Global equilibrium at $\mathcal{O}(\hbar^2)$: pseudo-gauge ambiguity, Maxwell relations, and thermodynamic consistency

Asaad Daher, David Wagner, Masoud Shokri, Dirk H. Rischke

thanks to

Victor Ambrus, Francesco Becattini, Ashutosh Dash, Xin-li Sheng

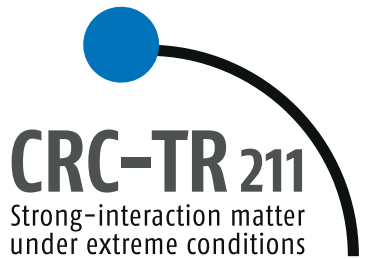
based on

[arXiv:2609.31819[nucl-th]]

"QCD Critical End Point and Effective Field Theory methods"
C^{3}NT, Wuhan, China, October 22, 2026





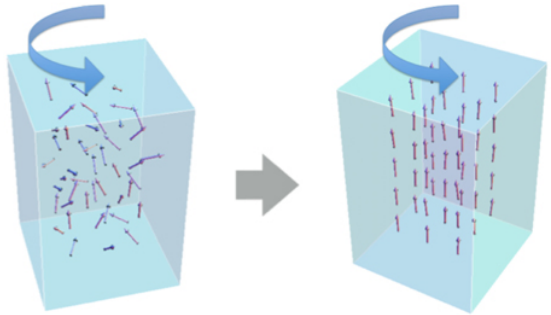


Rotation leads to polarization

Condensed matter: **Barnett effect**

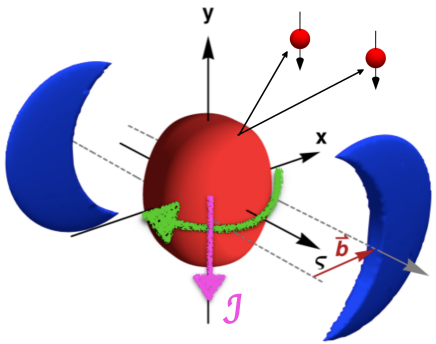
$\Longrightarrow$ spins align under rotation
(due to spin-orbit coupling)

$\Longrightarrow$ non-vanishing magnetization



Non-central heavy-ion collisions:
strong orbital angular momentum

Barnett effect in heavy-ion collisions?



$\Longrightarrow$ hadrons are polarized by vorticity of matter

Z.-T. Liang, X.-N. Wang, PRL 94 (2005) 102301

PRL **94,** 102301 (2005) PHYSICAL REVIEW LETTERS week ending 18 MARCH 2005

Globally Polarized Quark-Gluon Plasma in Noncentral $A + A$ Collisions

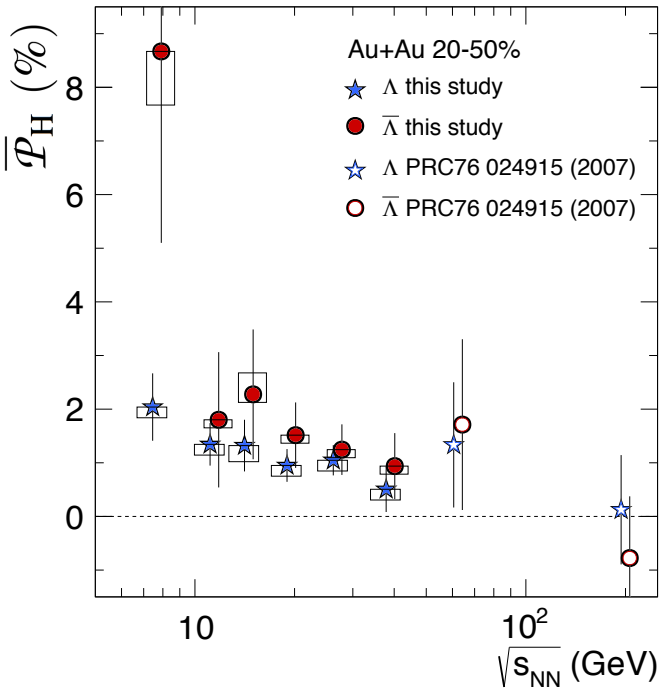
Zuo-Tang Liang[1] and Xin-Nian Wang[2,1]
[1]*Department of Physics, Shandong University, Jinan, Shandong 250100, China*
[2]*Nuclear Science Division, MS 70R0319, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*
(Received 25 October 2004; published 14 March 2005)





Experimental observation - Λ polarization

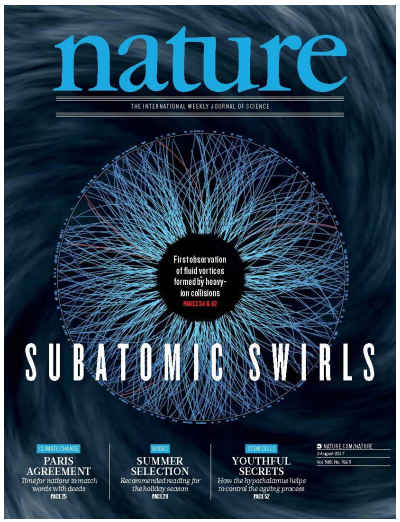
Λ polarization along angular-momentum direction ("global polarization")



L. Adamczyk et al. (STAR), Nature 548 (2017) 62

$\Longrightarrow$ QGP is "most vortical fluid ever observed"

$\omega \simeq (9+1) \times 10^{21} \mathrm{s}^{-1}$



For comparison:

- Great Red Spot of Jupiter $\omega \simeq 10^{-4}\,\mathrm{s}^{-1}$
- turbulent flow in superfluid He-II $\omega \sim 150\,\mathrm{s}^{-1}$
- superfluid nanodroplets $\omega \sim 10^{7}\,\mathrm{s}^{-1}$

Assuming local equilibrium on freeze-out hypersurface $\Sigma_{\text{f.o.}}$, a formula inspired by hydrodynamics describes global polarization and local longitudinal polarization (when neglecting temperature gradients on $\Sigma_{\text{f.o.}}$) quite well

$$\Pi^\mu \sim \epsilon^{\mu\nu\rho\sigma} k_\nu \int_{\Sigma_{\text{f.o.}}} \mathrm{d}\Sigma \cdot k \left(\varpi_{\rho\sigma} + 2\hat{t}_\rho \xi_{\sigma\lambda} \frac{k^\lambda}{k_0} \right) f_{0\mathbf{k}}$$

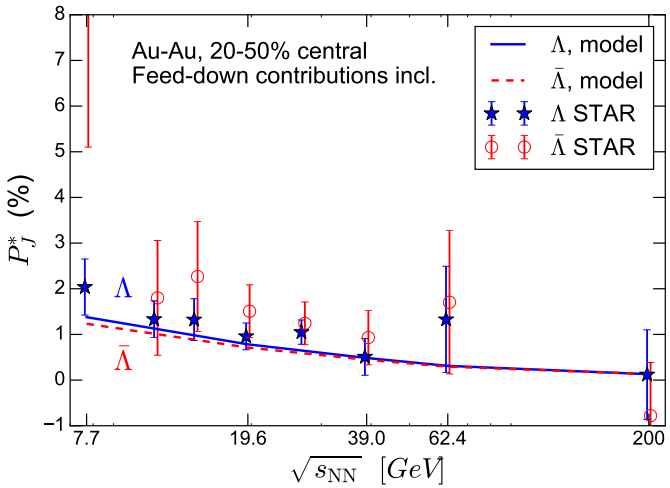
where $\varpi_{\rho\sigma} \equiv -\frac{1}{2}\partial_{[\rho}\beta_{\sigma]}$ thermal vorticity

