Critical fluctuation evolution with *second-order baryon diffusion* coupled to a *chiral condensate*

[Linearized *n-σ-v* system]

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Mainly based on **Sakai:2025xmo** (A. Sakai, K. Murase, H. Fujii, T. Hirano, Phys. Rev. C **112** (2025) 2, 024907 [2502.17791 [nucl-th]]).

CEP and EFT 2026, C3NT, CCNU, 2026-10-22

This study

SPDE for **Linearized** (n, σ, v) system

(Stochastic PDE)

Dynamical degrees (state variables)

- $n(x)$: baryon density *fluctuation*

$$n := \sum_f \tfrac{1}{3}[\langle \bar{q}_f \gamma^0 q_f \rangle_J - \langle \bar{q}_f \gamma^0 q_f \rangle_0]$$

- $v(x)$: baryon diffusion current
- $\sigma(x)$: chiral condensate *fluctuation*

$$\sigma := \sum_f [\langle \bar{q}_f q_f \rangle_J - \langle \bar{q}_f q_f \rangle_0]$$

Defined to be deviations from the equilibrium values *in this talk*

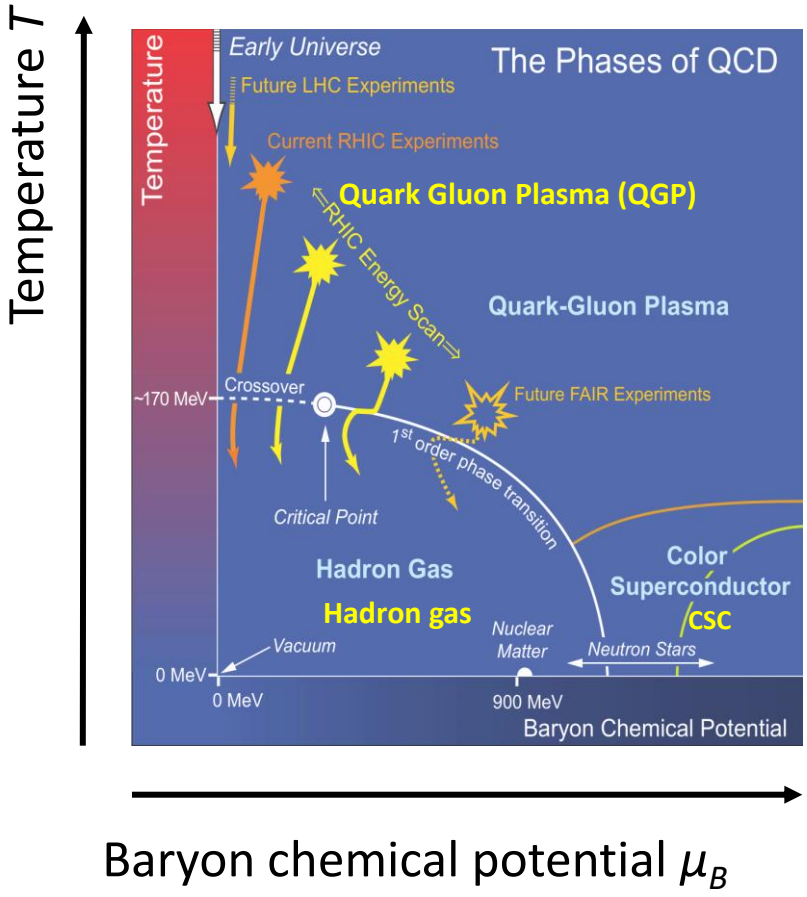
Non-linear couplings (such as n^2, $n\sigma$, σ^2, n^3, etc. in constitutive equations) are neglected.

- No UV issues, no renormalization, etc.
- Only 2pt correlations

Motivation

Critical-point search in HIC

From the 2007 NSAC Long Range Plan



Dynamical modeling of HIC

for high-baryon density

Initialization models

Pre-equilibrium dynamics

Dynamical initialization

Relativistic hydrodynamics

EoS modeling

Critical fluctuations

Phase-transition dynamics

Dynamical integration

Microscopic transport

Long-range dynamics near critical point

Dynamical universality class [Hohenberg and Halperin (1977)]

- QCD critical point

  Model H: Long-range dynamics is governed conserved charge density n + momentum $\boldsymbol{\pi}$

  M. Stephanov and Y. Yin (Hydro+) (2018-), Chattopadhyay, Ott, Schaefer, and Skokov (2024), etc.

- Simplified models

  Model A: Order-parameter field σ (with linear σ etc.)

  Nahrgang et al (2011-), Mukherjee, Venugopalan, Y. Yin (2015), S. Wu, H. Song et al (2019-), L. Jiang, F. Gao, and Y.-x. Liu (2023), etc.

  Model B: Conserved charge density n

  Sakaida, Asakawa, Fujii, Kitazawa (2017), Nahrgang, Bluhm, et al (2017-), S. Wu and H. Song (2019), Pihan, Bluhm, Kitazawa, Sami, Nahrgang (2023), etc.

