Late-Time Relaxation from Landau Singularities

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Based on arXiv:2605.04020 with Shi Pu

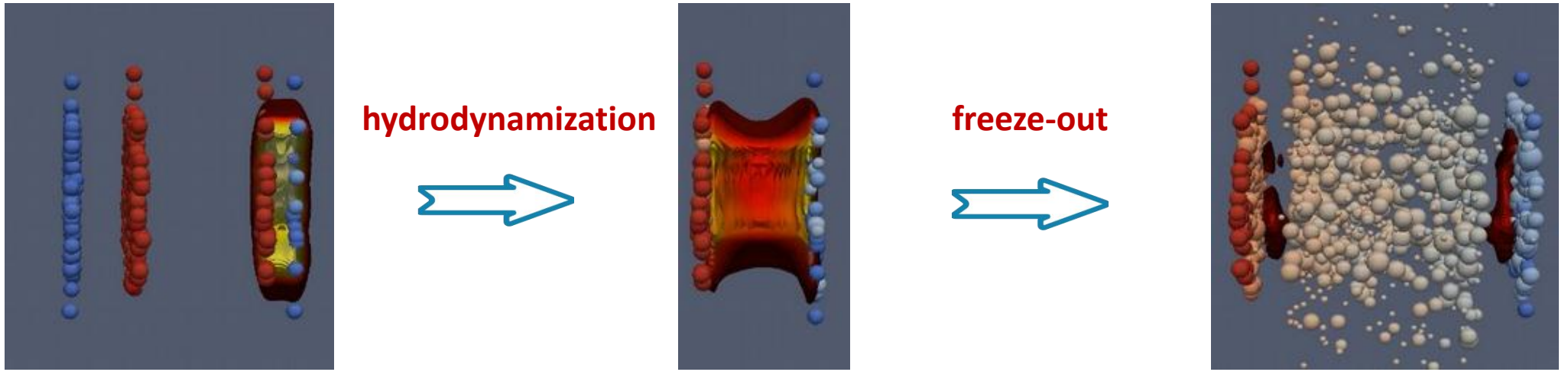
Workshop on QCD Critical End Point and Effective Field Theory Methods
C3NT, Wuhan / October 21, 2026

Outline

- Introduction
- Perturbative expansion of self-energy
- Determine singularities without integration
- Late-time relaxation
- Summary

Introduction

Hydro-method for heavy ion collisions



Initial state

Description of early time-dynamics and thermalization

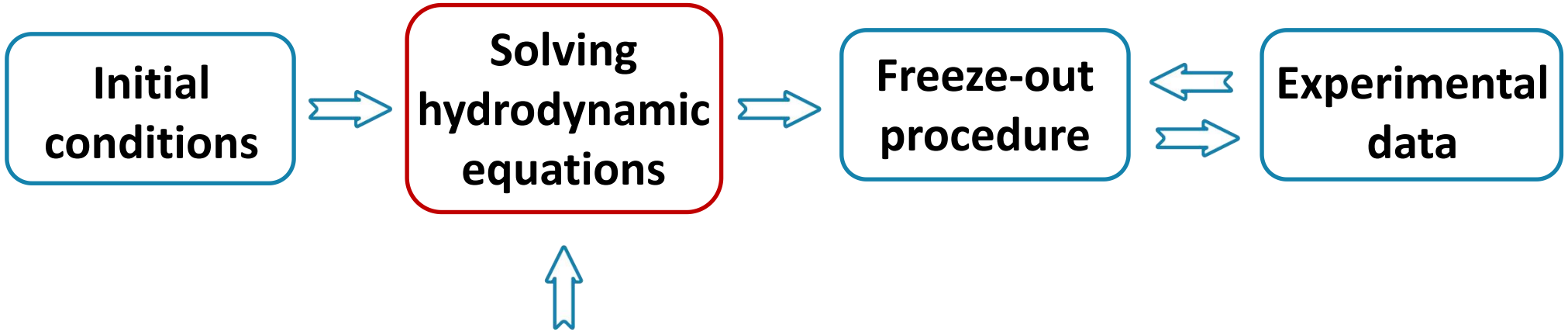
Hydrodynamic expansion of quark-gluon-plasma (QGP)

Description of QGP as a relativistic dissipative fluid

Transport description of hadron gas

Description of hadronic rescattering and decays

Hydro-method for heavy ion collisions



- **Hydrodynamic equations come from the conservation laws:**

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\mu j^\mu = 0$$

Energy-momentum conservation **Particle number conservation**

Fluctuations exist in dissipative systems

- **According to the fluctuation-dissipation theorem, every system that dissipates must also fluctuate:** Callen and Welton (1951); Kubo (1966)

$$\text{Fluctuation} \propto \text{Dissipation}$$

- **Introduce stochastic noises in hydrodynamic equations:**

Stochastic

$$j^\mu \to \underline{j^\mu_{\rm d}} + j^\mu_{\rm s}$$

Deterministic

$$\partial_\mu j^\mu_{\rm d} = -\partial_\mu j^\mu_{\rm s} \equiv \theta$$

Average over stochastic noises

- **Using the Martin-Siggia-Rose-De Dominicis-Jansses (sometimes called MSR) approach, the average of an observable can be expressed as:** Martin, Siggia and Rose (1973); De Dominicis (1976); Janssen (1976)

$$\langle \mathcal{O}[\varphi] \rangle = \int \mathcal{D}\theta \mathcal{O}[\varphi_\theta] e^{-W[\theta]}$$

$$= \int \mathcal{D}\varphi \mathcal{D}\varphi_a \mathcal{O}[\varphi] e^{iS_{\rm eff}[\varphi,\varphi_a]}$$

$$S_{\rm eff}[\varphi,\varphi_a] = \int d^dx \left[\varphi_a(\partial_\mu j^\mu_{\rm d}[\varphi]) + i\mathcal{S}\varphi_a^2\right]$$