$\xi_{\sigma\lambda} \equiv \frac{1}{2}\partial_{(\sigma}\beta_{\lambda)}$ thermal shear tensor

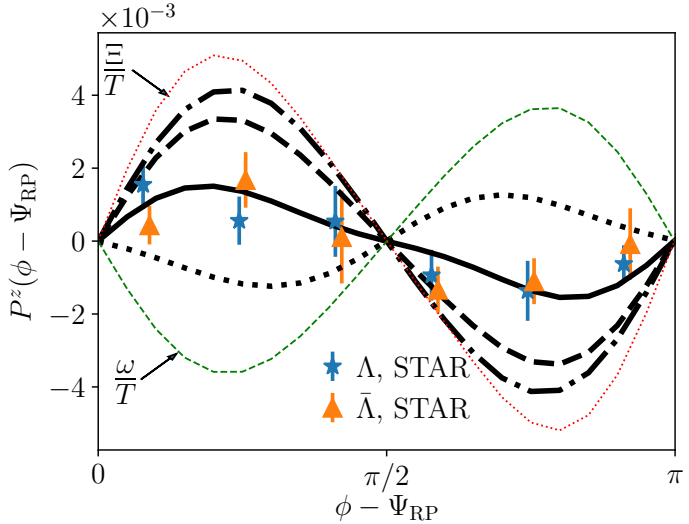
$\beta^\mu \equiv u^\mu / T$ thermal 4-velocity, $f_{0\mathbf{k}}$ local-equilibrium distribution function

$a_{[\mu} b_{\nu]} \equiv a_\mu b_\nu - a_\nu b_\mu$

$a_{(\mu} b_{\nu)} \equiv a_\mu b_\nu + a_\nu b_\mu$



I. Karpenko, F. Becattini, NPA 967 (2017) 764



F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, A. Palermo, PRL 127 (2021) 272302

(see also S.Y.F. Liu, Y. Yin, JHEP 07 (2021) 188)

$\Longrightarrow$ Second-order dissipative spin hydrodynamics

N. Weickgenannt, D. Wagner, E. Speranza, DHR, PRD 106 (2022) 096014, PRD 106 (2022) L091901

D. Wagner, PRD 111 (2025) 016008

Conservation equations

$$\partial_\mu J^\mu = 0$$
$$\partial_\mu T^{\mu\nu} = 0$$
$$\hbar\,\partial_\mu S^{\mu,\nu\lambda} = T^{[\lambda\nu]}$$

Relaxation equations for dissipative quantities

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\,\theta + \dots$$
$$\tau_n \Delta^{\mu\nu}\dot{n}_\nu + n^\mu = \varkappa\,\nabla^\mu\alpha + \dots$$
$$\tau_\pi \Delta^{\mu\nu}_{\alpha\beta}\dot{\pi}^{\alpha\beta} + \pi^{\mu\nu} = 2\eta\,\sigma^{\mu\nu} + \dots$$
$$\tau_t \Delta^{\mu\nu}_{\alpha\beta}\dot{t}^{\alpha\beta} + t^{\mu\nu} = \frac{\mathfrak{d}}{T}\sigma^{\mu\nu} + \dots$$

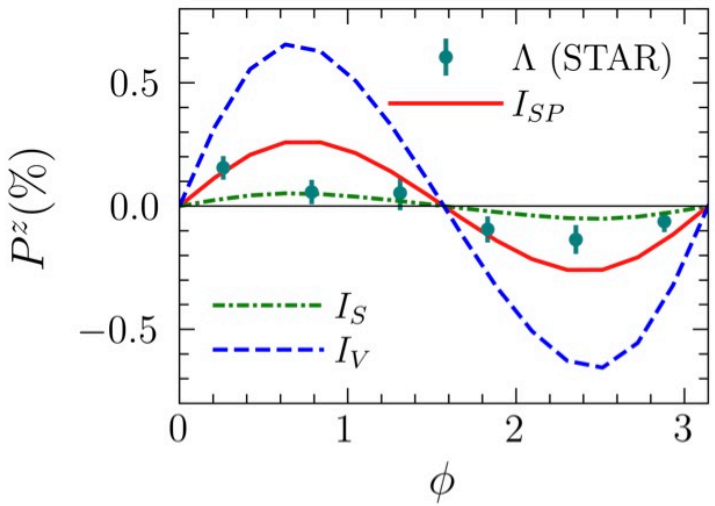
$\dot{A} \equiv u\cdot\partial A$, $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$, $\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}\Delta^{(\mu}{}_\alpha\Delta^{\nu)}{}_\beta - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$

$\Longrightarrow$ dynamical solution of spin hydrodynamics

Sapna, S. K. Singh, D. Wagner,

PRC 112 (2025) 5, 054902

$\Longrightarrow$ spin hydrodynamics can describe local longitudinal polarization!



Solving spin hydrodynamics

Equations of motion (EoM's) of spin hydrodynamics are solved perturbatively in $\hbar$

$\Longrightarrow$ Decompose $T^{\mu\nu} = \frac{1}{2} T^{(\mu\nu)} + \frac{1}{2} T^{[\mu\nu]}$

where: $\frac{1}{2} T^{(\mu\nu)} = \varepsilon\, u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \sim \mathcal{O}(1)$

while: $\hbar\, S^{\mu,\nu\lambda} \sim \mathcal{O}(\hbar^2)$, $T^{[\mu\nu]} = -\frac{\hbar}{2} \int \mathrm{d}\Gamma\, \Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}}\, \mathfrak{C}[f] \sim \mathcal{O}(\hbar^2)$

$\Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}} \equiv -\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} k_\alpha \mathfrak{s}_\beta$ dipole moment tensor, $\mathfrak{C}[f]$ collision term in Boltzmann equation

EoM's to $\mathcal{O}(\hbar)$: $\partial_\mu J^\mu = 0$, $\partial_\mu T^{(\mu\nu)} = 0$, plus relaxation equations for Π, n^μ, $\pi^{\mu\nu}$

Note: effect of spin d.o.f.'s enters equation of state (EoS) at order $\mathcal{O}(\hbar^2)$!

$\Longrightarrow$ To order $\mathcal{O}(\hbar)$, EoM's are identical to EoM's of ordinary dissipative fluid dynamics, spin dynamics does not affect bulk evolution!

Spin dynamics at order $\mathcal{O}(\hbar^2)$: $\hbar\, \partial_\mu S^{\mu,\nu\lambda} = T^{[\lambda\nu]}$, $\tau_{\mathfrak{t}} \Delta^{\mu\nu}_{\alpha\beta} \dot{\mathfrak{t}}^{\alpha\beta} + \mathfrak{t}^{\mu\nu} = \frac{\mathfrak{d}}{T} \sigma^{\mu\nu} + \ldots$

$\Longrightarrow$ solve spin dynamics on background given by ordinary dissipative fluid dynamics!

However: EoM's of spin hydrodynamics are not solved self-consistently to order $\mathcal{O}(\hbar^2)$

$\Longrightarrow$ Need to account for $T^{[\mu\nu]} \sim \mathcal{O}(\hbar^2)$ in $\partial_\mu T^{\mu\nu} = 0$

$\Longrightarrow$ Need to account for $\mathcal{O}(\hbar^2)$ corrections in EoS $\Longrightarrow$ this talk!