Scale separation in HIC

Long-range behavior (Model B)

- n: slow mode (hydrodynamic mode)
- σ, v: faster mode (non-hydrodynamic modes)

  irrelevant in global equilibrium (long range/time)

Heavy-ion collisions (Finite size)

Scale separation is non-trivial

- Relaxation time $\tau_R \sim 1/T \sim 1$ fm
- Lifetime and size of the system $\sim$ 1-10 fm

v is already solved within dynamical models for stability and causality (MIS-type 2nd order hydro)

→ coupled dynamics of (n, v, σ)

n-σ-*ν* system

n-σ system

Son and Stephanov (2004), Fujii and Ohtani (2004)

$$\frac{\partial}{\partial t}\begin{pmatrix} n \\ \sigma \end{pmatrix} = -\begin{pmatrix} -\lambda\nabla^2 & -\tilde{\lambda}\nabla^2 \\ -\tilde{\lambda}\nabla^2 & \Gamma \end{pmatrix}\begin{pmatrix} \frac{\delta F}{\delta n} \\ \frac{\delta F}{\delta \sigma} \end{pmatrix} + \begin{pmatrix} \xi_n \\ \xi_\sigma \end{pmatrix}$$

Kinetic coefficients (Onsager coefficients) — Force — Noise

Ginzburg-Landau functional (effective action)

$$F[n,\sigma] = \int d^3\boldsymbol{x}\left[\frac{a}{2}(\partial_i\sigma)^2 + b\partial_i\sigma\partial_i n + \frac{c}{2}(\partial_i n)^2 + V(n,\sigma)\right],$$

$$V(n,\sigma) = \frac{A}{2}\sigma^2 + B\sigma n + \frac{C}{2}n^2$$

Fluctuation-dissipation relation (FDR)

$$\langle\xi_n(x)\xi_n(x')\rangle = 2T(-\lambda\nabla^2)\delta^4(x-x')$$
$$\langle\xi_n(x)\xi_\sigma(x')\rangle = 2T(-\tilde{\lambda}\nabla^2)\delta^4(x-x')$$
$$\langle\xi_\sigma(x)\xi_\sigma(x')\rangle = 2T\Gamma\delta^4(x-x')\ .$$

n-σ system

Son and Stephanov (2004), Fujii and Ohtani (2004)

Effective potential (lowest order = no nonlinear coupling)

$$V(n,\sigma) = \frac{A}{2}\sigma^2 + B\sigma n + \frac{C}{2}n^2$$

Prob ~ exp[-βF]

susceptibilities

$$\chi_{nn} = \frac{TA}{\Delta}, \quad \chi_{n\sigma} = -\frac{TB}{\Delta}, \quad \chi_{\sigma\sigma} = \frac{TC}{\Delta} \qquad \Delta = AC - B^2$$

Approximation up to the leading order in ∇

$$\frac{\partial n}{\partial t} = \nabla^2[(\lambda C + \tilde{\lambda} B)n + (\lambda B + \tilde{\lambda} A)\sigma] + \xi_n,$$
$$\frac{\partial \sigma}{\partial t} = -\Gamma B n - \Gamma A \sigma + \xi_\sigma.$$

Note: this breaks Onsager's reciprocal relations

n-σ-*v* system

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Extension to include the relaxation of diffusion current

$$\frac{\partial n}{\partial t} = -\nabla \cdot \boldsymbol{\nu} \ ,$$

$$\frac{\partial \sigma}{\partial t} = -\Gamma B n - \Gamma A \sigma + \xi_\sigma \ ,$$

$$\tau_{\rm R} \frac{\partial \boldsymbol{\nu}}{\partial t} + \boldsymbol{\nu} = -\nabla \left[(\tilde{\lambda} B + \lambda C) n + (\tilde{\lambda} A + \lambda B) \sigma \right] + \boldsymbol{\xi}_\nu \ ,$$

New term. $\tau_R = 0$ reproduces the *n*-σ system

FDR (essentially the same as *n*-σ, rewritten for *v*)

$$\langle \xi_\nu^i(x) \xi_\nu^j(x') \rangle = -\Delta^{ij} 2T\lambda \delta^4(x - x') \ ,$$

$$\langle \xi_\nu^i(x) \xi_\sigma(x') \rangle = -\Delta^{ij} 2T\tilde{\lambda} \nabla_j \delta^4(x - x') \ ,$$

$$\langle \xi_\sigma(x) \xi_\sigma(x') \rangle = 2T\Gamma \delta^4(x - x') \ ,$$

n-σ-v system Sakai:2025xmo

Dispersion relations

At small k

$$\omega_\sigma = -\frac{i}{\tau_\sigma}\,, \qquad \omega_{\mathrm{R}} = -\frac{i}{\tau_{\mathrm{R}}}\,, \qquad \omega_D = -iD\boldsymbol{k}^2$$

relaxation **New, relaxation** **diffusive**

$\tau_\sigma := 1/\Gamma A$ Not present in n-σ $D := \lambda\Delta/A$

At large k

$$\omega^2 = \frac{\tilde{\lambda}B + \lambda C}{\tau_{\mathrm{R}}}\boldsymbol{k}^2 =: v^2\boldsymbol{k}^2 \qquad \omega = -i\frac{\lambda\Gamma\Delta}{\tilde{\lambda}B + \lambda C}\,.$$

2 propagating **relaxation**

n-ν system and *n* system

***n-ν* system:** $\tau_\sigma = 1/\Gamma A \to 0$

$$\sigma = -(B/A)n$$

$$\frac{\partial n}{\partial t} = -\nabla \cdot \boldsymbol{\nu} \ ,$$

$$\tau_{\rm R}\frac{\partial \boldsymbol{\nu}}{\partial t} + \boldsymbol{\nu} = -D\nabla n + \boldsymbol{\xi}_\nu$$

Relaxation mode ω_σ (small k) and ω (large k) vanishes.

Reduces to KM (2013, 2019), Kapusta and Young (2014), Navid Abbasi, Xin An, Shanjin Wu (2026) (talks on Tuesday), etc.

