Kinetic Equations for Diffusive Fluctuations

— From the Keldysh Ansatz to Statistical Evolution

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CEP-EFT 2026 | C3NT, Wuhan | 19–22 October 2026

WHAT EVOLVES OUT OF EQUILIBRIUM?

How do prepared density correlations evolve, and how do interactions make that evolution depend on history?

1. Stochastic dynamics: conservation, dissipation and thermal noise.

2. Two-time statistical evolution: MSRJD, Keldysh parametrization and the Dyson equation for F.

3. Physical reconstruction: from F to G^K and equal-time covariance; optional reductions and their scope.

STOCHASTIC HYDRODYNAMICS: PHYSICAL GOAL

A finite, expanding QCD fluid carries thermal fluctuations:

$$\partial_\mu(J^\mu_{\rm det} + \xi^\mu) = 0, \qquad \partial_\mu(T^{\mu\nu}_{\rm det} + \Xi^{\mu\nu}) = 0.$$

Deterministic hydrodynamics propagates and damps existing perturbations. Kovtun, J. Phys. A 45, 473001 (2012)

Thermal noise continually regenerates fluctuations. Dissipative transport (η, ζ, κ) also sets the thermal-noise strengths.

Kapusta, Müller & Stephanov, PRC 85, 054906 (2012)

A stable diffusive mode: without noise, its initial variance decays; with thermal noise, it approaches a nonzero equilibrium variance.

Near criticality, enhanced fluctuations relax slowly; a finite-lived fluid may not equilibrate.

Berdnikov & Rajagopal, PRD 61, 105017 (2000)

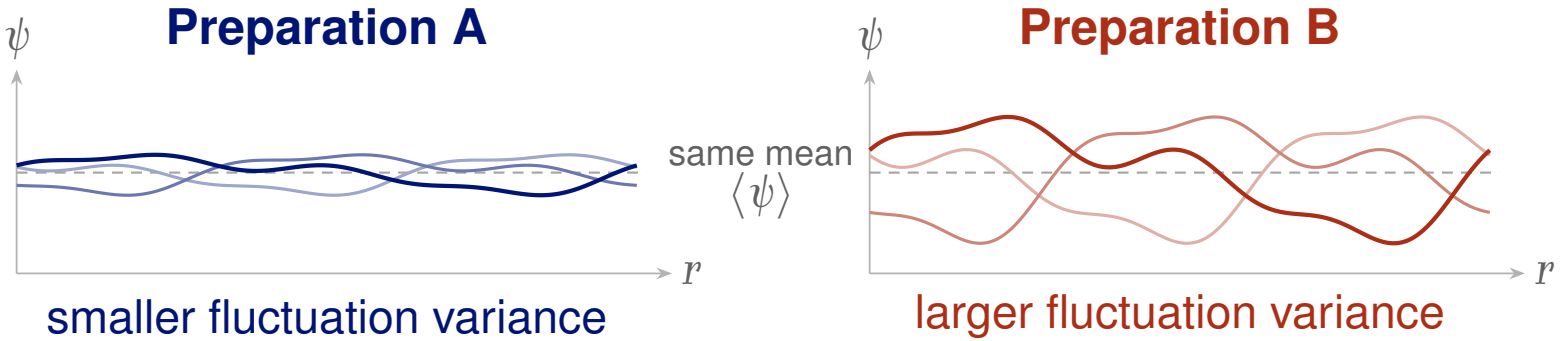
Goal: derive how fluctuation correlations evolve from stochastic dynamics.

NB: one conserved diffusive scalar, rather than the full relativistic fluid.

WHY A STATISTICAL EVOLUTION EQUATION?

Same stochastic dynamics; different initial statistical preparations:

$$\langle\psi\rangle_A = \langle\psi\rangle_B, \qquad \langle\delta\psi_q\delta\psi_{-q}\rangle_A \neq \langle\delta\psi_q\delta\psi_{-q}\rangle_B.$$



A and B are initial ensembles: same equations, transport and subsequent noise law.

A finite-lived fluid may retain this preparation information.

Same mean field, different statistical states.

Goal: evolve the two-point correlation directly.

Stephanov & Yin, PRD 98, 036006 (2018); An et al., PRC 102, 034901 (2020)

From Conserved Diffusion to the MSRJD Action

Density fluctuation $\psi = n - \bar{n}$; fixed uniform $\bar{n}$, constant $\kappa > 0$.

$$\mathcal{F}_h[\psi] = \int d^3x \left[\frac{1}{2}(\nabla\psi)^2 + \frac{m^2}{2}\psi^2 + \frac{\lambda_3}{3!}\psi^3 - h\psi\right].$$

Conservation gives diffusion:

$$\partial_t\psi + \nabla\cdot\mathbf{J} = 0, \qquad \mathbf{J} = -\kappa\nabla\left(\frac{\delta\mathcal{F}_h}{\delta\psi}\right) + \mathbf{J}_{\rm noise}.$$

$$\mathcal{E}_h[\psi] \equiv \partial_t\psi - \kappa\nabla^2\left(\frac{\delta\mathcal{F}_h}{\delta\psi}\right) = \theta, \qquad \theta = -\nabla\cdot\mathbf{J}_{\rm noise}.$$

Gaussian white noise: $x_i \equiv (t_i, \mathbf{r}_i)$, $N = -2\kappa T\nabla^2$.

$$\langle\theta(x_1)\theta(x_2)\rangle = -2\kappa T\nabla^2_{\mathbf{r}_1}\delta(t_1 - t_2)\,\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2).$$

Constraint → auxiliary field → noise average:

$$S_{\rm MSRJD} = \int dt\, d^3x \left[\widetilde{\psi}\,\mathcal{E}_h[\psi] - \frac{1}{2}\widetilde{\psi}\,(N\widetilde{\psi})\right], \qquad e^{-S_{\rm MSRJD}}.$$

Martin et al., PRA 8, 423; Chao & Schäfer, JHEP 06 (2023) 057; Chao & Schäfer, JHEP 01 (2021) 071

GAUSSIAN BASELINE: RESPONSE AND CORRELATION

Homogeneous Gaussian reference: $\lambda_3 = 0$, $q > 0$, $\Gamma_q > 0$; Fourier kernel $e^{+i\omega\tau - i\mathbf{q}\cdot\mathbf{r}}$.

$$\boxed{(\Gamma_q - i\omega)\psi_{\omega,q} = \kappa q^2 h_{\omega,q} + \theta_{\omega,q}, \qquad \Gamma_q = \kappa q^2(m^2 + q^2).}$$

Response to a physical source

$$G_0^R(x_1, x_2) = \langle \psi(x_1)\widetilde{\psi}(x_2)\rangle_0,$$
$$G_0^A(x_1, x_2) = \langle \widetilde{\psi}(x_1)\psi(x_2)\rangle_0.$$

$$G_0^R = \frac{1}{\Gamma_q - i\omega}, \qquad G_0^A = (G_0^R)^*,$$

$$\boxed{R_h \equiv \frac{\delta\langle\psi_q\rangle}{\delta h_q} = \kappa q^2 G_0^R,}$$

Static-response check

$$R_h(0, q) = \frac{\kappa q^2}{\Gamma_q} = \frac{1}{m^2 + q^2} = \chi_q.$$

Conserved source vertex: κq^2.

