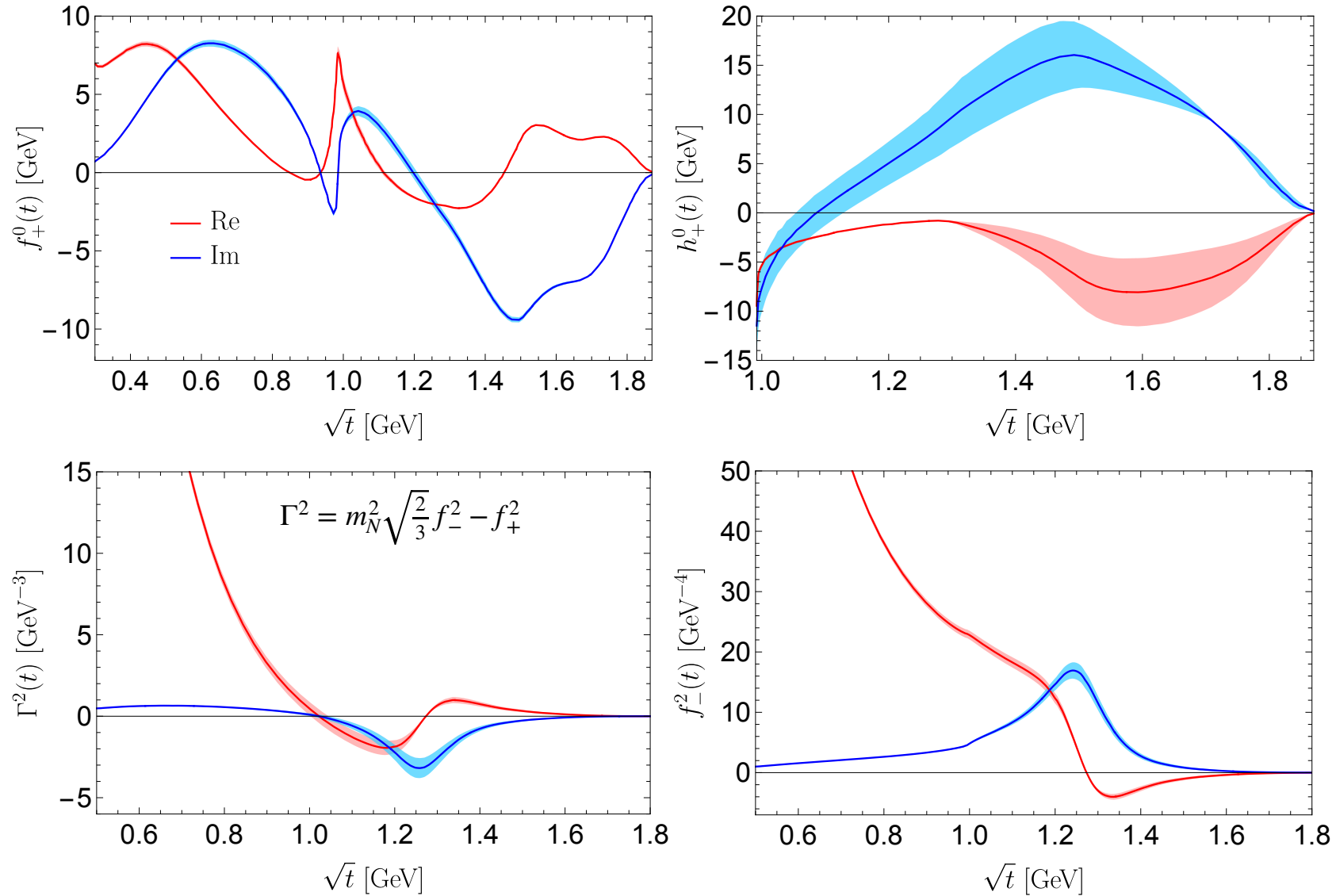
C. Ditsche, et al., JHEP (2012); M. Hoferichter et al., JHEP (2012); M. Hoferichter, et al., PRL 115, 092301(2015); PRL 115, 192301 (2015); Phys. Rept. (2016); PLB (2016); EPJA (2016); J. Ruiz de Elvira et al., JPG (2018); M. Hoferichter, et al., PRL (2018); XHC, et al., JHEP (2022); M. Hoferichter, et al., PLB (2024)...

Inputs from Roy-Steiner equations

- $\pi\pi/K\bar{K} \rightarrow N\bar{N}$ partial-wave amplitudes $f_{\pm}^{0,2}$, h_{+}^0 from Roy-Steiner analyses

G.E. Hite ,F. Steiner (1973)

M. Hoferichter et al., Phys. Rept. 625 (2016) 1; PLB 853 (2024) 138698; X.-H. Cao et al, JHEP 12 (2022) 073

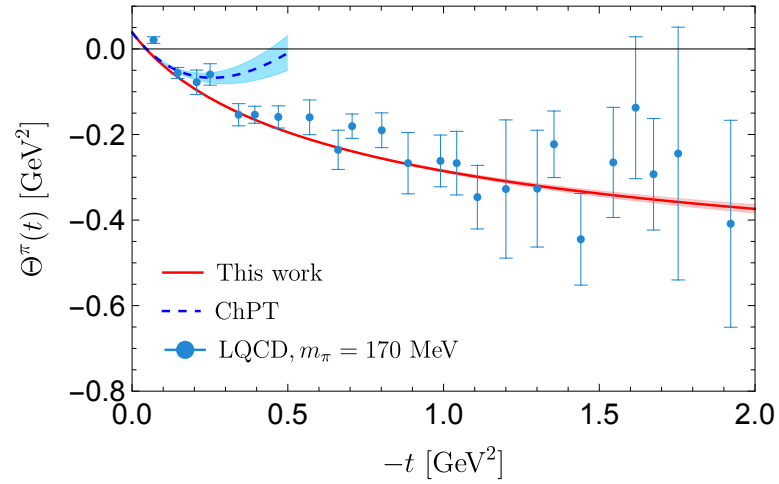


Inputs of Pion and Kaon GFFs

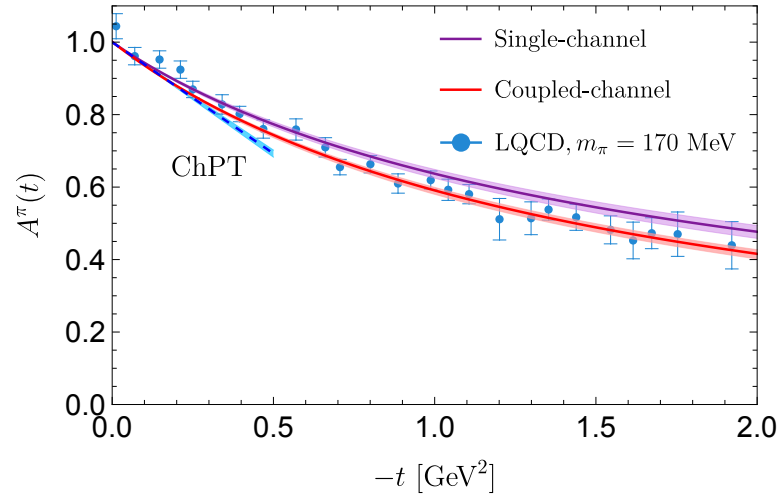
Predictions:

