functional QCD in magnetic field

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based on arXiv: 2604.24410, 2606.23736

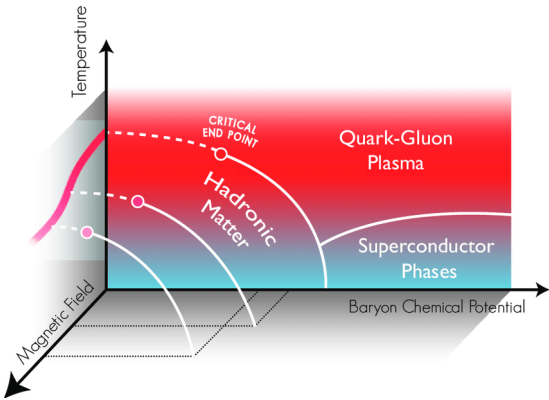


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QCD phase structure in magnetic field

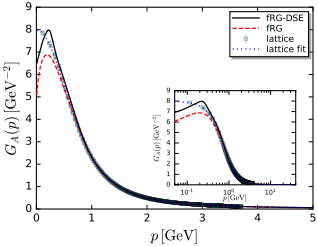
During the heavy ion collision, a large magnetic field up to $10^{15} \sim 10^{16}$ T is created and in company with the QCD phase transitions. This causes new phenomena of QCD matter:

- Magnetic catalysis at low temperature
  Inverse magnetic catalysis around chiral phase transition
- New anomalous phenomena such as CME/CSE.
- What is the mechanism behind these phenomena? Where does the CEP go at large magnetic field?



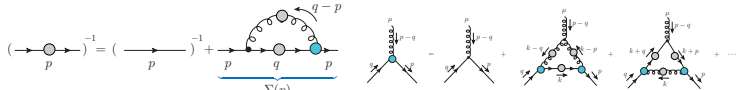
The minimal QCD scheme of Dyson-Schwinger equation

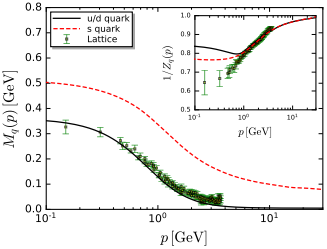
The Yang-Mills sector is relatively separable. One can apply the data in vacuum and compute the difference between finite T/μ and vacuum.



Lattice:
A. G. Duarte et al, PRD 94, 074502 (2016),
P. Boucaud et al, PRD 98, 114515 (2018),
S. Zafeiropoulos et al, PRL122, 162002 (2019)
fRG:
W.-j. Fu et al, PRD 101, 054032 (2020)
Cyrol, Fister, Mitter, Pawlowski, Strodthoff, PRD 94 (2016) 5, 054005

Solve the DSEs of quark propagator and quark gluon vertex:





lattice: P. O. Bowman et al, PRD71, 054507 (2005) **fRG**: W.-j. Fu et al, PRD 101, 054032 (2020) **DSE**: FG et al, PRD 103, 094013(2021)

Dyson-Schwinger equation at finite magnetic field

With the presence of a constant magnetic field, one can decompose the kinetic term as:

$$\begin{aligned} & \not{D} + m = (\partial_x \gamma_1 + i q_f B x \gamma_2 + i\vec{\not{p}} + m_0) \\ = \quad & \Sigma^+(i\not{p}_{\parallel} + m_0) + \Sigma^-(i\not{p}_{\parallel} + m_0) + I_+(\partial_{x_{p_y}} + q_f B x_{p_y}) + I_-(\partial_{x_{p_y}} - q_f B x_{p_y}), \end{aligned}$$

The quark propagator is no longer the eigenfunction in the momentum representation. Instead, the Ritus basis based on the Hermite funciton is applied.

One can further obtain the Schwinger propagator by applying a proper time formalism to the propagator in Ritus formalism, making it possible to sum up the Hermite function.