***n* system:** $\tau_\sigma = 1/\Gamma A \to 0$ & $\tau_{\rm R} \to 0$

$$\frac{\partial n}{\partial t} = D\nabla^2 n + \xi_n$$

M. Sakaida, M. Asakawa, H. Fujii, M. Kitazawa (2017), Nahrgang, Bluhm, et al (2017-)

n-σ, *n-ν*, *n* systems are special cases of *n-σ-ν*

System	σ relaxation	ν relaxation
n–σ–ν	Finite τ_σ	Finite $\tau_{\rm R}$
n–ν	$\tau_\sigma \to 0$	Finite $\tau_{\rm R}$
n–σ	Finite τ_σ	$\tau_{\rm R} \to 0$
n	$\tau_\sigma \to 0$	$\tau_{\rm R} \to 0$

1+1 one-dimensionally expanding model

One-dimensionally expanding system

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1+1 evolution on top of 0+1 Bjorken background

After appropriate transformation to the (1+1) Milne coordinates,

$$(\tilde{n}, \nu)^{\mathrm{T}} := \sqrt{-g}(n, \nu^{\eta_\mathrm{s}})^{\mathrm{T}} \qquad \xi_\nu := \tau \xi_\nu^{\eta_\mathrm{s}}$$

EoM

$$\frac{\partial \tilde{n}}{\partial \tau} = -\frac{\partial \nu}{\partial \eta_\mathrm{s}} ,$$
$$\frac{\partial \sigma}{\partial \tau} = -\Gamma A \sigma - \Gamma B \frac{\tilde{n}}{\tau} + \xi_\sigma ,$$
$$\tau_\mathrm{R} \frac{\partial \nu}{\partial \tau} + \nu = -\frac{1}{\tau} \frac{\partial}{\partial \eta_\mathrm{s}} \left[(\tilde{\lambda} A + \lambda B)\sigma + (\tilde{\lambda} B + \lambda C) \frac{\tilde{n}}{\tau} \right] + \xi_\nu ,$$

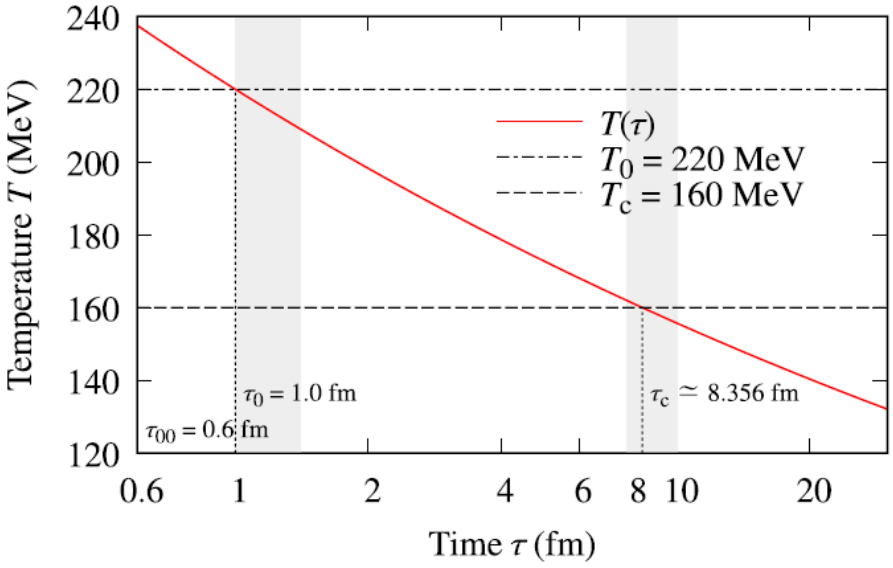
FDR (with Gaussian cutoff *w*)

$$\langle \xi_\sigma(\tau, \eta_\mathrm{s}) \xi_\sigma(\tau', \eta_\mathrm{s}') \rangle = 2T\Gamma \delta(\tau - \tau') \delta_w(\eta_\mathrm{s} - \eta_\mathrm{s}') ,$$
$$\langle \xi_\nu(\tau, \eta_\mathrm{s}) \xi_\nu(\tau', \eta_\mathrm{s}') \rangle = 2T\lambda \delta(\tau - \tau') \delta_w(\eta_\mathrm{s} - \eta_\mathrm{s}') ,$$
$$\langle \xi_\sigma(\tau, \eta_\mathrm{s}) \xi_\nu(\tau', \eta_\mathrm{s}') \rangle = 2T\tilde{\lambda} \frac{1}{\tau} \frac{\partial}{\partial \eta_\mathrm{s}} \delta(\tau - \tau') \delta_w(\eta_\mathrm{s} - \eta_\mathrm{s}') ,$$

δ_w: *mollified* delta function = Gaussian of width ($\sqrt{2} w \tau S_\perp$), w = 0.02

Coefficients in EoMs

- Relaxation time τ_R is varied among 0, 2, 4 fm (constant)
- Kinetic and potential parameters (λ,λ~,Γ,A,B,C) are given as functions of temperature T
- Temperature T is specified as a function of time τ