Pion

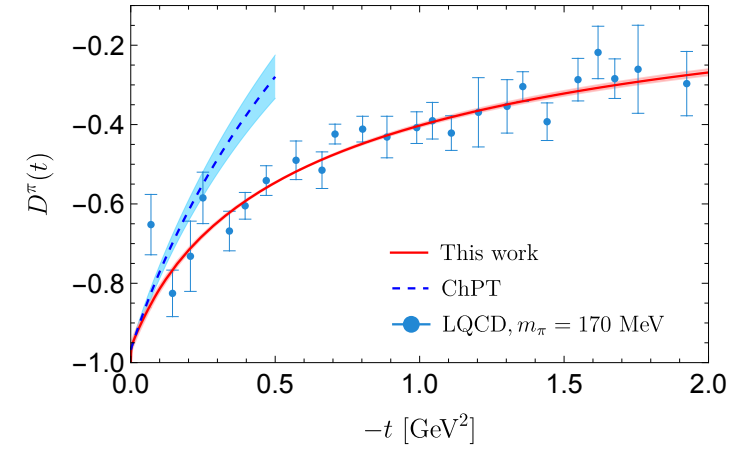
$\Theta(t)$



$A(t)$

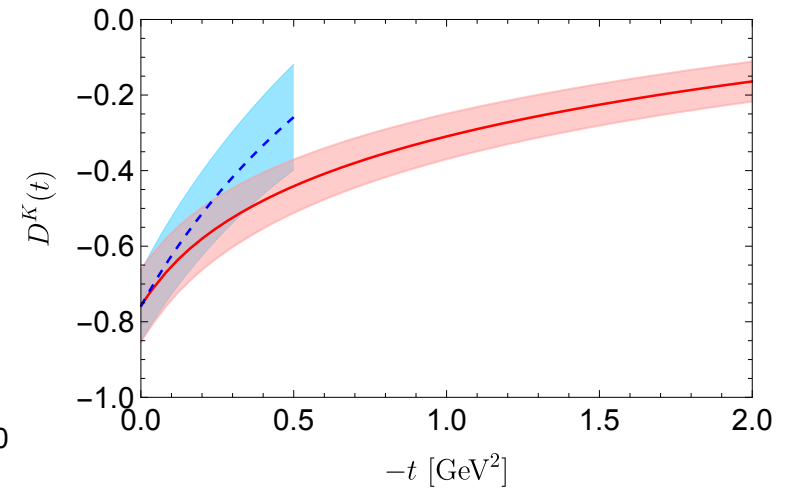
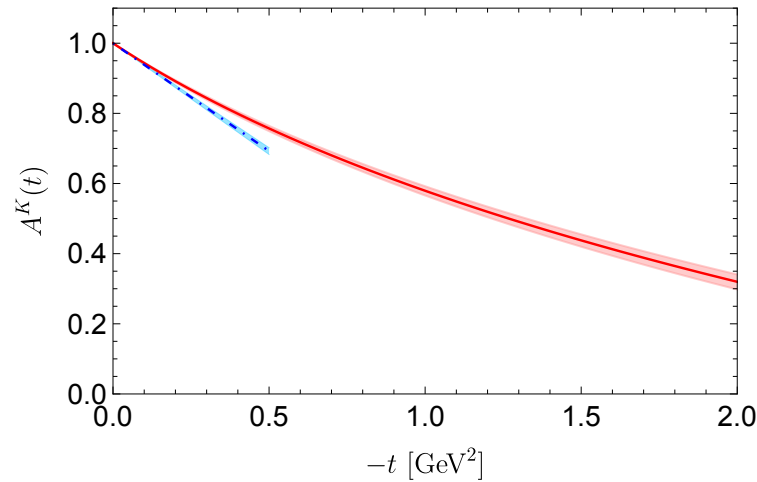
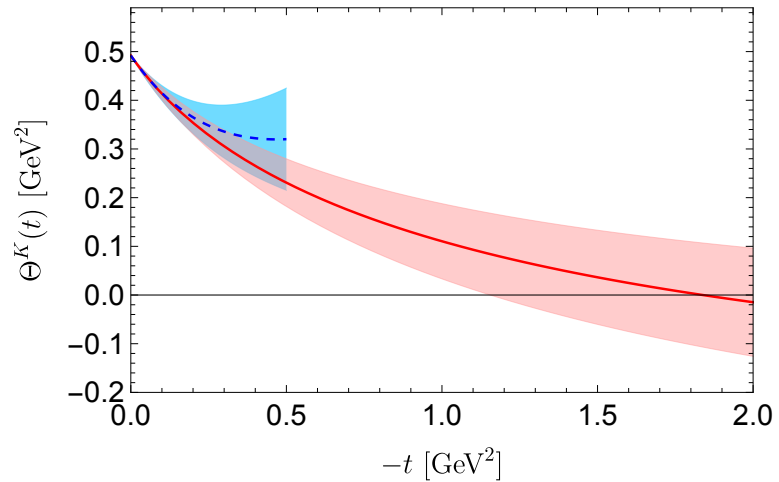


$D(t)$



LQCD ($m_\pi = 170$ MeV): D.C. Hackett et al., PRL 132 (2024) 251904

Kaon



Nucleon GFFs

- Sum rules for the nucleon GFFs

- Normalisation

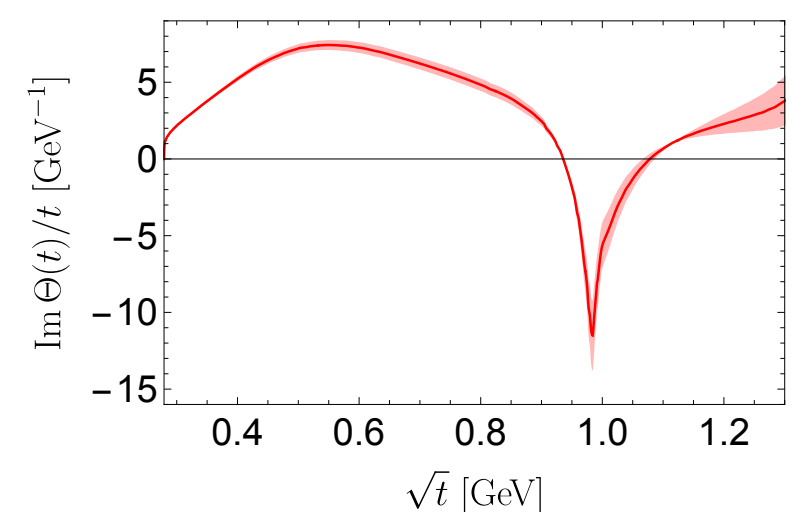
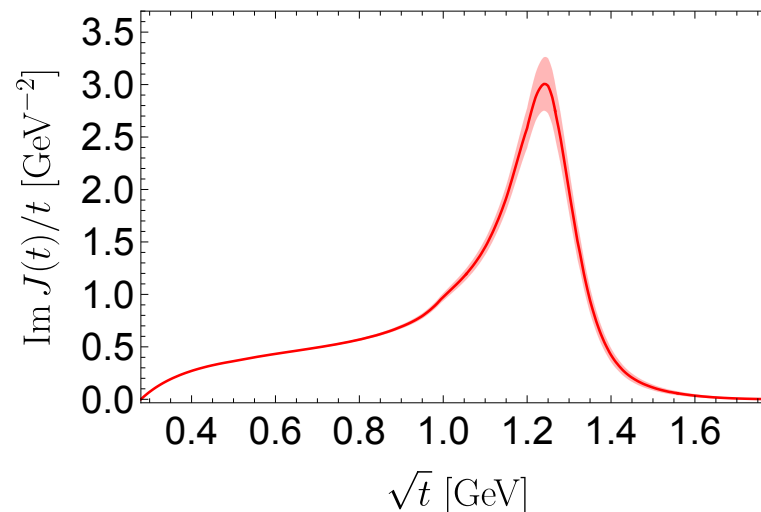
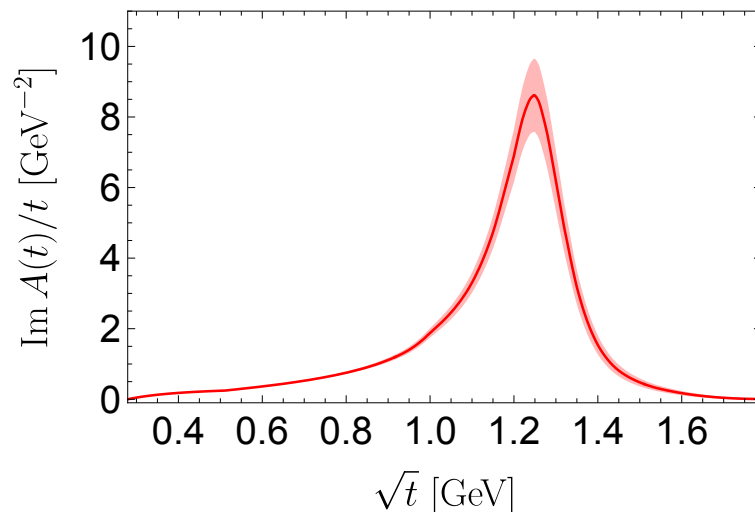
M.A. Belushkin et al., PRC 75 (2007) 035202; M. Hoferichter et al., EPJA 52 (2016) 331

$$(A, J, \Theta)(t) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} dt' \frac{\text{Im}(A, J, \Theta)(t')}{t' - t} \quad \longrightarrow \quad \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} dt' \frac{\text{Im}(A, J, \Theta)(t')}{t'} = \left(1, \frac{1}{2}, m_N\right)$$

- Introduce S-wave (0^{++}) and D-wave (2^{++}) poles to the spectral functions: $\pi c_{S,D} m_{S,D}^2 \delta(t - m_{S,D}^2)$ to satisfy the sum rules