Retarded response is causal; $\langle \widetilde{\psi}(x_1)\widetilde{\psi}(x_2)\rangle_0 = 0$.

Correlations generated by noise

Zero source; stationary preparation.

$$\langle \theta_q(t)\theta_{-q}(t')\rangle = N_{0q}\delta(t - t'),$$
$$G_0^K(x_1, x_2) = \langle \delta\psi(x_1)\delta\psi(x_2)\rangle_0.$$

$$\psi_q(t) = \int_{-\infty}^{t} dt'\, G_0^R(t - t', q)\, \theta_q(t'),$$

$$\boxed{G_0^K = G_0^R N_{0q} G_0^A = \frac{N_{0q}}{\omega^2 + \Gamma_q^2},}$$

Noise sets the stationary variance

$$\int \frac{d\omega}{2\pi}\, G_0^K = \frac{N_{0q}}{2\Gamma_q}.$$

White-noise strength N_{0q} stays symbolic.

For details, see backup slide B1

At thermal equilibrium, what ties N_{0q} to the dissipative rate?

TIME REVERSAL AND FLUCTUATION–DISSIPATION

Equilibrium detailed balance fixes the response-field shift. $h = 0$.

$$\psi^{\mathrm{TR}}(t,\mathbf{r}) = \psi(-t,\mathbf{r}), \qquad \text{require: } S^{\mathrm{TR}} = S - \tfrac{\Delta\mathcal{F}}{T},$$
$$\widetilde{\psi}^{\mathrm{TR}}(t,\mathbf{r}) = -\widetilde{\psi}(-t,\mathbf{r}) + \frac{1}{T}\frac{\delta\mathcal{F}}{\delta\psi(\mathbf{r})}\bigg|_{\psi=\psi(-t)}.$$

Deterministic drift: $\boxed{L \equiv -\kappa\nabla^2}$. Equilibrium TR requires drift–noise matching.

$C_{\mathrm{eq}}(\tau,q) \equiv \langle \delta\psi_q(t+\tau)\,\delta\psi_{-q}(t)\rangle_{\mathrm{eq}}$; $\mathcal{F} = \mathcal{F}_{h=0}$, $\Delta\mathcal{F} = \mathcal{F}_f - \mathcal{F}_i$.

Causality + equilibrium TR $\Longrightarrow$ FDT (details in B2); $\tau = t_{\mathrm{response}} - t_{\mathrm{source}} > 0$.

$$\boxed{R_h(\tau,q) = -\frac{\Theta(\tau)}{T}\,\partial_\tau C_{\mathrm{eq}}(\tau,q)} \quad\Longrightarrow\quad \boxed{G^K_{\mathrm{eq}}(\omega,q) = \frac{2T}{\omega}\mathrm{Im}\,R_h(\omega,q).}$$

Gaussian FDT consistency via G_0^K, R_h, $\mathrm{Im}\,G_0^R$:

$$G_0^R N_{0q} G_0^A = \frac{2T}{\omega}\kappa q^2 \mathrm{Im}\,G_0^R = 2T\kappa q^2 G_0^R G_0^A \quad\Longrightarrow\quad \boxed{N_{0q} = 2\kappa T q^2.}$$

Operator form: $\boxed{N = 2TL = -2\kappa T\nabla^2}$. This reproduces the thermal noise matching.

Gaussian equilibrium covariance: $C_{0,\mathrm{eq}}(q) \equiv T\chi_q = \dfrac{T}{m^2+q^2} = \dfrac{N_{0q}}{2\Gamma_q}$.

Equilibrium TR fixes thermal noise; Gaussian FDT confirms the matching.

Janssen, Z. Phys. B 23, 377; Chao & Schäfer, JHEP 01 (2021) 071

THE CLASSICAL WORKING DESCRIPTION

For thermal slow modes in a matched classical regime,

$$\epsilon_Q = \frac{\hbar\omega_{\rm char}}{k_B T} \ll 1 \qquad \text{in the target frequency band.}$$

Four distinct choices organize the description:

Classical statistics	Thermal fluctuations and response
Markov approximation	Replace temporal history by local terms
Spatial gradients	Resolve a slowly varying background
Loop truncation	Choose the interaction diagrams retained

A classical stochastic theory with nonlinear interactions, white and/or colored noise and nonlocal self-energies.

Liu & Glorioso, PoS TASI2017, 008

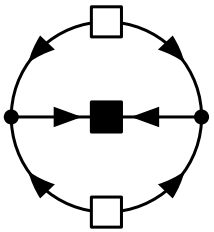
Taking a classical limit does not require a short-memory approximation.

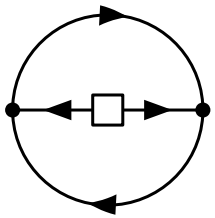
From Cubic Interactions to 2PI Self-Energies

Quadratic MSRJD: bare Gaussian response and statistical propagators.
Cubic interaction: nonlinear mode coupling and self-energy corrections.

$$\varphi_a = (\tilde{\psi}, \psi), \qquad G_{ab} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} 0 & G^A \\ G^R & G^K \end{pmatrix}.$$

$\Phi[G]$: interaction/skeleton part of the 2PI effective action, built from full G lines.





$$\Sigma_{ab}(x_1, x_2) = 2\frac{\delta\Phi[G]}{\delta G_{ba}(x_2, x_1)}$$

Opening a line: two-loop 2PI skeleton $\longrightarrow$ one-loop self-energies.