Critical temperature T_c is assumed at 160 MeV

Potential parameters A, B, C

$$V(\sigma, n) = \frac{A}{2}\sigma^2 + B\sigma n + \frac{C}{2}n^2$$

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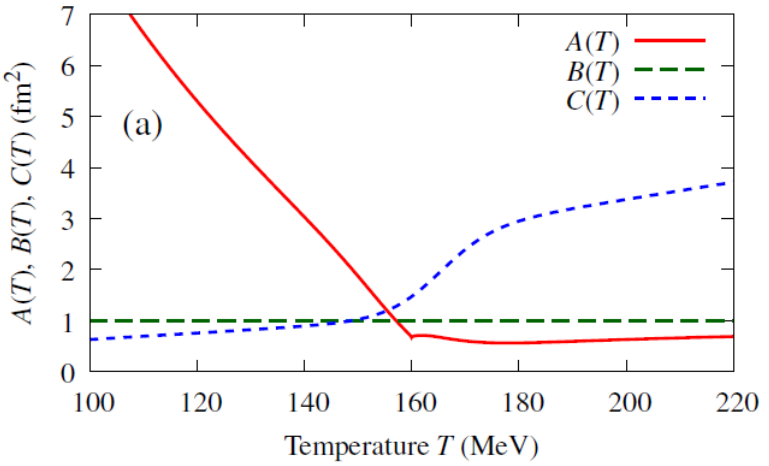
We determine A, B, C by the following conditions:

$$\left.\begin{aligned} \frac{TA(T)}{\Delta} &= \chi^{tot}(T) \\ \frac{T}{C(T)} &= \chi^{\mathrm{reg}}(T) \end{aligned}\right\}$$ Constraints by susceptibility

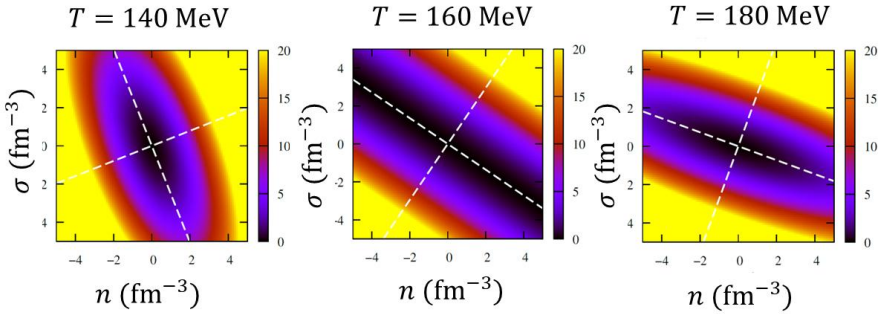
$$B(T) = 1$$ for simplicity

χ^{reg} and χ^{tot} are given by the mapping to the Ising model as described in Sakaida et al (2017).

Resulting (A, B, C)



Potential landscape



Potential parameters A, B, C

$$V(\sigma, n) = \frac{A}{2}\sigma^2 + B\sigma n + \frac{C}{2}n^2$$

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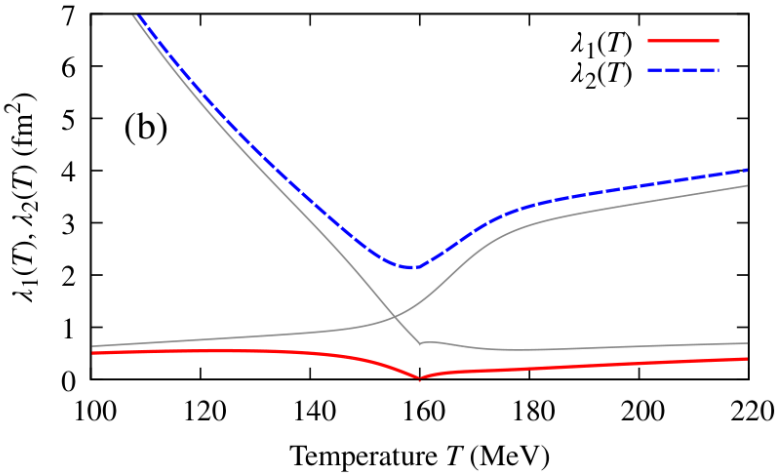
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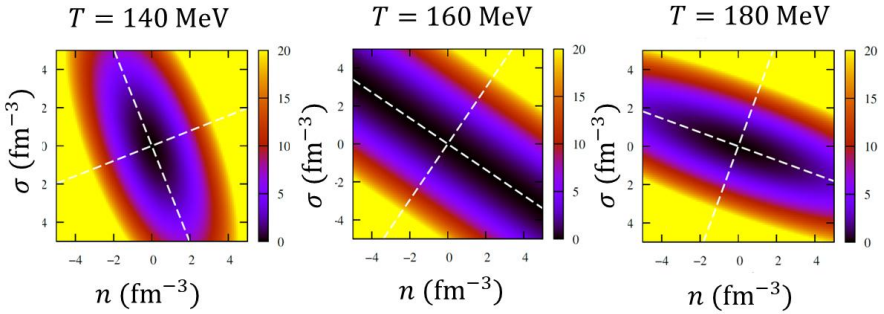
$$B(T) = 1$$ for simplicity

χ^{reg} and χ^{tot} are given by the mapping to the Ising model as described in Sakaida et al (2017).

Mode masses (eigenval of Hessian)