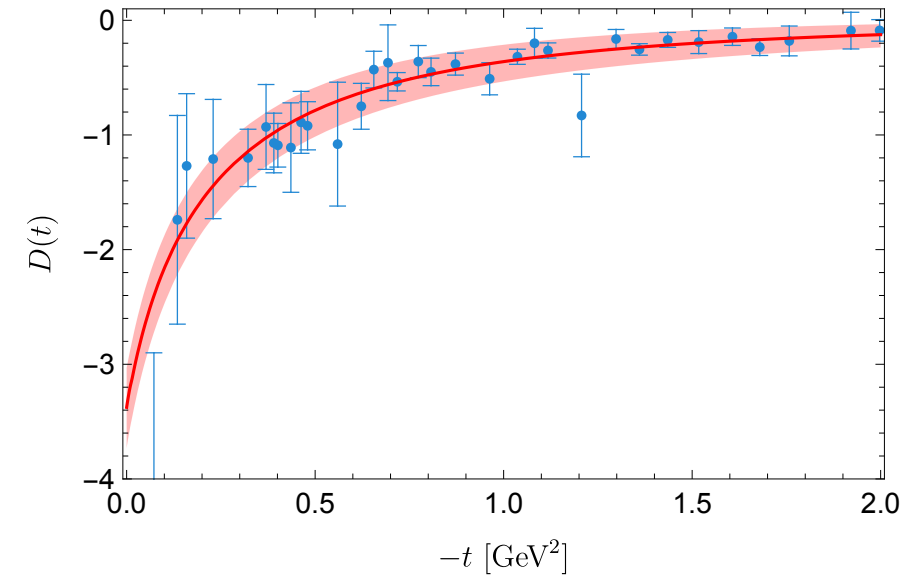
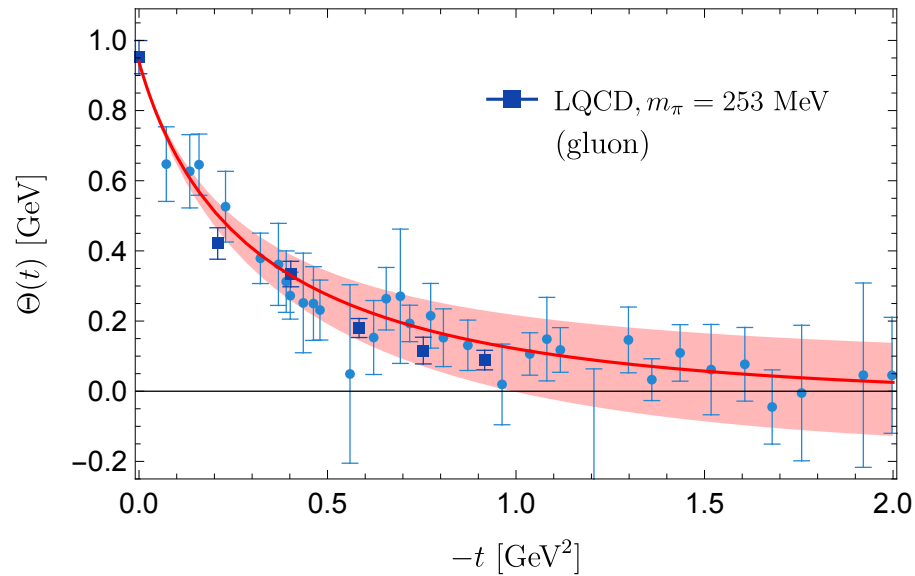
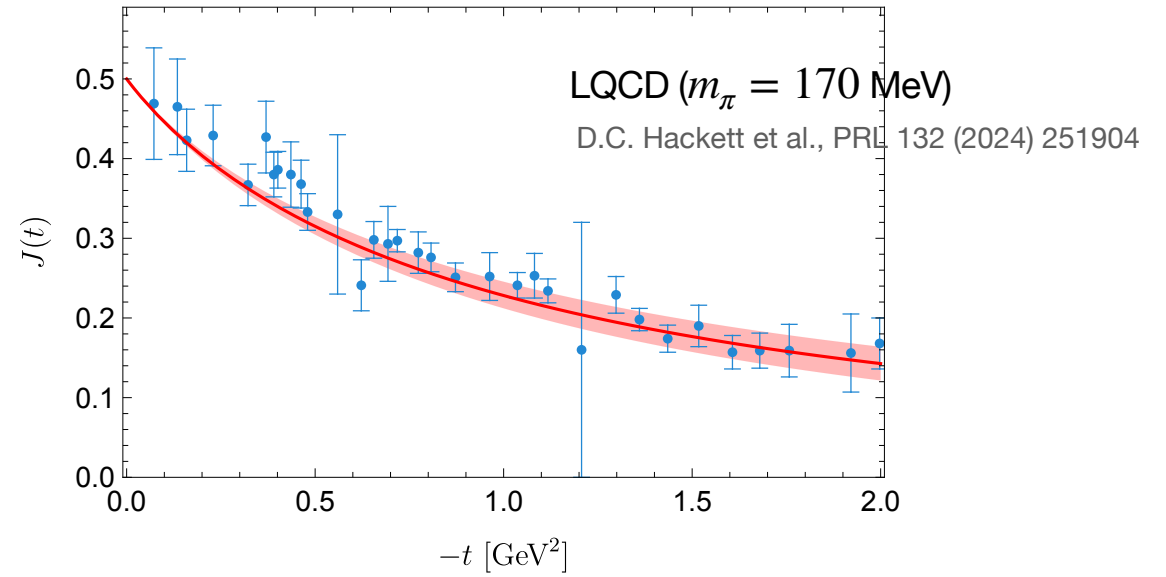
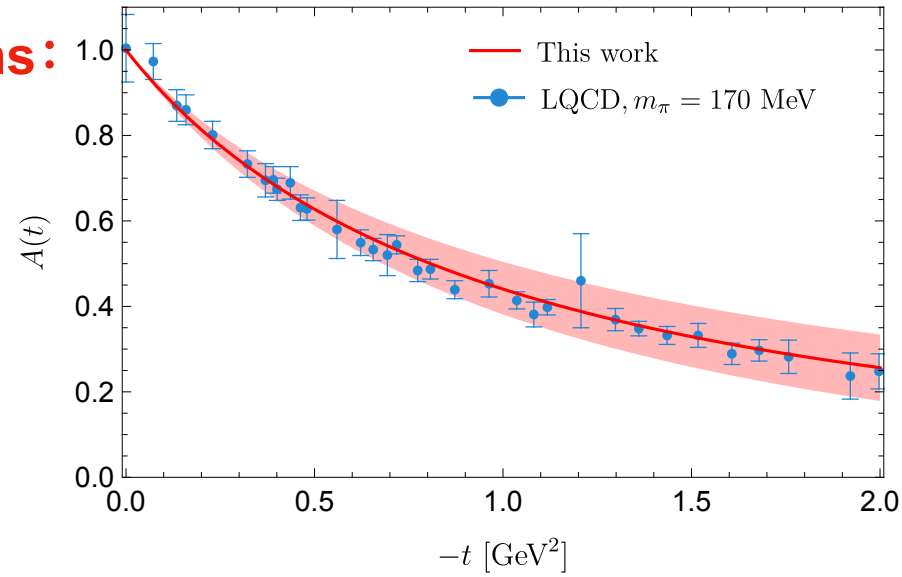
✓ 0^{++} : $m_S \in (1.5, 1.8)$ GeV to cover $f_0(1500)$ and $f_0(1710)$

✓ 2^{++} : $m_D \in (1.5, 2.2)$ GeV to cover $f_2(1500)$, $f_2(1950)$ and $f_2(2010)$



Nucleon GFFs

Predictions:



LQCD ($m_\pi = 253$ MeV), gluon part only: B. Wang et al. [χ QCD], PRD 109 (2024) 094504

Nucleon GFFs: D-term & radii

- D-term: $D \equiv D(0)$

- **Various radii in the Breit frame**

- From the trace FF:

$$\langle r_{\Theta}^2 \rangle = 6A'(0) - \frac{9D}{2m_N^2}$$

- Radius of energy density:

$$\langle r_{\text{Ener}}^2 \rangle = 6A'(0) - \frac{3D}{2m_N^2}$$

- Mechanical radius

$$\langle r_{\text{Mech}}^2 \rangle = \frac{6D}{\int_{-\infty}^0 dt D(t)}$$

- Radius of the density $J(t) + \frac{2}{3}t \frac{d}{dt} J(t)$

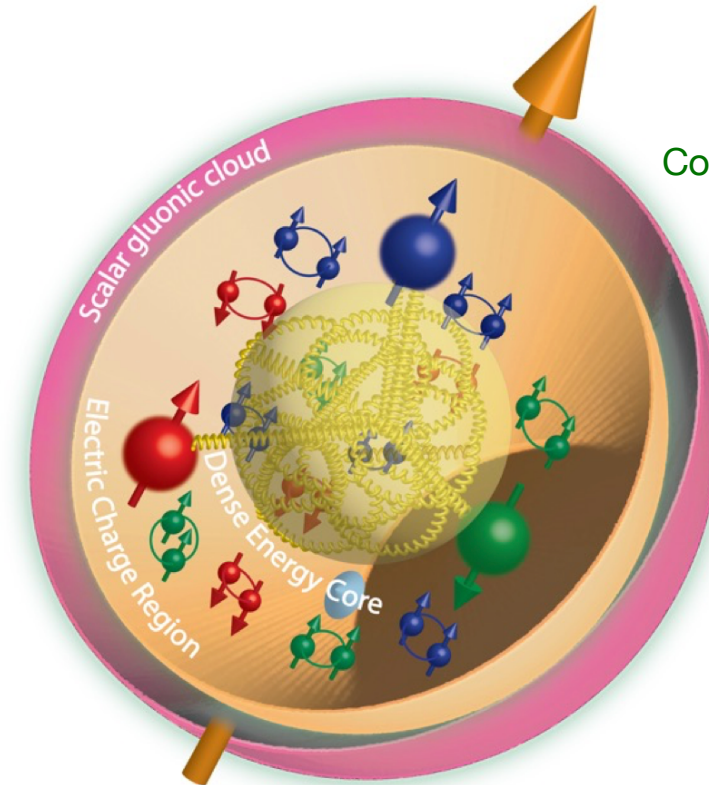
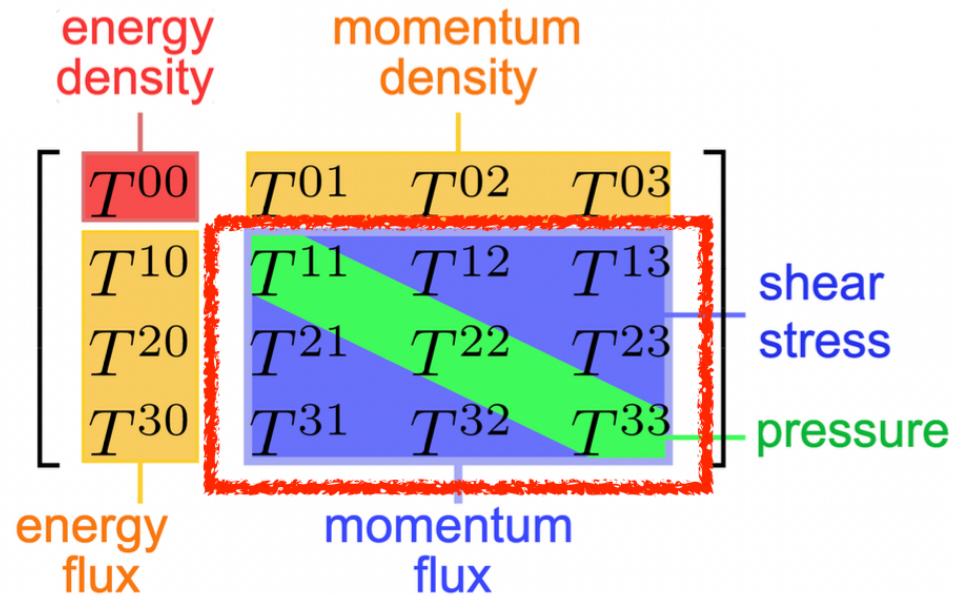
$$\langle r_J^2 \rangle = 20J'(0)$$

