Is First-Order Hydrodynamics Really Unstable? — Mode Analysis as an IR Effective Theory

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Roadmap

1. The stability puzzle
2. Hydrodynamics as a rigorous PDE vs an IR EFT
3. EFT for dissipative fluids
4. Frame-invariant mode analysis
5. Boosted frames
6. Conclusions



Motivation: the stability puzzle

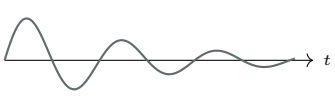
First, the textbook problem: first-order theory looks unstable.

Step 1– Plane-wave expansion of perturbations around equilibrium:

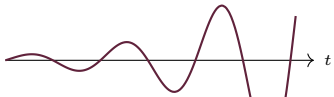
$$\delta\Phi \propto e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Step 2– Solve the dispersion relation and find

$$\omega = \omega(k).$$



$\mathrm{Im}\,\omega < 0$
stable



$\mathrm{Im}\,\omega > 0$
unstable

[Hiscock, Lindblom (1985)]

Instability in first order hydrodynamics

Evolve a dissipative relativistic fluid as a closed system of PDEs built from conservation laws:

$$\nabla_\mu T^{\mu\nu} = 0, \qquad \nabla_\mu J^\mu = 0.$$

Then, this is unstable.

	Hydrodynamic frame		
Inertial frame	**Landau**	**Eckart**	**others**
comoving	✓ **stable**	× unstable	× unstable
boosted	× unstable	× unstable	× unstable

[Hiscock, Lindblom (1985)]

Choose the hydrodynamic frame judiciously so that a genuinely *first-order* PDE system is

- stable,
- causal, and
- well-posed,

without introducing any new dissipative variables.

> BDNK is a prescription for turning first-order hydrodynamics into a well-behaved PDE system.

[Bemfica, Disconzi, Noronha (2018); Kovtun (2019)]



Reframing hydrodynamics as an IR effective theory

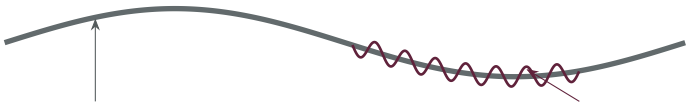
The fix: stop treating hydrodynamics as exact — treat it as an IR EFT.

Hydrodynamics is not a fundamental theory

Solving the hydrodynamic PDEs exactly treats hydrodynamics as a fundamental theory, not as an IR effective theory.

Nobody solves truncated chiral perturbation theory as an exact PDE.

Is first-order hydrodynamics any different?



Background
fluid configuration itself

Perturbations
d.o.f. around a given configuration

$\mathcal{O}(\varepsilon)$ for the constitutive relation $\not\Longrightarrow$ $\mathcal{O}(\varepsilon)$ for the perturbation EoM

Questions as physics:

- Is the effective theory of perturbations on a fluid background stable at first order in the derivative expansion?
- Does its behavior depend on the hydrodynamic frame or the inertial frame?

[AO (2026)]



Effective theory for dissipative hydrodynamics

Write the most general first-order theory; its signs are fixed by the UV.

Hydrodynamics as an EFT of Nambu–Goldstone modes

	Chiral perturbation theory	**Hydrodynamic EFT**
UV theory	QCD	microscopic U(1) QFT
Symmetry	$SU(N_f)_L \times SU(N_f)_R$	(Poincaré $\times U(1))_\pm$
Broken by	$\langle \bar{q} q \rangle$	β^μ: thermal state selects a rest frame
SSB pattern	$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$	?
NG modes	$U = e^{i\pi^a \lambda^a / f_\pi}$	$\beta^\mu = e^{-\pounds_\pi} \beta^\mu_c$
Scales	$f_\pi, \ \Lambda_\chi \sim 4\pi f_\pi$	$M^4_{\rm SSB} \sim \rho + P, \ M^3_{\rm diss} \sim \eta$
Expansion	p/Λ_χ	$\varepsilon \sim \omega M^3_{\rm diss}/M^4_{\rm SSB}$

[Gasser, Leutwyler (1984); Dubovsky, Hui, Nicolis, Son (2011); Crossley, Glorioso, Liu (2017)]

Which symmetries does a thermal state break?