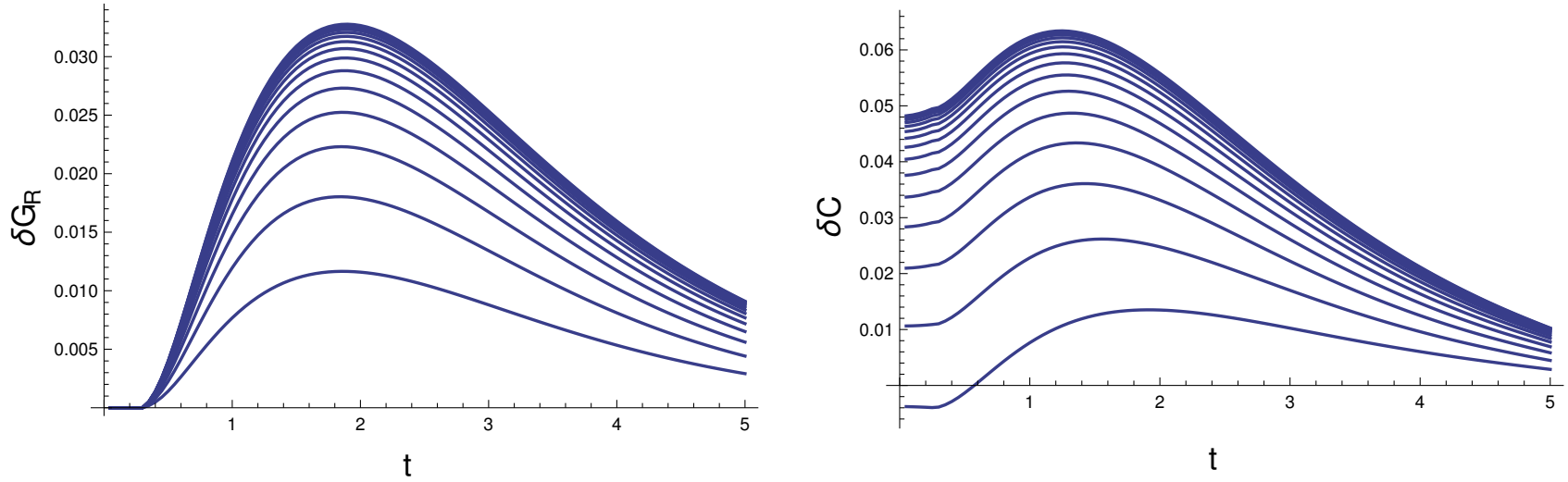
$$\boxed{\widehat{G}^{-1} = \widehat{G}_0^{-1} + \widehat{\Sigma}[G]} \qquad \text{(plus-Dyson)}.$$

Cornwall et al., PRD 10, 2428; Chao & Schäfer, JHEP 06 (2023) 057

Dyson corrects both response and statistical propagators.

From Stationary Dyson to Statistical Evolution

Chao & Schäfer, JHEP 06 (2023) 057, Fig. 5



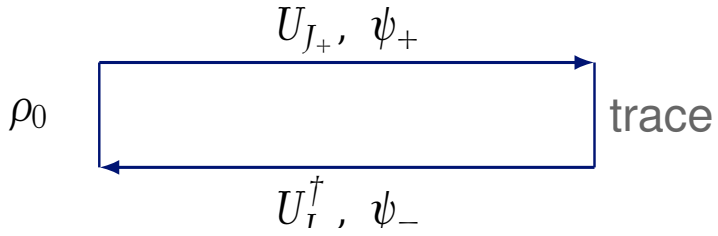
Left: $\delta G^R = G^R - G_0^R$; right: $\delta C(\tau) = C(\tau) - C_{\rm bare}(\tau)$ (classical G^K).
Successive stationary Dyson iterations; plotted $t = \tau$ is *relative time*, not center time.
These are propagator/correlation corrections, not $\Sigma^{R,K}$ plots.

These stationary solutions describe relative-time correlations, not the center-time evolution of a prepared statistical state.

MSRJD AND CTP: A CLASSICAL DICTIONARY

1. CTP density-matrix trace

$$Z[J_+, J_-] = \mathrm{Tr}(U_{J_+}\rho_0 U_{J_-}^\dagger).$$



Adjoint branch, not a TR symmetry.

2. Average and difference fields

$$\psi_r = \frac{\psi_+ + \psi_-}{2}, \qquad \psi_a = \psi_+ - \psi_-.$$

Classical matching: $\psi_r \longleftrightarrow \psi$;
$\psi_a \longleftrightarrow$ rotated/rescaled $\widetilde{\psi}$.
r/a label fields; R/A label response functions.

3. Action difference $\longrightarrow$ classical response

$\mathcal{S}$: real-time action, distinct from $\mathcal{F}_h$ and $\mathcal{S}_{\mathrm{MSRJD}}$; $\psi_\pm = \psi_r \pm \psi_a/2$.

$$\mathcal{S}[\psi_r + \tfrac{1}{2}\psi_a] - \mathcal{S}[\psi_r - \tfrac{1}{2}\psi_a] = \int d^{d+1}x\, \psi_a(x)\, \frac{\delta \mathcal{S}[\psi_r]}{\delta \psi_r(x)} + O(\psi_a^3).$$

$$\text{CTP: } \psi_a \times \mathrm{EOM} \quad \longleftrightarrow \quad \text{MSRJD: } \widetilde{\psi}\, \mathcal{E}_h[\psi].$$

The shared $G^{R,A,K}$ response/statistical organization lets us adapt the familiar Keldysh parametrization to MSRJD variables.

Dissipation and thermal noise require coarse-grained effective-action matching.

Kamenev & Levchenko, Adv. Phys. 58, 197 (2009); Liu & Glorioso, PoS TASI2017, 008

THE KELDYSH ANSATZ: WHY INTRODUCE F?

1. Exact *local* Gaussian starting point

$$\boxed{\begin{aligned} t_1 > t_2: \quad & G_0^K(t_1, t_2; q) = G_0^R(t_1, t_2; q)\, C_q(t_2), \\ t_2 > t_1: \quad & G_0^K(t_1, t_2; q) = C_q(t_1)\, G_0^A(t_1, t_2; q). \end{aligned}}$$

Response carries propagation/time ordering; $C_q(t)$ carries the statistical amplitude.