Potential landscape



Transport coefficient $\Gamma, \tilde{\lambda}, \lambda$

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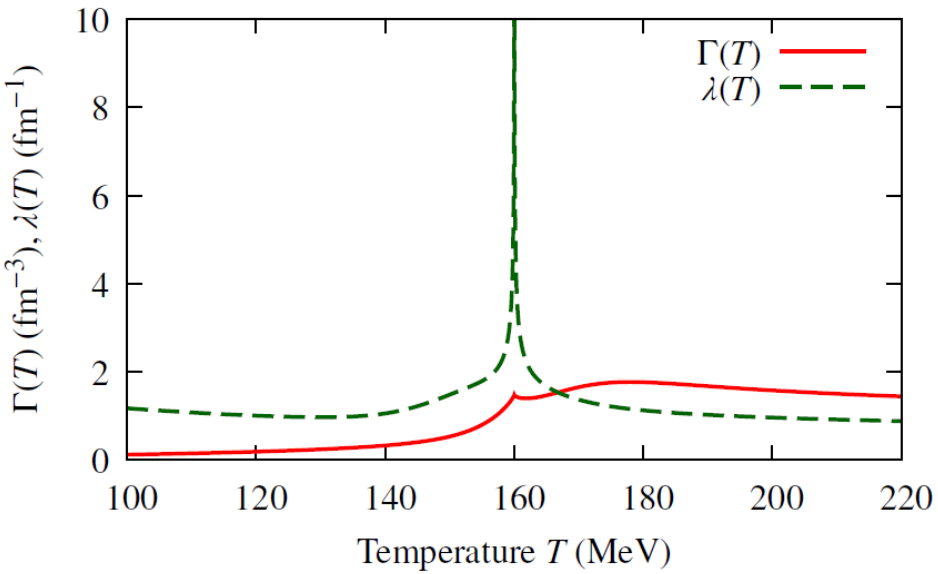
Determine λ to match the diffusion coefficient (in the long range)

$$\frac{\lambda(T)\,\Delta}{A(T)} = D(T)$$ Diffusion coefficient

$$\left.\begin{aligned}\Gamma(T)A(T) &= 1\ \text{fm}^{-1}\\ \tilde{\lambda}(T) &= 0\end{aligned}\right\}$$ For simplicity

D(T) is determined as described in Sakaida el al. (2017). E.g. Critical contribution is defined as

$$D^{\mathrm{cr}}(T) = d_c \left[\frac{\chi^{\mathrm{cr}}(T)}{\chi^{\mathrm{H}}}\right]^{-1+\frac{\chi\lambda}{2-\eta}}$$



Note: The present model is linearized model

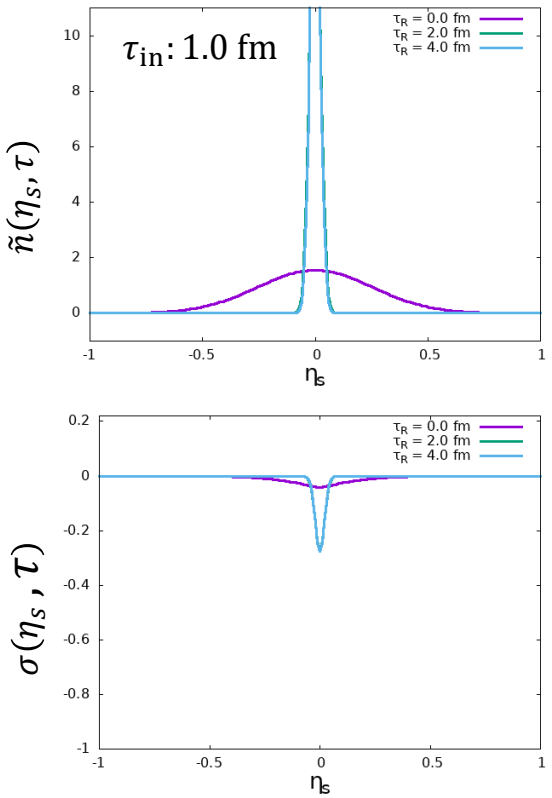
→ No renormalization. With the present matching, divergence at infinite V is directly reflected to the bare coupling at the 1/T scale.

Non-linear coupling with finite coupling should be considered in the future.

Green’s functions

Green's function of n and σ above T_c

This is a linearized system → Green's function contains the full information on the time evolution. Green's function = response to an impulse (the delta function perturbation)



Initial condition (1 example)

Impulse in charge density *n* at (τ=1 fm, η=0)
of (Gaussian of width $w\tau S_\perp$)

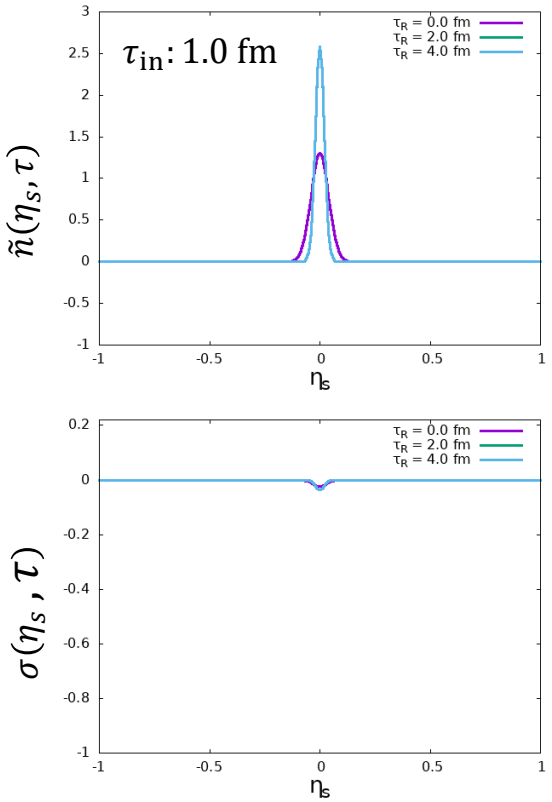
→ Impulse at early time. Spread before reaching the critical temperature

- $\sigma \neq 0$ at $\tau > \tau_0$
  →negative σ is induced by the positive n
- $\tau_R = 0$ fm: $\tilde{n}$ instantly diffuses to large η_s
- $\tau_R > 0$ fm: $\tilde{n}$ diffuses at a finite speed

✔ Two peaks move to the $\pm\eta_s$ directions in a finite speed v with finite relaxation time τ_R