- Great advantages in numerical calculations, as it can be expressed in momentum representation besides of a factor called Schwinger phase.
- This can be achieved if the interaction kernel contains only the transferred momentum, as now the Schwinger phase can be completely factored out.

V. P. Gusynin, V. A. Miransky, I. A. Shovkovy, PRL 83 (1999) 1291-1294

Under this circumstance, the quark propagator can be expressed as:

$$S^{-1}(x,y) = e^{i\Phi(x,y)}\bar{S}(x-y), \quad S(x,y) = e^{i\Phi(x,y)}S(x-y),$$

with $\Phi(x,y)$ the Schwinger phase, and $\bar{S}(x-y)$ can be parametrized as in momentum representation as:

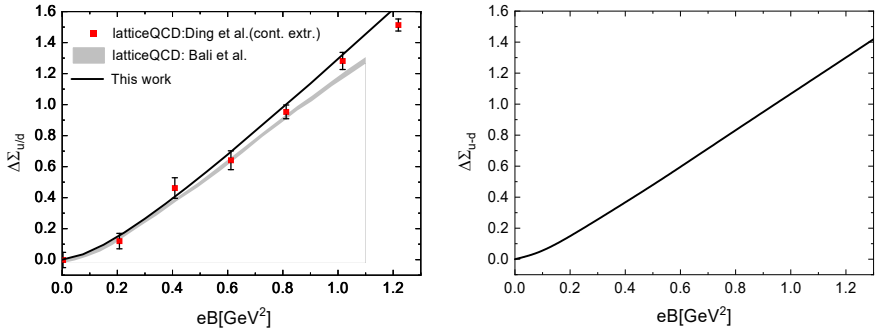
$$\bar{S}(x-y) = \int \frac{d^4p}{(2\pi)^4}\bar{S}(p)e^{ip(x-y)}, \; S(x-y) = \int \frac{d^4p}{(2\pi)^4}S(p)e^{-ip(x-y)},$$

One can then parametrize the full propagator as:

$$\begin{aligned}
\bar{S}(p) &= \Sigma^+(M^+(p) + Z_{\parallel}^+(p)\not{p}_{\parallel}) + Z_{\perp}(p)\not{p}_{\perp} + \Sigma^-(M^-(p) + Z_{\parallel}^-(p)\not{p}_{\parallel}),\\
S(p) &= \Sigma^+(W_{M^+}(p) + W_{Z_{\parallel}^+}(p)\not{p}_{\parallel}) + W_{Z_{\perp}}(p)\not{p}_{\perp} + \Sigma^-(W_{M^-}(p) + W_{Z_{\parallel}^-}(p)\not{p}_{\parallel}).
\end{aligned}$$

The relation between $W_{Z,M}$ and Z, M now become more complicated since the relation between the inverse quark propagator $\bar{S}(p)$ is not simply the inverse of the propagator $S(p)$, as they have the same dependence on the Schwinger phase which is not reversed.

The condensate at T=0 as a function of magnetic field:



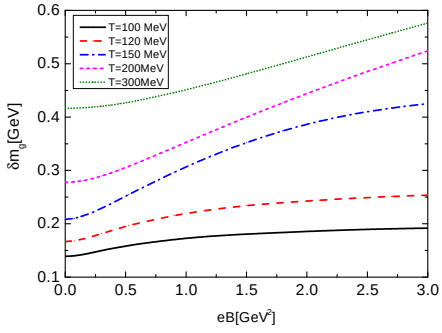
- In good agreement with lattice QCD.
- MC arises from the direct enhancement of the quark mass, more enhanced with larger quark mass.
- This direct enhancement persists at finite temperature if one neglects the magnetic field dependence in the gluon propagator