2. Embed *local* statistics as a two-time kernel

$$F_{\rm loc}(t_1, t_2; q) = C_q(t_1)\, \delta(t_1 - t_2) \qquad G_0^K = G_0^R \circ F_{\rm loc} + F_{\rm loc} \circ G_0^A.$$

3. Allow general relative-time statistical structure

$$F_{\rm loc}(t_1, t_2; q) \longrightarrow F(t_1, t_2; q) \quad \Longrightarrow \quad \boxed{G^K = G^R \circ F + F \circ G^A.}$$

Kamenev & Levchenko, Adv. Phys. 58, 197 (2009)

$$(A \circ B)(x_1, x_2) = \int dt_3\, d^3\mathbf{x}_3\, A(x_1, x_3) B(x_3, x_2).$$

G^K: physical correlation; F: evolving statistical kernel, not the memory kernel.

F is generally not an equal-time covariance.

THE STATISTICAL BLOCK OF THE INVERSE

Dyson acts on the full two-field operator; we need its inverse statistical block.

$$\widehat{G}^{-1} = \begin{pmatrix} [\widehat{G}^{-1}]^K & (G^R)^{-1} \\ (G^A)^{-1} & 0 \end{pmatrix}.$$

In $\widehat{G} \circ \widehat{G}^{-1} = \mathbf{1}$, the lower-left block vanishes:

$$G^R \circ [\widehat{G}^{-1}]^K + G^K \circ (G^A)^{-1} = 0.$$

$$\boxed{[\widehat{G}^{-1}]^K = -(G^R)^{-1} \circ G^K \circ (G^A)^{-1}.}$$

Insert the Keldysh ansatz; cancel only adjacent inverse pairs:

$$\boxed{[\widehat{G}^{-1}]^K = -(G^R)^{-1} \circ \left(G^R \circ F + F \circ G^A\right) \circ (G^A)^{-1}.}$$

$$= -\underbrace{(G^R)^{-1} \circ G^R}_{\mathbf{1}} \circ F \circ (G^A)^{-1} - (G^R)^{-1} \circ F \circ \underbrace{G^A \circ (G^A)^{-1}}_{\mathbf{1}}.$$

$$\boxed{[\widehat{G}^{-1}]^K = -(G^R)^{-1} \circ F - F \circ (G^A)^{-1}.}$$

$[\widehat{G}^{-1}]^K$: a block of the inverse full matrix.

Kamenev & Levchenko, Adv. Phys. 58, 197 (2009)

For details, see backup slide B3

THE KINETIC IDENTITY FOR F

With plus-Dyson,

$$(G^{R,A})^{-1} = (G_0^{R,A})^{-1} + \Sigma^{R,A}, \qquad [\widehat{G}^{-1}]^K = -N_0 + \Sigma^K.$$

N_0: bare noise operator symbol.

Insert these into the block identity

$$\underbrace{(G_0^R)^{-1} \circ F + F \circ (G_0^A)^{-1}}_{\text{bare dynamics}} = \underbrace{N_0 - \Sigma^K}_{\text{statistical/noise source}} - \underbrace{[\Sigma^R \circ F + F \circ \Sigma^A]}_{\text{response feedback}}.$$

Kamenev & Levchenko, Adv. Phys. 58, 197 (2009)

No gradient, Markov, or on-shell approximation is introduced in this algebraic step.

For details, see backup slide B3

An identity becomes an evolution problem only after kernels and preparation are specified.

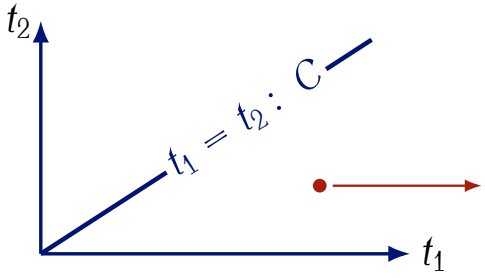
WIGNER TRANSFORM: KEEPING THE EVOLUTION TIME

$$\mathcal{T} = \frac{t_1 + t_2}{2}, \quad \tau = t_1 - t_2, \qquad \mathbf{X} = \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}, \quad \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2.$$

Fourier transform the relative coordinates; keep the center coordinates.

$$F_\omega(\mathcal{T}, \omega; \mathbf{X}, \mathbf{q}) = \int d\tau\, d^3r\, e^{i\omega\tau - i\mathbf{q}\cdot\mathbf{r}} \times F\left(\mathcal{T} + \frac{\tau}{2}, \mathbf{X} + \frac{\mathbf{r}}{2}; \mathcal{T} - \frac{\tau}{2}, \mathbf{X} - \frac{\mathbf{r}}{2}\right).$$

Kamenev & Levchenko, Adv. Phys. 58, 197 (2009)



$\mathcal{T} = (t_1 + t_2)/2, \quad \tau = t_1 - t_2$
center: evolving statistical state / background
relative coordinates: $\tau \to \omega,\ \mathbf{r} \to \mathbf{q}$

$G^K(t_1, t_2)$ off diagonal

Diagram: physical G^K with covariance C on its diagonal.

A change of representation:

The center time $\mathcal{T}$ remains the evolution variable.

ω resolves relative-time structure; $\mathbf{q}$ resolves spatial fluctuation scales.

How Two-Time Operators Transform

$$[A \circ B]_W = A_W \star B_W.$$

$$\star = \exp\left\{ \frac{i}{2} \left(\overleftarrow{\partial}_\omega \overrightarrow{\partial}_\mathcal{T} - \overleftarrow{\partial}_\mathcal{T} \overrightarrow{\partial}_\omega + \overleftarrow{\nabla}_X \cdot \overrightarrow{\nabla}_q - \overleftarrow{\nabla}_q \cdot \overrightarrow{\nabla}_X \right) \right\}.$$

W: Wigner transform; arrows indicate which factor is differentiated.

Exact operator composition; no gradient truncation.

Kamenev & Levchenko, Adv. Phys. 58, 197 (2009)

Chain rule, followed by the relative-time Fourier transform:

$$\partial_{t_1} = \tfrac{1}{2}\partial_\mathcal{T} + \partial_\tau \quad \Longrightarrow \quad [\partial_{t_1} F]_W = (\tfrac{1}{2}\partial_\mathcal{T} - i\omega) F_\omega,$$

$$\partial_{t_2} = \tfrac{1}{2}\partial_\mathcal{T} - \partial_\tau \quad \Longrightarrow \quad [\partial_{t_2} F]_W = (\tfrac{1}{2}\partial_\mathcal{T} + i\omega) F_\omega.$$

Advancing both observation times changes the center time, not their separation.

$$\partial_{t_1} + \partial_{t_2} = \partial_\mathcal{T}.$$

Relative derivatives cancel before Fourier transformation; F_ω may still depend on ω.