Quantity	Result	Error budget
D-term	$-3.38^{+0.34}_{-0.35}$	$+(0.18)_{\text{ChPT}}(0.12)_{\text{PWA}}(0.26)_{\text{eff}}$ $-(0.16)_{\text{ChPT}}(0.12)_{\text{PWA}}(0.29)_{\text{eff}}$
$\langle r_{\Theta}^2 \rangle$ [fm]	$0.97^{+0.03}_{-0.03}$	$+(0.01)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.03)_{\text{eff}}$ $-(0.02)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.26)_{\text{eff}}$
$\langle r_{\text{Ener}}^2 \rangle$ [fm]	$0.70^{+0.03}_{-0.04}$	$+(0.02)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.02)_{\text{eff}}$ $+(0.02)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.03)_{\text{eff}}$
$\langle r_{\text{Mech}}^2 \rangle$ [fm]	$0.72^{+0.09}_{-0.08}$	$+(0.02)_{\text{ChPT}}(0.00)_{\text{PWA}}(0.09)_{\text{eff}}$ $-(0.03)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.07)_{\text{eff}}$
$\langle r_J^2 \rangle$ [fm]	$0.70^{+0.02}_{-0.02}$	$+(0.01)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.01)_{\text{eff}}$ $-(0.01)_{\text{ChPT}}(0.00)_{\text{PWA}}(0.02)_{\text{eff}}$

- ChPT: NLO ChPT inputs
- paw: $\pi\pi/K\bar{K} \rightarrow N\bar{N}$
- eff: effective poles m_S, m_D

Nucleon GFFs: D-term & radii

- A modern structural view



Courtesy of Meziani

$$\sqrt{\langle r_{\Theta}^2 \rangle} = 0.97_{-0.03}^{+0.03} \text{ fm} > \sqrt{\langle r_{E,p}^2 \rangle} \simeq 0.84 \text{ fm} > \sqrt{\langle r_{\text{Ener}}^2 \rangle} = 0.70_{-0.04}^{+0.03} \text{ fm}$$

proton electric charge

Summary and Outlook

- The nucleon GFFs are precisely determined using dispersive method with inputs:
 - ▶ Coupled-channel $\pi\pi-K\bar{K}$, $\pi\pi/K\bar{K} \rightarrow N\bar{N}$ amplitudes
 - ▶ Low energy: NLO ChPT with LECs estimated using resonance saturation, improvable with lattice calculations
 - ▶ High energy: highly excited meson resonances

- Nucleon static properties:

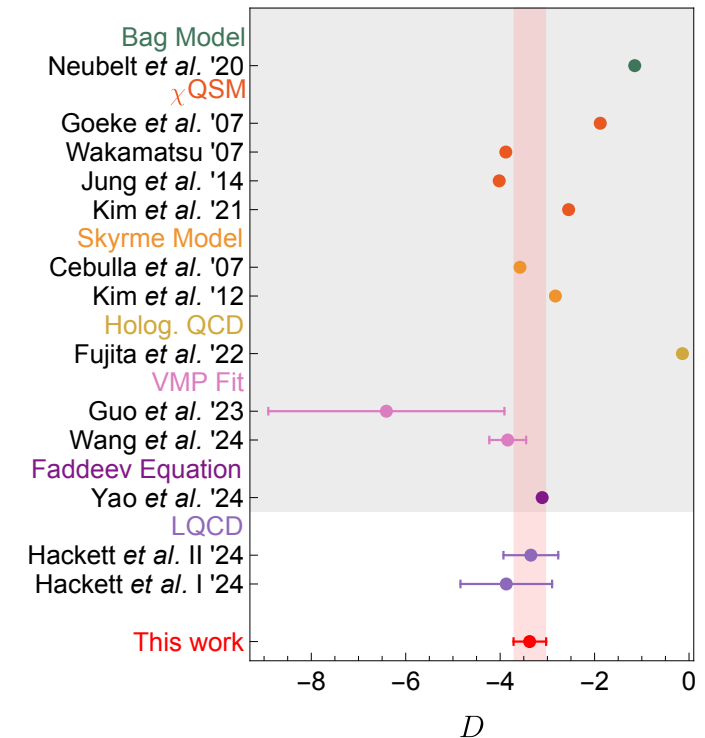
- ▶ Precise determination of D term: $D^N = -3.38^{+0.34}_{-0.35}$
- ▶ Suggesting a modern structural view of the nucleon:

$$\sqrt{\langle r_{\Theta}^2 \rangle} = 0.97^{+0.03}_{-0.03} \text{ fm} > \sqrt{\langle r_{E,p}^2 \rangle} \simeq 0.84 \text{ fm} > \sqrt{\langle r_{\text{Ener}}^2 \rangle} = 0.70^{+0.03}_{-0.04} \text{ fm}$$

proton electric charge

- Outlook:

- ▶ Pion mass dependence
- ▶ Extension to hyperon GFFs



Thank you very much!

Backups

Unitarity relation for the pion GFFs

- Discontinuity of A^π and D^π : associated only with t_0^0, t_2^0 partial waves.

$$\begin{aligned} \text{Disc} \left\langle \pi^a(p') \pi^b(p) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle &= \frac{\delta^{ab}}{2} \left[\text{Disc} A^\pi(t) \Delta^\mu \Delta^\nu + \text{Disc} D^\pi(t) (P^\mu P^\nu - t g^{\mu\nu}) \right] \\ &= 2i \frac{2p_\pi}{\sqrt{t}} \frac{\delta^{ab}}{2} \left[(A^\pi(t))^* \left(\frac{4}{3t} p_\pi^2 (t_0^0(t) - t_2^0(t)) (P^\mu P^\nu - t g^{\mu\nu}) + t_2^0(t) \Delta^\mu \Delta^\nu \right) + (D^\pi(t))^* t_0^0(t) (P^\mu P^\nu - t g^{\mu\nu}) \right] \end{aligned}$$

- Partial-wave unitarity

$$\begin{aligned} \text{Im } A^\pi(t) &= \frac{2p_\pi}{\sqrt{t}} (t_2^0(t))^* A^\pi(t) \\ \text{Im } D^\pi(t) &= \frac{2p_\pi}{\sqrt{t}} \left[\frac{4}{3} \frac{p_\pi^2}{t} (t_0^0(t) - t_2^0(t))^* A^\pi(t) + (t_0^0(t))^* D^\pi(t) \right] \end{aligned} \quad \Rightarrow \quad \text{Im } \Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}} (t_0^0(t))^* \Theta^\pi(t)$$

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$ matrix elements (conserved separately):

$$\left\langle \pi^a(p') \pi^b(p) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle = \delta^{ab} \left\{ \frac{1}{3} \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) \Theta^\pi(t) + \left[\Delta^\mu \Delta^\nu + \frac{\Delta^2}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] A^\pi(t) \right\}$$

Trace part: $\left\langle \pi^a(p') \pi^b(p) \left| \hat{T}_\mu^\mu(0) \right| 0 \right\rangle = \delta^{ab} \Theta^\pi(t), \quad \Theta^\pi(t) = -\frac{1}{2} (4p_\pi^2 A^\pi(t) + 3t D^\pi(t))$

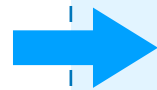


$\pi\pi-K\bar{K}$ coupled channels

- $\pi\pi$ phase shifts known precisely from Roy(-like) equation analyses
- Generalisation to coupled channels: isoscalar, scalar $\pi\pi-K\bar{K}$; $f_0(500)$, $f_0(980)$ mesons
 - Unitarity relation for $\Theta^\pi(t) \Rightarrow$ Matrix relation for coupled channels (both pion and kaon trace GFFs):

$$\text{Im } \Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}} (t_0^0(t))^* \Theta^\pi(t)$$

Omnes solution



$$\text{Im } \Theta(t) = [\mathbf{T}_0^0(t)]^* \Sigma_0^0(t) \Theta(t)$$

$$\Theta(t) = \begin{pmatrix} \Theta^\pi(t) \\ \frac{2}{\sqrt{3}} \Theta^K(t) \end{pmatrix}$$

Phase-space factor

$$\Sigma_0^0(t) \equiv \text{diag}(\sigma_\pi \theta(t - t_\pi), \sigma_K \theta(t - t_K))$$

$$\text{With } \sigma_i(t) = \sqrt{1 - 4m_i^2/t} \quad (i = \pi, K)$$

$\pi\pi-K\bar{K}$ T-matrix

$$\mathbf{T}_0^0(t) = \begin{pmatrix} \frac{\eta_0^0(t) e^{2i\delta_0^0(t)} - 1}{2i\sigma_\pi} & |g_0^0(t)| e^{i\Psi_0^0(t)} \\ |g_0^0(t)| e^{i\Psi_0^0(t)} & \frac{\eta_0^0(t) e^{2i(\Psi_0^0(t) - \delta_0^0(t))} - 1}{2i\sigma_K} \end{pmatrix}.$$

$$\eta_0^0(t) = \sqrt{1 - 4\sigma_\pi \sigma_K |g_0^0(t)|^2 \theta(t - t_K)}$$

Muskhelishvili-Omnes problem

Omnes representation

- **Single-channel:** Watson's theorem $\Rightarrow$ **phase of FF = scattering phase shift**

