# The Holographic Critical Point is Sensitive to Quark Flavors

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ITP-CAS, Beijing

Workshop on QCD Critical End Point and Effective Field Theory Methods

C3NT — Wuhan — 20 October 2026

[MJ, Mitra 2507.08087; Jokela, MJ, Mitra, Piispa 2609.39598;...]

# Outline

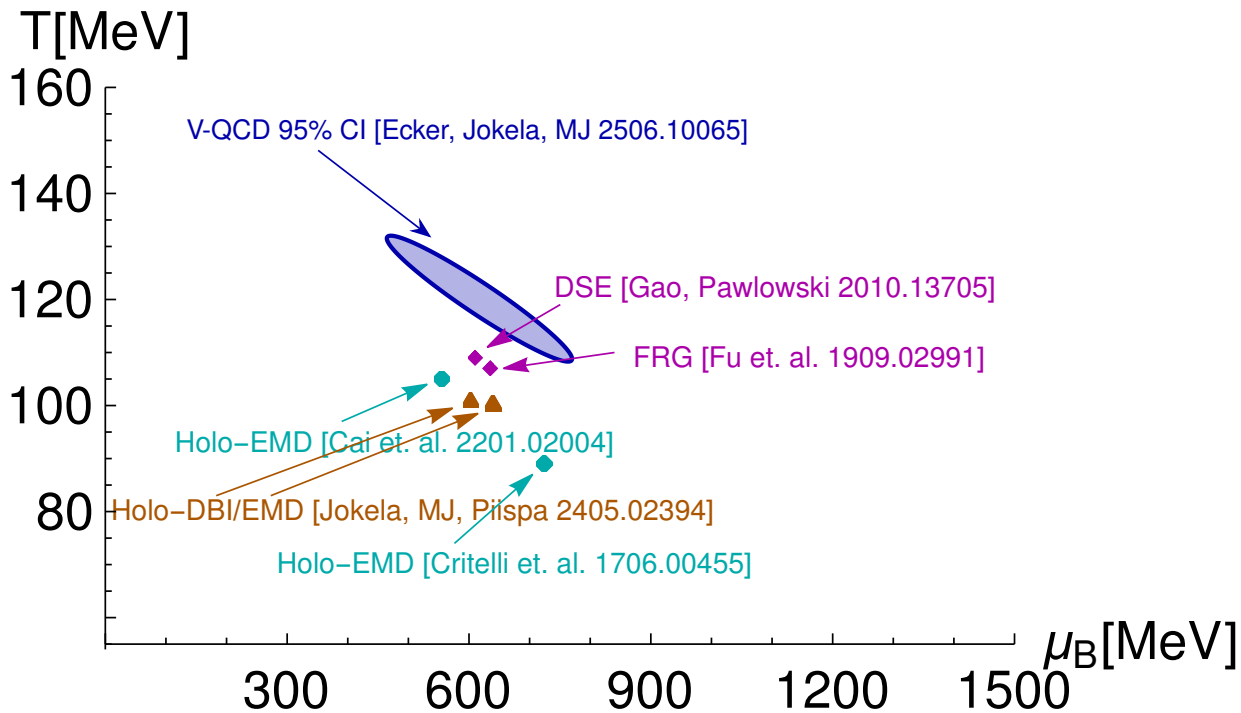
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2. Quark flavors and the V-QCD model
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4. Inhomogeneous instability
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# Holographic critical point: lightning review

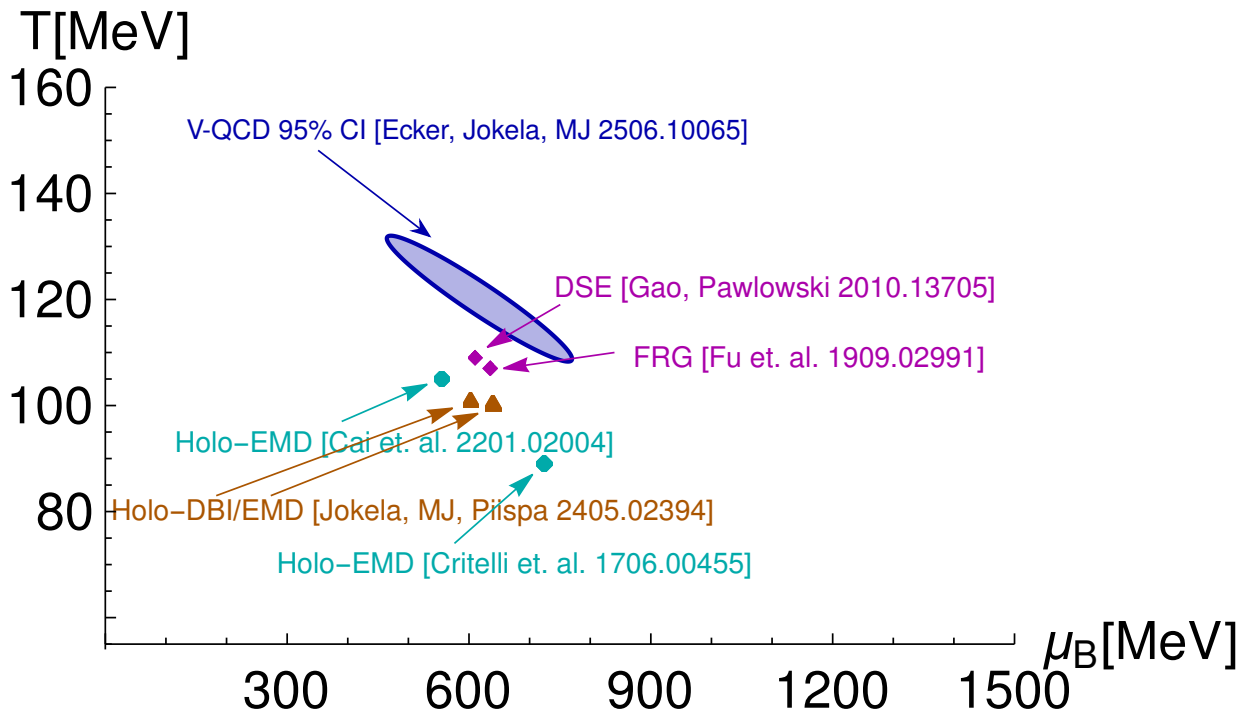
Wide literature on the QCD critical point in bottom-up holography: fit to lattice data, extrapolate to higher density ...




- Results show nice convergence! (Note: $T$-range zoomed in)
- Also good agreement with Dyson-Schwinger and functional renormalization group computations in QCD
- This includes "old" V-QCD results (not result of this talk)

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- Results show nice convergence! (Note: $T$-range zoomed in)
- Also good agreement with Dyson-Schwinger and functional renormalization group computations in QCD
- This includes "old" V-QCD results (not result of this talk)

However, things may not be working as smoothly as it seems ...

# Holographic critical point: issues

- Flavor dependence and flavors on lattice
  - Most studies use baryon number susceptibilities only — there is also data for other susceptibilities
  - Adding flavors and fitting this additional data changes results dramatically
  - Holographic models typically predict zero off-diagonal quark number susceptibilities, $\chi_{us} = 0$ — lattice shows $\chi_{us} \neq 0$

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II Inhomogeneous (Nakamura-Ooguri-Park) instability
- Appears in all previous models—consistently extended to implement full chiral symmetry and flavor anomalies
- Destroys the critical point?! [Cruz Rojas,Demircik,MJ 2405.02399; Demircik,Jokela,MJ,Piispa 2405.02392]
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III Modeling of the confined region
- Pressure of confined phase arises from free meson (pion) gas
- Tricky in holography — would require string loops ...
- Not obvious that this can be captured by pure gravity analysis

I will cover some of these topics ...