Gaussian noise: $W[\theta] \sim \int d^dx\theta^2(x)$

Solution of stochastic equation: $\varphi\theta$

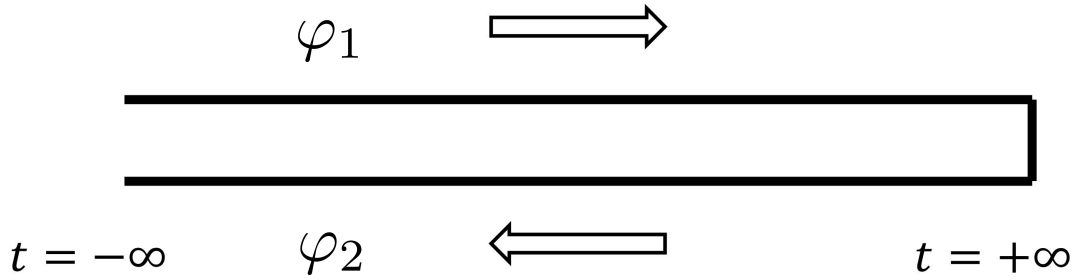
Auxiliary field: φ_a

Effective field theory for hydrodynamics

- **The classical MSR approach can be viewed as a special case of the modern Schwinger-Keldysh approach.**

Classical fields $\longrightarrow$ $\varphi_r = \frac{1}{2}(\varphi_1 + \varphi_2)$

Noise fields $\longrightarrow$ $\varphi_a = \varphi_1 - \varphi_2$



- **Generally, the effective Lagrangian has the form:**

$$\mathcal{L}_{\text{eff}} = \varphi_a(\mathcal{R}\varphi_r + \mathcal{S}\varphi_a) + \sum_{m\geq 1, n\geq 0}^{m+n\geq 3} \mathcal{P}_m \varphi_a^m \mathcal{Q}_n \varphi_r^n$$

Haehl, Loganayagam and Rangamani, JHEP (2017); Crossley, Glorioso and Liu, JHEP (2017); Jensen, Pinzani-Fokeeva and Yarom, JHEP (2018).

Interaction effects

➢ **Two-point correlation functions satisfy the Dyson equation:**

$$\underline{G^{-1}} = \underline{(G^{(0)})^{-1}} - \underline{\Sigma}$$

Exact propagator **Free propagator** **Self-energy**

$$G_{ar} \equiv \langle \varphi_a \varphi_r \rangle \quad \textbf{advanced}$$
$$G_{ra} \equiv \langle \varphi_r \varphi_a \rangle \quad \textbf{retarded}$$
$$G_{rr} \equiv \langle \varphi_r \varphi_r \rangle \quad \textbf{symmetric}$$

➢ **The free propagator $G^{(0)}_{ra}(t, \mathbf{k})$ decays exponentially for $t > 0$:**

$$G_{ra}(t, \mathbf{k}) = G^{(0)}_{ra}(t, \mathbf{k}) \sim e^{-i\mathfrak{w}(\mathbf{k})t}$$

$$\mathrm{Re}\,\eta \geq 0, \quad c, \Gamma \in \mathbb{R},$$
$$\Gamma > 0 \text{ if } \mathrm{Re}\,\eta = 0.$$

$$\mathfrak{w}(\mathbf{k}) = -i\eta - c|\mathbf{k}| - i\Gamma|\mathbf{k}|^2 + \ldots$$

Interaction effects

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$G_{rr} \equiv \langle \varphi_r \varphi_r \rangle$ **symmetric**

➢ **The corrections from interactions are given by**

$$\Delta G \equiv G - G^{(0)} = G^{(0)} \Sigma G^{(0)} + \ldots$$

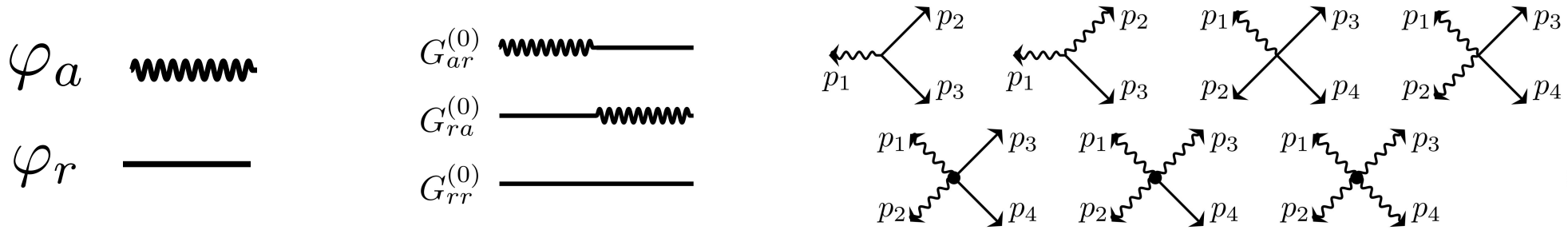
➢ **How do these corrections evolve?**

Answering this requires analyzing the analytic structure of Σ.