Green's function of n and σ above T_c

This is a linearized system → Green's function contains the full information on the time evolution. Green's function = response to an impulse (the delta function perturbation)



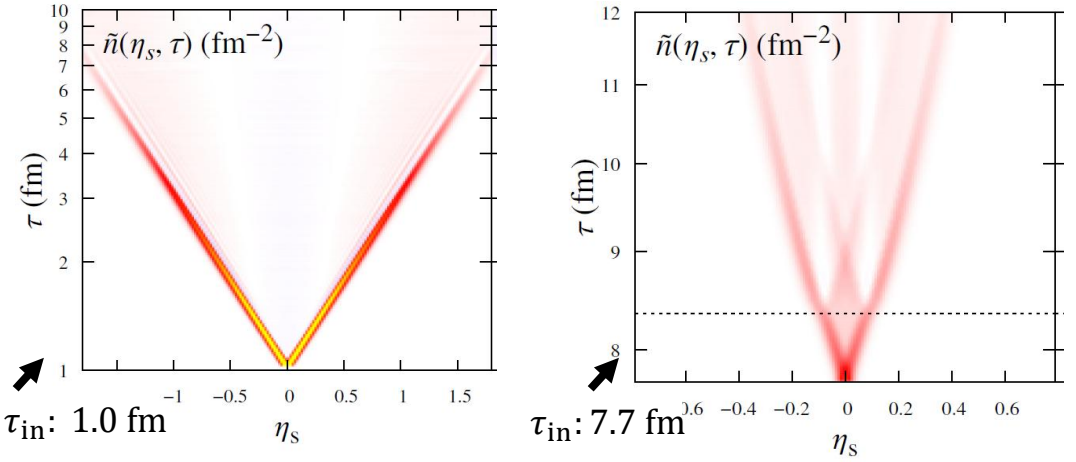
Initial condition

Impulse in charge density n at (τ=7.7 fm, η=0)

of (Gaussian of width $w\tau S_\perp$)

→ Impulse just before critical temperature.

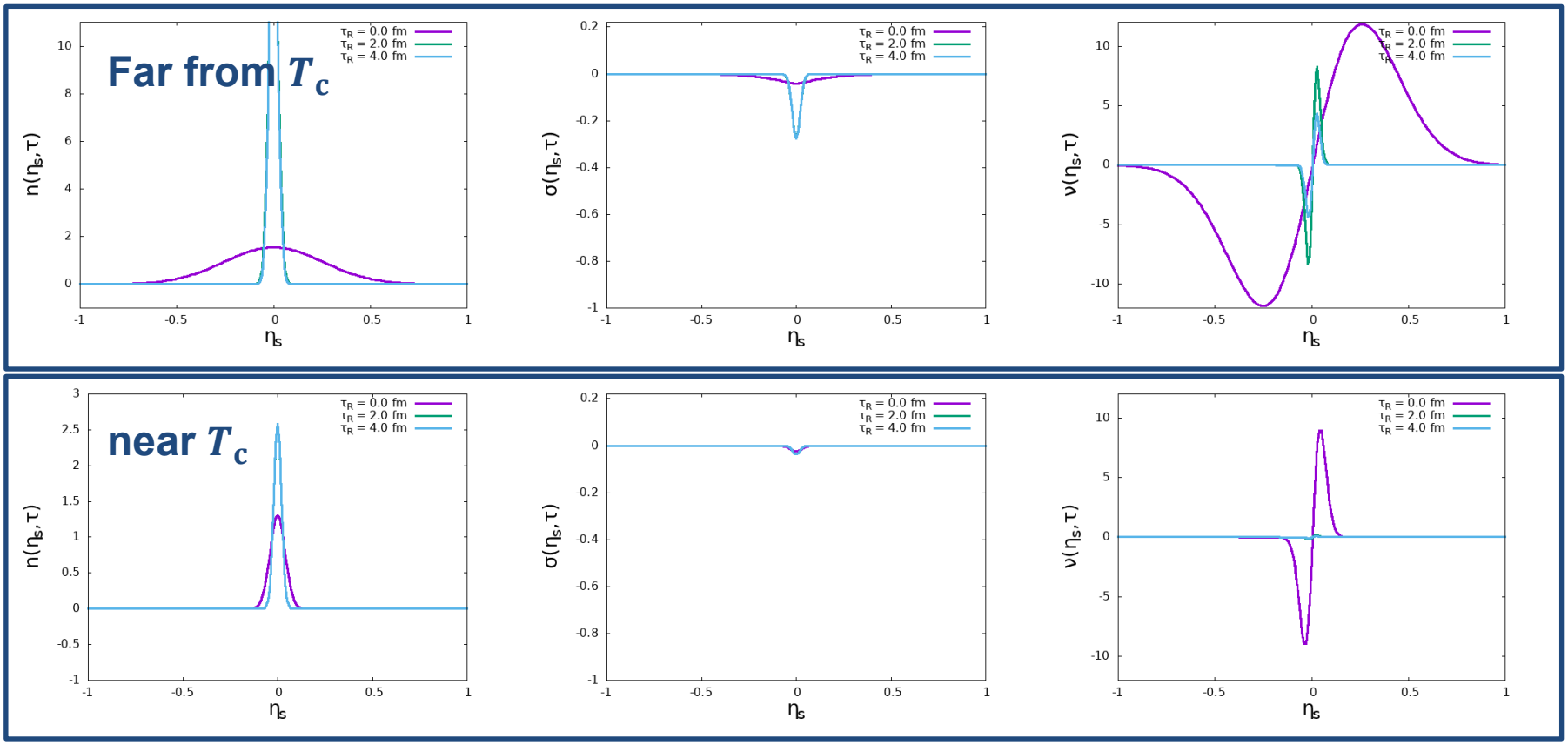
Comparison of different impulse times Sakai:2025xmo