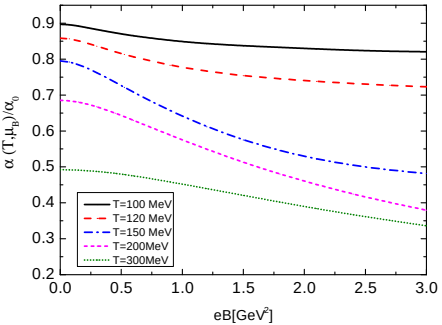
Quark loop in the gluon self energy

The self-energy in the difference DSE for the gluon propagator is given by the quark vacuum polarization which induces the magnetic field dependence of gluon:

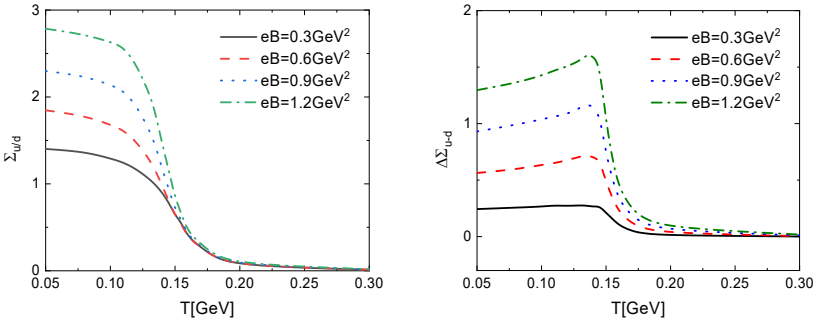
$$\Delta\Pi^{\vec{v},\vec{v}_0}_{A,\mu\nu}(k) = -\int_q \mathrm{tr}\left[g_s\gamma_\mu S^m_f(\tilde{q})\Gamma^m_\nu(\tilde{q},\tilde{q}-k)S^m_f(\tilde{q}-k)\right]\Big|^{m=|q_fB|}_{m=0},$$

The gluon mass increases for larger magnetic field:





The condensate as a function of temperature:

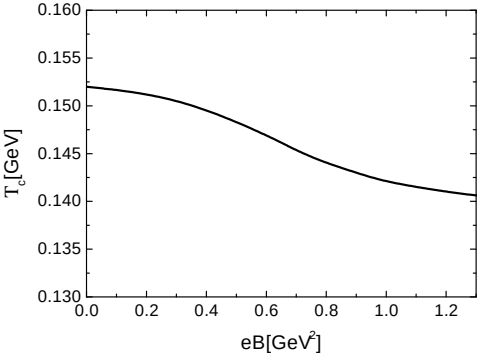


A competing mechanism between the quark and gluon.

- MC For low temperature, the enhancement of the quark mass is dominant.
- IMC For larger temperature, in particular close to T_c:
  - the gluon enhancement increases at larger temperature;
  - the quark mass function is small, and thus the direct enhancement effect on quark mass is small.

The chiral phase transition temperature

The chiral phase transition temperature (pseudo) as a function of magnetic field:



- A crossover up to $eB \approx 2$ GeV^2
- Drops mildly from $T_c = 152$ MeV at $eB = 0$ to $T_c = 142$ MeV for $eB \approx 1$ GeV^2.

- Neglecting the quark loop in the gluon propagator, the quark mass simply increases with the magnetic field strength, T_c rises accordingly.
- Due to the difference in the electric charges of the up and down quarks, the chiral condensate of up quark always increases while the suppression is mainly from the down quark
- **A potential splitting:** T_c from up quark remains the same, decreasing T_c comes solely from the decreasing of the down quark condensate.

There exists many anomalous phenomena: Chiral anomaly/ABJ anomaly in association with pseudoscalar meson; axial anomaly/$U_A(1)$ anomaly that persists in chiral symmetric phase above T_c. Both are related to $\partial_\mu J^5_\mu = \tilde{F}_{\mu\nu}F_{\mu\nu}$, but the first one is related to the scalar operator while the second is to the vector operator.

More anomalous phenomena with the presence of magnetic field: chiral magnetic effect (CME), chiral separation effect (CSE)....

