Correlation functions in stable first-order relativistic hydrodynamics

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Why Do Fluctuations Matter Near the QCD Critical Point?

Near the QCD critical point, fluctuations become enhanced and slow, and can no longer be treated as negligible corrections to hydrodynamic evolution.

In particular, higher-order fluctuations are directly connected to experimentally relevant observables such as cumulants of conserved charges.

This motivates studying not only the deterministic evolution of the fluid, but also the dynamics and correlation functions of its fluctuations.

$$\boxed{\text{critical fluctuations} \longrightarrow \text{correlation functions and cumulants}}$$

Although we do not study the QCD critical point itself, nonlinear fluctuations are expected to play an important role in its vicinity. This motivates us to isolate nonlinear mode coupling and investigate how it shapes the correlation functions in causal relativistic hydrodynamics.

M. A. Stephanov, *Phys. Rev. Lett.* 102, 032301 (2009); N. Abbasi and M. Kaminski, *Phys. Rev. D* 106, 016004 (2022)

Motivation

Hydrodynamics provides an effective description of relativistic systems in the long-wavelength and low-frequency regime.

However, conventional first-order relativistic hydrodynamics, such as the Eckart and Landau–Lifshitz formulations, suffers from fundamental problems:

- **instability** of thermal equilibrium,
- **acausal** propagation of perturbations.

These problems are associated with the behavior of the theory beyond the hydrodynamic regime, and therefore with its **ultraviolet behavior**.

This motivates the search for a first-order relativistic formulation that is both stable and causal.

$$\boxed{\text{conventional first-order hydro} \longrightarrow \text{instability} + \text{acausality}}$$

W. A. Hiscock and L. Lindblom, *Ann. Phys.* 151, 466 (1983); *Phys. Rev. D* 31, 725 (1985)

A. Ota, *Phys. Rev. D* 114, L011902 (2026); *Phys. Rev. D* 114, 016001 (2026).

BDNK: Deterministic Equations and Linear Modes

BDNK modifies first-order constitutive relations to obtain stable and causal relativistic hydrodynamics.

For a conformal fluid,

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p\, g^{\mu\nu} + \Pi^{\mu\nu}, \qquad \partial_\mu T^{\mu\nu} = 0.$$

Here, $\Pi^{\mu\nu}$ contains gradient corrections controlled by $\bar{\eta}$, $\bar{\theta}$, and $\bar{\pi}$.

Linearizing around equilibrium gives two channels:

Sound	*and*	Shear

Besides the usual hydrodynamic modes, BDNK also has non-hydrodynamic modes.