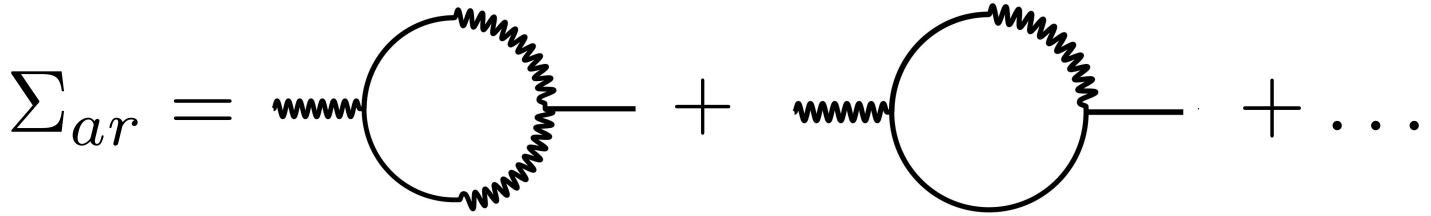
Perturbative expansion of self-energy

Perturbative expansion of self-energy

- **Three types of free propagators and a lot of interaction vertices:**



- **Self-energy is the sum of 1PI diagrams:**



- **The number of Feynman diagrams is large.**

Example: Nonlinear Brownian motion

- **Effective Lagrangian for nonlinear Brownian motion:**

  Glorioso and Liu, PoS TASI2017 (2018)

$$\mathcal{L}_{\text{eff}} = -x_a(\partial_t + \eta)x_r + iT_0 x_a^2 - \lambda x_a x_r^3 + \ldots$$

- **The corresponding equation of motion is:**

$$\partial_t x_r + \eta x_r + \lambda x_r^3 = \text{noise terms}$$

Damping coefficient: η **Nonlinear coupling: λ** **Background temperature: T_0**

Example: Nonlinear Brownian motion

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$$\mathcal{L}_{\text{eff}} = -x_a(\partial_t + \eta)x_r + iT_0 x_a^2 - \lambda x_a x_r^3 + \ldots$$

- **The number of 1PI Feynman diagrams:**

Phi4tools →

Number of loops	two	three	four	five	six	seven	eight
Phi4-theory	1	2	6	19	75	317	1622
Brownian motion (Σ_{ar})	1	5	31	283	3201	43611	692260

← DLW and Pu, in preparation

Sberveglieri and Spada, JHEP (2024)

Complicated loop integrals

➢ **The loop integral for a 1PI diagram contributing to Σ, with L loops and E internal edges (propagators), has the form**

$$I^d_{LE}(k) = \int_{-\infty}^{\infty} [\mathrm{d}p]\Omega \prod_{e=1}^{E} G_e^{(0)}(q_e)$$

$$= \int_{-\infty}^{\infty} [\mathrm{d}p]\Omega' \int_0^1 \prod_{e=1}^{E} d\alpha_e \frac{\delta(\Sigma_{e=1}^{E}\alpha_e - 1)}{\mathcal{W}^E}$$

$$[\mathrm{d}p] \equiv \prod_{l=1}^{L} \frac{d^{1+d}p_l}{(2\pi)^{1+d}}$$

$$\mathcal{W} \equiv \sum_{e=1}^{E} \alpha_e \mathrm{deno}[G_e^{(0)}(q_e)]$$

denominator

$$\alpha_e \in [0,1]$$

$$\alpha_1 + \ldots + \alpha_E = 1$$

- p_l denotes the l-th internal momentum.
- $G_e(q_e)$ denotes the e-th propagator carrying momentum q_e.
- Ω and Ω' are functions of coupling constants and momenta.