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# V-QCD model

An advanced holographic bottom-up model for QCD: V-QCD

- Bottom-up, but trying to follow principles from string theory closely
- Many parameters: effective description of QCD
- Comparison with QCD data essential
- Includes full backreaction of quarks to glue dynamics
- 5D dilaton gravity (IHQCD) + tachyonic DBI action

[MJ, Kiritsis arXiv:1112.1261]

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Earlier analysis mostly at zero quark mass, identical quark flavors

- Ongoing work: generalize the model to distinct quark flavors and nonzero quark masses
- In particular, include effect of the strange quark mass: $N_f = 2 + 1$ model

# Quark flavors in V-QCD

Dictionary (with $i, j = 1, \ldots N_f$; only vectorial gauge fields)

- Dilaton $\lambda \leftrightarrow \mathrm{tr} G^2$ — source = 't Hooft coupling
- Tachyon $T_{ij} \leftrightarrow \bar{q}_i q_j$ — sources = quark masses
- Gauge fields $A^{\mu}_{ij} \leftrightarrow \bar{q}_i \gamma^{\mu} q_j$ — sources = chemical potentials

$$\mathcal{S}_{\mathrm{V-QCD}} = S_{\mathrm{IHQCD}} + S_{\mathrm{DBI}}$$

$$= N_c^2 M^3 \int d^5x \sqrt{-\det g} \left[ R - \frac{4}{3} \frac{(\partial\lambda)^2}{\lambda^2} + V_g(\lambda) \right]$$

$$- N_c M^3 \int d^5x \, \mathrm{Tr} \left[ V_f(\lambda, T) \sqrt{-\det(g_{ab} + \kappa(\lambda) \partial_a T \partial_b T^{\dagger} + w(\lambda, T) F_{ab})} \right]$$

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Trace over flavor indices — insert diagonal Ansatz in flavors $T_{ij} = \tau_i(r)\delta_{ij}$, $A_{ij} = A^{(i)}(r)\delta_{ij}$:

$$S_{\text{DBI}} \propto \sum_{i=1}^{N_f} \int d^5x \, V_f(\lambda, \tau_i) \sqrt{-\det \left( g_{ab} + \kappa(\lambda) \delta^r_a \delta^r_b (\tau_i')^2 + w(\lambda, \tau_i) F^{(i)}_{ab} \right)}$$

- Take $N_f = 2 + 1$; two massless and one massive quark

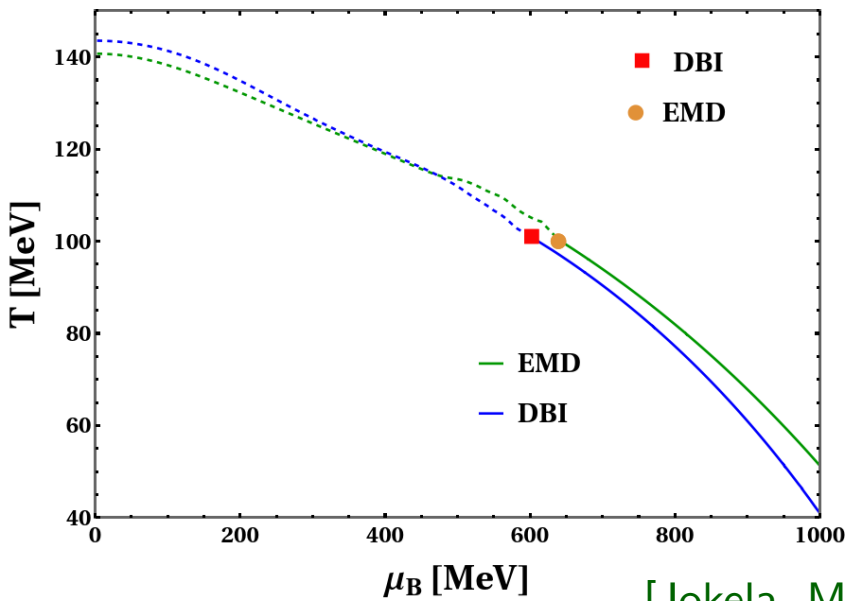
[MJ, Mitra 2507.08087]

# Using DBI versus Maxwell

Note that I am using a DBI action for gauge fields — literature mostly uses Maxwell action, i.e. Einstein-Maxwell-dilaton (EMD)

[Song He's talk]

- However, at small chemical potentials, gauge fields are small
  $\Rightarrow$ expanding the DBI leads to Maxwell
- I.e., the actions are essentially the same
- Actually, agreement even at relatively high chemical potential (in a bit simpler setting than this talk, without tachyon)




[Jokela, MJ, Piispa 2405.02394]

$\Rightarrow$ Expect findings of this talk to also hold in EMD-like approach

# Adjusting the potentials

Functions $V_g$, $V_f$, $\kappa$ and $w$ determined by comparing to QCD

- Require confinement, discrete spectrum, linear meson trajectories ...
- Working potentials turn out to have power law asymptotics
- E.g. $V_g \sim \lambda^{4/3}$, $w \sim \lambda^{-4/3}$

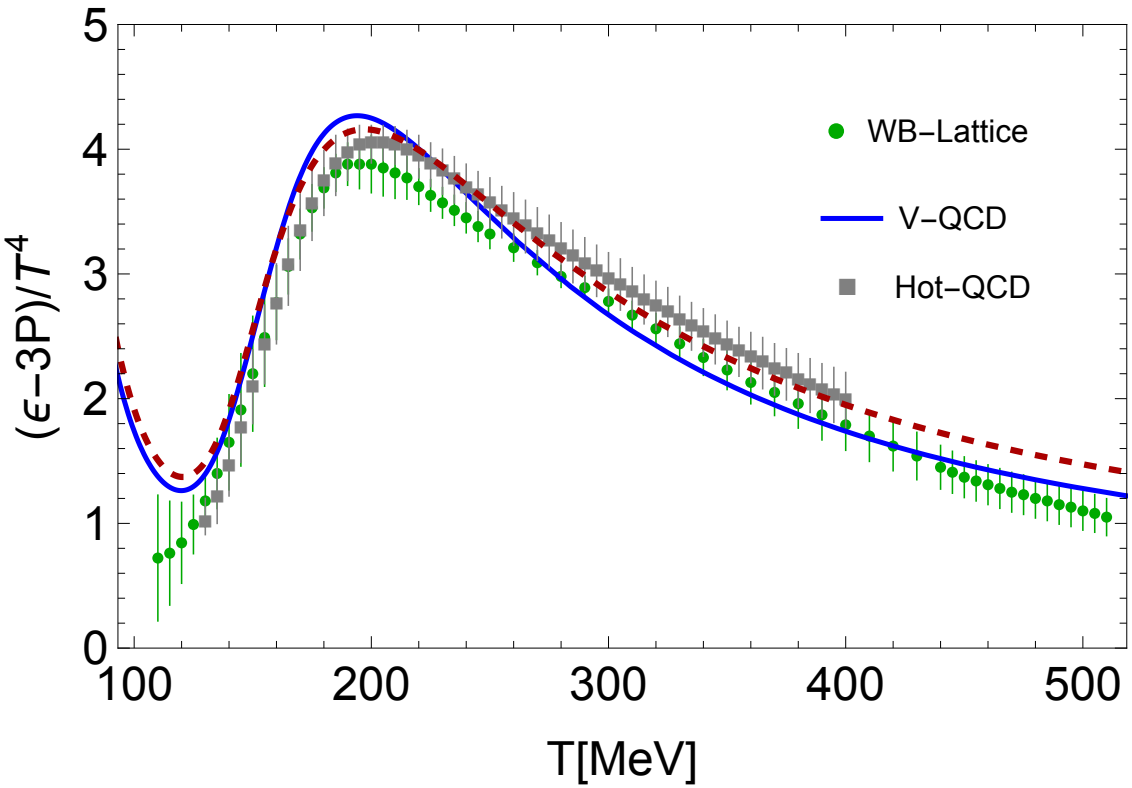
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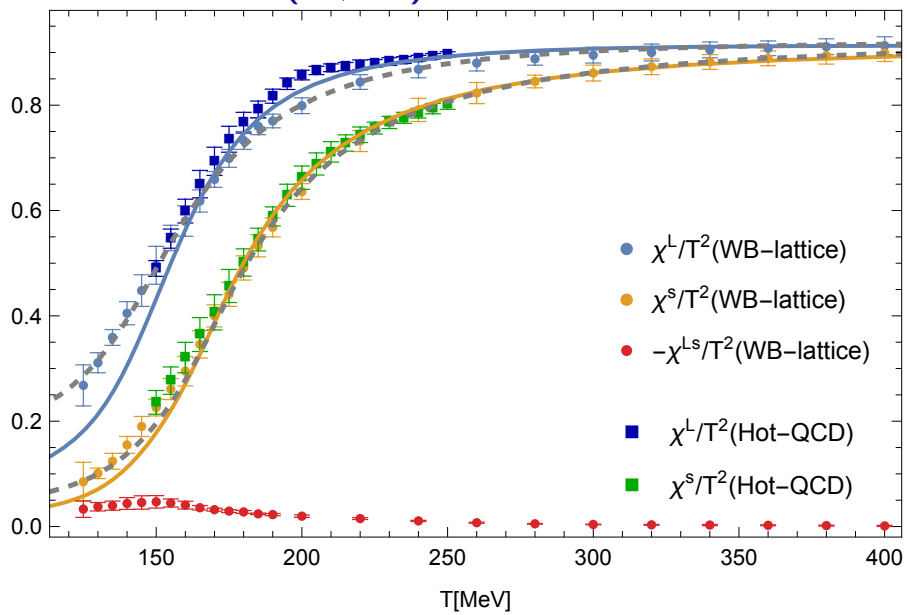
Rest fitted to lattice data sequentially, e.g. $V_f$ to EOS at $\mu = 0$