First of all, one has the typical chiral/axial anomaly which becomes explicit within the presence of the magnetic field:

$$\partial_\mu J^5_\mu = \tilde{F}_{\mu\nu}F_{\mu\nu} \sim E \cdot B$$

This has been argued to be related to the non trivial topology, however,
A possible classification of anomaly:

- non trivial topology of time-space manifold (materials, Berry phase),
- of internal structure of the field (soliton/instanton, CME),
- and of higher order Green function (ABJ anomaly from triangle diagram)

The Ritus basis of the quark propagator

Fermion propagator in the finite magnetic field in Ritus basis:

$$S^{-1}(p_\parallel, p_\perp) = e^{-\frac{p_\perp^2}{|q_f B|}} \sum_{n=0}^{\infty} 2(-1)^n \Big[2i\not{p}_\perp Z_\parallel(p_\parallel, n) L_{n-1}^1(\alpha_\perp)$$

$$- \Sigma^+(-i\not{p}_\parallel Z_\parallel^+(p_\parallel, n) + M^+(p_\parallel, n)) L_{n-1}^0(\alpha_\perp) + \Sigma^-(-i\not{p}_\parallel Z_\parallel^-(p_\parallel, n) + M^-(p_\parallel, n)) L_n^0(\alpha_\perp) \Big]$$

$$S(p_\parallel, p_\perp) = e^{-\frac{p_\perp^2}{|q_f B|}} \sum_{n=0}^{\infty} 2(-1)^n \Big[2i\not{p}_\perp W_{Z_\parallel}(p_\parallel, n) L_{n-1}^1(\alpha_\perp)$$

$$- \Sigma^+(-i\not{p}_\parallel W_{Z_\parallel^+}(p_\parallel, n) + W_{M^+}(p_\parallel, n)) L_{n-1}^0(\alpha_\perp) + \Sigma^-(-i\not{p}_\parallel W_{Z_\parallel^-}(p_\parallel, n)$$

$$+ W_{M^-}(p_\parallel, n)) L_n^0(\alpha_\perp) \Big]$$

All of the anomaly comes from the splitting between spin up and down, corresponding to the Dirac structure $\not{p}(\Sigma^+ - \Sigma_-) = \not{p}\gamma_1\gamma_2$

Axial vector current

The axial current can be calculated as:

$$J_5 = <\bar{\psi}\gamma^3\gamma_5\psi> = \frac{|q_f B|T}{4\pi} n_0, \quad n_0 = T\sum_m \int \frac{dp_3}{2\pi} \frac{-4i\tilde{p}_0}{p_\parallel^2 + m_f^2}.$$

It is not difficult to calculate the summation and the integration, which gives:

$$J_5 = \frac{|q_f B|}{2\pi^2} \int_0^\infty dp_3 \left(\frac{1}{e^{\frac{\sqrt{p_3^2+m_f^2}-\mu_q}{T}}+1} - \frac{1}{e^{\frac{\sqrt{p_3^2+m_f^2}+\mu_q}{T}}+1}\right)$$
$$\xrightarrow{m_f \to 0} \frac{|q_f B|}{2\pi^2}\mu_q.$$

The situation for the asymmetry of number density n_5 is subtle. One can still calculate n_5 directly from the free quark propagator as:

$$n_5 = <\bar{\psi}\gamma^0\gamma_5\psi> = \frac{|q_f B|}{4\pi} T\sum_m \int \frac{dp_3}{2\pi} \frac{4ip_3}{p_\parallel^2 + m_f^2} = 0,$$

It can be non vanishing with nonzero μ_5:

$$n_5 = \frac{|q_f B|}{2\pi^2} T \sum_{m,n} \int dp_3 \left[\frac{ip_3 + \mu_5}{p_\parallel^2 + \mu_5^2 + 2n|q_f B|} - \frac{ip_3 - \mu_5}{p_\parallel^2 + \mu_5^2 + 2(n+1)|q_f B|} \right]$$

$$\xrightarrow{\text{small } \mu_5} \frac{|q_f B| T \mu_5}{\pi} \left[-\sum_{m=0}^{\infty} \frac{1}{(2m+1)\pi T} + \sum_{n,m=0}^{\infty} \frac{2}{\sqrt{((2m+1)\pi T)^2 + 2n|q_f B|}} \right].$$