➢ **Each diagram involves a complicated loop integral.**

Determine singularities without integration

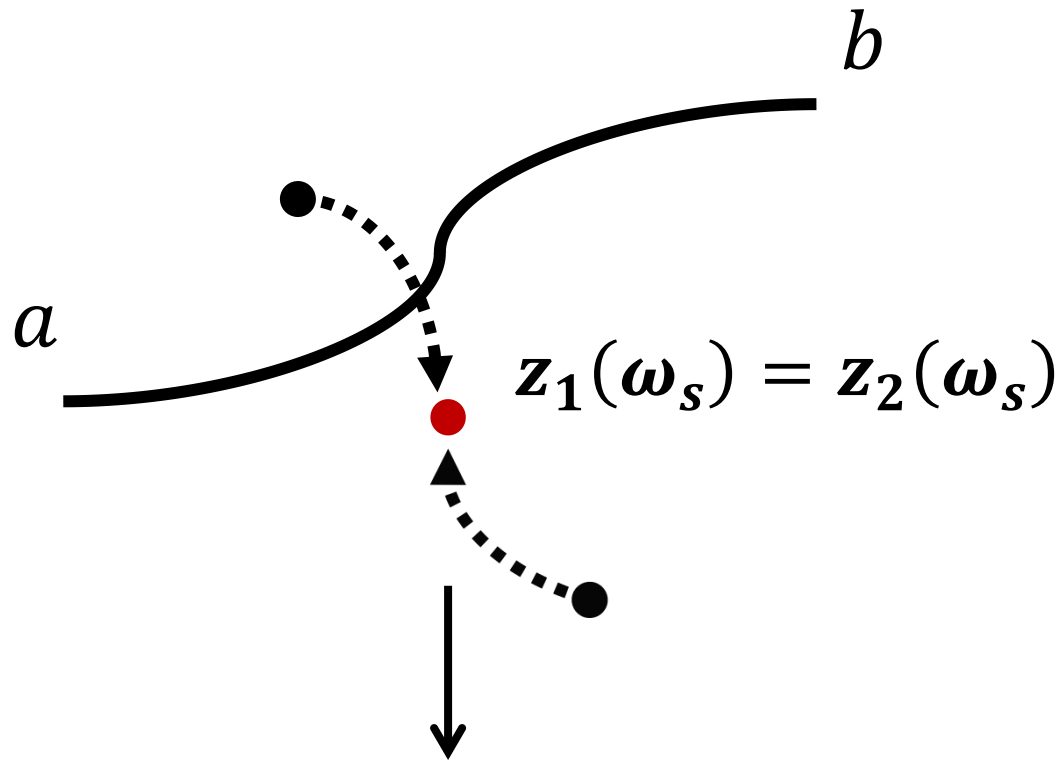
Singularities of a function

- Consider a function defined by

$$f(\omega) = \int_{C_{ab}} dz h(z, \omega)$$

- Suppose that the integrand has two poles at $z_1(\omega)$ and $z_2(\omega)$.

- When $\omega \to \omega_s$, the poles $z_1(\omega)$ and $z_2(\omega)$ approach each other from opposite sides of integration path.



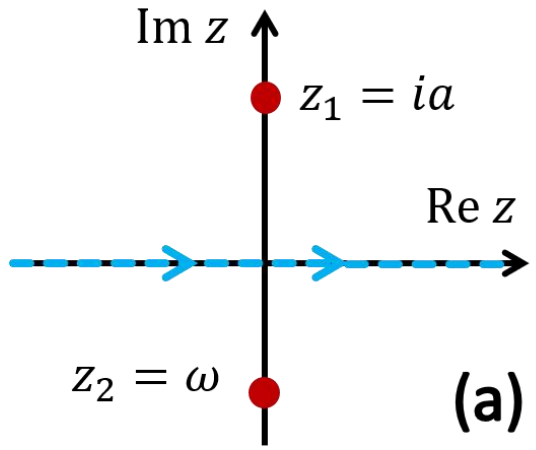
ω_s is a singularity of $f(\omega)$

Eden et al. *The Analytic S-Matrix* (1966)

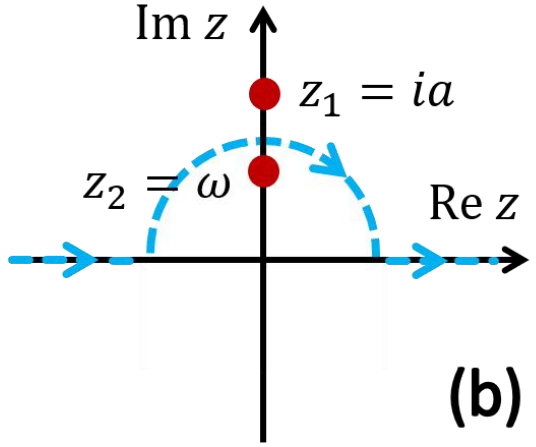
A simple example

- **Consider the integral:**

$$I(\omega) = \int_{-\infty}^{\infty} dz \frac{1}{(z - ia)(\omega - z)} = \frac{2\pi i}{\omega - ia}$$



- **Continuously change ω from $\mathrm{Im}\ \omega < 0$ to $\mathrm{Im}\ \omega > 0$ and deform the integration path (blue) to avoid the poles (red).**



- **As $\omega \to \mathrm{i}a$, the integration path becomes pinched between two poles. $\omega = \mathrm{i}a$ is a singularity.**

Landau equations

➢ **Singularities of $I^{d}_{LE}(k)$ are solutions of the Landau equations:**

Landau (1959); Eden et al. *The Analytic S-Matrix* (1966).

$$\alpha_e D_e(q_e) = 0, \quad e = 1, ..., E,$$

$$\sum_{e=1}^{E} \alpha_e \frac{\partial}{\partial p_l^{\mu}} D_e(q_e) = 0, \quad l = 1, ..., L.$$

k : external momentum,
p_l : loop momentum,
q_e : internal-line momentum,
$D_e(q_e)$ = denominator of $G_e^{(0)}(q_e)$

➢ **To determine the locations of singularities, it suffices to consider only solutions with $\alpha_e \in (0, 1)$ for all e.**

Landau equations

- **Singularities of $I^{d}_{LE}(k)$ are solutions of the Landau equations:**

  Landau (1959); Eden et al. *The Analytic S-Matrix* (1966).