F. S. Bemfica, M. M. Disconzi and J. Noronha, *Phys. Rev. D* 100, 104020 (2019); *Phys. Rev. X* 10, 021044 (2020).

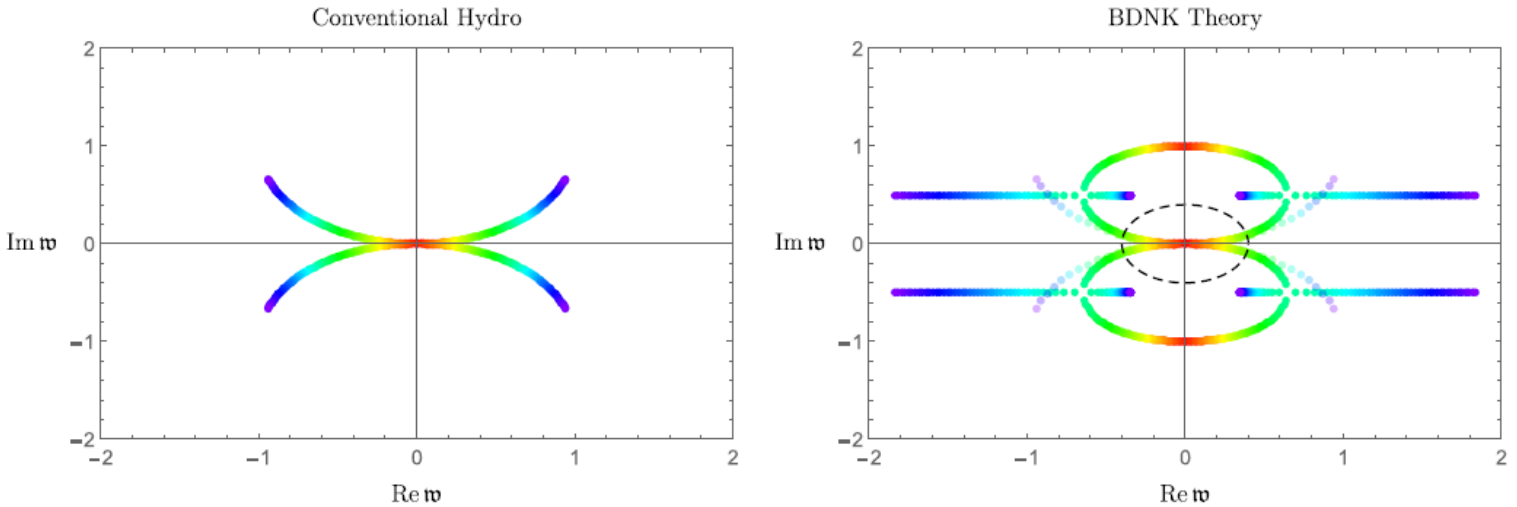
Linearized BDNK Modes

Linearizing around thermal equilibrium, the equations separate into sound and shear channels.

The poles of the correlation functions reveal the hydrodynamic and non-hydrodynamic modes.

We compare conventional hydrodynamics with BDNK, which introduces additional modes to ensure stable and causal evolution.

Linear-Response Pole Structure



The correlation functions contain retarded poles below and advanced poles above the real axis.

Left: conventional hydrodynamics

where the hydrodynamic mode is $\mathfrak{w} = -iD_\eta \mathfrak{q}^2$, for small $\mathfrak{q}$.

Right: BDNK

The first red mode, at the origin, reproduces the hydrodynamic mode shown on the left. The second red mode, lower on the imaginary axis, is an additional non-hydrodynamic BDNK mode.

The dashed circle highlights the matching hydrodynamic mode. The additional modes are associated with the modifications that allow stable and causal evolution.

Why Fluctuations and Correlation Functions?

So far, we have considered the deterministic BDNK equations and their linear excitation spectrum.

However, deterministic evolution alone does not characterize the thermal fluctuations of a hydrodynamic system.

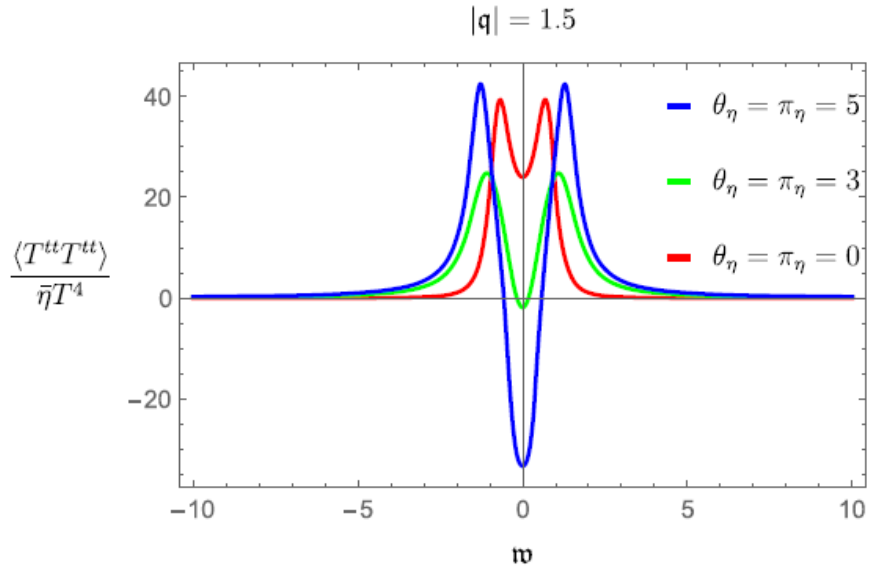
Their correlations can be described by

$$G_{AB}(\omega, \mathbf{k}) = \langle \delta A(\omega, \mathbf{k}) \delta B(-\omega, -\mathbf{k}) \rangle \, .$$

Correlation functions quantify the amplitudes and correlations of fluctuations, providing information beyond the locations of the deterministic modes.