Conservation laws

Charge	$\partial_\mu \hat{J}^\mu = 0$
Energy-momentum	$\partial_\mu \hat{T}^{\mu\nu} = 0$
Angular momentum	$\partial_\mu \hat{J}^{\mu,\nu\lambda} = 0$

with: $\hat{J}^{\mu,\nu\lambda} = x^\nu \hat{T}^{\mu\lambda} - x^\lambda \hat{T}^{\mu\nu} + \hbar \hat{S}^{\mu,\nu\lambda}$

$\Longrightarrow \quad \hbar \hat{S}^{\mu,\nu\lambda} = \hat{T}^{[\lambda\nu]}$

Conserved charges

$$\hat{\mathbb{Q}} = \int_\Sigma \mathrm{d}\Sigma_\mu \hat{J}^\mu$$

$$\hat{\mathbb{P}}^\nu = \int_\Sigma \mathrm{d}\Sigma_\mu \hat{T}^{\mu\nu}$$

$$\hat{\mathbb{J}}^{\nu\lambda} = \int_\Sigma \mathrm{d}\Sigma_\mu \hat{J}^{\mu,\nu\lambda}$$

with Σ Cauchy hypersurface

remain invariant under:

Pseudo-gauge transformations

$$\hat{J}'^\mu = \hat{J}^\mu + \hbar\, \partial_\lambda \hat{B}^{\lambda\mu}$$

$$\hat{T}'^{\mu\nu} = \hat{T}^{\mu\nu} + \frac{\hbar}{2}\, \partial_\lambda \left(\hat{\Phi}^{\lambda,\mu\nu} - \hat{\Phi}^{\mu,\lambda\nu} - \hat{\Phi}^{\nu,\lambda\mu} \right)$$

$$\hat{S}'^{\mu,\nu\lambda} = \hat{S}^{\mu,\nu\lambda} - \hat{\Phi}^{\mu,\nu\lambda} + \hbar\, \partial_\rho \hat{Z}^{\nu\lambda,\mu\rho}$$

$\hat{B}^{\lambda\mu} = -\hat{B}^{\mu\lambda}$, $\hat{\Phi}^{\lambda,\mu\nu} = -\hat{\Phi}^{\lambda,\nu\mu}$, $\hat{Z}^{\nu\lambda,\mu\rho} = -\hat{Z}^{\lambda\nu,\mu\rho} = -\hat{Z}^{\nu\lambda,\rho\mu}$

$$\mathcal{Z} = \mathrm{Tr}\exp\left[-\int_\Sigma \mathrm{d}\Sigma_\mu \left(\hat{T}^{\mu\nu}\mathfrak{b}_\nu - \alpha\hat{J}^\mu - \frac{1}{2}\Omega_{\nu\lambda}\hat{J}^{\mu,\nu\lambda}\right)\right]$$

F. Becattini,
PRL 108 (2012) 244502

$\mathfrak{b}_\nu$, α, $\Omega_{\nu\lambda}$ Lagrange multipliers to control mean values of energy-momentum, charge, and angular momentum

(α reduced chemical potential, $\Omega_{\nu\lambda}$ reduced spin potential)

$\Longrightarrow$ introduce thermal 4-velocity $\beta_\nu = \mathfrak{b}_\nu + \Omega_{\nu\lambda}x^\lambda$

$$\mathcal{Z} = \mathrm{Tr}\exp\left[-\int_\Sigma \mathrm{d}\Sigma_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \alpha\hat{J}^\mu - \frac{\hbar}{2}\Omega_{\nu\lambda}\hat{S}^{\mu,\nu\lambda}\right)\right]$$

$\Longrightarrow$ Density operator $\hat{\rho} = \frac{1}{\mathcal{Z}}\exp\left[-\int_\Sigma \mathrm{d}\Sigma_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \alpha\hat{J}^\mu - \frac{\hbar}{2}\Omega_{\nu\lambda}\hat{S}^{\mu,\nu\lambda}\right)\right]$

$\Longrightarrow$ Define thermodynamic-potential 4-current ϕ^μ via $\ln\mathcal{Z} = \int_\Sigma \mathrm{d}\Sigma_\mu\phi^\mu - \text{v.e.v.}$

F. Becattini, D. Rindori, PRD 99 (2019) 125011

v.e.v. $= 0$ for Boltzmann statistics

$$\alpha = \text{const.}\,, \quad \mathfrak{b}_\nu = \text{const.}\,, \quad \Omega_{\nu\lambda} = \text{const.}$$

$\Longrightarrow$ thermal 4-velocity is Killing vector: $\xi_{\nu\lambda} = \frac{1}{2}\partial_{(\nu}\beta_{\lambda)} = 0$

$\Longrightarrow$ thermal vorticity equals reduced spin potential: $\varpi_{\nu\lambda} = -\frac{1}{2}\partial_{[\nu}\beta_{\lambda]} = \Omega_{\nu\lambda} = \text{const.}$

$\Longrightarrow$ define fluid 4-velocity u_ν via $\beta_\nu \equiv \beta u_\nu \quad \Longrightarrow \quad u \cdot u = 1$

$\Longrightarrow$ inverse temperature: $\beta \equiv \sqrt{\beta \cdot \beta} \equiv 1/T$

$\Longrightarrow$

$$\mathcal{Z} = \mathrm{Tr}\exp\left(-\mathfrak{b}_\nu \hat{\mathbb{P}}^\nu + \alpha\hat{\mathbb{Q}} + \frac{1}{2}\,\Omega_{\nu\lambda}\hat{\mathbb{J}}^{\nu\lambda}\right) \qquad \hat{\rho} = \frac{1}{\mathcal{Z}}\,\exp\left(-\mathfrak{b}_\nu \hat{\mathbb{P}}^\nu + \alpha\hat{\mathbb{Q}} + \frac{1}{2}\,\Omega_{\nu\lambda}\hat{\mathbb{J}}^{\nu\lambda}\right)$$

$$\mathbb{Q} = \mathrm{Tr}\left(\hat{\rho}\,\hat{\mathbb{Q}}\right) = \left.\frac{\partial \ln \mathcal{Z}}{\partial \alpha}\right|_{\mathfrak{b}_\rho, \Omega_{\sigma\tau}} = \int_\Sigma \mathrm{d}\Sigma_\mu \left.\frac{\partial \phi^\mu}{\partial \alpha}\right|_{\mathfrak{b}_\rho, \Omega_{\sigma\tau}}$$

$$\mathbb{P}^\nu = \mathrm{Tr}\left(\hat{\rho}\,\hat{\mathbb{P}}^\nu\right) = -\left.\frac{\partial \ln \mathcal{Z}}{\partial \mathfrak{b}_\nu}\right|_{\alpha, \Omega_{\sigma\tau}} = -\int_\Sigma \mathrm{d}\Sigma_\mu \left.\frac{\partial \phi^\mu}{\partial \mathfrak{b}_\nu}\right|_{\alpha, \Omega_{\sigma\tau}}$$

$$\mathbb{J}^{\nu\lambda} = \mathrm{Tr}\left(\hat{\rho}\,\hat{\mathbb{J}}^{\nu\lambda}\right) = \left.\frac{\partial \ln \mathcal{Z}}{\partial \Omega_{\nu\lambda}}\right|_{\alpha, \mathfrak{b}_\rho} = \int_\Sigma \mathrm{d}\Sigma_\mu \left.\frac{\partial \phi^\mu}{\partial \Omega_{\nu\lambda}}\right|_{\alpha, \mathfrak{b}_\rho}$$

Note: $\mathcal{Z}$ pseudo-gauge invariant!