✔ Each peak splits into two peaks at T_c and propagates to $\pm\eta_s$ caused by the divergence of the kinetic coefficient λ

Green's function for n impulse (n, σ, and v)

Purple = *n*-σ ($\tau_R = 0$), Green/Blue = *n*-σ-*v* ($\tau_R \neq 0$)



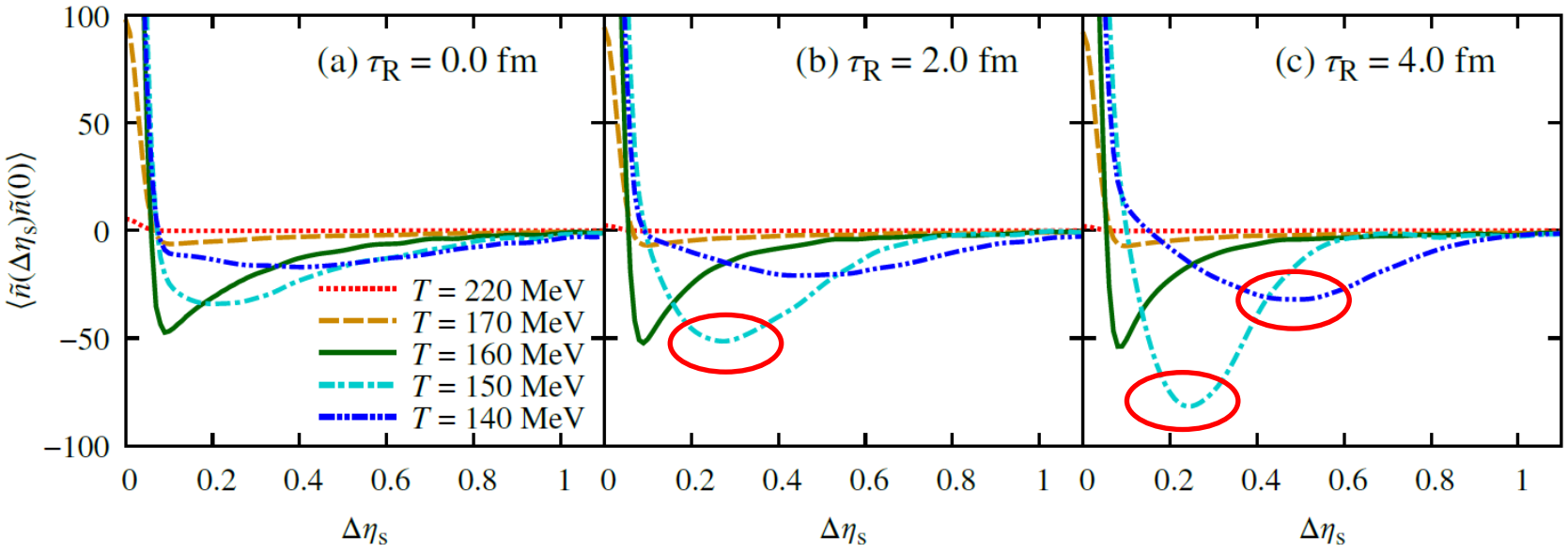
- ✔ Significant difference in behavior with dynamical diffusion current *v*
- ✔ In the *n*-σ-*v* system, critical point induces spatial structures

Two-point correlations

Two-point correlation functions

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$\langle n(\Delta\eta_s)n(0)\rangle$