For details, see backup slide B3

GAUSSIAN BALANCE: TWO LEGS AND A NOISE SOURCE

Homogeneous Gaussian baseline: constant parameters, additive white noise, $q > 0$, $\Sigma = 0$; matched preparation and boundaries.

$$(\tfrac{1}{2}\partial_{\mathcal{T}} - i\omega + \Gamma_q)F_\omega + (\tfrac{1}{2}\partial_{\mathcal{T}} + i\omega + \Gamma_q)F_\omega = N_{0q}.$$

$$\boxed{\partial_{\mathcal{T}} F_\omega = -2\Gamma_q(F_\omega - C_{0,\mathrm{eq}}).}$$

Two half derivatives $\to \partial_{\mathcal{T}}$; two legs $\to 2\Gamma_q$; $\pm i\omega$ cancel. No frequency projection.

Itô check for the physical variance $C_q(t) = \langle \psi_q(t)\psi_{-q}(t)\rangle$:

$$d(\psi_q\psi_{-q}) = (d\psi_q)\,\psi_{-q} + \psi_q\,(d\psi_{-q}) + d\psi_q\,d\psi_{-q}.$$

$$\boxed{\partial_t C_q = \underbrace{-\Gamma_q C_q - \Gamma_q C_q}_{\text{two relaxing legs}} + \underbrace{N_{0q}}_{\text{one noise source}}.} \qquad \tau_{\mathrm{cov,q}} = \frac{1}{2\Gamma_q}.$$

Plugging $N_{0q} = 2\Gamma_q C_{0,\mathrm{eq}}$ into the covariance balance:

$$\delta C_q \equiv C_q - C_{0,\mathrm{eq}} \quad \Longrightarrow \quad \boxed{\partial_t \delta C_q = -2\Gamma_q\,\delta C_q.}$$

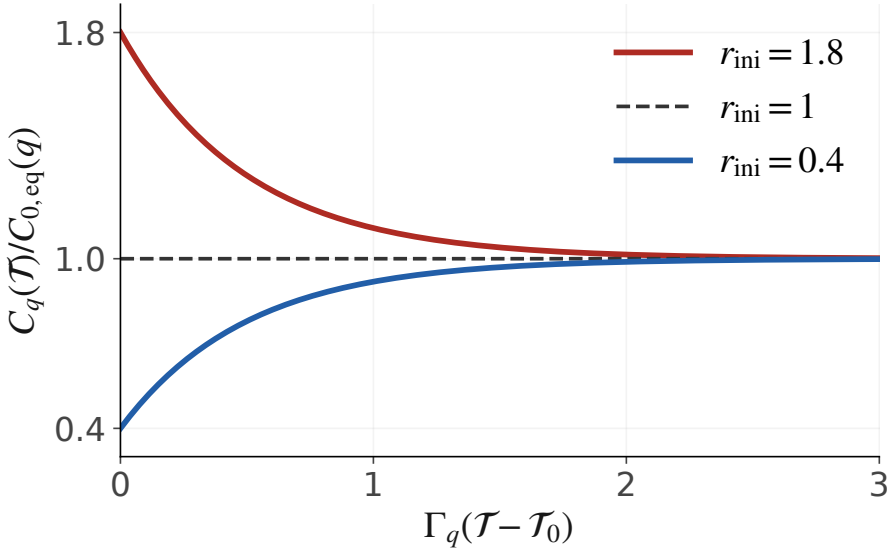
Two relaxing legs give $2\Gamma_q$; the Itô term supplies one noise source.

An Analytic Relaxation Benchmark

Homogeneous Gaussian baseline: constant parameters, $q > 0$, $\Sigma = 0$.

$$F_\omega(\mathcal{T}, \omega, q) = C_{0,\mathrm{eq}}(q) + \left[F_\omega(\mathcal{T}_0, \omega, q) - C_{0,\mathrm{eq}}(q)\right] e^{-2\Gamma_q(\mathcal{T}-\mathcal{T}_0)}.$$

$$C_{q,\mathrm{ini}} \equiv C_q(\mathcal{T}_0), \qquad r_{\mathrm{ini}} \equiv C_{q,\mathrm{ini}}/C_{0,\mathrm{eq}}(q).$$



One equilibrium, different preparations.
Excess variance decays; noise replenishes a deficit.
The plot uses the matched local Gaussian construction, $F_\omega = C_q$.

The Gaussian baseline relaxes exponentially at rate $2\Gamma_q$.

A Nonlocal Self-Energy Samples the Past

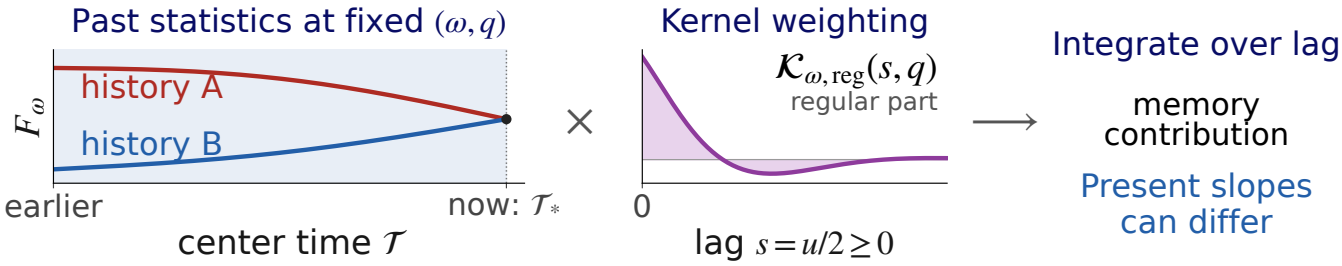
$$(\Sigma^R \circ F_\tau)(t_1, t_2; q) = \int_{-\infty}^{t_1} dt_3\, \Sigma^R(t_1 - t_3, q) F_\tau(t_3, t_2; q).$$

Retarded support: $u = t_1 - t_3 \geq 0$; t_2 is an external spectator.

$$(\Sigma^R \circ F_\tau)(\mathcal{T}, \tau; q) = \int_0^\infty du\, \Sigma^R(u, q) F_\tau\left(\mathcal{T} - \frac{u}{2}, \tau - u; q\right).$$

$\tau \to \omega$, $s = u/2$, combine $R + A$: $\Sigma^{R,A} \to \mathcal{K}_\omega^{\text{full}}$. Regular part shown: $\mathcal{K}_{\omega,\text{reg}} \to \mathcal{M}_{\omega,\text{reg}}$.

$$\mathcal{M}_{\omega,\text{reg}}(\mathcal{T}, q) \equiv \int_0^\infty ds\, F_\omega(\mathcal{T} - s, \omega, q)\, \mathcal{K}_{\omega,\text{reg}}(s, q).$$



Same present value, same kernel, different histories: slopes can differ.