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k : external momentum,
p_l : loop momentum,
q_e : internal-line momentum,
$D_e(q_e)$ = denominator of $G_e^{(0)}(q_e)$

- **This method was originally developed in quantum field theory. It has recently also been applied to hydrodynamics.**

  Abbasi, Davody, and Tahery, PRD (2024); Abbasi and Rischke, JHEP (2024);
  Abbasi, Kaminski, and Rischke, JHEP (2025)

Solving Landau equations: gapped mode

- **Apply this method to nonlinear Brownian motion, whose propagators are**

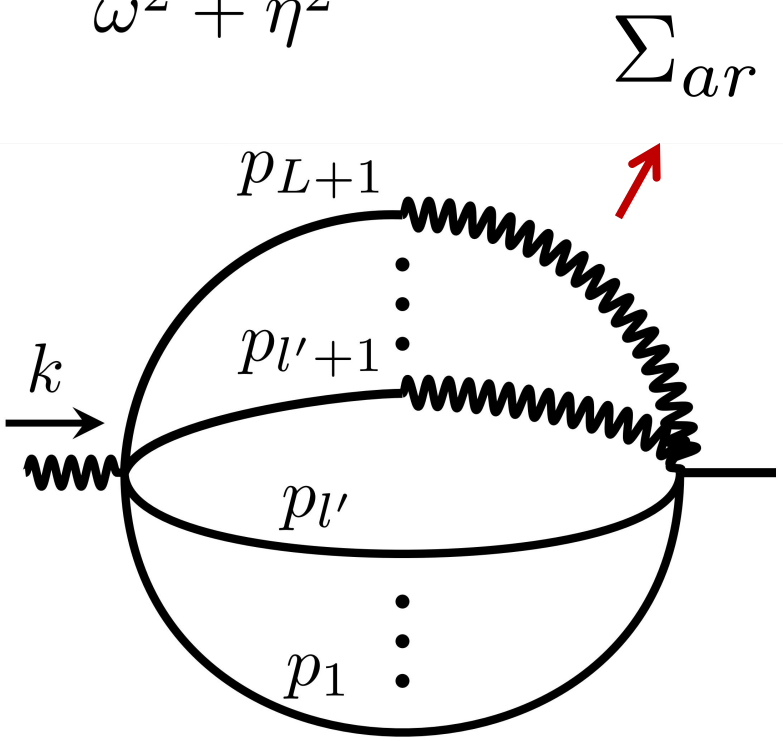
$$G^{(0)}_{ra}(\omega) = \frac{1}{\omega + i\eta}, \qquad G^{(0)}_{rr}(\omega) = \frac{c}{\omega^2 + \eta^2}$$

- **The linear mode is** $\mathfrak{w} = -i\eta$

- **Solving the Landau equations for Banana diagrams, we obtain the singularities:**

$$\omega_s = -i(L+1)\eta$$

Number of loops



Solving Landau equations: gapped mode

- Apply this method to nonlinear Brownian motion, whose free propagators are

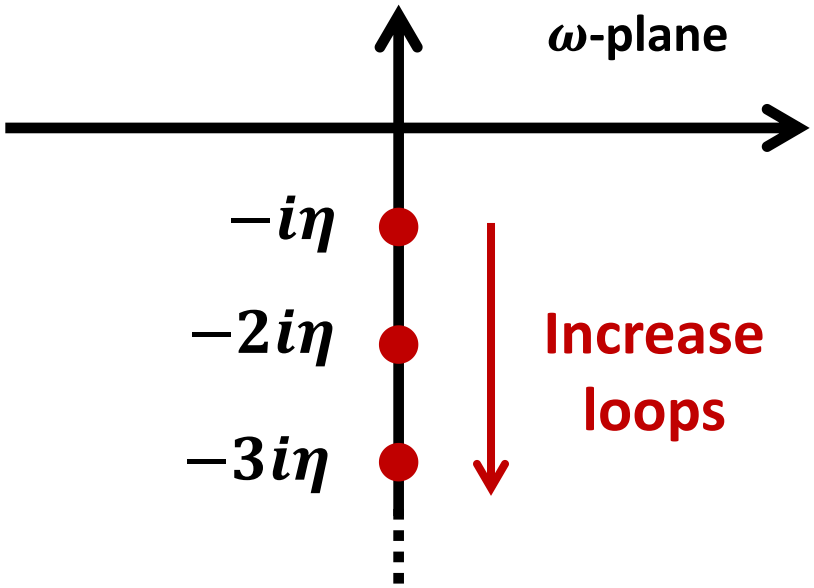
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Solving Landau equations: diffusion mode