For small $q_f B$, it is analytic continuation of Hurwitz zeta function and one has:

$$n_5 = -4\mu_5 T^2 \left[\zeta(-1) + \frac{1}{2}\zeta(0) \right] = \frac{4}{3}\mu_5 T^2.$$

For large $q_f B$ with lowest Landau level approximation:

$$n_5 = \frac{\mu_5 |q_f B|}{2\pi^2} \sum_m \frac{1}{m + \frac{1}{2}} = -\frac{\mu_5 |q_f B| \psi^{(0)}(\frac{1}{2})}{4\pi^2} \approx \frac{\mu_5 |q_f B|}{2\pi^2}.$$

From the above definition, one can write down the axial vector conservation directly:

$$\nabla_\mu J_5^\mu = \tilde{F}_{\mu\nu}(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\phi(x),$$

with $\tilde{F}_{\mu\nu}$ the field strength tensor coming from the magnetic field.
With only $z > 0$ direction which corresponds solely to the outgoing magnetic flux.

$$\begin{aligned}
&\int_{z>0} d^3x \int_0^\beta d\tau \partial_\mu J_5^\mu = \frac{i|q_f B|}{2\pi^2} \int dxdy \int_0^\infty dz \int_0^\infty dp_3 \sin(p_3 z) \\
&\times \left(\frac{\mu_q}{p_3 - \mu_q} \frac{e^{(p_3-\mu_q)\beta} - 1}{e^{(p_3-\mu_q)\beta} + 1} + \frac{\mu_q}{p_3 + \mu_q} \frac{e^{(-p_3-\mu_q)\beta} - 1}{e^{(-p_3-\mu_q)\beta} + 1}\right) \\
= \quad & -\frac{|q_f B| S}{2\pi^2} \sum_\pm [\psi^{(0)}(\frac{1}{2} \pm i\frac{\mu_q}{2\pi T}) - \psi^{(0)}(\frac{1}{2})] \xrightarrow{\text{small}\,\mu} -\frac{|q_f B| S}{2\pi^2} \frac{\mu_q^2 \psi^{(2)}(\frac{1}{2})}{4\pi^2 T^2},
\end{aligned}$$

Equivalently, a non vanishing μ_5 which gives nontrivial internal structure as it represents the splitting between spin up and down also leads to the axial anomaly.

CME is a mixing propagating from 0th component to 3th component, which encodes the above splitting

$$|q_f B| C_{\rm CME} = \frac{\partial J_3}{\partial \mu_5}\Big|_{\mu_5=0} = \int d^4y < \bar{\psi}_f(x)\gamma_3\psi_f(x)\bar{\psi}_f(y)\gamma_0\gamma_5\psi_f(y) >$$

$$\begin{aligned} C^0_{\rm CME} = & \quad \frac{T}{\pi}\int \frac{dp_3}{2\pi}\sum_{m,n}[\frac{\tilde{p}_0^2 - p_3^2 + m_f^2}{(p_\parallel^2 + m_f^2 + 2n|q_f B|)^2} - \frac{\tilde{p}_0^2 - p_3^2 + m_f^2}{(p_\parallel^2 + m_f^2 + 2(n+1)|q_f B|)^2}] \\ = & \quad \frac{1}{2\pi^2}\frac{\sqrt{1-(m_f/\epsilon)^2}\sinh(\epsilon/T)}{\cosh(\epsilon/T)+\cosh(\mu/T)}\Big|_{\epsilon=m_f}^{\epsilon=+\infty} = \frac{1}{2\pi^2} \end{aligned}$$