FIXED-KERNEL EVOLUTION OF F WITH MEMORY

Closure: stationary one-loop kernels, evaluated once on Gaussian equilibrium.

$$\Sigma^{R,A,K} \equiv \Sigma^{R,A,K}_{1\mathrm{L}}[\widehat{G}_{0,\mathrm{eq}}], \qquad \partial_{\mathscr{T}}\Sigma^{R,A,K} = 0.$$

No dynamical 2PI update or short-memory expansion.

$$\partial_{\mathscr{T}} F_\omega + 2\Gamma_q\big(F_\omega - C_{0,\mathrm{eq}}(q)\big) = -\Sigma^K(\omega,q) + \int_0^\infty ds\, \mathscr{K}^{\mathrm{full}}_\omega(s,q) F_\omega(\mathscr{T}-s,\omega,q).$$

$$\mathscr{K}^{\mathrm{full}}_\omega(s,q) = -2\left[\mathrm{e}^{2i\omega s}\Sigma^R(2s,q) + \mathrm{e}^{-2i\omega s}\Sigma^A(-2s,q)\right].$$

Full relative-frequency dependence; contact terms $s = 0$ treated consistently.

Matched stationary reference of the same equation:

$$F_{\omega,\mathrm{eq}}(\omega,q) = \frac{N_{0q} - \Sigma^K(\omega,q)}{2\Gamma_q + \Sigma^R(\omega,q) + \Sigma^A(\omega,q)}.$$

Subtract the matched stationary equation $\Longrightarrow$ closed evolution

$$\delta F_\omega \equiv F_\omega - F_{\omega,\mathrm{eq}},$$
$$\partial_{\mathscr{T}}\delta F_\omega + 2\Gamma_q \delta F_\omega = \mathscr{K}^{\mathrm{full}}_\omega *_{\mathscr{T}} \delta F_\omega.$$

Same kernels and history prescription: $-\Sigma^K$ cancels; noise fixes $F_{\omega,\mathrm{eq}}$ and FDT.

$*_{\mathscr{T}}$: causal center-time convolution.

For details, see backup slide B4

From F Back to Physical Correlations

The two-time kernel contains more information than a single equal-time observable.

Full-frequency route: reconstruct, then take equal time

$$F \xrightarrow{G^R \circ F + F \circ G^A} G^K \xrightarrow{\tau = 0} C_K.$$

$$C_K(\mathcal{T}, q) = G^K_\tau(\mathcal{T}, 0, q) = \int \frac{d\omega}{2\pi}\, G^K_\omega(\mathcal{T}, \omega, q).$$

Equal time is a relative-frequency integral of the physical correlator G^K.

Optional reduction: the zero-relative-frequency component

Select one component of the statistical kernel.

$$W(\mathcal{T}, q) = F_\omega(\mathcal{T}, 0, q) = \int d\tau\, F_\tau(\mathcal{T}, \tau, q).$$

This selects information from F; center-time evolution and momentum remain.

Solve the full frequency family, or select a reduced sector.

For details, see backup slide B5

THREE PROJECTIONS, DIFFERENT PHYSICS

On-shell quasiparticles

How is a propagating mode occupied?

Occupation near a narrow real-frequency peak.

$\gamma_q \ll |E_q|$; separated modes and controlled gradients.

Kamenev & Levchenko, Adv. Phys. 58, 197 (2009)

Equal-time hydrokinetics

How strong are the fluctuations now?

Direct evolution of $C_K(\mathcal{T}, q)$: same-time spatial fluctuations.

Physical initial statistics and a hierarchy closure.

An et al., PRC 102, 034901 (2020)

Optional zero-frequency sector

How do statistics evolve with memory?

$W = F_\omega(\mathcal{T}, 0, q)$.

Fixed stationary kernels; one member of a closed frequency family.

Less frequency work; retain memory, center time and momentum.

Closure is specified first; the zero-frequency choice reduces the calculation.

NB: for the Gaussian diffusion mode ($q > 0$), zero real frequency is neither the complex pole $\omega_{\rm pole} = -i\Gamma_q$ nor equal time.

When Can W Equal an Equal-Time Covariance?

Matched local Gaussian dynamics

Constant coefficients, additive white noise, compatible preparation; $\kappa > 0$, $q > 0$.

$$F_\tau(\mathcal{T}, \tau, q) = C_K(\mathcal{T}, q)\delta(\tau) \implies F_\omega = C_K \implies W = C_K.$$

Responses carry relative-time propagation; F carries the evolving amplitude.

$$\boxed{\partial_{\mathcal{T}} C_K = -2\Gamma_q(C_K - C_{0,\mathrm{eq}}), \qquad W = C_K.}$$

Same stochastic dynamics *and* matched initial variance.

$m > 0$, $q \ll m$: local diffusion limit. An et al., PRL 127, 072301 (2021)

Approximate matching

With Gaussian response and a controlled temporal-Wigner approximation, a frequency average can be represented by W if F_ω is sufficiently flat.

With nonlocal memory

W alone may not determine covariance.

Full-frequency reconstruction remains available within the same closure.

For details, see backup slide B5

What Closes the Evolution Problem?

Present closure: fixed kernels

Evaluate the one-loop self-energies on Gaussian equilibrium and keep them fixed. Evolve F across the chosen frequency domain; W is optional.

$$\widehat{G}_{0,\mathrm{eq}} \longrightarrow \Sigma_{1\mathrm{L}}^{R,A,K} \longrightarrow F_\omega.$$

No self-consistent update; full time-lag dependence is retained.

Extension: self-consistent feedback

Evolving response and statistical propagators update the self-energies.
The updated self-energies change the propagators in return.

$$\widehat{G} \rightleftarrows \Sigma[\widehat{G}].$$

Joint evolution of response, statistics and self-energies.

The earlier result solves stationary self-consistent relative-time Dyson equations; evolving nonequilibrium feedback is a future extension.