In V-QCD, solve black hole thermodynamics numerically



# Fitting the susceptibilities

Fitting susceptibilities $\chi_{ij} = \frac{\partial^2 p}{\partial \mu_i \partial \mu_j}$ more tricky — constrains gauge field coupling function $w(\lambda, \tau_i)$

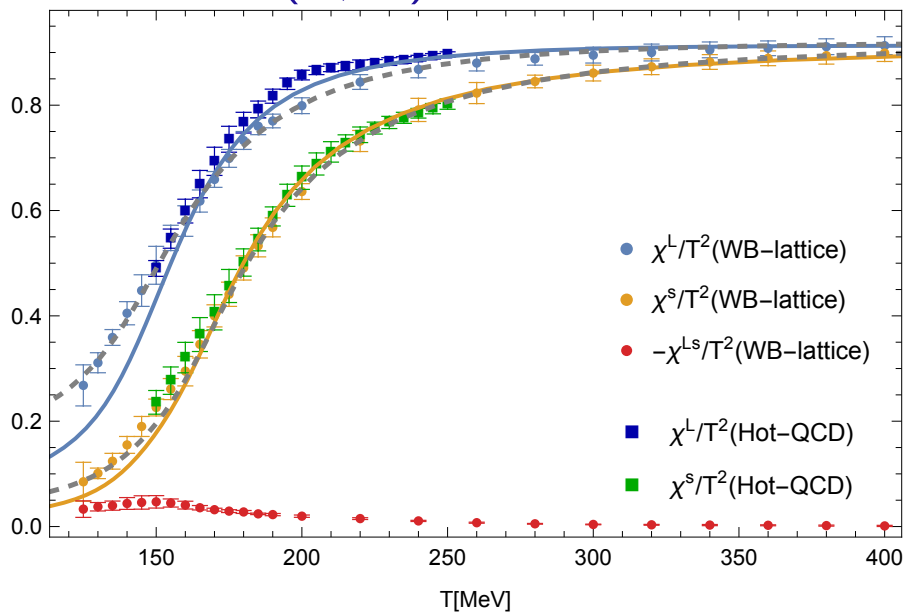



[WB 1112.4416]

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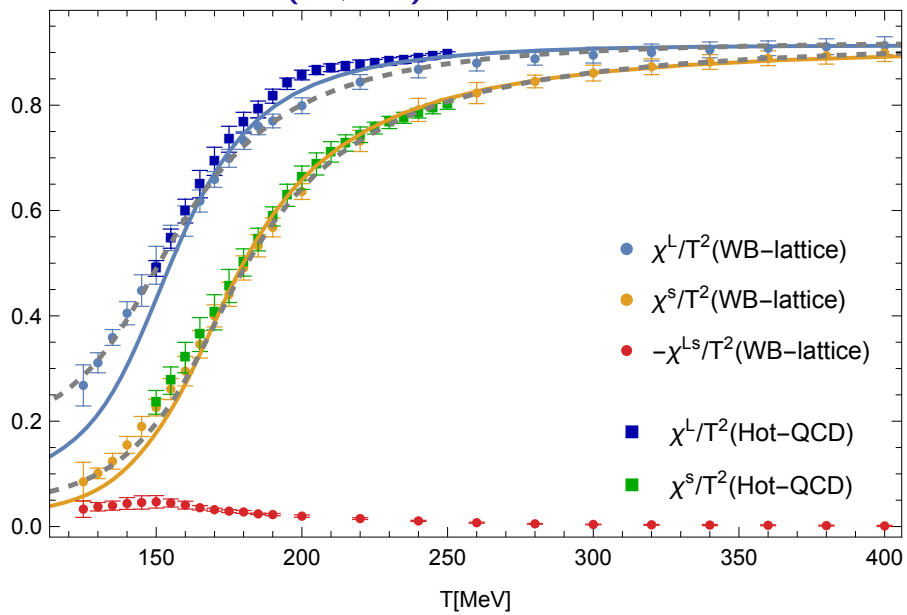

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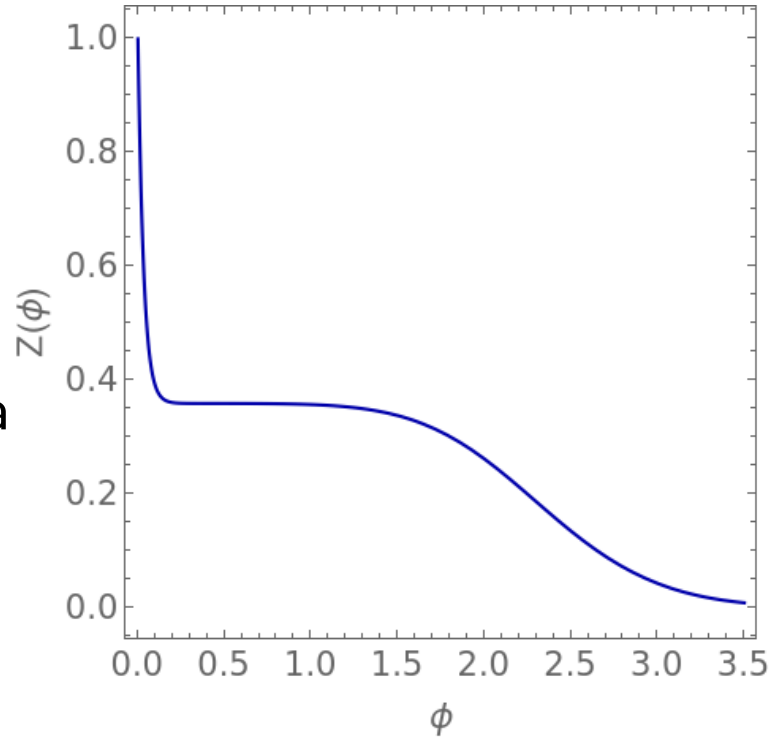
- Lattice results for light/strange quarks significantly different
- Strange quark mass sources tachyons — tachyon dependence of potentials modifies $\chi$'s, making fit possible
- N.B. We fit only diagonal $\chi_{ij}$ and NOT $\chi_B$: it depends on off-diagonal $\chi_{ij}$ (red data points) which vanish in our model

# A remark on the $\chi$ fit: a widespread issue

In EMD models,

$$\int d^5x\sqrt{-g}\left[R-(\partial\phi)^2+V(\phi)-Z(\phi)F^2\right]$$

$Z(\phi)$ fitted to $\chi_B$ often shows a spike . . .