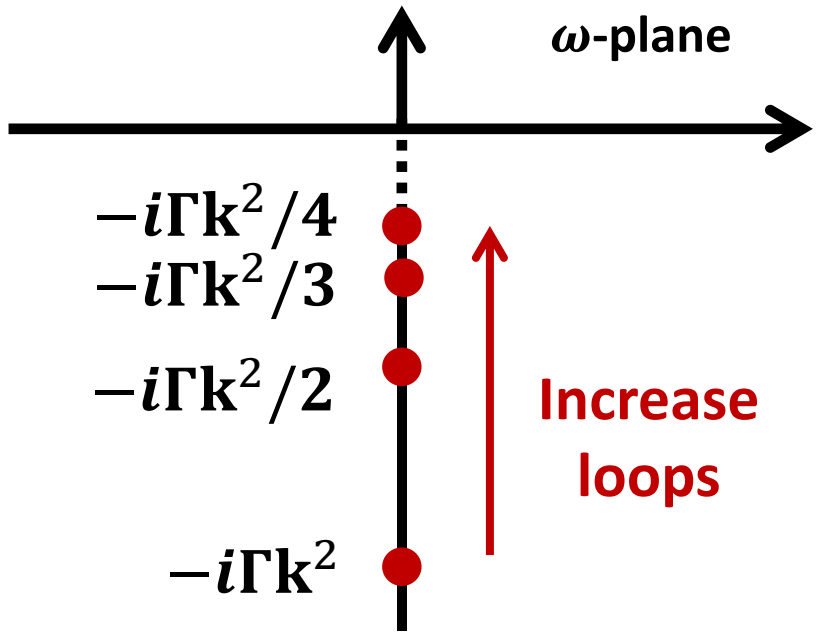
- In the diffusion theory with one diffusive charge, the linear mode is

$$\mathfrak{w}(\mathrm{p}) = -i\Gamma \mathrm{p}^2$$

↑ diffusive constant

- Solving the Landau equations for Banana diagrams, we obtain the singularities:

$$\omega_s = -i\frac{\Gamma}{L+1}\mathbf{k}^2$$



See also:
Delacrétaz, SciPost Phys (2020);
Grozdanov, Lemut, Pelaič, and Soloviev, PRD (2024)

Solving Landau equations: general mode

- **Generally, the linear mode has the following form, where the index e labels different lines in a diagram**

$$\Gamma_e \neq 0$$

$$\mathfrak{w}_e(\mathbf{p}) = -i\eta_e - c_e|\mathbf{p}| - i\Gamma_e|\mathbf{p}|^2 + \ldots$$

- **The singularities corresponding to Banana diagrams are**

$$\omega_s = -i\eta_{(s)} - c_{(s)}|\mathbf{k}| - i\Gamma_{(s)}|\mathbf{k}|^2 + \ldots$$

$$\eta_{(s)} = \sum_e \left[\eta_e + \frac{(c_e - c_{(s)})^2}{4\Gamma_e}\right], \quad c_{(s)} = \Gamma_{(s)} \sum_e \frac{c_e}{\Gamma_e}, \quad \Gamma_{(s)}^{-1} = \sum_e \Gamma_e^{-1}$$

Beyond Banana diagrams

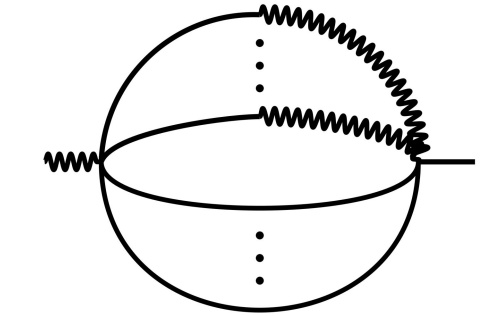
- **Suppose that η_e and c_e vanish. Singularities corresponding to general 1PI diagrams are:**

$$\omega_s = -\frac{i}{\mathcal{N}}\mathbf{k}^2, \quad \mathcal{N} = \min_{\lambda,\phi} \sum_{e:i\to j} \frac{1}{\Gamma_e} \frac{(\phi_j - \phi_i)^2}{\lambda_j - \lambda_i}$$

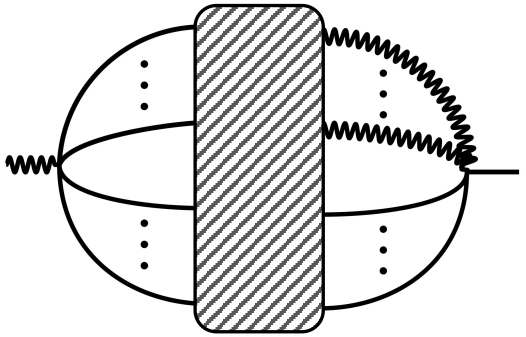
λ_i, ϕ_i : auxiliary functions of vertices,

$$\lambda_A, \phi_A = 0, \quad \lambda_B, \phi_B = 1,$$

$$\lambda_j - \lambda_i > 0 \text{ for } e : i \to j$$



diagrams with 2 vertices



diagrams with more vertices

Late-time relaxation

Results for (1+0)D systems

- In (1+0)-dimensional systems, all poles of the self-energy Σ_{ar} should be integer multiples of the basic pole:

$$\Sigma_{ar}(\omega) \propto \prod_n \frac{1}{(\omega + in\eta)^{k_n}} \rightarrow k_n\text{: integer}$$