They therefore provide an independent test of the physical consistency of the fluctuation sector.

A Problem Revealed by the Linear Correlator



Energy density correlation function in the Landau-Lifshitz frame (red), in an unstable frame (green), and in a stable frame (blue).

Energy-density correlator:

$$G_{T_{tt}T_{tt}}(\mathfrak{w}, \mathfrak{q})$$

At fixed momentum, the correlator becomes negative as

$$\mathfrak{w} \to 0.$$

For a sufficient stable and causal parameter choice,

$$\boxed{\bar{\theta} > 4\bar{\eta}, \qquad \bar{\pi} > 4\bar{\eta}}$$

the small-frequency limit is negative for any fixed $\mathfrak{q}^2$.

Central issue

$$\boxed{\text{stability} + \text{causality} \not\Rightarrow \text{consistent fluctuation theory}}$$

This negative correlator is a pathology of the BDNK fluctuation sector, indicating an inconsistency despite the theory's deterministic stability and causality.

N. Abbasi, A. Davody and S. Tahery, *Phys. Rev. D* 109, 036006 (2024)

From Linear Correlators to Nonlinear Fluctuations

How does the additional BDNK non-hydrodynamic shear mode modify nonlinear mode coupling?

$$\text{Nonlinear mode coupling} \longrightarrow \text{Branch cuts} \longrightarrow \text{Long-time tails}$$

Near the QCD critical point, nonlinear fluctuation effects become particularly important, motivating our study of nonlinear mode coupling even without critical dynamics.

Xin An, *Nucl.Phys.A* 1005 (2021) 121957

- **Why nonlinearities matter**
  At nonlinear order, hydrodynamic fluctuations interact with each other. Correlation functions therefore receive loop contributions.
- **For Gaussian fluctuations**, four-point functions factorize:

$$\langle A_1 A_2 A_3 A_4 \rangle \sim \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle + \cdots$$

  This reduces the nonlinear contribution to products of linear two-point functions.
- **Loop structure**

$$G_{\text{nonlinear}} \sim \int d\omega'\, d^3q\, G_1(\omega', q)\, G_2(\omega - \omega', |\mathbf{k} - \mathbf{q}|).$$

  The singularities of these loop integrals determine the non-analytic structure of the correlation functions. *We aim to determine the analytic structure without explicitly evaluating the loop integral.* Let's consider the shear sector as an example. In this case, G_1 becomes the two-point function $\langle T_{xy} T_{xy} \rangle$, which has been studied by Kovtun in linear response theory and expanded to first order. To study the nonlinear case, we need to expand it to second order, as shown in the next slide.

Nonlinear contribution

For the nonlinear contribution considered here, the relevant part of the shear stress starts at quadratic order in the velocity fluctuations:

$$T_{xy} = 4\bar{p}T^4\,\delta v_x\,\delta v_y.$$

Correlation function

The corresponding two-point function becomes a four-point function of the velocity fluctuations:

$$\langle T_{xy}(x)T_{xy}(0)\rangle \sim \langle \delta v_x \delta v_y \delta v_x \delta v_y\rangle\,.$$

For Gaussian fluctuations, this factorizes into products of linear two-point functions:

$$\langle \delta v_x \delta v_y \delta v_x \delta v_y\rangle \sim \langle \delta v_x \delta v_x\rangle \langle \delta v_y \delta v_y\rangle + \cdots$$

For the first time, we develop this loop-integral approach to nonlinear hydrodynamic correlation functions.