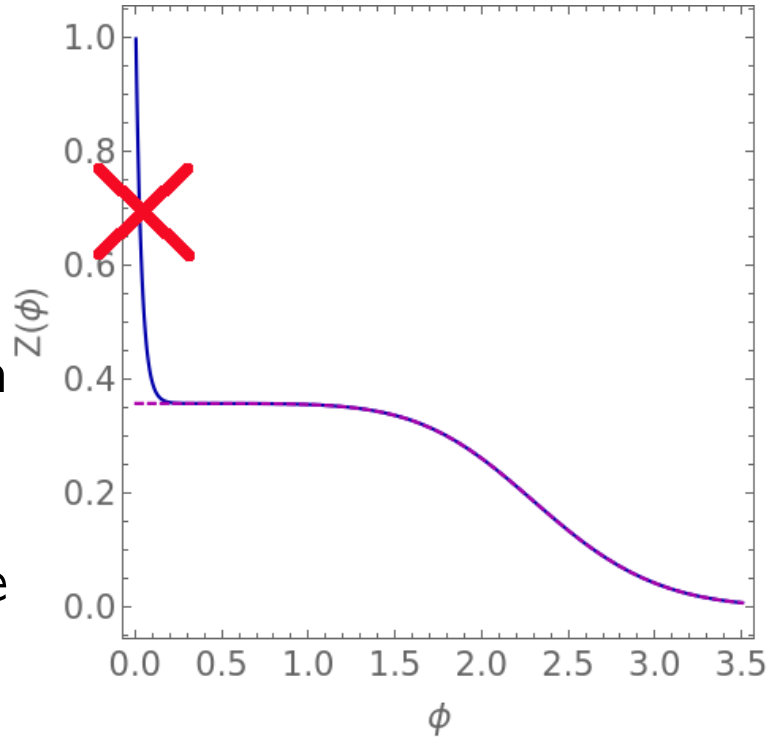
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Analysis technically correct, but the spike is useless




- Spike was inadvertly introduced to fix a computational error in early literature — mistake was copied to later works
- Does not affect (low-$T$) results, effects $< 1\%$ [MJ unpublished]
- So lattice fitting works equally well without it
- Does not have any real physical function
- Only screws up thermodynamics at very high $T$

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# Setting up thermodynamics

We use the flavored $N_f = 2 + 1$ V-QCD model

- Phase diagram depends on $T$, $\mu_u$, $\mu_d$, $\mu_s$ . . . cannot explore the whole 4D phase diagram and 4D equation of state

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Choose physically motivated slices

1. $\beta$-equilibrium (without leptons):
   - Equilibrium with respect to flavor changing exchange $\Rightarrow$ $\mu_d = \mu_s$
   - Require charge neutrality $\Rightarrow$ quark densities satisfy $n_u = (n_d + n_s)/2$
2. Heavy-ion conditions
   - Strangeness neutrality $\Rightarrow$ $n_s = 0$
   - Fixed charge to baryon number ratio, $n_Q/n_B = 0.4$

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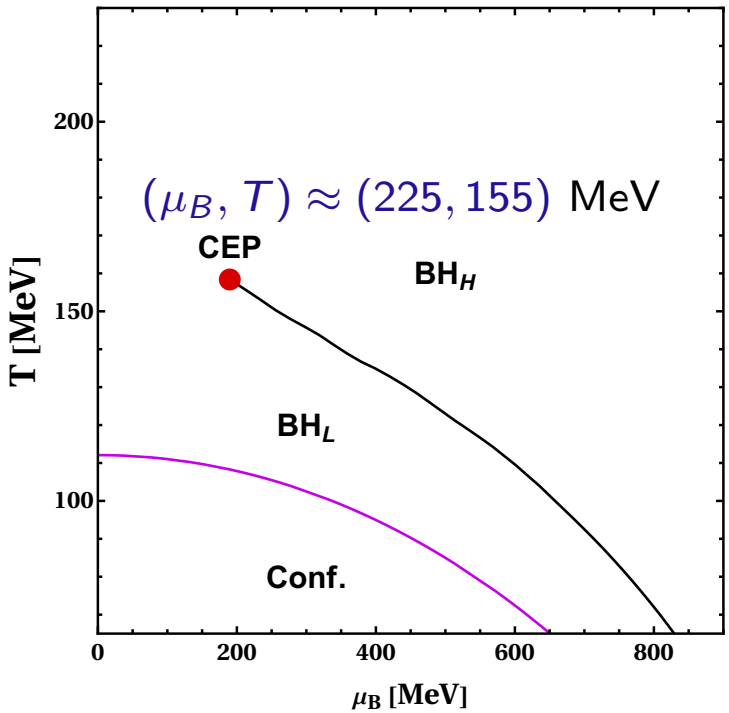
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Plot on ($T$,$\mu_B$)-plane

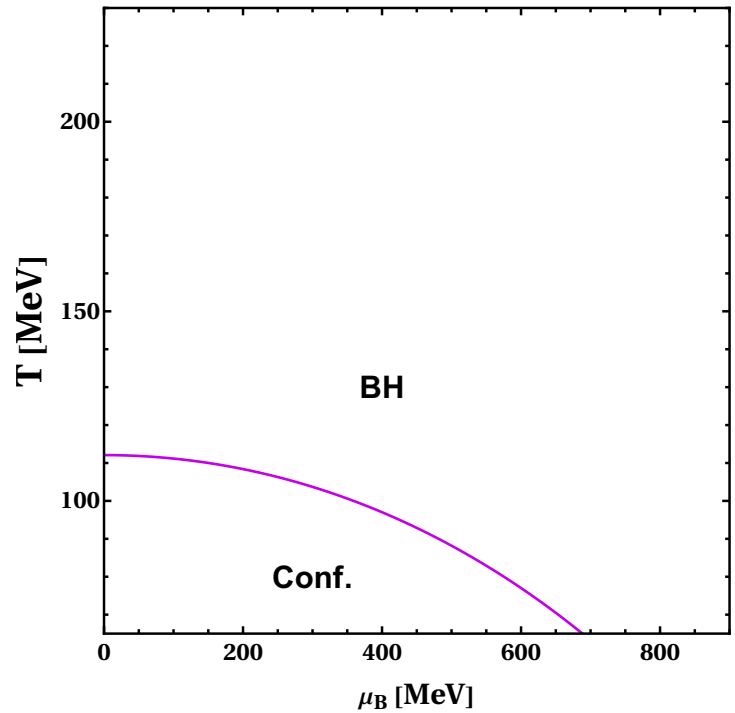
[Jokela, MJ, Mitra, Piispa 2609.39598]

# Results: phase diagram

$\beta$-equilibrium




Heavy ions

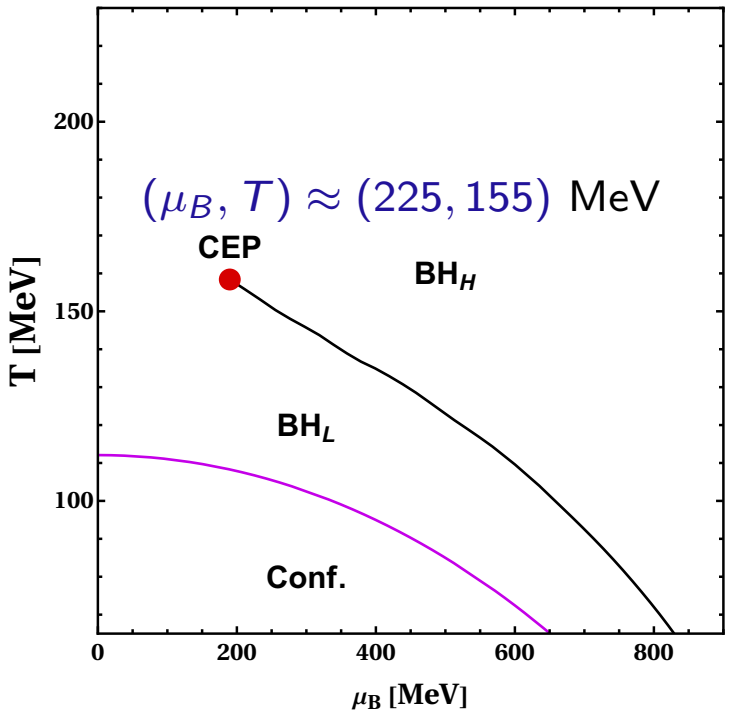



Surprising results, at odds with earlier studies:

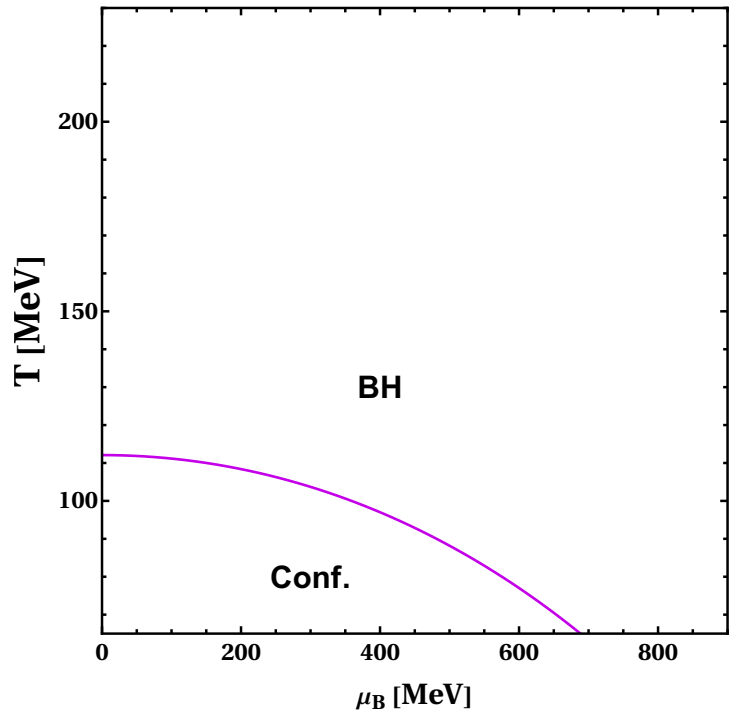
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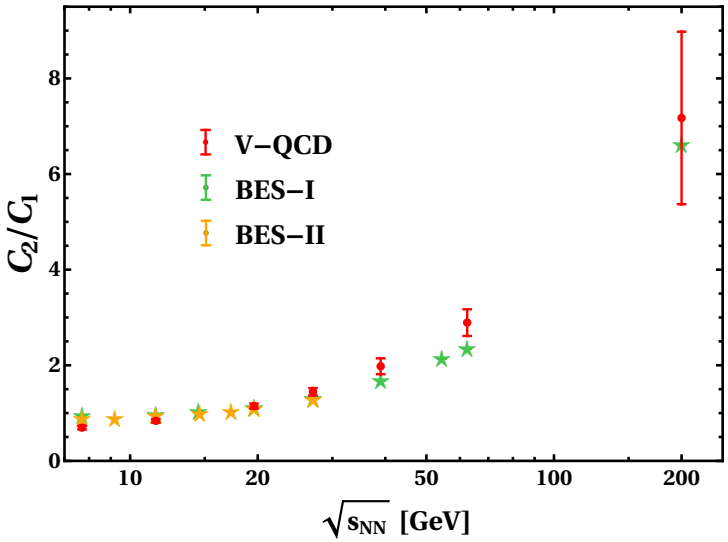


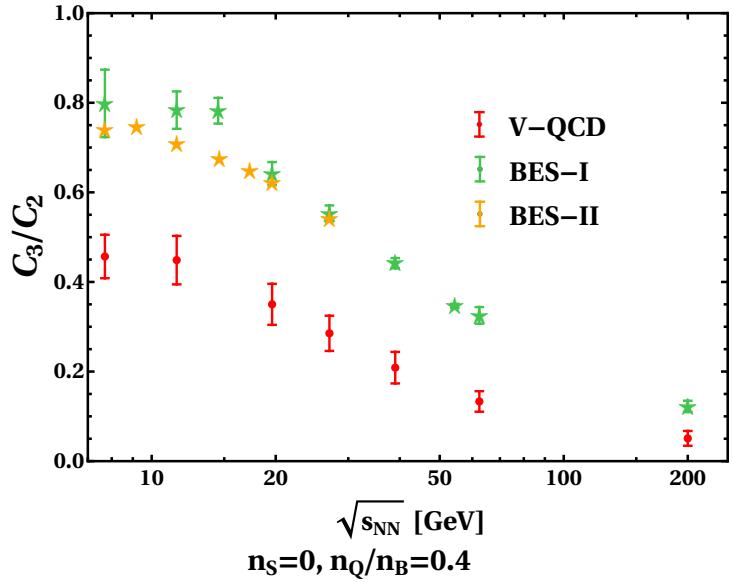

Surprising results, at odds with earlier studies:

1. At $\beta$-equilibrium, critical point emerges at unexpectedly low $\mu_B$ — much lower than earlier V-QCD and other studies
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- Extra transition (black curve): chiral restoration transition for strange quarks, $\langle \bar{s}s \rangle$ jumps
- Low pure-V-QCD confinement $T$ reflects missing hadron gas

# Results: cumulants

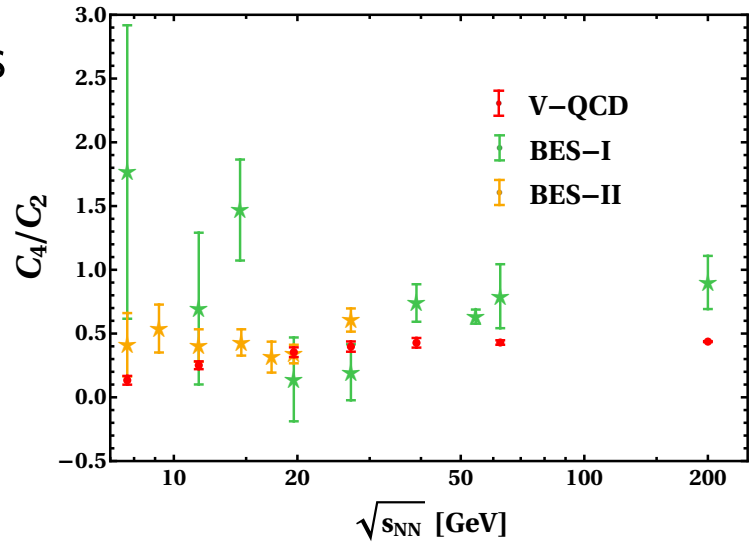



Compare to RHIC data for cumulants

$$C_n \propto \chi_n^B$$

at freezeout points




- $C_2/C_1$: good agreement
- $C_3/C_2$: systematic normalization offset
- Linked to missing off-diagonal susceptibilities
- No pronounced non-monotonic/sign-changing structure

# Towards a hybrid equation of state

Pure V-QCD computation does not capture the pressure of the meson gas in confined phase — low confinement critical $T$

- We will not try to include meson (and baryon) gas in V-QCD (would require string loops)
- Instead we will use an independent approach for hadron gas, and compare to (deconfined) V-QCD

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- Choose two hadron resonance gas (HRG) models:
  1. Free hadron gas with excluded volume corrections for baryons
  2. A model with full van-der-Waals interaction for baryons