$G_+ \times G_-$: strong symmetry
G_{diag}: weak symmetry

Symmetry	**Weak**	**Strong**	$\mathcal{O}(\varepsilon^0)$	$\mathcal{O}(\varepsilon^1)$
time translation	unbroken	broken	energy conservation	–
space translations	unbroken	broken	momentum conservation	shear diffusion
rotations	unbroken	broken	–	–
$U(1)$	unbroken	broken	charge conservation	charge–energy diffusion
boosts	broken	broken	sound	diffusive sound

[Huang, Qi, Zhang, Lucas (2025); Dubovsky, Hui, Nicolis, Son (2011)]

Schwinger–Keldysh quadratic action in the gradient expansion

KMS: thermodynamic relation

$$I^{(2)} = \underbrace{\int d^4x \Big[-(\rho + P)\, \dot{\pi}_{Ai} \dot{\pi}^i - \beta P_\beta\, \partial^i \pi_{Ai}\, \dot{\pi}^0 + \cdots \Big]}_{\mathcal{O}(\varepsilon^0)\text{: ideal fluid}}$$

$$+ \underbrace{\int d^4x \Big[\eta\, \partial_j \pi_{Ai} \big(\partial^j \dot{\pi}^i + \partial^i \dot{\pi}^j \big) + \kappa_2\, \dot{\pi}_{Ai} \ddot{\pi}^i + \cdots \Big]}_{\mathcal{O}(\varepsilon^1)\text{: dissipation}}$$

$$+ \underbrace{i \int d^4x \Big[w_5\, (\partial_j \pi_{Ai})^2 + w_4\, (\partial^i \pi_{Ai})^2 + \cdots \Big]}_{\mathcal{O}(\varepsilon^1),\ \mathcal{O}(\pi_A^2)\text{: noise}}$$

KMS: FDR
$w_5 = \eta/\beta,\ \ldots$

unitarity + $\hat{D} \succeq 0$: $\mathrm{Im}\, I \geq 0$

[AO (2026)]

Hydrodynamic-frame invariants

Physical predictions must be built from frame-invariant combinations:

$$\tilde{\lambda} = \lambda_2 - \left(\frac{\partial P}{\partial \rho}\right)_n \epsilon - \beta \left(\frac{\partial P}{\partial n}\right)_\rho \nu_1, \qquad \tilde{\tau}_1 = \tau_1 - \frac{n}{\beta(\rho + P)}\kappa_1,$$

$$\tilde{\zeta} = \zeta - \left(\frac{\partial P}{\partial \rho}\right)_n \lambda_1 - \beta \left(\frac{\partial P}{\partial n}\right)_\rho \nu_2, \qquad \tilde{\tau}_2 = \tau_2 - \frac{n}{\beta(\rho + P)}\kappa_2,$$

$$\tilde{\nu} = \nu_5 - \left(\frac{\partial P}{\partial \rho}\right)_n \nu_4 - \beta \left(\frac{\partial P}{\partial n}\right)_\rho \nu_3, \qquad \tilde{\chi}_1 = \chi_1 - \frac{n}{\beta(\rho + P)}\tau_3,$$

$$\eta \quad \text{(invariant by itself)}, \qquad \tilde{\chi}_2 = \chi_2 - \frac{n}{\beta(\rho + P)}\tau_4.$$

[Kovtun (2019)]

KMS, unitarity, and state positivity of the UV theory imply

$$\tilde{\tau}_1 = \tilde{\tau}_2, \quad \eta \geq 0, \quad \tilde{\chi}_1 - \frac{n}{\beta(\rho + P)}\tilde{\tau}_2 \geq 0, \quad \tilde{\zeta} - \left(\frac{\partial P}{\partial \rho}\right)_n \tilde{\lambda} - \beta \left(\frac{\partial P}{\partial n}\right)_\rho \tilde{\nu} \geq 0.$$

These are exactly the ingredients that will guarantee stability later.