Branch Cuts from On-Shell Conditions

The nonlinear shear-stress correlator contains the loop integral

$$\mathcal{I}(\mathfrak{w}, \mathfrak{q}) \sim \int d\mathfrak{w}'\, d^3q'\, \frac{\mathcal{N}(\mathfrak{w}')}{\mathcal{D}_T(\mathfrak{w}', \mathfrak{q}')} \frac{\mathcal{N}(\mathfrak{w} - \mathfrak{w}')}{\mathcal{D}_T(\mathfrak{w} - \mathfrak{w}', \mathbf{q} - \mathbf{q}')}.$$

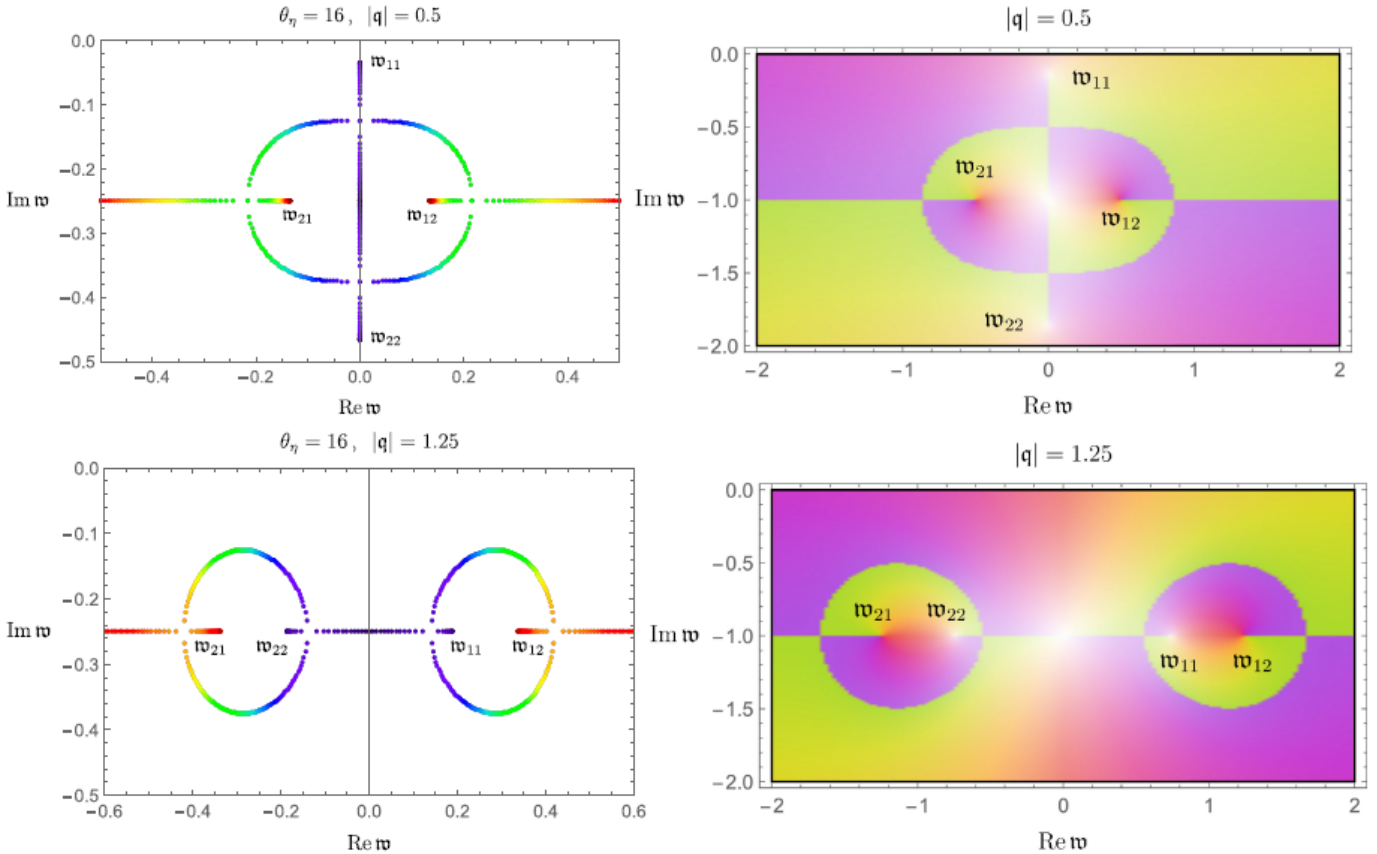
The singularities of the integrand occur when both internal lines go on shell:

$$\boxed{\mathcal{D}_T(\mathfrak{w}', \mathbf{q}') = 0, \qquad \mathcal{D}_T(\mathfrak{w} - \mathfrak{w}', \mathbf{q} - \mathbf{q}') = 0}$$