  [Vovchenko, Gorenstein, Stoecker 1609.03975]
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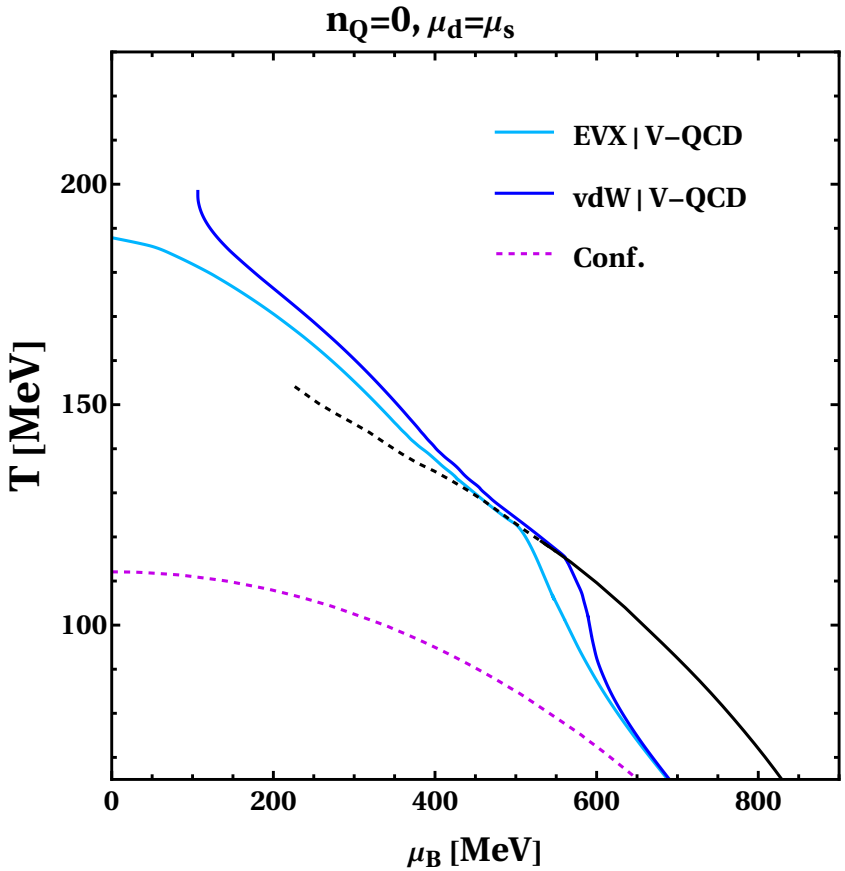
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- Focus here on results in $\beta$-equilibrium

# Hybrid phase diagram

Putting together V-QCD and HRG models: hybrid phase diagram

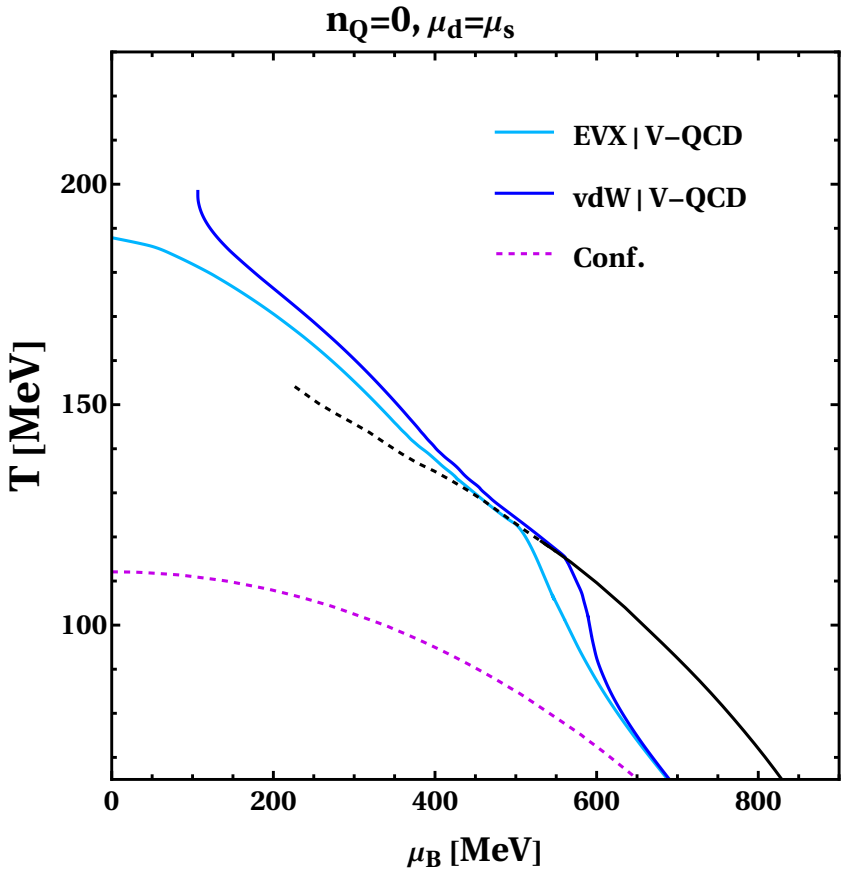
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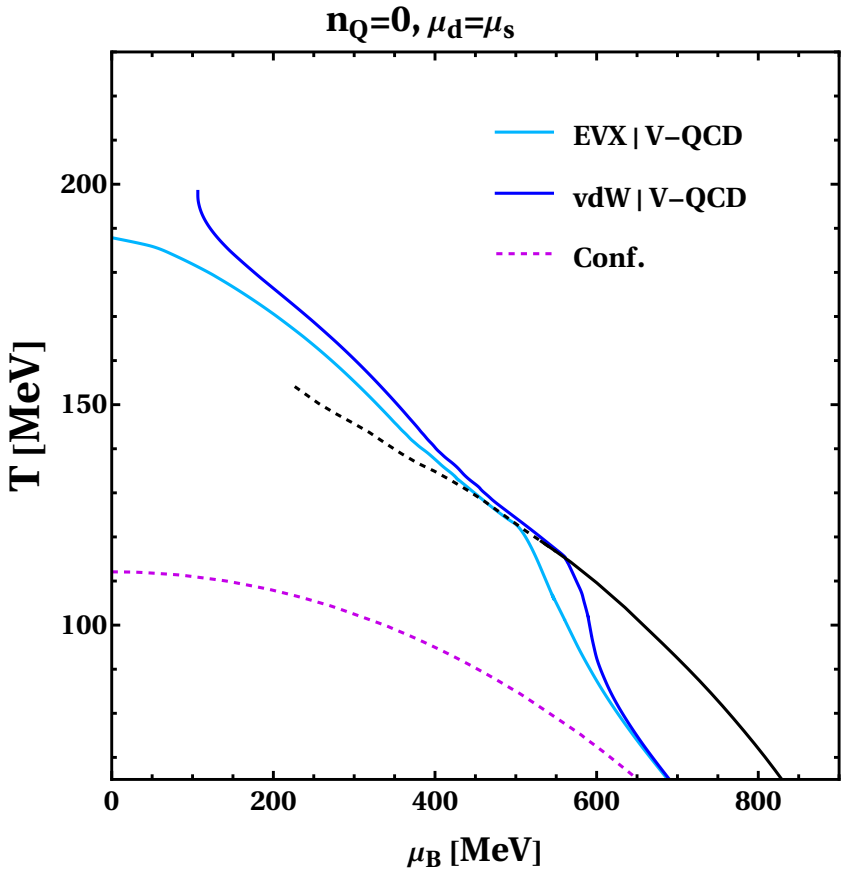
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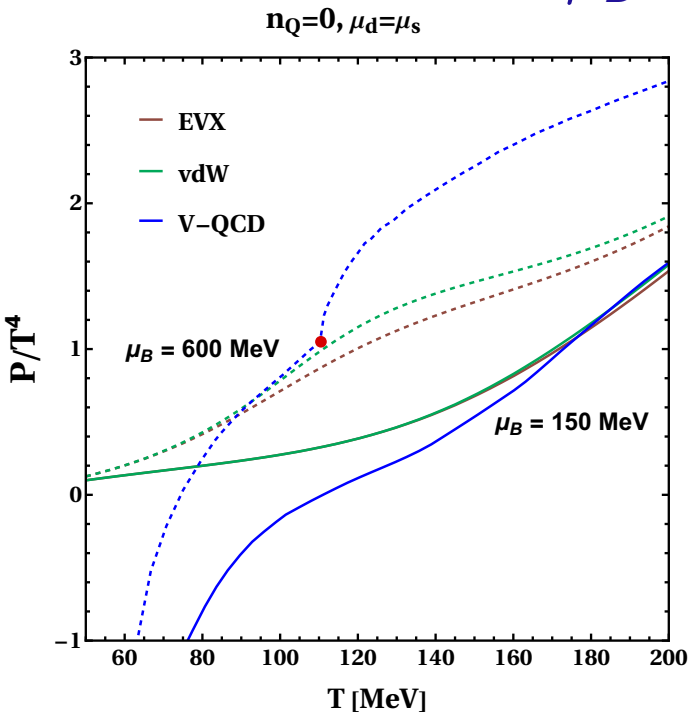
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- The pure V-QCD deconfinement transition (dashed purple) gets removed
- Replaced by the HRG to V-QCD transition lines at higher $T$
- Essentially, low $T$ V-QCD phases replaced by HRG
- Mild dependence on HRG model (excluded volume or van-der-Waals)