K.M. Watson (1952)

$$\text{disc } A^\pi(t) = 2iA^\pi(t)\theta(t - t_\pi)\sin\delta_2^0(t)e^{-i\delta_2^0(t)}$$

$$|t_2^0|e^{i\phi_2^0} = \frac{\eta_2^0 e^{2i\delta_2^0} - 1}{2i\sigma_\pi}$$

❖ Omnes solution

$$A^\pi(t) = P_2^\pi(t)\Omega_2^0(t), \quad \Omega_2^0(t) \equiv \exp\left\{\frac{t}{\pi} \int_{t_\pi}^{\infty} \frac{dt'}{t'} \frac{\phi_2^0(t')}{t' - t}\right\}$$

- ▶ δ_2^0, η_2^0 up to $E_0 \simeq 2$ GeV from latest dispersive analysis

P. Bydžovský et al., PRD 94 (2016) 116013

- ▶ Beyond matching point E_0 , B. Moussallam, EPJC 14 (2000) 111

$$\delta_2^0(t) = \pi + (\delta_2^0(E_0^2) - \pi) \frac{2}{1 + (\sqrt{t}/E_0)^3}, \quad (\text{also for } \eta_2^0)$$

- ▶ Polynomial: $P_2^\pi(t) = 1 + \alpha t$ matched to NLO ChPT

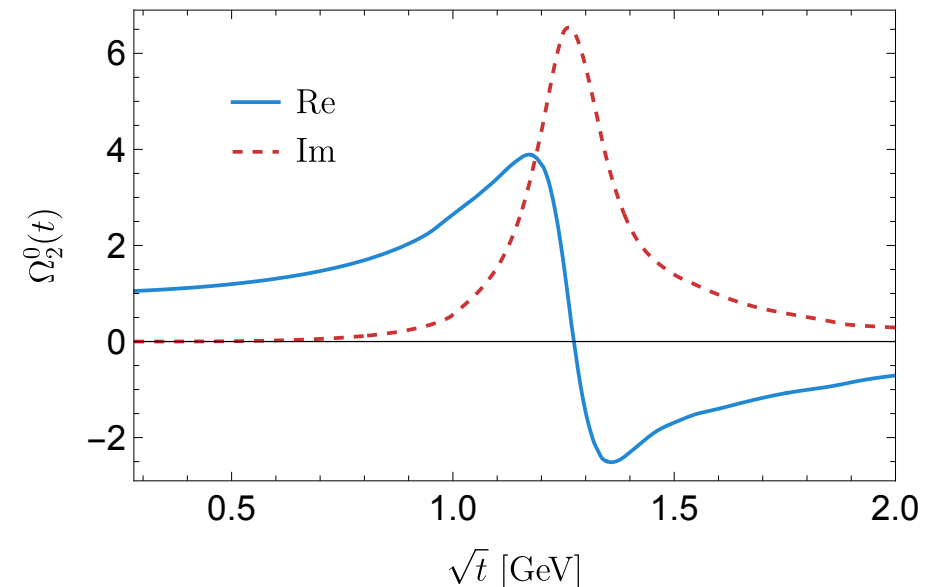
$$A^\pi(t) = 1 - \frac{2L_{12}^r}{F_\pi^2}t$$

Tensor-meson dominance estimate: $L_{12}^r = -\frac{F_\pi^2}{2m_{f_2}^2}$

$$P_2^\pi(t) = 1 + \left[\frac{1}{m_{f_2}^2} - \dot{\Omega}_2^0(0)\right]t \simeq 1 - (0.01 \text{ GeV}^{-2})t$$

δ_2^0 replaced by the phase of $\pi\pi$ partial wave ϕ_2^0 to account for inelasticity

M. Hoferichter et al., Phys. Rept. 625 (2016) 1



PDG average: $m_{f_2} = (1275.4 \pm 0.8) \text{ MeV}$

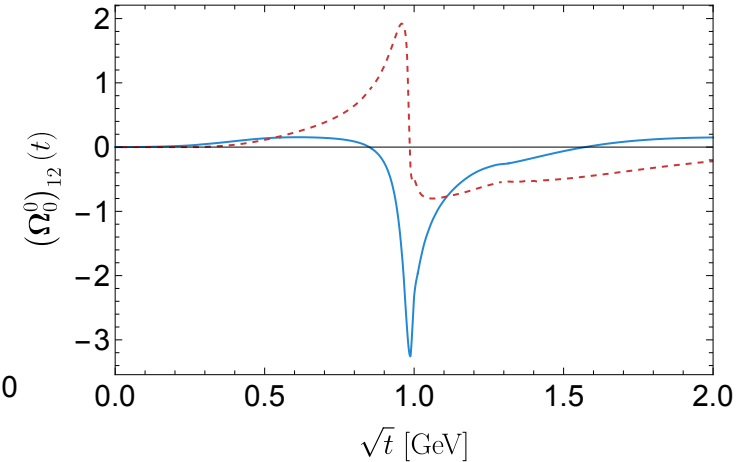
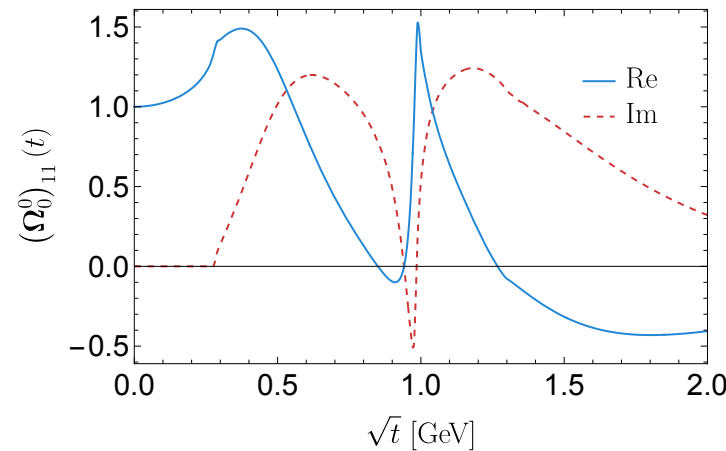
We take $m_{f_2} = (1275.4 \pm 20) \text{ MeV}$ for a conservative error estimate.

Muskhelishvili-Omnès representation

- **Coupled-channel:** solution known as the Muskhelishvili-Omnès (MO) representation

- ▶ The above can be generalised to $\pi\pi-K\bar{K}$ coupled channels (matching point: ~ 1.3 GeV)
- ▶ Take isoscalar scalar $\pi\pi-K\bar{K}$ for example

$$\Omega_0^0(t) = \frac{1}{\pi} \int_{t_\pi}^{\infty} \frac{dt'}{t' - t} [\mathbf{T}_0^0(t)]^* \Sigma_0^0(t) \Omega_0^0(t')$$



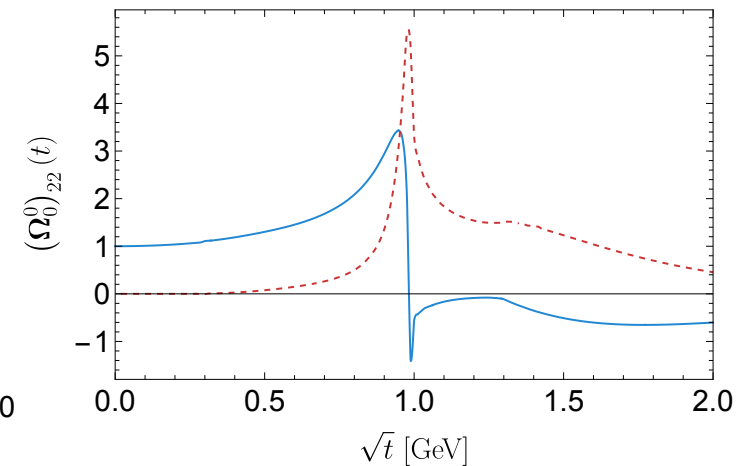
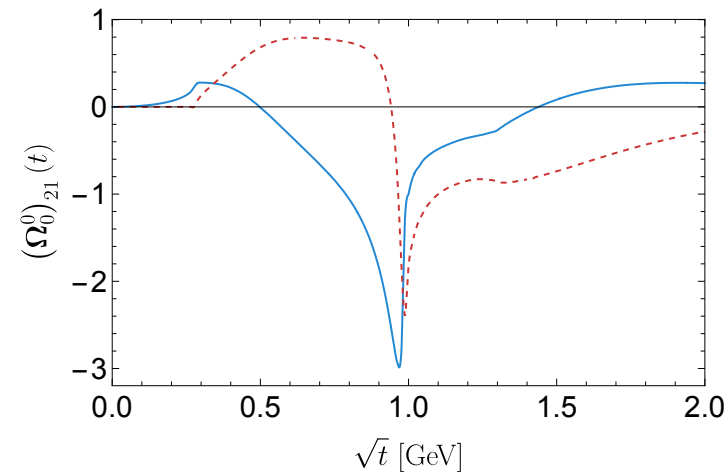
$\pi\pi$ phase shifts: Roy equation

I. Caprini et al. (2012);

$\pi\pi \rightarrow K\bar{K}$: Roy-Steiner equation

P. Büttiker et al. (2004);

M. Hoferichter et al., JHEP 06 (2012) 063



Muskhelishvili-Omnès representation

- Pion and Kaon trace GFFs

$$[\Theta(t)]^T = [P_0(t)]^T \Omega_0^0(t), \quad P_0(t) = \begin{pmatrix} 2m_\pi^2 + \beta_\pi t \\ \frac{2}{\sqrt{3}} (2m_K^2 + \beta_K t) \end{pmatrix}$$

$$\beta_\pi = \dot{\Theta}^\pi(0) - 2m_\pi^2 \left(\dot{\Omega}_0^0 \right)_{11}(0) - \frac{4m_K^2}{\sqrt{3}} \left(\dot{\Omega}_0^0 \right)_{12}(0),$$

$$\beta_K = \dot{\Theta}^K(0) - \sqrt{3}m_\pi^2 \left(\dot{\Omega}_0^0 \right)_{21}(0) - 2m_K^2 \left(\dot{\Omega}_0^0 \right)_{22}(0),$$