Chao & Schäfer, JHEP 06 (2023) 057;
Jingyi Chao, Jianing Yao and Thomas Schaefer, in preparation

The kernels specify the dynamics;
preparation supplies the initial state and history.

SUMMARY

1. Two-time statistical evolution

   MSRJD and the Keldysh parametrization organize prepared two-point statistics into center-time evolution of $F_\omega(\mathcal{T}, \omega, q)$.

2. Causal memory

   Nonlocal self-energies make evolution depend on prepared history. Fixed kernels retain memory without dynamical self-consistent feedback.

3. Physical reconstruction

   The full relative-frequency equation supports reconstruction of G^K and covariance. Zero frequency is optional; exact covariance matching holds in the matched local Gaussian benchmark.

Further work: full-frequency reconstruction; dynamical self-consistent feedback.

Thank You for Your Attention!

CONVENTIONS, NOISE AND PHYSICAL RESPONSE

For e^{-S}, integrate $\widetilde{\psi}$ along the imaginary response contour. At fixed q; space-time integrations are suppressed.

$$S_2 = \tfrac{1}{2}(\widetilde{\psi}, \psi) \begin{pmatrix} -N_{0q} & \mathscr{D}_q^R \\ \mathscr{D}_q^A & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\psi} \\ \psi \end{pmatrix}, \qquad \mathscr{D}_q^{R,A} \equiv \pm\partial_t + \Gamma_q.$$

$$(G_0^{R,A})^{-1} = \mathscr{D}_q^{R,A}, \qquad [\widehat{G}_0^{-1}]^K = -N_{0q}.$$

N_{0q}: noise strength defined on S05, thermally matched on S06.

Define a physical source by $\mathscr{F}_h = \mathscr{F} - \int h\psi$; for constant κ,

$$R_h = \frac{\delta\langle\psi\rangle}{\delta h} = \kappa q^2 G^R, \qquad G^K_{\rm eq}(\omega, q) = \frac{2T}{\omega}\mathrm{Im}\, R_h(\omega, q).$$

State-dependent mobility requires a composite source vertex.

Chao & Schäfer, JHEP 01 (2021) 071

The retarded/Itô response-loop prescription is distinct from B5's continuous physical equal-time limit. Multiplicative noise also needs a matched measure.

Back to S05: response/correlation Back to S06: FDT

WHY CAUSALITY + EQUILIBRIUM TR GIVES FDT

Chao & Schäfer, JHEP 01 (2021) 071, Appendix A. Project convention.
$L = -\kappa\nabla^2 > 0$ ($q > 0$), $N = 2TL$: constant mobility, equilibrium at $h = 0$.
FDT within the thermally matched theory; S06 checks Gaussian consistency.

1. Additive Gaussian noise: Onsager–Machlup action and source variation.

$$\mathcal{F}_h = \mathcal{F} - \int h\psi, \qquad U = L\frac{\delta\mathcal{F}}{\delta\psi}, \qquad \mathcal{E}_h = \dot{\psi} + U - Lh, \qquad \boxed{S_{\mathrm{OM},h} = \frac{1}{4T}\int \mathcal{E}_h L^{-1}\mathcal{E}_h.}$$

Normalized expectation: $x_i = (t_i, \mathbf{r}_i)$, $\delta S_{\mathrm{OM},h}/\delta h = -\mathcal{E}_h/(2T)$.

$$R_h(x_1,x_2) \equiv \frac{\delta\langle\psi(x_1)\rangle}{\delta h(x_2)} = -\left\langle \psi(x_1)\frac{\delta S_{\mathrm{OM},h}}{\delta h(x_2)}\right\rangle_c = \frac{1}{2T}\langle\psi(x_1)[\dot{\psi}+U](x_2)\rangle_c.$$

2. Apply equilibrium TR to the forbidden-order causality relation.

Reverse both times in that relation: ψ, U are TR-even; $\dot{\psi}$ is TR-odd.
$A(x_1,x_2) = \langle\psi(x_1)\dot{\psi}(x_2)\rangle_c$, $\quad B(x_1,x_2) = \langle\psi(x_1)U(x_2)\rangle_c$.

$$t_1 < t_2: \quad R_h = 0 \implies A(x_1,x_2) + B(x_1,x_2) = 0,$$
$$t_1 > t_2: \quad -A(x_1,x_2) + B(x_1,x_2) = 0 \implies A = B \implies R_h = A/T.$$

Equal OM terms; $\tau = t_{\mathrm{response}} - t_{\mathrm{source}}$, positive Fourier phase $e^{+i\omega\tau}$.

$$\boxed{R_h(\tau,q) = -\frac{\Theta(\tau)}{T}\partial_\tau C_{\mathrm{eq}}(\tau,q)} \quad\Longrightarrow\quad \boxed{G^K_{\mathrm{eq}}(\omega,q) = \frac{2T}{\omega}\mathrm{Im}\,R_h(\omega,q).}$$

Matched preparation and contact prescription; zero mode excluded from L^{-1}.

Back to S06: time reversal and FDT

BLOCK ALGEBRA AND BARE SPATIAL OPERATORS

Ordered multiplication, with time convolutions understood, gives

$$\begin{pmatrix} 0 & G^A \\ G^R & G^K \end{pmatrix} \begin{pmatrix} -(G^R)^{-1} \circ G^K \circ (G^A)^{-1} & (G^R)^{-1} \\ (G^A)^{-1} & 0 \end{pmatrix} = \mathbf{1}.$$

$$[\widehat{G}^{-1}]^K = -(G^R)^{-1} \circ G^K \circ (G^A)^{-1} \neq (G^K)^{-1}.$$

The ansatz and plus-Dyson fix both response feedback signs to minus.

Local diffusion only: $m > 0$, $q \ll m$, constant $D = \kappa m^2$. With matched bulk boundaries, the right advanced action gives

$$\int d3\, F_{13}(-\partial_{t_3} - D\nabla_3^2)\delta_{32} = (\partial_{t_2} - D\nabla_2^2)F_{12}.$$

$$Dq^2 \star_X F = Dq^2 F - iD\mathbf{q}\cdot\nabla_X F - \tfrac{D}{4}\nabla_X^2 F,$$

$$F \star_X Dq^2 = Dq^2 F + iD\mathbf{q}\cdot\nabla_X F - \tfrac{D}{4}\nabla_X^2 F,$$

$$Dq^2 \star_X F + F \star_X Dq^2 = 2Dq^2 F - \tfrac{D}{2}\nabla_X^2 F.$$

First-order left/right gradients cancel; the second-order center gradient remains. The homogeneous Gaussian benchmark sets $\nabla_X F = 0$. The full q^4 spatial operator is outside this check.