- In (1+0)-dimensional systems, interactions induce new decaying modes with rates that are integer multiples of the basic decay rate:

$$\Delta G(t) \propto \sum_n g_n e^{-n\eta t}$$

Singular behavior in (1+d)D

- **The singular part of the loop integral $I(\boldsymbol{k})$ arises from the integration region around the solution of Landau equations.**

  Polkinghorne and Screaton, Nuovo Cim. (1960)

- **The dominant singular part takes the form:** $\sigma \equiv Ld - V + \kappa$

$$I(\omega_s + \delta\omega, \mathbf{k}) \propto \begin{cases} (\delta\omega)^{\sigma/2} \ln \delta\omega & \sigma = 0, 2, 4, \ldots, \\ (\delta\omega)^{\sigma/2} & \text{otherwise.} \end{cases}$$

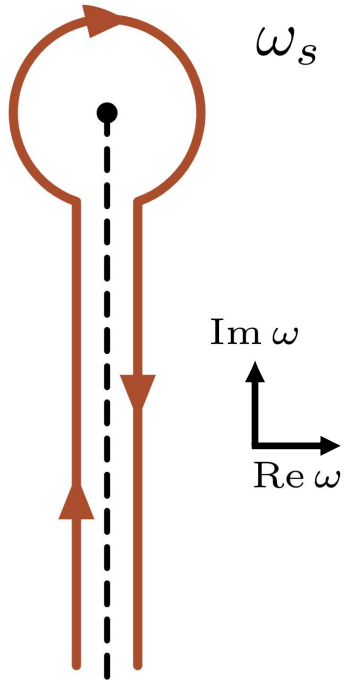
L: number of loops, V: number of vertices, κ: number of vanishing α_e

- **In most cases, the singularities are branch points.**

Late-time relaxation in (1+d)D

- **Performing the inverse Fourier transform from ω to t and deforming the contour, we obtain the late-time behavior**

$$\Delta G(t,\mathbf{k}) \simeq \sum_{L,V,\kappa,\omega_s} \sum_{j\geq 0} \mathcal{C}(\mathbf{k}) t^{-1-j-\sigma/2} e^{-i\omega_s(\mathbf{k})t}$$



- **For gapless singularities, as $\mathbf{k}\to\mathbf{0}$, the dominant contribution of each singularity is**

$$\propto t^{-1-\sigma/2} \qquad \sigma \equiv Ld - V + \kappa$$

L: number of loops, V: number of vertices, κ: number of vanishing α_e

Late-time relaxation in (1+d)D

- As $\mathbf{k} \to \mathbf{0}$, the contribution from the banana diagram ($V = 2$) with the minimum number of loops ($L = L_{\min}$) and the leading Landau singularity ($\kappa = 0$) is:

$$\propto t^{-L_{\min} d/2}$$

- The number, $L_{\min}$, is also related to the lowest-order nonlinearity in the equation of motion:

$$\mathcal{R}\varphi_r + \boxed{\lambda \varphi_r^{1+L_{\min}}} + \mathcal{O}(\varphi^{2+L_{\min}}) = 0$$

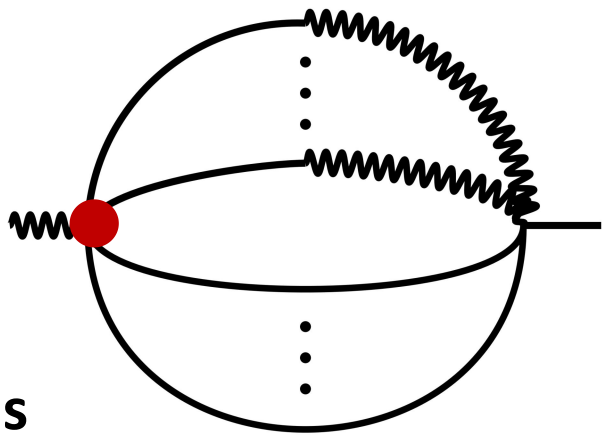
Late-time relaxation in (1+d)D

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$$\mathcal{R}\varphi_r + \boxed{\lambda\varphi_r^{1+L_{\min}}} + \mathcal{O}(\varphi^{2+L_{\min}}) = 0$$

- **A nonlinear term in the equation of motion corresponds to an interaction term in the Lagrangian:**

$$\mathcal{L}_{\text{int}} = \boxed{\lambda\varphi_a\varphi_r^{1+L_{\min}}} + \dots$$



One a-field

$1 + L_{\min}$ r-fields

Summary

Summary

- Interactions generate nonzero self-energy, whose singularities correspond to nonlinear relaxation modes.
- Landau singularity analysis identifies the singularities by solving algebraic equations, without explicit loop integration.
- Nonlinear nonhydro modes decay faster than their linear counterparts, whereas nonlinear hydro modes decay more slowly.
- The power-law decay exponent is determined by spatial dimension and lowest nonlinearity in equations of motion.