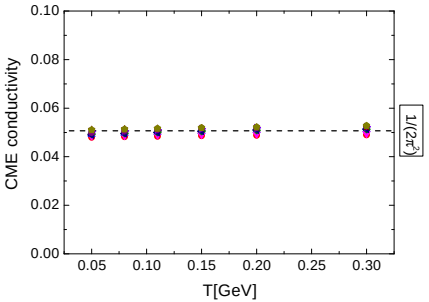
It is actually the transverse Ward Takahashi Identity(tWTI):

$$\int d^4x \epsilon^{\mu\nu\rho\lambda}\left(\bar{\psi}(x)\sigma_{\rho\lambda'}\gamma_5\nabla_{\lambda'}\psi(x)\bar{J}_\lambda(y) + m_f J^5_\rho(x)\bar{J}_\lambda(y)\right) = \frac{m_f}{2\pi^2}F^{\mu\nu}$$

The first term satisfies another tWTI:

$$\int \epsilon^{\mu\nu\rho\lambda}\bar{\psi}(x)[\gamma_\rho\gamma_5, \not{\nabla}]\psi(x)\tilde{J}_\lambda(y) - \bar{\psi}(x)[\sigma_{\mu\nu}, \not{\nabla}]\psi(x)J(y) = 0$$

Numerical results with full dressed propagator:



The CME current becomes:

$$J_3 = \mu_5 |q_f B| \frac{1}{2\pi^2} = \begin{cases} \frac{3|q_f B|}{8\pi^2 T^2} n_5, & |q_f B| \ll T^2 \\ n_5, & |q_f B| \gg T^2 \end{cases}$$

Different scenarios could be detected in experiments

QCD phase transition in magnetic field:

- The quark mass response for the magnetic field is positive as Magnetic Catalysis.
- The gluon mass response is also positive, however, it then suppresses the quark gluon coupling, resulting in the Inverse Magnetic Catalysis.

More interesting phenomena in magnetic field about anomaly and topology:

- An exact formula for the CME current in small and large magnetic field.
- CME conductivity is a constant against interaction, temperature, chemical potential as it is the transverse Ward Takahashi identity without nontrivial topology of time space.
- A further and deeper question: the classification of topology in various anomalous phenomena.

Thank you!

A further simplification on the quark gluon vertex:

Quark gluon vertex In Landau gauge:

$$\Gamma^{\mu}(q,p) = \sum_{i=1}^{8} t_i(q,p) P^{\mu\nu}(q-p) \mathcal{T}_i^{\nu}(q,p)\,,$$

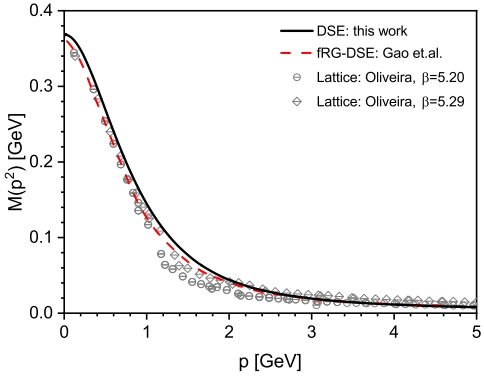
The dominant structures are Dirac and Pauli term:

$$\mathcal{T}_1(p,q) = -i\gamma^{\mu}\,, \mathcal{T}_4^{\mu}(p,q) = \sigma_{\mu\nu}(p-q)^{\nu}\,,$$

$$t_1(p,q) = F(k^2)\frac{A(p^2)+A(q^2)}{2}$$

$$t_4(p,q) = \left[Z(k^2)\right]^{-1/2}\frac{B(p^2)-B(q^2)}{p^2-q^2}$$

All quantities are expressed by the running of two point functions.
The Quark Mass function:



L. Chang, YX Liu, and C. D. Roberts, PRL 106, 072001(2011)
SX Qin, L. Chang, YX Liu, C. D. Roberts, S. M. Schmidt, PLB 722, 384(2013)
FG, J. Papavassiliou, J. Pawlowski, PRD 103.094013 (2021).
Y, Lu, **FG**, YX Liu, J. Pawlowski, PRD 110, 014036 (2024)