- Matching to NLO ChPT

$$\dot{\Theta}^\pi(0) = 1 - 4L_{12}' \frac{m_\pi^2}{F_\pi^2} - 24(L_{11}' - L_{13}') \frac{m_\pi^2}{F_\pi^2} - \frac{3}{2} \frac{m_\pi^2}{F_\pi^2} I_\pi + \frac{m_\pi^2}{2F_\pi^2} I_\eta = 0.98(2),$$

$$\dot{\Theta}^K(0) = 1 - 4L_{12}' \frac{m_K^2}{F_\pi^2} - 24(L_{11}' - L_{13}') \frac{m_K^2}{F_\pi^2} - \frac{m_K^2}{F_\pi^2} I_\eta = 0.94(14).$$

$$\Rightarrow \text{chiral logs: } I_i = \frac{1}{48\pi^2} \left[\ln \frac{\mu^2}{m_i^2} - 1 \right]$$

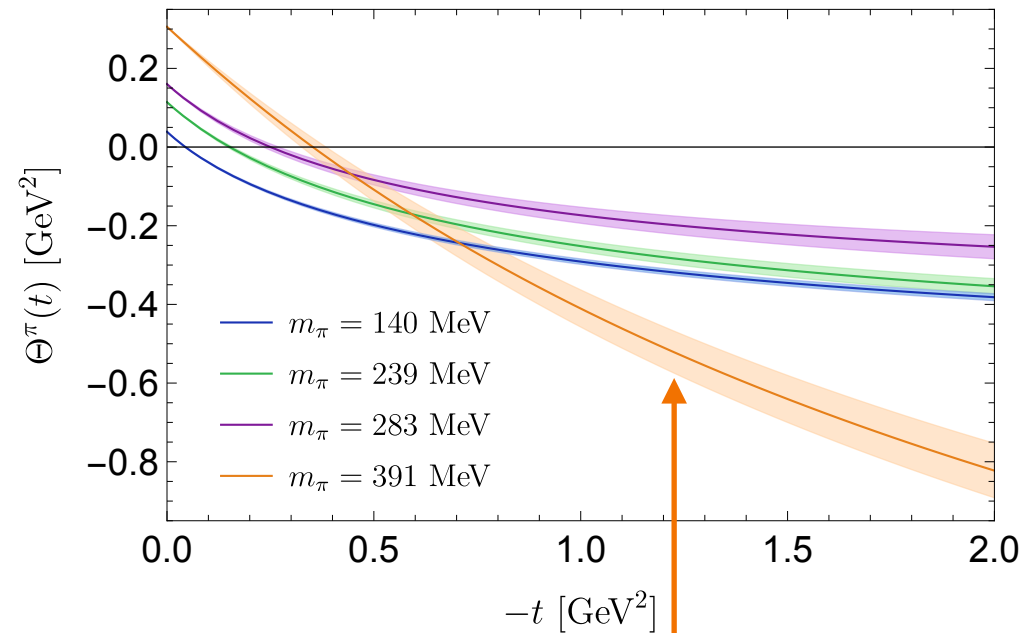
- Similar coupled-channel analysis for D-wave $\pi\pi-K\bar{K}$

$\Rightarrow$ coupled channel results for A^π, A^K and D^π, D^K

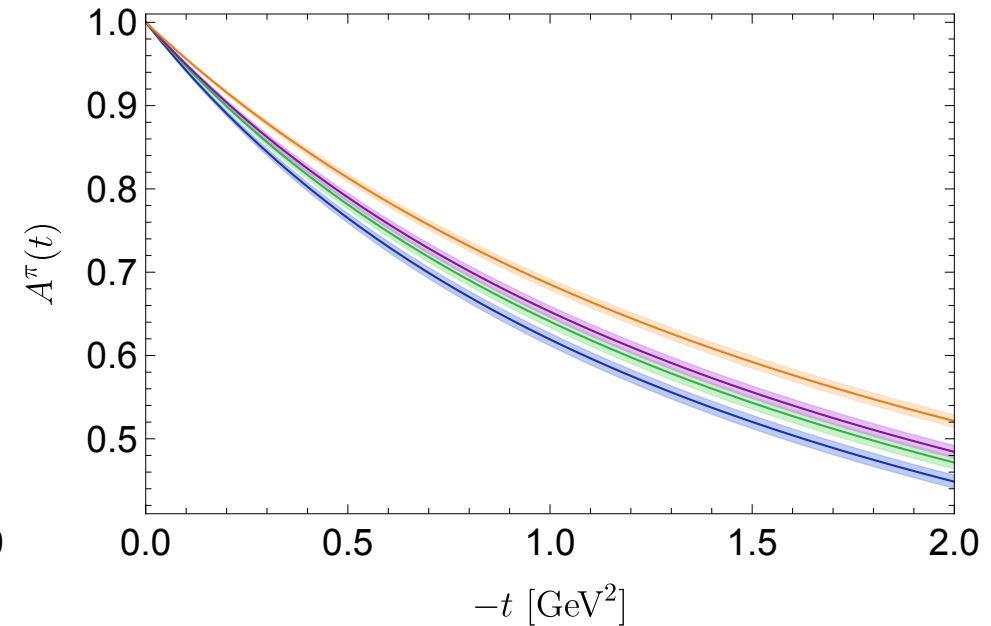
Pion GFFs: pion mass dependence

- Using the $\pi\pi$ scattering phase shifts at unphysical pion masses (239 MeV, 283 MeV, 391 MeV) obtained from Roy equation analyses

X.-H. Cao et al., PRD 108 (2023) 034009; A. Rodas et al., PRD 109 (2024) 034513



Fast change before $m_\pi = 391$ MeV:
 σ meson becomes a $\pi\pi$ bound state!



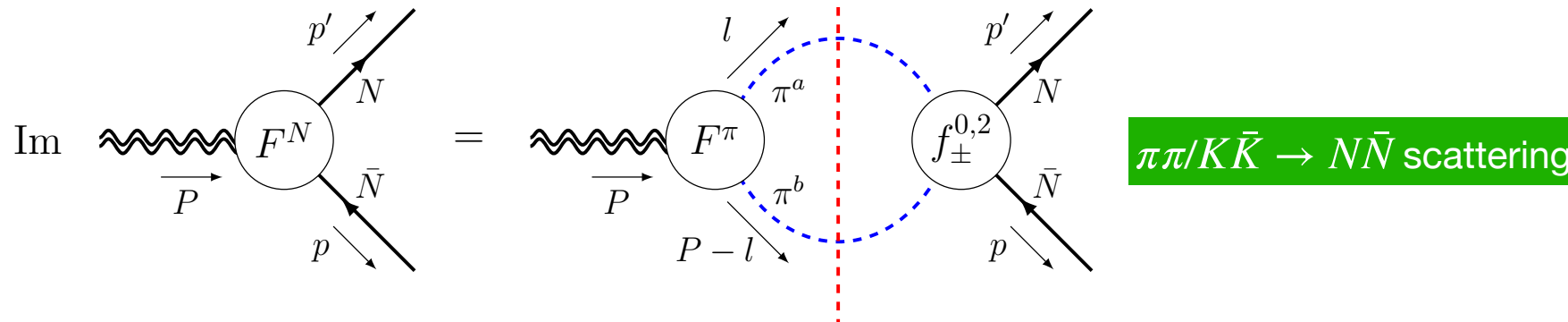
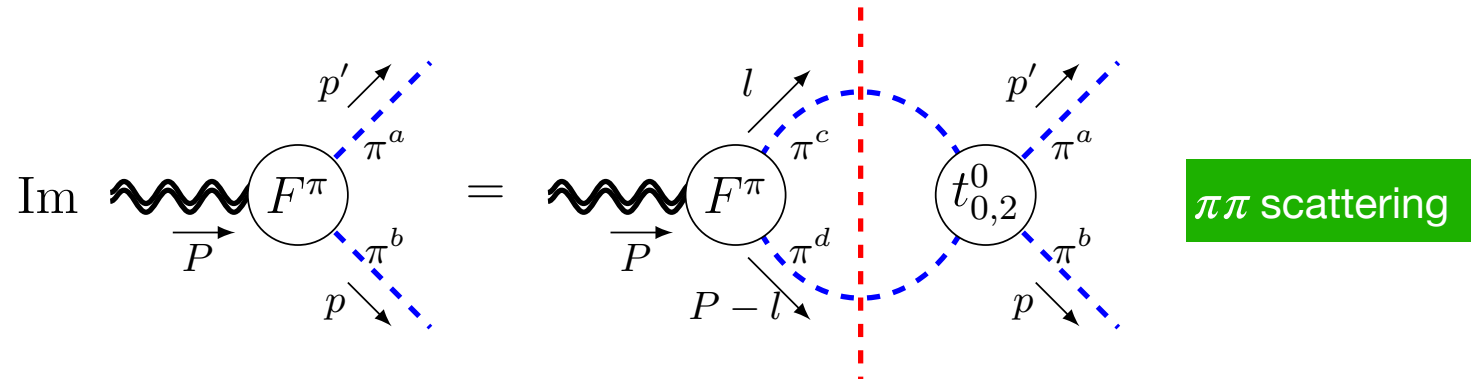
R.A. Briceno et al. [HadSpec], PRL 118 (2017) 022002

Unitarity relation for nucleon GFFs

- Discontinuity

$$\text{Disc} \langle N(p') \bar{N}(p) | \hat{T}^{\mu\nu}(0) | 0 \rangle \propto \sum_n \langle N(p') \bar{N}(p) | n \rangle \langle n | \hat{T}^{\mu\nu}(0) | 0 \rangle^* \delta^4(p + p' - p_n)$$

- In the region $t \in (t_\pi, 16t_\pi)$, only $\pi\pi$ intermediate state



Unitarity relation for nucleon GFFs

- Partial-wave amplitudes for $\pi\pi \rightarrow N\bar{N}$

W. Frazer, J. Fulco (1960); G. Höhler (1983)

$$A^I(t, s) = -\frac{8\pi}{p_N^2} \sum_{J=0}^{\infty} (J + \frac{1}{2})(p_\pi p_N)^J \left[P_J(\cos\theta) f_+^J(t) - \frac{m_N \cos\theta}{\sqrt{J(J+1)}} P_J'(\cos\theta) f_-^J(t) \right]$$

$$B^I(t, s) = 8\pi \sum_{J=1}^{\infty} \frac{J + \frac{1}{2}}{\sqrt{J(J+1)}} (p_\pi p_N)^{J-1} P_J'(\cos\theta) f_-^J(t)$$

$I = + / -$ for even/odd J ;
 $f_{\pm}^J$: $\pi\pi \rightarrow N\bar{N}$ partial-wave amp. with $+/-$ for parallel/
anti-parallel $N\bar{N}$ helicities.