$\Longrightarrow$ Define local quantities via

$$J^{\mu}(x) \equiv \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\Sigma_{\mu}} = \mathrm{Tr}\left[\hat{\rho}\,\hat{J}^{\mu}(x)\right] = \left.\frac{\partial\phi^{\mu}(x)}{\partial\alpha}\right|_{\mathfrak{b}_{\rho},\Omega_{\sigma\tau}}$$

$$T^{\mu\nu}(x) \equiv \frac{\mathrm{d}\mathbb{P}^{\nu}}{\mathrm{d}\Sigma_{\mu}} = \mathrm{Tr}\left[\hat{\rho}\,\hat{T}^{\mu\nu}(x)\right] = -\left.\frac{\partial\phi^{\mu}(x)}{\partial\mathfrak{b}_{\nu}}\right|_{\alpha,\Omega_{\sigma\tau}} \qquad (\star)$$

$$J^{\mu,\nu\lambda}(x) \equiv \frac{\mathrm{d}\mathbb{J}^{\nu\lambda}}{\mathrm{d}\Sigma_{\mu}} = \mathrm{Tr}\left[\hat{\rho}\,\hat{J}^{\mu,\nu\lambda}(x)\right] = \left.\frac{\partial\phi^{\mu}(x)}{\partial\Omega_{\nu\lambda}}\right|_{\alpha,\mathfrak{b}_{\rho}}$$

Using

$$\left.\frac{\partial}{\partial\alpha}\right|_{\mathfrak{b}_{\rho},\Omega_{\sigma\tau}} = \left.\frac{\partial}{\partial\alpha}\right|_{\beta_{\rho},\Omega_{\sigma\tau}}, \quad \left.\frac{\partial}{\partial\mathfrak{b}_{\nu}}\right|_{\alpha,\Omega_{\sigma\tau}} = \left.\frac{\partial}{\partial\beta_{\nu}}\right|_{\alpha,\Omega_{\sigma\tau}},$$

$$\left.\frac{\partial}{\partial\Omega_{\nu\lambda}}\right|_{\alpha,\mathfrak{b}_{\rho}} = \left.\frac{\partial}{\partial\Omega_{\nu\lambda}}\right|_{\alpha,\beta_{\rho}} - x^{\nu}\left.\frac{\partial}{\partial\mathfrak{b}_{\lambda}}\right|_{\alpha,\Omega_{\sigma\tau}} + x^{\lambda}\left.\frac{\partial}{\partial\mathfrak{b}_{\nu}}\right|_{\alpha,\Omega_{\sigma\tau}}$$

$\Longrightarrow$

$$J^{\mu}(x) = \left.\frac{\partial\phi^{\mu}(x)}{\partial\alpha}\right|_{\beta_{\rho},\Omega_{\sigma\tau}}, \quad T^{\mu\nu}(x) = -\left.\frac{\partial\phi^{\mu}(x)}{\partial\beta_{\nu}}\right|_{\alpha,\Omega_{\sigma\tau}}, \quad \hbar\,S^{\mu,\nu\lambda}(x) = \left.\frac{\partial\phi^{\mu}(x)}{\partial\Omega_{\nu\lambda}}\right|_{\alpha,\beta_{\rho}}$$

$(\star\star)$

Caveat: holds if one neglects contributions from boundary of Σ!

$\Longrightarrow$ Equations $(\star), (\star\star)$ give rise to:

Gibbs-Duhem relation

$$\mathrm{d}\phi^\mu = J^\mu \mathrm{d}\alpha - T^{\mu\nu}\mathrm{d}\mathfrak{b}_\nu + \frac{1}{2} J^{\mu,\nu\lambda}\mathrm{d}\Omega_{\nu\lambda} = J^\mu \mathrm{d}\alpha - T^{\mu\nu}\mathrm{d}\beta_\nu + \frac{\hbar}{2} S^{\mu,\nu\lambda}\mathrm{d}\Omega_{\nu\lambda} \qquad (+)$$

Define entropy 4-current S^μ via: $S = -\mathrm{Tr}\,(\hat{\rho}\,\ln\hat{\rho}) = \int_\Sigma \mathrm{d}\Sigma_\mu\, S^\mu$

$\Longrightarrow$

Euler relation

$$\phi^\mu = S^\mu + J^\mu\alpha - T^{\mu\nu}\mathfrak{b}_\nu + \frac{1}{2} J^{\mu,\nu\lambda}\Omega_{\nu\lambda} = S^\mu + J^\mu\alpha - T^{\mu\nu}\beta_\nu + \frac{\hbar}{2} S^{\mu,\nu\lambda}\Omega_{\nu\lambda} \qquad (\nabla)$$

$\Longrightarrow$

First law of thermodynamics

$$\mathrm{d}S^\mu = -\alpha\,\mathrm{d}J^\mu + \mathfrak{b}_\nu\,\mathrm{d}\,T^{\mu\nu} - \frac{1}{2}\,\Omega_{\nu\lambda}\,\mathrm{d}J^{\mu,\nu\lambda} = -\alpha\,\mathrm{d}J^\mu + \beta_\nu\,\mathrm{d}\,T^{\mu\nu} - \frac{\hbar}{2}\,\Omega_{\nu\lambda}\,\mathrm{d}S^{\mu,\nu\lambda} \qquad (\Delta)$$

In global equilibrium, $\partial_\mu\phi^\mu = 0$, and conservation laws imply $\partial_\mu S^\mu = 0$

$\Longrightarrow$ $\ln\mathcal{Z}$, S independent of hypersurface Σ!

As antisymmetric tensor, reduced spin potential can be decomposed as:

$$\Omega_{\mu\nu} = \beta \left(u^{[\mu} \varkappa^{\nu]} + \epsilon^{\mu\nu\alpha\beta} u_\alpha \omega_\beta \right)$$

$\varkappa^\mu = -T \Omega^{\mu\nu} u_\nu$ acceleration-like 4-vector ("electric"), $\varkappa^2 \equiv -\varkappa \cdot \varkappa$

$\omega^\mu = \frac{T}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \Omega_{\alpha\beta}$ rotation-like 4-vector ("magnetic"), $\omega^2 \equiv -\omega \cdot \omega$

$\Longrightarrow$ tetrad $u^\mu,\ \omega^\mu,\ \ell^\mu \equiv \frac{L^\mu}{\sqrt{-L \cdot L}},\ \xi^\mu \equiv \varkappa^\mu + \frac{\varkappa \cdot \omega}{\omega^2} \omega^\mu$ $\quad L^\mu \equiv \epsilon^{\mu\nu\alpha\beta} u_\nu \varkappa_\alpha \omega_\beta$

$\Longrightarrow$ $\phi^\mu = \phi_u\, u^\mu + \phi_\xi\, \xi^\mu + \phi_\omega\, \omega^\mu + \phi_\times\, \ell^\mu$

One can show Daher, Wagner, Shokri, DHR [arXiv:2609.31819[nucl-th]] that $\phi_\xi = \phi_\omega = 0$

Denoting $\phi_u \equiv \beta\, P$, $\phi_\times \equiv \beta P_\times$ $\Longrightarrow$ $\phi^\mu = \beta\, P\, u^\mu + \beta\, P_\times\, \ell^\mu$ $(\square)$

In homogeneous equilibrium, $\Omega_{\nu\lambda} = \ell^\mu = 0$, $\beta = \text{const.}$, $P(\alpha, \beta) = \text{const.}$

$\Longrightarrow$ $\ln \mathcal{Z} = \int_\Sigma \mathrm{d}\Sigma_\mu\, \phi^\mu = \beta\, P \int_\Sigma \mathrm{d}\Sigma_\mu\, u^\mu \equiv \beta\, P\, V$

$\Longrightarrow$ P is thermodynamic pressure!