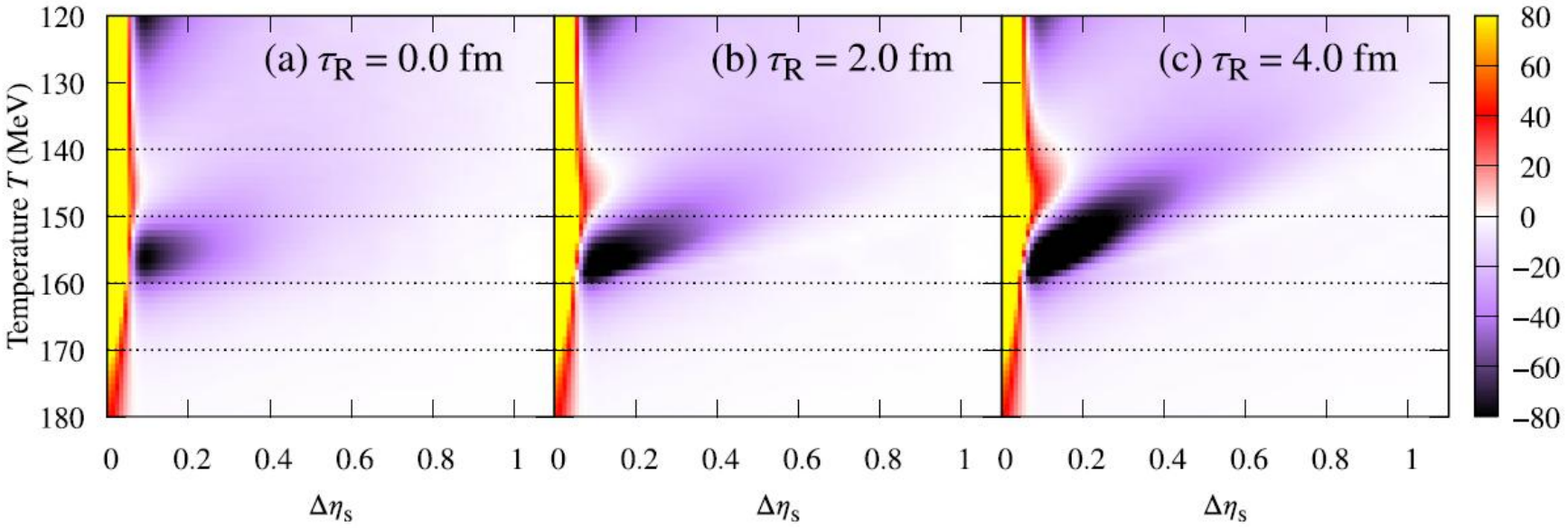


- Maximum at $\Delta\eta_s = 0$
- Negative dip
  - Integration of $\tilde{n}$ is conserved
  - More significant below the T_c with larger τ_R

Two-point correlation functions

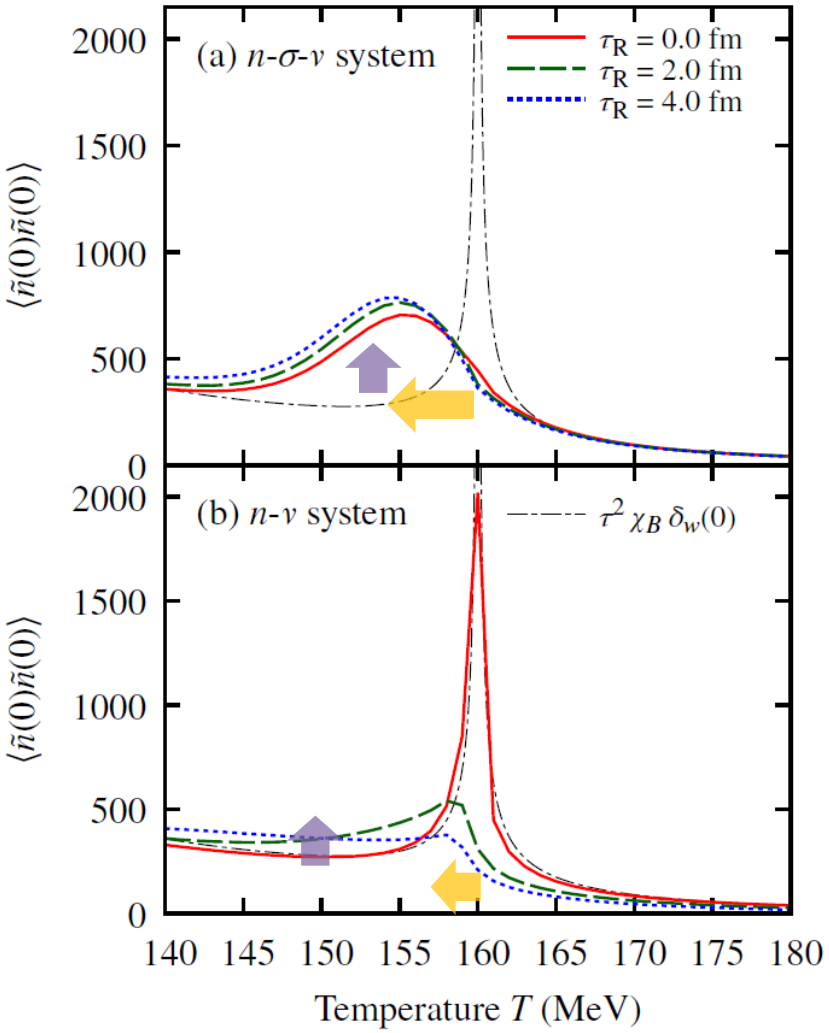
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$\langle n(\Delta\eta_s)n(0)\rangle$



- Maximum at $\Delta\eta_s = 0$
- Negative dip
  - Integration of $\tilde{n}$ is conserved
  - More significant below the T_c with larger τ_R

Evolution of fluctuations $\langle n(0)^2 \rangle$



Comparison of different systems

- **Upper red: the n-σ system**
- **Upper others: the n-σ-v system**
- **Lower red: the n system**
- **Lower others: the n-v system**
- **Gray dot-dash:** Equilibrium value

$$\langle \tilde{n}(0)\tilde{n}(0) \rangle_{\mathrm{eq}} = \tau^2 \langle n(0)n(0) \rangle_{\mathrm{eq}} = \tau^2 \chi^{\mathrm{tot}} \delta_w(0)$$

χ^{tot}: Susceptibility
$\delta_w(\Delta\eta_s)$: Smeared delta

Peaks shift to a lower temperature by τ_R
$\langle \tilde{n}(0)\tilde{n}(0) \rangle$ is larger with τ_R at low temperature

Enhancement remains at lower temperature

Summary

SPDE for **Linearized** *n*-*σ*-*v* system

Dynamical degrees (state variables)

- $n(x)$: baryon density *fluctuation*
- $v(x)$: baryon diffusion current
- $\sigma(x)$: chiral condensate *fluctuation*

Generalization of

- *n*-*σ* system Son Stephanove, Fujii Ohtani…
- *n*-*v system*
  2nd-order diffusion / Maxwell-Cattaneo
- *n system* Model B

Modeling 1+1 evolution on top of 0+1 Bjorken BG

- Potential parameters are determined to reproduce suscep. χ^{cr} *and* χ^{reg}
- Transport coefficients are determined to reproduce diffusion const *D(T)*

spatial structures

Significant difference in qualitative behavior by τ_R

- τ_R induces propagating wave structures in the dynamics
- τ_R makes the critical effect remain in later times