- Discontinuity of the nucleon GFFs

$$\text{Im } A^s(t) = \frac{3p_\pi^5}{\sqrt{6t}} \left[f_-^2(t) + \sqrt{\frac{3}{2}} \frac{m_N}{p_N^2} \Gamma^2(t) \right] A^\pi(t)$$

$$\text{Im } J^s(t) = \frac{3p_\pi^5}{2\sqrt{6t}} [f_-^2(t)]^* A^\pi(t)$$

$$\text{Im } D^s(t) = -\frac{3m_N p_\pi}{2p_N^2 \sqrt{t}} \left[\frac{4p_\pi^2}{3t} \left((f_+^0(t))^* - (p_\pi p_N)^2 (f_+^2(t))^* \right) A^\pi(t) + (f_+^0(t))^* D^\pi(t) \right]$$

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$ matrix elements

$$\langle N(p') \bar{N}(p) | \hat{T}^{\mu\nu}(0) | 0 \rangle = \bar{u}(p') \left(T_S^{\mu\nu} + T_T^{\mu\nu} \right) v(p)$$

$$T_S^{\mu\nu} = \frac{1}{3} \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) \Theta^s(t)$$

$$T_T^{\mu\nu} = \frac{1}{4m_N} \left[\Delta^\mu \Delta^\nu + \frac{\Delta^2}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] A^s(t) + \left[i \Delta^{\{\mu} \sigma^{\nu\} \rho} P_\rho + \frac{2i \sigma^{\rho\kappa} \Delta_\rho P_\kappa}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] J^s(t)$$

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$$\text{Im } J^s(t) = \frac{3p_\pi^5}{2\sqrt{6t}} [f_-^2(t)]^* A^\pi(t)$$



$$\text{Im } \Theta^s(t) = -\frac{3p_\pi}{4p_N^2 \sqrt{t}} [f_+^0(t)]^* \Theta^\pi(t)$$

$$\text{Im } D^s(t) = -\frac{3m_N p_\pi}{2p_N^2 \sqrt{t}} \left[\frac{4p_\pi^2}{3t} \left((f_+^0(t))^* - (p_\pi p_N)^2 (f_+^2(t))^* \right) A^\pi(t) + (f_+^0(t))^* D^\pi(t) \right]$$

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$$\text{Im } D^s(t) = -\frac{3m_N p_\pi}{2p_N^2 \sqrt{t}} \left[\frac{4p_\pi^2}{3t} \left((f_+^0(t))^* - (p_\pi p_N)^2 (f_+^2(t))^* \right) A^\pi(t) + (f_+^0(t))^* D^\pi(t) \right]$$



Coupled-channel

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$ matrix elements

$$\langle N(p') \bar{N}(p) | \hat{T}^{\mu\nu}(0) | 0 \rangle = \bar{u}(p') (T_S^{\mu\nu} + T_T^{\mu\nu}) v(p)$$

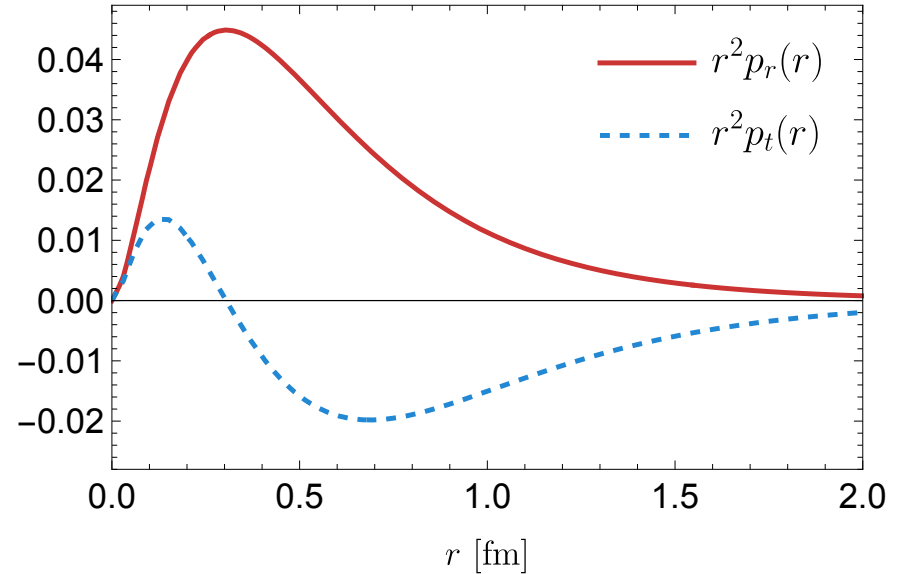
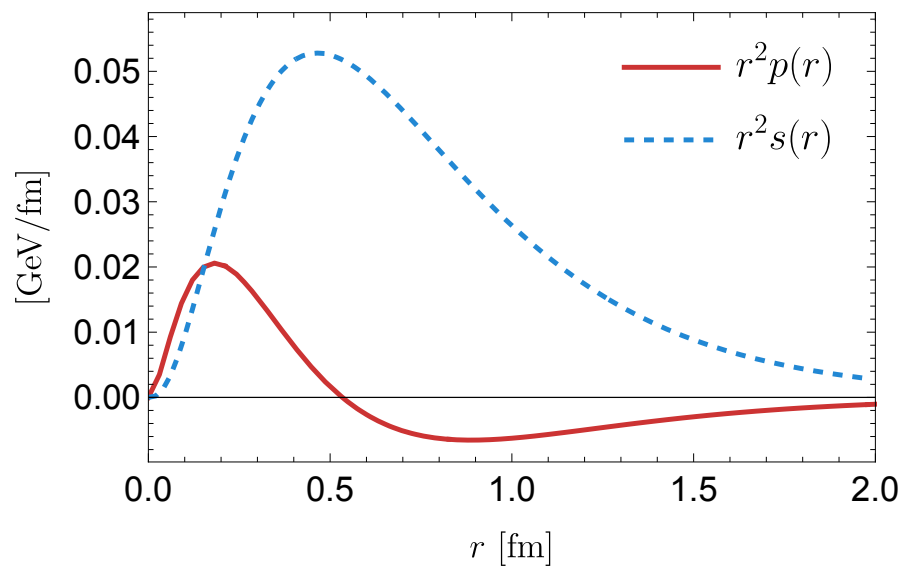
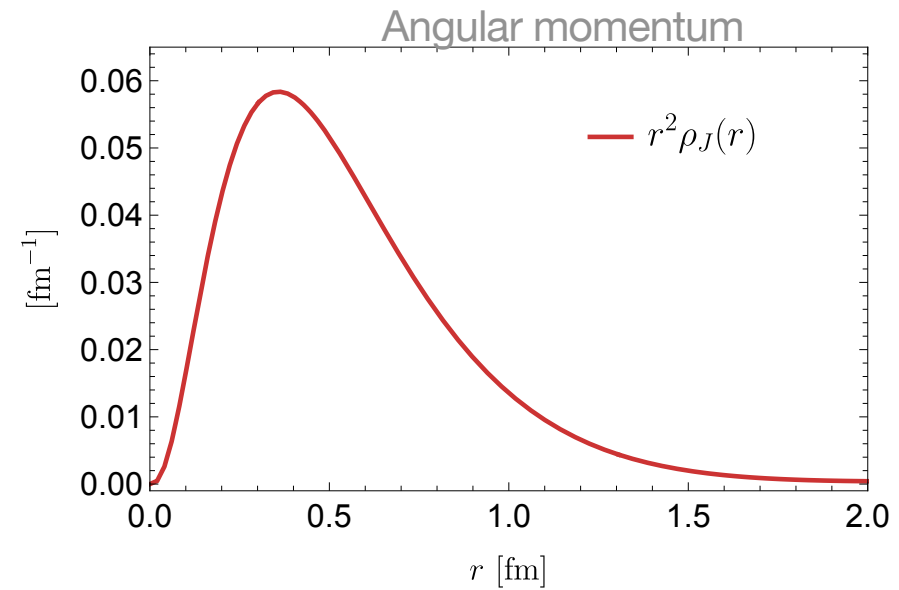
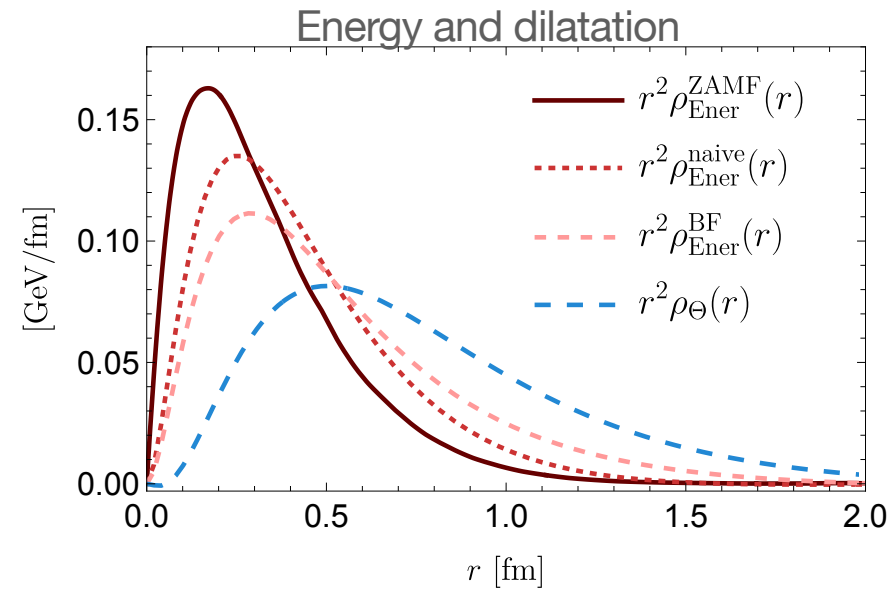
$$T_S^{\mu\nu} = \frac{1}{3} \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) \Theta^s(t)$$