These on-shell conditions determine the possible combinations of the two shear modes. Following Landau (1959), we use his method to locate branch points at $\mathfrak{w}_i(q)$. Landau provided conditions for finding these branch points, but did not explicitly calculate the resulting branch cuts.

L. D. Landau, *Nucl. Phys.* 13, 181 (1959).

Numerical Branch-Cut Structure



Left panel: branch-cut structure of $G^R_{T_{xy}T_{xy}}$, found from solving the on shell equations. Four branch point singularities $\mathfrak{w}_{11}$, $\mathfrak{w}_{22}$, $\mathfrak{w}_{12}$, and $\mathfrak{w}_{12}$ correspond to the four parts of $\mathcal{S}_\mathfrak{q}$, shown in Fig. 4. Right panel: branch-cut structure of G^R_{nn}, found from explicit loop calculations. The right panel plots are taken from Phys. Rev. Lett. **132**, 131602

$\mathfrak{w} = \mathfrak{w}_{ij}(\mathfrak{q}), \qquad i,j = 1,2.$ The branch cuts are obtained numerically from the on-shell conditions, without explicitly evaluating the response function. *Our branch cuts agree with the explicit loop-integral results of Navid Abbasi et al Phys.Rev. Lett.* ***132*** *131602.*

Landau conditions

Threshold singularities occur when the internal lines are on shell and the loop integration is pinched:

$$D_i(q_i) = 0, \qquad \sum_i \alpha_i \frac{\partial D_i}{\partial q^\mu} = 0, \qquad \alpha_i \geq 0.$$

Four mode combinations

Since the shear channel has two modes, the two internal lines can combine as $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$ giving four threshold singularities $\mathfrak{w}_{11}$, $\mathfrak{w}_{12}$, $\mathfrak{w}_{21}$, $\mathfrak{w}_{22}$.

Classification

The 11 and 22 singularities are normal thresholds, while 12 and 21 are second-type singularities.

Landau's 1959 equations were derived for relativistic scalar field theory, which is Lorentz covariant. In hydrodynamics, frequency $\mathfrak{w}$ and spatial momentum $\mathbf{q}$ play distinct roles, and the symmetry is reduced to spatial rotations, $SO(3)$. We extend Landau's method to hydrodynamic correlation functions, where the absence of Lorentz covariance substantially changes the analysis.

Among the threshold singularities, the $\mathfrak{w}_{11}$ branch point is closest to the real axis and therefore controls the late-time behavior.
For small momentum,

$$\boxed{\mathfrak{w}_{11}(\mathfrak{q}) = -i\,\frac{\mathfrak{q}^2}{2} + \mathcal{O}(\mathfrak{q}^4)}$$