[Crossley, Glorioso, Liu (2017); AO (2026)]



Mode analysis: a frame-invariant dispersion relation

Solve the dispersion relation order by order in gradients: the result is frame invariant.

Eliminating redundant modes

$\delta I^{(2)} = 0$ gives a formally sixth-order dispersion relation:

$$\sum_{n=0}^{6} d_n \, \omega^n = 0.$$

- Solving this exactly reproduces the known gapped, unstable mode with $\mathrm{Im}\,\omega \propto \kappa_2^{-1}$
- But this over-interprets a truncated first-order theory

We must instead solve *perturbatively* in the derivative-counting parameter ε:

$$\omega = \omega_0 + \varepsilon\,\omega_1 + O(\varepsilon^2).$$

[Hiscock, Lindblom (1985); AO (2026)]

Leading order: the sound speed

The leading-order solution is fixed purely by zeroth-order thermodynamics:

$$\omega_0 = 0, \pm c_{s/n}\, k, \qquad c_{s/n}^2 \equiv \left(\frac{\partial P}{\partial \rho}\right)_{s/n}.$$

- The adiabatic sound speed is completely insensitive to any first-order (dissipative) coefficient
- Thermodynamic stability implies $c_{s/n}^2 \geq 0$

Diffusive sound –

$$\omega_1 = -i\,\Gamma_L\,k^2,$$

with

$$\Gamma_L = \frac{1}{2(\rho+P)}\Bigg[\frac{4}{3}\underbrace{\eta}_{\geq 0} + \underbrace{\tilde{\zeta} - \left(\frac{\partial P}{\partial \rho}\right)_n \tilde{\lambda} - \beta\left(\frac{\partial P}{\partial n}\right)_\rho \tilde{\nu}}_{\geq 0}$$
$$+ \frac{\beta^2}{c_{s/n}^2}\left(\frac{\partial P}{\partial n}\right)_\rho^2 \underbrace{\left(\tilde{\chi}_1 - \frac{n}{\beta(\rho+P)}\tilde{\tau}_2\right)}_{\geq 0} + \frac{\beta}{c_{s/n}^2}\left(\frac{\partial P}{\partial n}\right)_\rho\left(\frac{\partial P}{\partial \rho}\right)_n \underbrace{(\tilde{\tau}_1 - \tilde{\tau}_2)}_{=0}\Bigg].$$

This is hydrodynamic-frame invariant. [AO (2026)]

Charge–energy diffusion – the $\omega_0 = 0$ branch of the longitudinal sector:

$$\omega_1 = -i\,\Gamma_D\,k^2, \qquad \Gamma_D = \frac{\beta^2(\rho+P)}{n_\mu\big(-\beta\,\partial\rho/\partial\beta\big)_n\,c_{s/n}^2}\underbrace{\Big(\tilde{\chi}_1 - \frac{n}{\beta(\rho+P)}\tilde{\tau}_2\Big)}_{\geq 0}.$$

Shear diffusion – the transverse mode, absent in a perfect fluid:

$$\omega_0 = 0, \qquad \omega_1 = -i\,\Gamma_T\,k^2, \qquad \Gamma_T = \frac{\eta}{\rho+P}, \quad \eta \geq 0.$$

These are also hydrodynamic-frame invariant. [AO (2026)]

The main result: stability from thermodynamics and positivity

Thermodynamic stability is also derived from the KMS, unitarity and the state positivity:

$$\mathrm{Im}[I[A_\pm, g_\pm]] \geq 0 \to \begin{pmatrix} \big(-\beta\,\partial\rho/\partial\beta\big)_{\beta\mu} & \rho_\mu & 0 \\ \rho_\mu & n_\mu & 0 \\ 0 & 0 & (\rho+P)\delta_{ij} \end{pmatrix} \succeq 0.$$

For a nondegenerate state, this is equivalent to

$$n_\mu > 0, \qquad \Big(-\beta\frac{\partial\rho}{\partial\beta}\Big)_n > 0, \qquad \rho + P > 0.$$

$$\Gamma_L,\ \Gamma_T,\ \Gamma_D \geq 0 \implies \mathrm{Im}\,\omega \leq 0 \text{ in any hydrodynamic frame}$$

[AO (2026), AO in prep.]