$$T_T^{\mu\nu} = \frac{1}{4m_N} \left[\Delta^\mu \Delta^\nu + \frac{\Delta^2}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] A^s(t) + \left[i \Delta^{\{\mu} \sigma^{\nu\} \rho} P_\rho + \frac{2i \sigma^{\rho\kappa} \Delta_\rho P_\kappa}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] J^s(t)$$

$$\text{Im } \Theta^s(t) = -\frac{3}{4p_N^2 \sqrt{t}} \left\{ p_\pi [f_+^0(t)]^* \Theta^\pi(t) \theta(t - t_\pi) + \frac{4}{3} p_K [h_+^0(t)]^* \Theta^K(t) \theta(t - t_K) \right\}$$

$K\bar{K} \rightarrow N\bar{N}$ amplitude

Spatial density profiles



Pressure and shear force

Radial and tangential force

Unitarity relation for the pion GFFs

- Discontinuity of A^π and D^π : associated only with t_0^0, t_2^0 partial waves.

$$\begin{aligned} \text{Disc} \left\langle \pi^a(p') \pi^b(p) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle &= \frac{\delta^{ab}}{2} \left[\text{Disc} A^\pi(t) \Delta^\mu \Delta^\nu + \text{Disc} D^\pi(t) (P^\mu P^\nu - t g^{\mu\nu}) \right] \\ &= 2i \frac{2p_\pi}{\sqrt{t}} \frac{\delta^{ab}}{2} \left[(A^\pi(t))^* \left(\frac{4}{3t} p_\pi^2 (t_0^0(t) - t_2^0(t)) (P^\mu P^\nu - t g^{\mu\nu}) + t_2^0(t) \Delta^\mu \Delta^\nu \right) + (D^\pi(t))^* t_0^0(t) (P^\mu P^\nu - t g^{\mu\nu}) \right] \end{aligned}$$

- Partial-wave unitarity

$$\begin{aligned} \text{Im} A^\pi(t) &= \frac{2p_\pi}{\sqrt{t}} (t_2^0(t))^* A^\pi(t) \\ \text{Im} D^\pi(t) &= \frac{2p_\pi}{\sqrt{t}} \left[\frac{4}{3} \frac{p_\pi^2}{t} (t_0^0(t) - t_2^0(t))^* A^\pi(t) + (t_0^0(t))^* D^\pi(t) \right] \end{aligned}$$

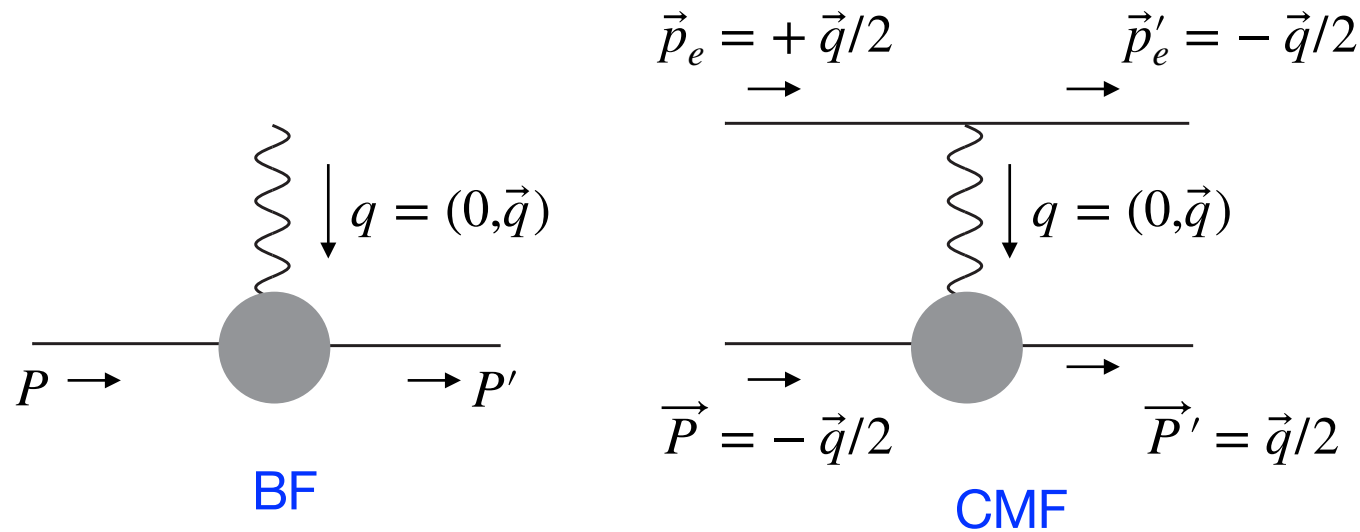
- Decomposition into $J^{PC} = 0^{++}, 2^{++}$ matrix elements (conserved separately):

$$\left\langle \pi^a(p') \pi^b(p) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle = \delta^{ab} \left\{ \frac{1}{3} \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) \Theta^\pi(t) + \left[\Delta^\mu \Delta^\nu + \frac{\Delta^2}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] A^\pi(t) \right\}$$

Trace part: $\left\langle \pi^a(p') \pi^b(p) \left| \hat{T}_\mu^\mu(0) \right| 0 \right\rangle = \delta^{ab} \Theta^\pi(t), \quad \Theta^\pi(t) = -\frac{1}{2} (4p_\pi^2 A^\pi(t) + 3t D^\pi(t))$

Breit frame (BF)

- The BF, for elastic electron scattering, coincides with electron-nucleon center-of-mass frame (CMF)



- BF matrix elements $\rightarrow$ Three-dimensional Fourier transformation $\rightarrow$ Charge distribution & Magnetic Density

$$\langle N_{s'}(\vec{q}/2) | J_{\text{EM}}^0(0) | N_s(-\vec{q}/2) \rangle = 2m_N G_E(\vec{q}^2) \delta_{s',s},$$

$$\langle N_{s'}(\vec{q}/2) | \vec{J}_{\text{EM}}(0) | N_s(-\vec{q}/2) \rangle = G_M(\vec{q}^2) \chi_{s'}^\dagger (i\vec{\sigma} \times \vec{q}) \chi_s.$$



Spatial density profiles

- Consider various densities

M. Polyakov, PLB 555 (2003) 57; M. Polyakov, P. Schweitzer, IJMPA 33 (2018) 183005;
C. Lorcé et al., EPJC 79 (2019) 89; C. Lorcé et al., PLB 776 (2018) 38;
X. Ji, Front. Phys. (Beijing) 16 (2021) 64601; D.E. Kharzeev, PRD 104 (2021) 054015; ...

$$\rho_{\text{Ener}}(r) = m_N \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir \cdot \Delta} \left[A(t) - \frac{t}{4m_N^2} [A(t) - 2J(t) + D(t)] \right] = \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir \cdot \Delta} \left[\Theta(t) + \frac{t}{2m_N} D(t) \right],$$

$$\rho_{\Theta}(r) = m_N \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir \cdot \Delta} \left[A(t) - \frac{t}{4m_N^2} [A(t) - 2J(t) + 3D(t)] \right] = \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir \cdot \Delta} \Theta(t),$$

$$\rho_J(r) = \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir \cdot \Delta} \left[J(t) + \frac{2}{3} t \frac{d}{dt} J(t) \right],$$

$$p_r(r) = m_N \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir \cdot \Delta} \left[-\frac{1}{r^2} \frac{1}{\sqrt{t} m_N^2} \frac{d}{dt} t^{\frac{3}{2}} D(t) \right] \equiv p(r) + \frac{2}{3} s(r),$$

$$p(r) = \frac{1}{6m_N} \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} \tilde{D}(r), \quad s(r) = -\frac{1}{4m_N} r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} \tilde{D}(r)$$

Zero-average-momentum frame (ZAMF)

E. Epelbaum et al., PRL 129 (2022) 012001; J.Y. Panteleeva et al., EPJC 83 (2023) 617; ...

$$\rho_{\text{Ener}}^{\text{ZAMF}}(r) = \frac{m_N}{4\pi r} \int_0^\infty d\Delta \Delta \sin(\Delta r) \int_{-1}^1 d\alpha A[(\alpha^2 - 1)\Delta^2]$$

$$\rho_{\text{Ener}}^{\text{naive}}(r) = m_N \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir \cdot \Delta} A(t)$$