Thus the dominant singularity approaches the origin as

$$\mathfrak{q} \to 0.$$

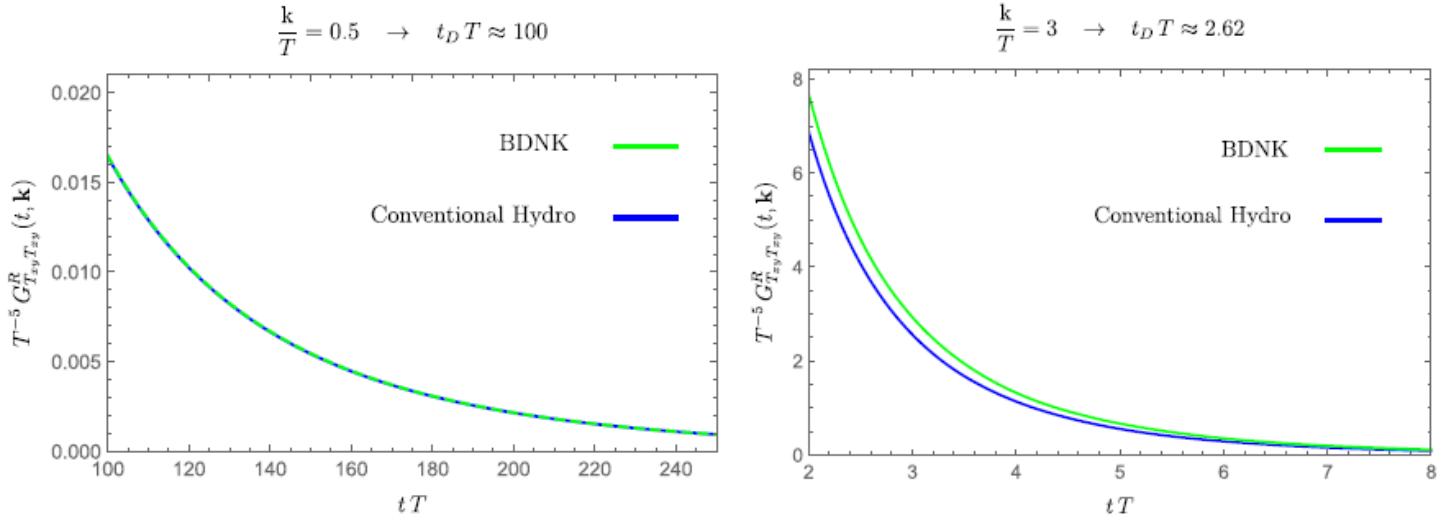
The corresponding contribution in the time domain is therefore

$$G^R(t, \mathbf{q}) \sim e^{-i\mathfrak{w}_{11}(\mathfrak{q})t} \times (\text{power law}),$$

showing that the nonlinear fluctuations generate a parametrically long-lived contribution.

Xinyi Chen-Lin, Luca V. Delacrétaz, Sean A. Hartnoll *Phys.Rev.Lett.* 122 (2019) 9, 091602

Long-Time Tail: BDNK vs Conventional Hydrodynamics



Long-time tail comparison between BDNK and the conventional first-order relativistic hydrodynamics for $\frac{\theta}{s} = \frac{4\eta}{s} = \frac{1}{\pi}$. Both panels have been illustrated for times larger than the diffusion time of the transverse momentum, i.e., $t \gtrsim t_D$. Left panel: for relatively small momenta, nonlinear effects in BDNK theory are consistent with the same effects in the conventional hydro. Right panel: at large momentum, the nonlinear effects in BDNK theory lead to slower decay of the response function than those in conventional hydro.

Hydrodynamic regime: At small momentum, the leading long-time tail is the same in BDNK and conventional hydrodynamics: $G^R_{T_{xy}T_{xy}}(t, \mathbf{k}) \sim w^{1/2} \frac{e^{-\frac{1}{2}\gamma_\eta \mathbf{k}^2 t}}{(\gamma_\eta t)^{3/2}}$. Thus, despite the pathology identified in the linear correlator, it does not manifest itself in the leading nonlinear long-time tail in the hydrodynamic regime.

Beyond the hydrodynamic regime: At large momentum, the UV regulator modifies the decay, but this lies outside the validity range of hydrodynamics.

Conclusion

- Causality and stability of deterministic hydrodynamics do not guarantee physically consistent fluctuations. The energy-density correlator becomes negative in the fixed-momentum, small-frequency limit.
- The additional BDNK non-hydrodynamic shear mode modifies the nonlinear analytic structure, producing additional threshold singularities and branch cuts. Nevertheless, the leading hydrodynamic long-time tail agrees with conventional hydrodynamics.

The nonlinear calculation presented here does not resolve the linear-correlator pathology; a fully consistent treatment of nonlinear fluctuations remains an important open question.

N. Mullins, M. Hippert, L. Gavassino and J. Noronha, *Phys. Rev. D* 108, 116019 (2023);

L. Gavassino, N. Mullins and M. Hippert, *Phys. Rev. D* 109, 125002 (2024)


Thank you for your attention.