In accelerating and rotating equilibrium, however,
$\beta = \beta(x)$, $P = P(x)$, $P_\times = P_\times(x)$, $u^\mu = u^\mu(x)$, $\ell^\mu = \ell^\mu(x)$

Longitudinal and transverse Euler relations

Projecting Euler relation (∇) onto u^μ

$\Longrightarrow$ **Longitudinal Euler relation**

$$\varepsilon + P = T\, s + \mu\, n + \frac{\hbar}{2}\, \omega_{\nu\lambda} S^{\nu\lambda}$$

$\varepsilon \equiv u_\mu T^{\mu\nu} u_\nu$ energy density
$s \equiv u_\mu S^\mu$ entropy density
$n \equiv u_\mu J^\mu$ charge density
$S^{\nu\lambda} \equiv u_\mu S^{\mu,\nu\lambda}$ spin tensor
$\mu = T\alpha$ chemical potential
$\omega_{\nu\lambda} = T\Omega_{\nu\lambda}$ spin potential

Projecting Euler relation (∇) onto ℓ^μ

$\Longrightarrow$ **Transverse Euler relation**

$$q_\times + P = T\, s_\times + \mu\, n_\times + \frac{\hbar}{2}\, \omega_{\nu\lambda} S_\times^{\nu\lambda}$$

$q_\times \equiv -\ell_\mu T^{\mu\nu} u_\nu$
$s_\times \equiv -\ell_\mu S^\mu$
$n_\times \equiv -\ell_\mu J^\mu$
$S_\times^{\nu\lambda} \equiv -\ell_\mu S^{\mu,\nu\lambda}$

cf. also W. Florkowski, M. Hontarenko, PRL 134 (2025) 082302

Projecting Gibbs-Duhem relation $(+)$ onto u^μ

Longitudinal Gibbs-Duhem relation

$$\Longrightarrow \quad \mathrm{d}P = s\,\mathrm{d}T + n\,\mathrm{d}\mu + \frac{\hbar}{2} S^{\nu\lambda}\mathrm{d}\omega_{\nu\lambda} + \left(P_\times \ell^\mu - u_\nu T^{\nu\lambda}\Delta_\lambda{}^\mu\right)\mathrm{d}u_\mu$$

New term! cf. A.D. Gallegos, U. Gursoy, A. Yarom, [arXiv:2203.05044 [hep-th]]

$\Longrightarrow$ fluid 4-velocity becomes independent thermodynamic variable!

$$\Longrightarrow \quad s = \left.\frac{\partial P}{\partial T}\right|_{\mu,\omega_{\sigma\tau},u_\rho}, \quad n = \left.\frac{\partial P}{\partial \mu}\right|_{T,\omega_{\sigma\tau},u_\rho}, \quad \hbar S^{\nu\lambda} = \left.\frac{\partial P}{\partial \omega_{\nu\lambda}}\right|_{T,\mu,u_\rho}$$

Note: T, $u_\rho = \text{const.} \iff \beta_\rho = \text{const.} \iff \varpi_{\sigma\tau} = \text{const.}$!

Projecting Gibbs-Duhem relation $(+)$ onto ℓ^μ

Transverse Gibbs-Duhem relation

$$\Longrightarrow \quad \mathrm{d}P_\times = s_\times\,\mathrm{d}T + n_\times\,\mathrm{d}\mu + \frac{\hbar}{2} S_\times^{\nu\lambda}\mathrm{d}\omega_{\nu\lambda} + \left(P\ell^\mu + \ell_\nu T^{\nu\lambda}\Delta_\lambda{}^\mu\right)\mathrm{d}u_\mu$$

$$\Longrightarrow \quad s_\times = \left.\frac{\partial P_\times}{\partial T}\right|_{\mu,\omega_{\sigma\tau},u_\rho}, \quad n_\times = \left.\frac{\partial P_\times}{\partial \mu}\right|_{T,\omega_{\sigma\tau},u_\rho}, \quad \hbar S_\times^{\nu\lambda} = \left.\frac{\partial P_\times}{\partial \omega_{\nu\lambda}}\right|_{T,\mu,u_\rho}$$

Integral identities for the pressure

With $s = \left.\frac{\partial P}{\partial T}\right|_{\mu,\omega_{\sigma\tau},u_\rho}$, $s_\times = \left.\frac{\partial P_\times}{\partial T}\right|_{\mu,\omega_{\sigma\tau},u_\rho}$ and the longitudinal/transverse Euler relations

$$\left.T^2\,\frac{\partial(P/T)}{\partial T}\right|_{\mu,\omega_{\sigma\tau},u_\rho} = -P + s = \varepsilon - \mu n - \frac{\hbar}{2}\,\omega_{\nu\lambda}S^{\nu\lambda}$$

$$\Longrightarrow \quad \left.T^2\,\frac{\partial(P_\times/T)}{\partial T}\right|_{\mu,\omega_{\sigma\tau},u_\rho} = -P_\times + s_\times = q_\times - \mu n_\times - \frac{\hbar}{2}\,\omega_{\nu\lambda}S_\times^{\nu\lambda}$$

$$\Longrightarrow \quad P = T\int_0^T \frac{\mathrm{d}T'}{T'^2}\left(\varepsilon - \mu n - \frac{\hbar}{2}\,\omega_{\nu\lambda}S^{\nu\lambda}\right)_{\mu,\omega_{\sigma\tau},u_\rho}$$

$$P_\times = T\int_0^T \frac{\mathrm{d}T'}{T'^2}\left(q_\times - \mu n_\times - \frac{\hbar}{2}\,\omega_{\nu\lambda}S_\times^{\nu\lambda}\right)_{\mu,\omega_{\sigma\tau},u_\rho}$$

F. Becattini, A. Daher, X.-L. Sheng, PLB 850 (2024) 138533

(I1)

On the other hand, with Eqs. ($\star\star$), ($\square$) and longitudinal/transverse Gibbs-Duhem rel.'s

$$\Longrightarrow \quad \varepsilon = -u_\mu u_\nu\,\left.\frac{\partial\phi^\mu}{\partial\beta_\nu}\right|_{\alpha,\Omega_{\sigma\tau}} = -\left.\frac{\partial(\beta P)}{\partial\beta}\right|_{\alpha,\Omega_{\sigma\tau},u_\rho}, \quad q_\times = -\ell_\mu u_\nu\,\left.\frac{\partial\phi^\mu}{\partial\beta_\nu}\right|_{\alpha,\Omega_{\sigma\tau}} = -\left.\frac{\partial(\beta P_\times)}{\partial\beta}\right|_{\alpha,\Omega_{\sigma\tau},u_\rho}$$

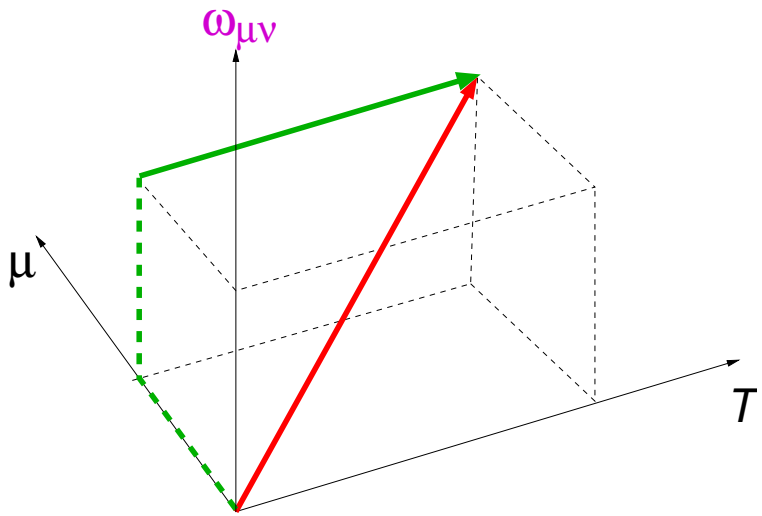
$$\Longrightarrow \quad P = \frac{1}{\beta}\int_\beta^\infty \mathrm{d}\beta'\;\varepsilon|_{\alpha,\Omega_{\sigma\tau},u_\rho}$$

$$P_\times = \frac{1}{\beta}\int_\beta^\infty \mathrm{d}\beta'\;q_\times|_{\alpha,\Omega_{\sigma\tau},u_\rho}$$