Backup

Regularization

- **Regularization in loop diagrams:**

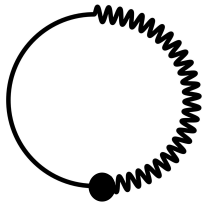
  Gao, Glorioso and Liu, JHEP (2020)

  r: a positive integer Λ: frequency cutoff

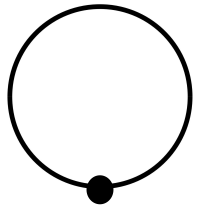
$$G^{(0)}_{ra}(\omega,\vec{p}) \to G^{(0)}_{ra}(\omega,\vec{p})\left(\frac{\Lambda}{\Lambda-i\omega}\right)^r \qquad G^{(0)}_{ar}(\omega,\vec{p}) \to G^{(0)}_{ar}(\omega,\vec{p})\left(\frac{\Lambda}{\Lambda+i\omega}\right)^r$$

- **Loop diagrams with only 1 vertex are regularized to zero:**

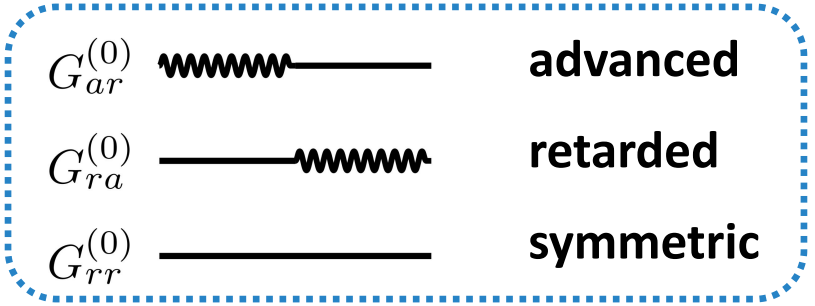
  Jain and Kovtun, PRL (2022)



$$\propto \int_{-\infty}^{\infty} d\omega G_{ra} \to 0$$



$$\propto \int_{-\infty}^{\infty} d\omega G_{rr} = \int_{-\infty}^{\infty} d\omega (c_1 G_{ra} + c_2 G_{ar}) \to 0$$



Simplify Landau equations

- **The original Landau equations are**

$$\alpha_e D_e(q_e) = 0, \quad e = 1, ..., E,$$

$$\sum_{e=1}^{E} \alpha_e \frac{\partial}{\partial p_l^\mu} D_e(q_e) = 0, \quad l = 1, ..., L.$$

- **If $D_e(q_e)$ is a polynomial of q_e, we can write it as**

$$D_e(q_e) = \prod_{i=1}^{n_e} f_e^{(i)}(q_e) \qquad f_e^{(i)}(q_e) = q_e^0 - \mathfrak{w}_e^{(i)}(\mathbf{q}_e)$$

- **Simple zeros:** $f_e^{(j)}(q_e) \implies f_e^{(i)}(q_e) \neq 0 \quad (i \neq j)$

Simplify Landau equations

- **For leading Landau singularities ($\alpha_e \neq 0$), the Landau equations become**

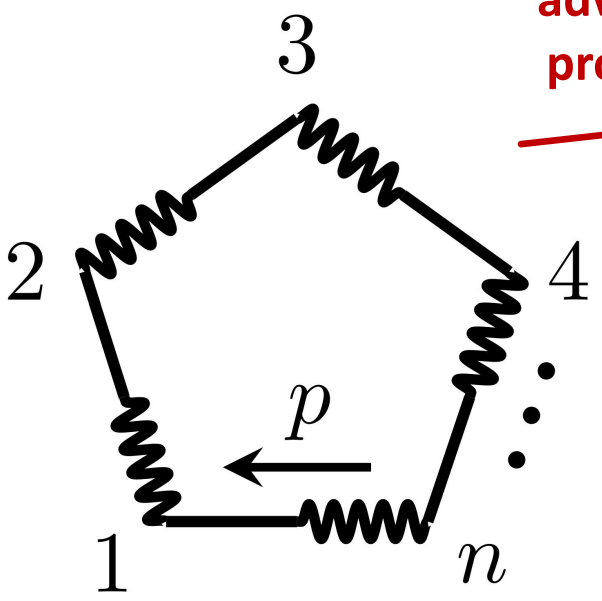
$$D_e(q_e) = 0 \;\Rightarrow\; f_e^{(j_e)}(q_e) = 0,$$

$$\sum_e \alpha_e \frac{\partial D_e(q_e)}{\partial p_l^\mu} = 0 \;\Rightarrow\; \sum_e \alpha_e^{(j_e)} \frac{\partial f_e^{(j_e)}(q_e)}{\partial p_l^\mu} = 0.$$

$$\alpha_e > 0, \quad \sum_e \alpha_e = 1 \qquad\qquad \alpha_e^{(j_e)} = \alpha_e \prod_{i \neq j_e} f_e^{(i)}(q_e)$$

Vanishing diagrams

- If a diagram contains a subloop which is made of only advanced propagators, then the corresponding loop integral is zero.



advanced property

$$t_1 < t_2 < t_3 < \ldots < t_n < t_1$$

$$\sim \int dt_1 \ldots dt_n G_{ar}(t_1, t_2) \ldots G_{ar}(t_n, t_1) = 0$$