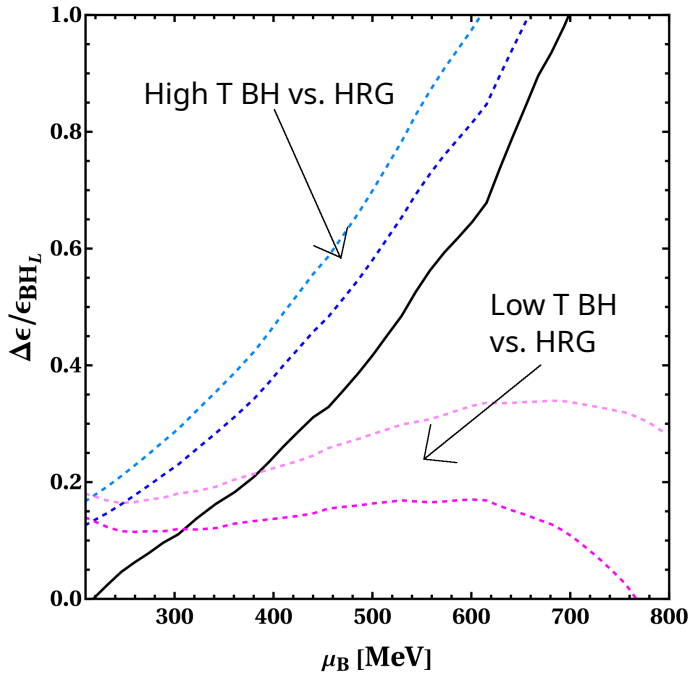


# Hybrid phase diagram: more detailed comparison

Pressure at fixed $\mu_B$



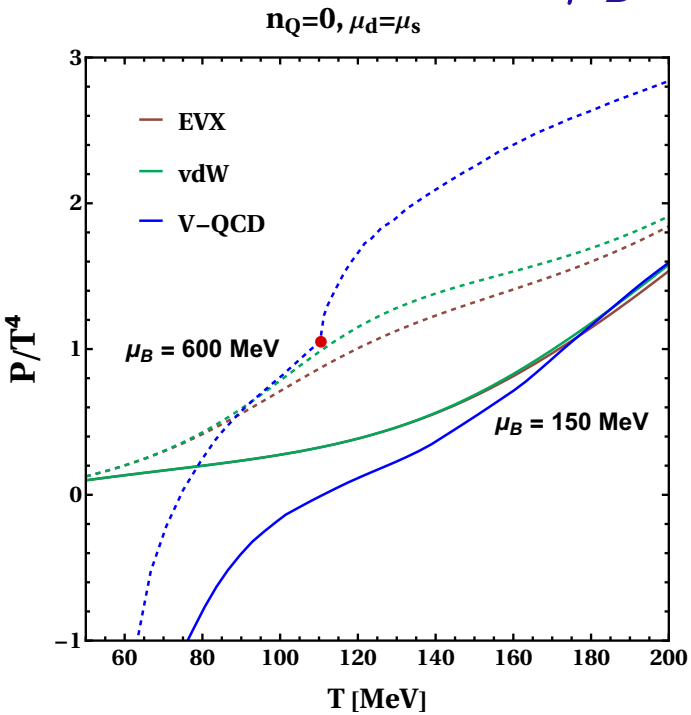

Jump in energy density



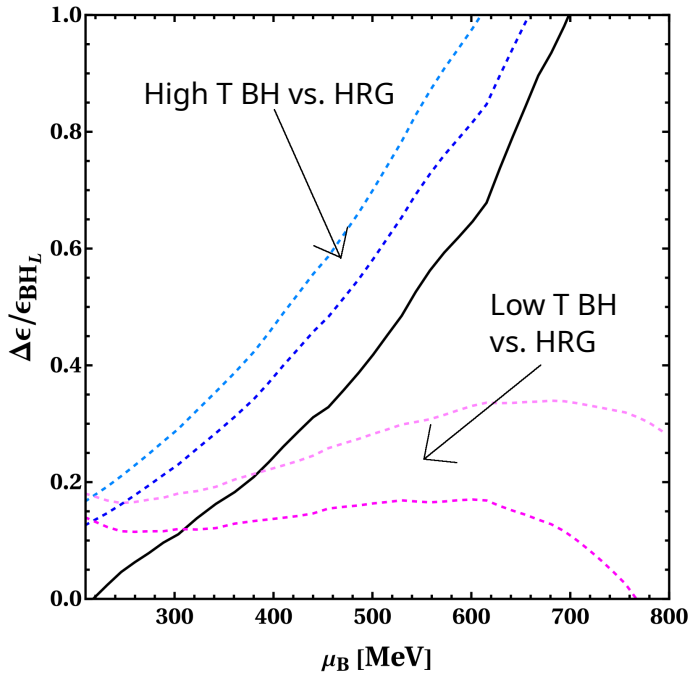

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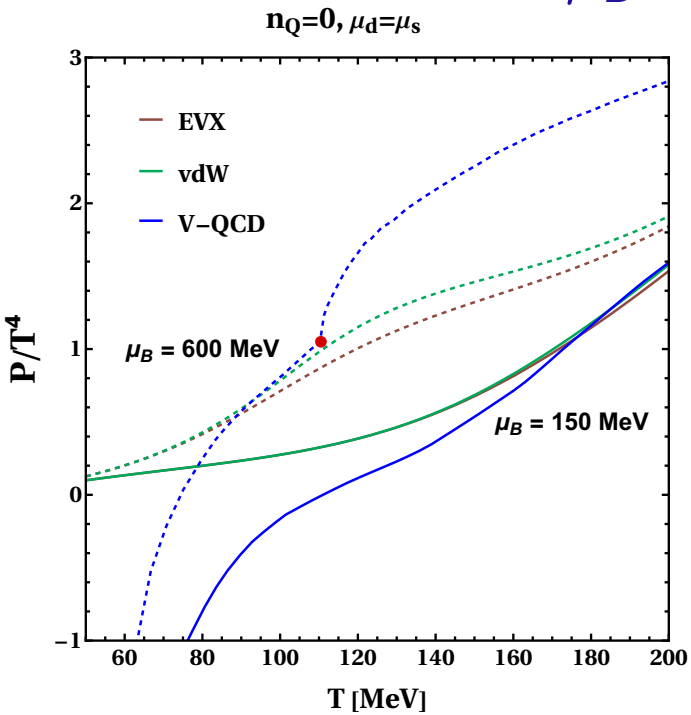