Back to S12: inverse block Back to S13: kinetic identity
Back to S15: Wigner operators

CAUSAL LAGS AND PREPARED HISTORY

Stationary scalar kernels: $u_{R,A} \geq 0$; no ordering between external times t_1, t_2.

$$t_3 = t_1 - u_R \leq t_1 : \quad F_\tau(\mathcal{T} - u_R/2, \tau - u_R)\, \Sigma^R(u_R),$$
$$t_3 = t_2 - u_A \leq t_2 : \quad F_\tau(\mathcal{T} - u_A/2, \tau + u_A)\, \Sigma^A(-u_A).$$

Phases $e^{\pm i\omega u}$: $u = 2s$ gives the 2 in $\mathcal{K}^{\rm full}_\omega$, a Jacobian distinct from $2\Gamma_q$ due to the two correlation legs.

Prepared history for $\mathcal{T} < \mathcal{T}_0$; the causal center-time convolution $*_\mathcal{T}$ splits as

$$(\mathcal{K}^{\rm full}_\omega *_\mathcal{T} \delta F_\omega)(\mathcal{T}) = \int_0^{\mathcal{T}-\mathcal{T}_0} ds\, \mathcal{K}^{\rm full}_\omega(s,q)\delta F_\omega(\mathcal{T} - s, \omega, q) + J_{\rm hist}(\mathcal{T}, \omega, q),$$

$$J_{\rm hist}(\mathcal{T}, \omega, q) = \int_{\mathcal{T}-\mathcal{T}_0}^{\infty} ds\, \mathcal{K}^{\rm full}_\omega(s,q)\delta F_{\omega,\rm hist}(\mathcal{T} - s, \omega, q).$$

Finite microscopic start $t_{\rm ini}$: $|\tau| \leq 2(\mathcal{T} - t_{\rm ini})$; windowed statistical source:

$$\int_{-2(\mathcal{T}-t_{\rm ini})}^{2(\mathcal{T}-t_{\rm ini})} d\tau\, e^{i\omega\tau}\Sigma^K(\tau, q).$$

Microscopic start $t_{\rm ini}$ differs in role from the center-time split $\mathcal{T}_0$. Boundary terms must also be matched; a remote-past split does not replace them.

Back to S18: causal time shift Back to S19: memory equation

Local Reconstruction and Initial Data

Local Gaussian scalar: $C_q(\mathcal{T}) \equiv C_K(\mathcal{T}, q)$; constant $\Gamma_q > 0$, $q > 0$, white noise and matched $C_{q,\mathrm{ini}}$.

$$F_{12} = C_q(t_1)\delta(t_1 - t_2), \qquad G^K_{12} = e^{-\Gamma q|t_1-t_2|}C_q(\min(t_1, t_2)) .$$

With $U(t, s) = e^{-\Gamma q(t-s)}$, propagated initial covariance plus accumulated noise gives

$$G^K_{12} = U(t_1, t_{\mathrm{ini}})C_{q,\mathrm{ini}}U(t_2, t_{\mathrm{ini}}) + \int_{t_{\mathrm{ini}}}^{\min(t_1,t_2)} ds\, U(t_1, s)N_{0q}U(t_2, s).$$

For $t_{1,2} > t_{\mathrm{ini}}$, future noise is independent of matched initial data. The continuous coincidence limit uses symmetric endpoint weights, not a new Itô loop prescription.

$$\begin{aligned} F_\omega(\mathcal{T}, \omega, q) &= C_K(\mathcal{T}, q), \\ W(\mathcal{T}, q) = F_\omega(\mathcal{T}, 0, q) &= C_K(\mathcal{T}, q), \\ C_K(\mathcal{T}, q) &= \int \frac{d\omega}{2\pi}\, G^K_\omega(\mathcal{T}, \omega, q). \end{aligned}$$

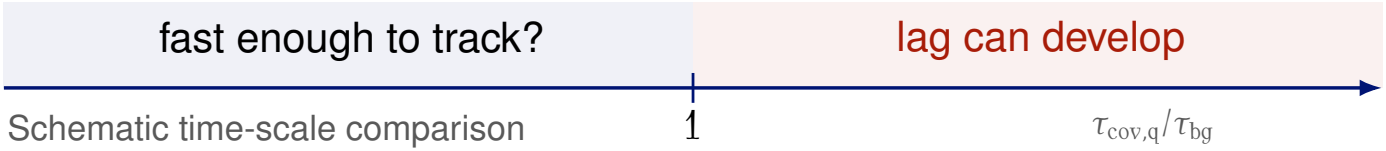
Exact only for matched local Gaussian dynamics and compatible preparation.
Finite start t_{ini}: physical domain $|\tau| \le 2(\mathcal{T} - t_{\mathrm{ini}})$. Full-axis transforms need an explicit extension/window and matched boundary terms; see B4.

Back to S20: reconstruction Back to S22: local matching

A Moving Target Can Lag in a Local Equation

Local illustration with positive $\Gamma_q(t)$ and instantaneous target $C_{q,\mathrm{inst}}(t)$:

$$\delta C_q = C_q - C_{q,\mathrm{inst}}, \qquad N(t,t';q) = 2\Gamma_q(t) C_{q,\mathrm{inst}}(t)\delta(t-t').$$
$$\dot{C}_q = -2\Gamma_q(t)(C_q - C_{q,\mathrm{inst}}), \qquad \partial_t \delta C_q = -2\Gamma_q \delta C_q - \partial_t C_{q,\mathrm{inst}}.$$



Schematic time-scale comparison

Schematic crossover, not a sharp boundary.

$$\tau_{\mathrm{cov,q}} = \frac{1}{2\Gamma_q}, \qquad \tau_{\mathrm{bg}} = \left|\partial_t \ln C_{q,\mathrm{inst}}\right|^{-1}.$$

Their ratio is a schematic diagnostic of tracking the moving target.

Moving-target lag is distinct from a nonlocal self-energy.

A nonlocal kernel can change relaxation even in a stationary background.

Stephanov & Yin, PRD 98, 036006 (2018); Akamatsu et al., PRC 95, 014909

Back to S16: Gaussian balance Back to S19: nonlocal memory