V.E. Ambrus, A. Gecić, [arXiv:2509.17640 [hep-th]]

(I2)

Visualizing integration in $(T, \mu, \omega_{\mu\nu})$ space



$$(I1)\ P = T \int_0^T \frac{\mathrm{d}T'}{T'^2} \left(\varepsilon - \mu n - \frac{\hbar}{2}\, \omega_{\nu\lambda} S^{\nu\lambda} \right)_{\mu, \omega_{\sigma\tau}, u_\rho}$$

$$(I2)\ P = \frac{1}{\beta} \int_\beta^\infty \mathrm{d}\beta'\ \ \varepsilon|_{\alpha, \Omega_{\sigma\tau}, u_\rho}$$

P twice cont. differentiable $\Longleftrightarrow$ Maxwell relations $\Longleftrightarrow$ integration path-independent

Maxwell relations

$$\left.\frac{\partial n}{\partial T}\right|_{\omega_{\sigma\tau}} = \left.\frac{\partial s}{\partial \mu}\right|_{\omega_{\sigma\tau}}, \quad \left.\frac{\partial s}{\partial \omega_{\mu\nu}}\right|_{\mu} = \hbar \left.\frac{\partial S^{\mu\nu}}{\partial T}\right|_{\mu}, \quad \left.\frac{\partial n}{\partial \omega_{\mu\nu}}\right|_{T} = \hbar \left.\frac{\partial S^{\mu\nu}}{\partial \mu}\right|_{T}$$

$\Longrightarrow$ Provided: (i) Maxwell relations hold,

(ii) $P(0, \mu, \omega_{\mu\nu}) = 0$ (which holds, e.g., for Boltzmann statistics),

$\Longrightarrow$ both integral identities yield identical results!

Note: not automatically fulfilled in every pseudo-gauge!

Wigner function

$$W_{\alpha\beta}(x,k) = \int \mathrm{d}^4 y\, e^{-ik\cdot y} \left\langle : \hat{\rho}\, \hat{\bar{\psi}}_\beta \left(x + \frac{y}{2}\right) \hat{\psi}_\alpha \left(x - \frac{y}{2}\right) : \right\rangle$$

Dyson-Schwinger equation

$$\left[k^\mu \left(\gamma_\mu + \frac{i\hbar}{2}\,\partial_\mu\right) - m\right] W(x,k) = I_{\text{coll}}$$

In equilibrium, $I_{\text{coll}} = 0$

Mass-shell constraint and equation of motion

$$\Longrightarrow \quad \left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right) W(x,k) = 0\,, \quad k\cdot\partial W(x,k) = 0 \qquad (\lozenge)$$

Clifford decomposition

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma_5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma_5\gamma^\mu\mathcal{A}_\mu + \frac{i}{4}\,[\gamma^\mu,\gamma^\nu]\,\mathcal{S}_{\mu\nu}\right)$$

$\Longrightarrow$ From Eqs. $(\lozenge)$: only $\mathcal{F}$ and components of $\mathcal{A}_\mu$ transverse to k^μ are independent!

Distribution function in extended phase space

Assign spin 4-vector $\mathfrak{s}^\mu$ to (anti-)particles, extend phase space:

$$\Longrightarrow \quad \mathrm{d}\Gamma \equiv \mathrm{d}K\,\mathrm{d}S(K)\,, \quad \mathrm{d}K \equiv \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\,, \quad \mathrm{d}S(K) \equiv \sqrt{\frac{k^2}{3\pi^2}}\,\mathrm{d}^4\mathfrak{s}\,\delta(k\cdot\mathfrak{s})\delta(\mathfrak{s}\cdot\mathfrak{s}+3)$$

$$\Longrightarrow \quad \int \mathrm{d}S(K) = 2\,, \quad \int \mathrm{d}S(K)\,\mathfrak{s}^\mu = 0\,, \quad \int \mathrm{d}S(K)\,\mathfrak{s}^\mu\mathfrak{s}^\nu = -2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$$

$$\Longrightarrow$$

Distribution function

$$\mathfrak{f}(x,k,\mathfrak{s}) = \frac{1}{2}\left[\mathcal{F}(x,k) - \mathfrak{s}^\mu\mathcal{A}_\mu(x,k)\right]$$

$$\mathcal{F}(x,k) \equiv \int \mathrm{d}S(K)\,\mathfrak{f}(x,k,\mathfrak{s})$$

$$\mathcal{A}_\mu(x,k) \equiv \int \mathrm{d}S(K)\,\mathfrak{s}_\mu\,\mathfrak{f}(x,k,\mathfrak{s}) \qquad (\clubsuit)$$

For on-shell particles:

$$\mathfrak{f}(x,k,\mathfrak{s}) = 4\pi\hbar\,m\,\theta(k_0)\,\delta\left(k^2 - m^2 - F(k)\right)\left[f^+(x,k,\mathfrak{s}) + f^-(x,k,\mathfrak{s})\right]$$

Assume global equilibrium and Boltzmann statistics:

$$f^\pm_{\mathrm{eq}}(x,k,\mathfrak{s}) \equiv \exp\left(-\beta_\mu k^\mu \pm \alpha + \frac{\hbar}{4}\,\Omega_{\mu\nu}\Sigma^{\mu\nu}_{k\mathfrak{s}}\right)$$

$\Sigma^{\mu\nu}_{k\mathfrak{s}} \equiv -\frac{1}{m}\,\epsilon^{\mu\nu\alpha\beta}\,k_\alpha\mathfrak{s}_\beta$ dipole-moment tensor

$\Longrightarrow$ From mass-shell constraint $(\diamondsuit)$: $F_{\mathrm{eq}}(k) \equiv -\frac{\hbar^2}{4}\varpi_\mu{}^\lambda\varpi_{\lambda\nu}k^\mu k^\nu$

$\Longrightarrow$ modified mass-shell condition:

$$m^2 = k^\mu\left(g_{\mu\nu} + \frac{\hbar^2}{4}\varpi_\mu{}^\lambda\varpi_{\lambda\nu}\right)k^\nu$$

Canonical

$$J^{\mu}_{C} = \int \mathrm{d}K \, \frac{\partial \mathcal{V}^{\mu}}{\partial \alpha}$$

$$T^{\mu\nu}_{C} = \int \mathrm{d}K \, k^{\nu} \, \mathcal{V}^{\mu}$$

$$S^{\lambda,\mu\nu}_{C} = -\frac{1}{2} \, \epsilon^{\lambda\mu\nu\alpha} \int \mathrm{d}K \, \mathcal{A}_{\alpha}$$

de Groot – van Leeuwen – van Weert (GLW)

$$J^{\mu}_{GLW} = \int \mathrm{d}K \, k^{\mu} \, \frac{\partial \mathcal{F}}{\partial \alpha}$$

$$T^{\mu\nu}_{GLW} = \int \mathrm{d}K \, k^{\nu} \left(\mathcal{V}^{\mu} + \frac{\hbar}{2m} \, \partial_{\lambda} \mathcal{S}^{\lambda\mu} \right)$$

$$S^{\lambda,\mu\nu}_{GLW} = -\frac{1}{2} \int \mathrm{d}K \left[\epsilon^{\lambda\mu\nu\alpha} \, \mathcal{A}_{\alpha} + \frac{1}{m} \left(k^{\mu} \mathcal{S}^{\nu\lambda} - k^{\nu} \mathcal{S}^{\mu\lambda} \right) \right]$$