Criterion of phase transitions

The Cornwall–Jackiw–Tomboulis (CJT) effective potential:

$$\Gamma(S) = -Tr[\ln(S_0^{-1}S) - S_0^{-1}S + 1] + \Gamma_2(S),$$

where S_0 and S stands for the bare and full quark propagator,Γ_2 is the 2PI contribution. Calculating the variation respective to quark propagator, we have:

$$\frac{\partial^2\Gamma}{\partial S^2} = S^{-2} + \frac{\partial\Gamma_2(S)}{\partial S}.$$

combing with the derivative on the quark propagator DSE as:

$$-S^{-2}\frac{\partial S}{\partial T} = 1 + \frac{\partial\Gamma_2(S)}{\partial S}\frac{\partial S}{\partial T},$$

The criterion is then given by[1]:

$$\frac{\partial S}{\partial T} = -\frac{1}{\partial^2\Gamma/\partial S^2}.$$

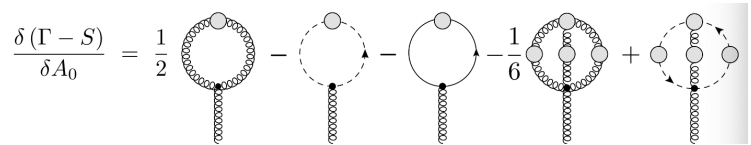
[1] **Fei Gao**, Yu-xin Liu. Phys. Rev. D 94 (2016) 7, 076009.

$$S_q^{-1}(p) = \mathrm{i}(\tilde{\omega}_n + gA_4)\gamma_4 Z_q^E(\boldsymbol{p}, \tilde{\omega}_n) + \mathrm{i}\gamma \cdot \boldsymbol{p} Z_q^M(\boldsymbol{p}, \tilde{\omega}_n) + Z_q^E(\boldsymbol{p}, \tilde{\omega}_n) M_q(\boldsymbol{p}, \tilde{\omega}_n)$$

Quark mass function for Chiral PT and Polyakov loop for deconfinement, and in background field approach is related to A_0^a condensate as:

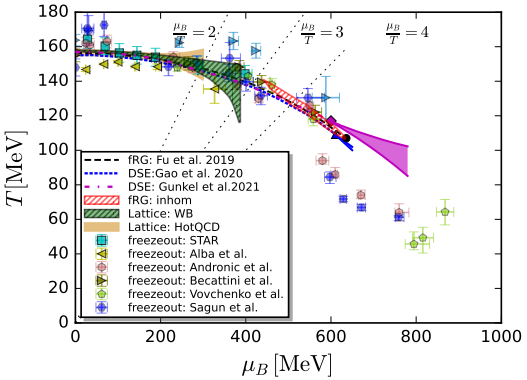
$$\mathcal{L}(A_0) = \frac{1}{N_c}\mathrm{tr}\,\mathcal{P}e^{\mathrm{i}g\int dx_0 A_0} = \frac{1}{3}\left[1 + 2\cos\left(g\beta A_0/2\right)\right]$$

One can obtain A_0 condensate by solving the The DSE of A_0^a as $\frac{\delta(\Gamma - S_A)}{\delta A_0} = 0$. The diagrammatic representation is:



QCD phase diagram up to CEP

Chiral Phase diagram for 2+1 flavour QCD can be directly obtained from quark propagator



The fQCD computations of chiral phase transition are converging:

- $T_c = 155\,\mathrm{MeV}$ and $\kappa \sim 0.016$
- Estimated range of CEP:
  $T \in (100, 120)$ MeV
  $\mu_B \in (600, 700)$ MeV

W.-j. Fu et al, PRD 101, 054032 (2020)

FG and J. Pawlowski, PRD 102, 034027 (2020)

FG and J. Pawlowski, PLB 820, 136584(2021)

P.J. Gunkel, C. S. Fischer, PRD 104, 054022 (2021).