Lorentz boosted frames

Check the same conclusion holds in any inertial frame, not just comoving.

Non comoving frame

Lorentz boost between comoving ($\tilde{k}_\mu$) and non comoving (k_μ) frames:

$$k_\mu \to \tilde{k}_\mu = \Lambda_\mu{}^\nu k_\nu.$$

For a boost in t-x plain ($\gamma \equiv 1/\sqrt{1-v^2}$),

$$\begin{aligned}
\tilde{\omega} &= \gamma\left(\omega - v k_x\right), \\
\tilde{k}_x &= \gamma\left(k_x - v\omega\right), \\
\tilde{k}_y &= k_y, \\
\tilde{k}_z &= k_z.
\end{aligned}$$

In comoving frame, we found

$$\tilde{\omega} = \pm c_{s/n}\tilde{k} - i\Gamma_L \tilde{k}^2$$

Then what about $\omega = \omega_0 + \varepsilon\omega_1 + \mathcal{O}(\varepsilon^2)$?

The leading order dispersion depends on the relative direction:

$$\omega_0 = \frac{v\left(1-c_{s/n}^2\right)k_x \;\pm\; c_{s/n}\sqrt{1-v^2}\,\sqrt{\left(1-v^2\right)k_x^2+\left(1-c_{s/n}^2v^2\right)k_\perp^2}}{1-c_{s/n}^2v^2}.$$

If $\vec{k}\propto\vec{x}$ and then $k_\perp = 0$,

$$\omega_0 = \frac{v\pm c_{s/n}}{1\pm c_{s/n}v}k_x.$$

The first order dispersion relation:

$$\omega_1 = -\frac{i\,\Gamma_{\mathrm{L}}\,\tilde{k}^2}{\gamma\left(1 \pm c_{s/n}\,v\cos\tilde{\theta}\right)},$$

with

$$\cos\tilde{\theta} \equiv \frac{\gamma\,(k_x - v\omega_0)}{\left|\tilde{k}\right|},$$

$$\tilde{k}^2\Big|_{\omega_0} \equiv \gamma^2\,(k_x - v\omega_0)^2 + k_\perp^2.$$

This is stable.



Conclusions

Put it together: first-order hydrodynamics is a healthy IR effective theory.

First-order hydrodynamics is a healthy IR effective theory

- Unstable modes arise only when the truncated gradient expansion is solved exactly
- The physical IR pole does not depend on the hydrodynamic frame
- Under KMS + unitarity + positivity $\Rightarrow \mathrm{Im}\,\omega \leq 0$
- Lorentz boosts maintain the stability of the IR spectrum.

As an IR effective theory, no instability issue in first-order hydrodynamics.

[AO (2026)]

Thank you for your attention.

References

- A. Ota, *Frame-independent analysis of relativistic hydrodynamics via effective theory*, Phys. Rev. D 114, L011902 (2026).
- A. Ota, *Mode analysis of Nambu-Goldstone modes in $U(1)$ charged first-order relativistic viscous hydrodynamics*, Phys. Rev. D 114, 016001 (2026).

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Outlook: nonlinear fluctuations and loop corrections

- Loop corrections sourced by the hydrodynamic modes themselves
- IR renormalization of transport coefficients; long-time tails
- A hydrodynamic-frame-invariant nonlinear effective action

Next goal: a frame-independent, nonlinear, quantum effective theory of relativistic fluids.