Using mass-shell constraint and equation of motion, Eq. (♢), as well as Eq. (♣), up to order $\mathcal{O}(\hbar^2)$ one obtains:

Canonical

$$J^{\mu}_{C} = \frac{1}{m} \int \mathrm{d}\Gamma \left[k^{\mu} \left(1 - \frac{\hbar^2}{4m^2} \Box \right) + \frac{\hbar}{2} \, \Sigma^{\mu\lambda}_{k\mathfrak{s}} \partial_{\lambda} \right] \frac{\partial \mathfrak{f}(x,k,\mathfrak{s})}{\partial \alpha}$$

$$T^{\mu\nu}_{C} = \frac{1}{m} \int \mathrm{d}\Gamma \, k^{\nu} \left[k^{\mu} \left(1 - \frac{\hbar^2}{4m^2} \Box \right) + \frac{\hbar}{2} \, \Sigma^{\mu\lambda}_{k\mathfrak{s}} \partial_{\lambda} \right] \mathfrak{f}(x,k,\mathfrak{s})$$

$$S^{\lambda,\mu\nu}_{C} = -\frac{1}{2} \, \epsilon^{\lambda\mu\nu\alpha} \int \mathrm{d}\Gamma \, \mathfrak{s}_{\alpha} \mathfrak{f}(x,k,\mathfrak{s})$$

GLW

$$J^{\mu}_{GLW} = \frac{1}{m} \int \mathrm{d}\Gamma \, k^{\mu} \, \frac{\partial \mathfrak{f}(x,k,\mathfrak{s})}{\partial \alpha}$$

$$T^{\mu\nu}_{GLW} = \frac{1}{m} \int \mathrm{d}\Gamma \, k^{\mu} k^{\nu} \, \mathfrak{f}(x,k,\mathfrak{s})$$

$$S^{\lambda,\mu\nu}_{GLW} = \frac{1}{2m} \int \mathrm{d}\Gamma \, k^{\lambda} \left[\Sigma^{\mu\nu}_{k\mathfrak{s}} - \frac{\hbar}{2m} \left(k^{\mu} \partial^{\nu} - k^{\nu} \partial^{\mu} \right) \right] \mathfrak{f}(x,k,\mathfrak{s})$$

Three pseudo-gauges II

Canonical

$$J^\mu_C = \frac{1}{m}\int \mathrm{d}\Gamma \left[k^\mu\left(1-\frac{\hbar^2}{4m^2}\Box\right)+\frac{\hbar}{2}\Sigma^{\mu\lambda}_{k\mathfrak{s}}\partial_\lambda\right]\frac{\partial \mathfrak{f}(x,k,\mathfrak{s})}{\partial\alpha}$$

$$T^{\mu\nu}_C = \frac{1}{m}\int \mathrm{d}\Gamma\, k^\nu \left[k^\mu\left(1-\frac{\hbar^2}{4m^2}\Box\right)+\frac{\hbar}{2}\Sigma^{\mu\lambda}_{k\mathfrak{s}}\partial_\lambda\right]\mathfrak{f}(x,k,\mathfrak{s})$$

$$S^{\lambda,\mu\nu}_C = -\frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\int \mathrm{d}K\,\mathfrak{s}_\alpha \mathfrak{f}(x,k,\mathfrak{s})$$

GLW

$$J^\mu_{GLW} = \frac{1}{m}\int \mathrm{d}\Gamma\, k^\mu \frac{\partial \mathfrak{f}(x,k,\mathfrak{s})}{\partial\alpha}$$

$$T^{\mu\nu}_{GLW} = \frac{1}{m}\int \mathrm{d}\Gamma\, k^\mu k^\nu\, \mathfrak{f}(x,k,\mathfrak{s})$$

$$S^{\lambda,\mu\nu}_{GLW} = \frac{1}{2m}\int \mathrm{d}\Gamma\, k^\lambda \left[\Sigma^{\mu\nu}_{k\mathfrak{s}} - \frac{\hbar}{2m}\left(k^\mu\partial^\nu - k^\nu\partial^\mu\right)\right]\mathfrak{f}(x,k,\mathfrak{s})$$

$\Longrightarrow$ Blue term in GLW pseudo-gauge can be eliminated by a further pseudo-gauge transformation! D. Wagner, PRD 111 (2025) 016008

kinetic theory

$\Longrightarrow$

$$J^\mu = \frac{1}{m}\int \mathrm{d}\Gamma\, k^\mu \frac{\partial \mathfrak{f}(x,k,\mathfrak{s})}{\partial\alpha}$$

$$T^{\mu\nu} = \frac{1}{m}\int \mathrm{d}\Gamma\, k^\mu k^\nu\, \mathfrak{f}(x,k,\mathfrak{s})$$

$$S^{\lambda,\mu\nu} = \frac{1}{2m}\int \mathrm{d}\Gamma\, k^\lambda \Sigma^{\mu\nu}_{k\mathfrak{s}}\mathfrak{f}(x,k,\mathfrak{s})$$

cf. W. Israel, Lett. Nuovo Cim. 7 (1973) 860

Note: although in global equilibrium $\Omega_{\mu\nu} \equiv \varpi_{\mu\nu}$, keep them distinct at first

Reason: partial derivatives of pressure performed at $\varpi_{\mu\nu} = \text{const.}$

$\Longrightarrow$ also T', β' integrations performed at $\varpi_{\mu\nu} = \text{const.}$

$\Longrightarrow$ Decompose $\varpi^{\mu\nu} = \beta \left(u^{[\mu} a^{\nu]} + \epsilon^{\mu\nu\alpha\beta} u_\alpha b_\beta \right)$, $a^2 \equiv -a \cdot a$, $b^2 \equiv -b \cdot b$

$\Longrightarrow$ General structure of conserved currents in kinetic-theory pseudogauge:

$$J^\mu = n\, u^\mu + \hbar^2 n^\mu_\times$$

$$T^{\mu\nu} = \varepsilon\, u^\mu u^\nu - P\, \Delta^{\mu\nu} + \hbar^2 q^{(\mu}_\times u^{\nu)} + \hbar^2 \Pi_\Omega \left[\varkappa^\mu \varkappa^\nu + \omega^\mu \omega^\nu + \Delta^{\mu\nu} \left(\varkappa^2 + \omega^2 \right) \right] + \hbar^2 \Pi_\varpi \left[a^\mu a^\nu + b^\mu b^\nu + \Delta^{\mu\nu} \left(a^2 + b^2 \right) \right]$$

$$\hbar\, S^{\lambda,\mu\nu} = \hbar^2 \left[u^\lambda \left(A\, u^{[\mu} \varkappa^{\nu]} + B\, \epsilon^{\mu\nu\alpha\beta} u_\alpha \omega_\beta \right) + C\, \Delta^{\lambda[\mu} \varkappa^{\nu]} + D\, u^{[\mu} \epsilon^{\nu]\lambda\alpha\beta} u_\alpha \omega_\beta \right]$$

$$n = n_0 + \hbar^2 \left[n_{2\varkappa}\, \varkappa^2 + n_{2\omega}\, \omega^2 + n_{2a}\, a^2 + n_{2b}\, b^2 \right]$$

$$\varepsilon = \varepsilon_0 + \hbar^2 \left[\varepsilon_{2\varkappa}\, \varkappa^2 + \varepsilon_{2\omega}\, \omega^2 + \varepsilon_{2a}\, a^2 + \varepsilon_{2b}\, b^2 \right]$$

$$n^\mu_\times = n_{\times L}\, L^\mu + n_{\times K}\, K^\mu, \quad q^\mu_\times = q_{\times L}\, L^\mu + q_{\times K}\, K^\mu, \quad K^\mu \equiv \epsilon^{\mu\nu\alpha\beta} u_\nu a_\alpha b_\beta$$

where n_0, $n_{2\varkappa}$, $n_{2\omega}$, n_{2a}, n_{2b}, ε_0, $\varepsilon_{2\varkappa}$, $\varepsilon_{2\omega}$, ε_{2a}, ε_{2b},

$n_{\times L}$, $n_{\times K}$, $q_{\times L}$, $q_{\times K}$, Π_Ω, Π_ϖ, A, B, C, D are functions of α, β

Consequences:

(i) $T^{\mu\nu} u_\nu = \varepsilon u^\mu + \hbar^2 q^\mu_\times$

$\Longrightarrow$ ε is not time-like eigenvalue of $T^{\mu\nu}$, u^μ is not corresponding eigenvector

$\Longrightarrow$ u^μ is not fluid 4-velocity in Landau frame

(ii) with ε, n, $S^{\mu\nu} \equiv u_\lambda S^{\lambda,\mu\nu}$, compute thermodynamic pressure P from Eqs. (I1), (I2)

$\Longrightarrow$ agreement!

(iii) $n = \beta P \tanh\alpha$ $\Longrightarrow$ ideal gas equation of state!

(iv) check validity of $n = \left.\dfrac{\partial P}{\partial \mu}\right|_{T,\omega_{\sigma\tau},u_\rho}$, $\hbar\, S^{\mu\nu} = \left.\dfrac{\partial P}{\partial \omega_{\mu\nu}}\right|_{T,\mu,u_\rho}$

$\Longrightarrow$ fulfilled!

(v) check consistency: $s = \left.\dfrac{\partial P}{\partial T}\right|_{\mu,\omega_{\sigma\tau},\varpi_{\sigma\tau}} \equiv \beta\left(\varepsilon + P - \mu n - \dfrac{\hbar}{2}\,\omega_{\nu\lambda} S^{\nu\lambda}\right)$

$\Longrightarrow$ fulfilled!

(vi) check validity of Maxwell relations

$\Longrightarrow$ fulfilled!

$\Longrightarrow$ kinetic-theory pseudo-gauge is thermodynamically consistent!

GLW pseudo-gauge:

(i) Since $n_{GLW} = n$, but $S^{\nu\lambda}_{GLW} \neq S^{\nu\lambda}$, third Maxwell relation not fulfilled!

(ii) Since $T^{\mu\nu}_{GLW} = T^{\mu\nu} \implies \varepsilon_{GLW} = \varepsilon \implies P_{GLW} = P$ if computed from Eq. (I2), but because $S^{\nu\lambda}_{GLW} \neq S^{\nu\lambda}$, $P_{GLW} \neq P$ if computed from Eq. (I1)

$\implies$ GLW pseudo-gauge is thermodynamically inconsistent!

Canonical pseudo-gauge:

(i) If $\Omega_{\mu\nu}$ and $\varpi_{\mu\nu}$ are strictly kept distinct, canonical pseudo-gauge is thermodynamically inconsistent!

(ii) However: by an educated application of $\Omega_{\mu\nu} = \varpi_{\mu\nu}$ in some selected terms, it can be made thermodynamically consistent!
(for details, see Daher, Wagner, Shokri, DHR [arXiv:2609.31819[nucl-th]])

Conclusions and Outlook

- Derived thermodynamic relations for system in global equilibrium with acceleration and rotation

  $\Longrightarrow$ thermodynamic-potential 4-current $\phi^\mu = \beta\, P\, u^\mu + \beta\, P_\times\, \ell^\mu$

  $\Longrightarrow$ longitudinal and transverse Euler and Gibbs-Duhem relations

  $\Longrightarrow$ fluid 4-velocity becomes thermodynamic variable

- Thermodynamic consistency requires validity of Maxwell relations

  $\Longrightarrow$ integral identities for thermodynamic pressure yield identical results

- Explicitly computed conserved currents, thermodynamical quantities for massive spin-1/2 Boltzmann particles in kinetic-theory, GLW, canonical pseudo-gauge

- Mass-shell modification at order $\mathcal{O}(\hbar^2)$: $m^2 = k^\mu \left(g_{\mu\nu} + \frac{\hbar^2}{4} \varpi_\mu{}^\lambda \varpi_{\lambda\nu} \right) k^\nu$

- In global equilibrium, $\mathcal{Z}$ pseudo-gauge invariant,
  while local thermodynamic quantities (n, ε, P, s, $S^{\mu\nu}$) depend on pseudo-gauge

- Thermodynamic consistency:
  - kinetic-theory pseudo-gauge ✓
  - GLW pseudo-gauge –
  - canonical pseudo-gauge (✓)

Outlook: Investigate role of boundary terms for thermodynamic consistency

Back-up

Projecting first law of thermodynamics (Δ) onto u^μ

Longitudinal first law

$$\Longrightarrow \quad \mathrm{d}\varepsilon = T\,\mathrm{d}s + \mu\,\mathrm{d}n + \frac{\hbar}{2}\,\omega_{\nu\lambda}\mathrm{d}S^{\nu\lambda} - \left(P_\times \ell^\mu - u_\nu\, T^{\nu\lambda}\Delta_\lambda{}^\mu\right)\mathrm{d}u_\mu$$

$$\Longrightarrow \quad T = \left.\frac{\partial\varepsilon}{\partial s}\right|_{n,S^{\sigma\tau},u_\rho}, \quad \mu = \left.\frac{\partial\varepsilon}{\partial n}\right|_{s,S^{\sigma\tau},u_\rho}, \quad \hbar\,\omega_{\nu\lambda} = \left.\frac{\partial\varepsilon}{\partial S^{\nu\lambda}}\right|_{s,n,u_\rho}$$

Projecting first law of thermodynamics (Δ) onto ℓ^μ

Transverse first law

$$\Longrightarrow \quad \mathrm{d}q_\times = T\,\mathrm{d}s_\times + \mu\,\mathrm{d}n_\times + \frac{\hbar}{2}\,\omega_{\nu\lambda}\mathrm{d}S_\times^{\nu\lambda} - \left(P\ell^\mu + \ell_\nu\, T^{\nu\lambda}\Delta_\lambda{}^\mu\right)\mathrm{d}u_\mu$$

$$\Longrightarrow \quad T = \left.\frac{\partial q_\times}{\partial s_\times}\right|_{n_\times,S_\times^{\sigma\tau},u_\rho}, \quad \mu = \left.\frac{\partial q_\times}{\partial n_\times}\right|_{s_\times,S_\times^{\sigma\tau},u_\rho}, \quad \hbar\,\omega^{\nu\lambda} = \left.\frac{\partial q_\times}{\partial S_\times^{\nu\lambda}}\right|_{s_\times,n_\times,u_\rho}$$