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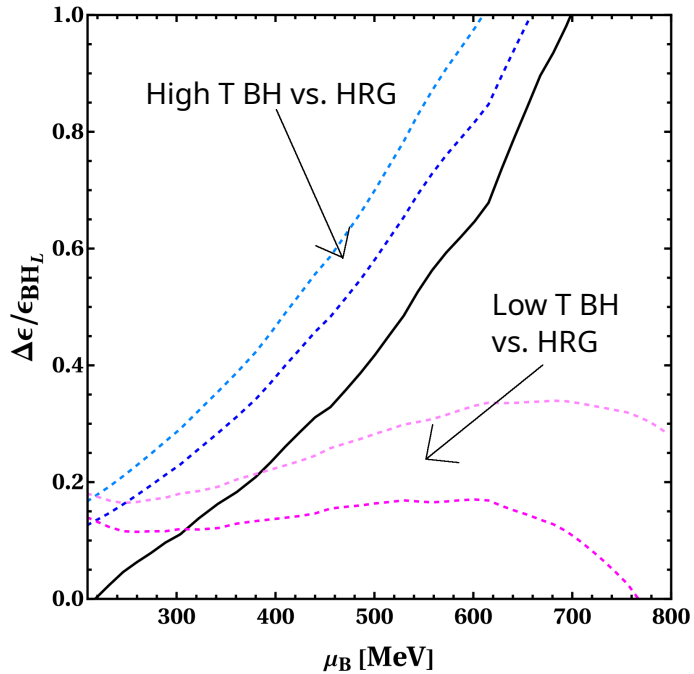

- HRG to low $T$ black hole "latent heat" $\ll 1$ in a wide range of $\mu$ — smooth matching lends support to both descriptions near the matching region
- Further supports the idea that low-$T$ V-QCD should be replaced by a confining, chirally broken phase
- No sign of critical point moving to higher $\mu$—remains at low $\mu$

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Pressure at fixed $\mu_B$




Jump in energy density




- HRG to low $T$ black hole "latent heat" $\ll 1$ in a wide range of $\mu$ — smooth matching lends support to both descriptions near the matching region
- Further supports the idea that low-$T$ V-QCD should be replaced by a confining, chirally broken phase
- No sign of critical point moving to higher $\mu$—remains at low $\mu$
- Heavy-ions (not shown): no critical point even with HRG

# Outline

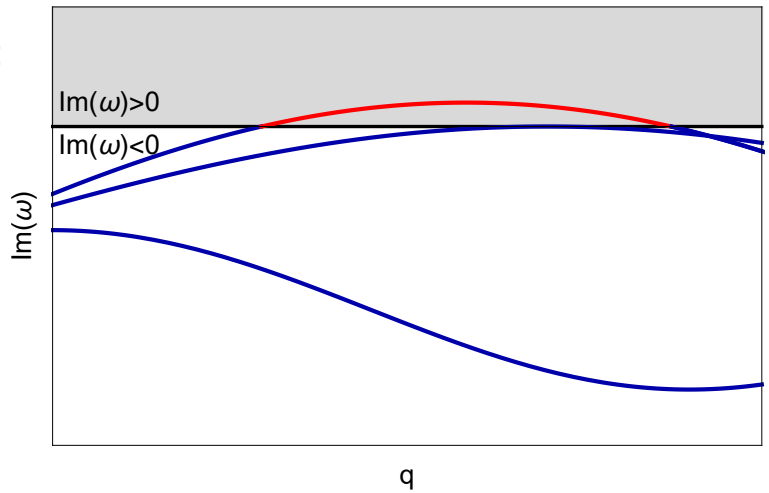
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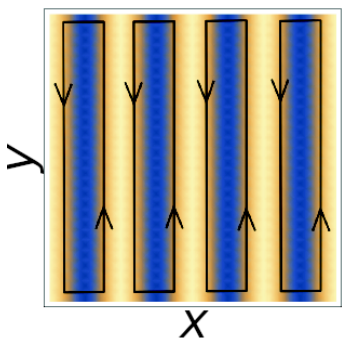
# Inhomogeneity in holographic plasma?

Spatially modulated phases

[Nakamura, Ooguri, Park 0911.0679;
Ooguri, Park 1011.4144]

- Fluctuate the black hole: quasi-normal modes
- Exponentially growing perturbation at $q \neq 0$: a mode with $\mathrm{Im}\,\omega > 0$
- The CS term, fixed by anomalies, can drive a modulated instability at finite density
- Modulated 5D gauge fields dual to modulated persistent chiral currents in field theory

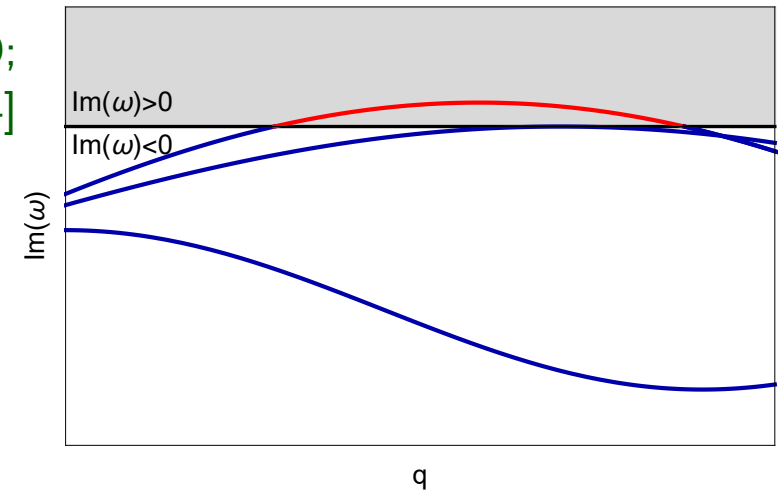


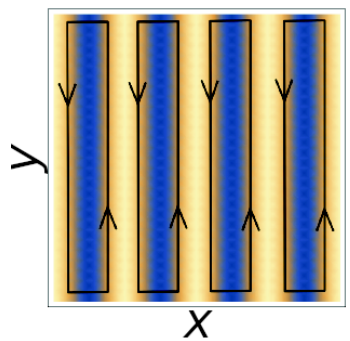
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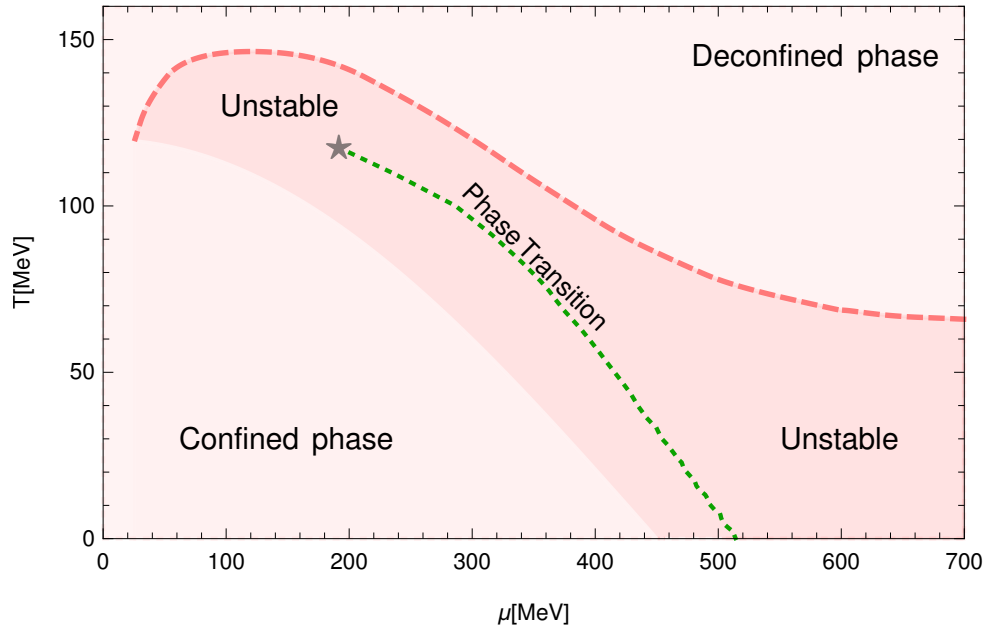



Schematic fluctuation equation

$$\psi''(r) + \underbrace{\frac{C\,q\,n}{w(\phi)^4}\,\psi(r)}_{\text{Ratio of CS \& DBI}} + \left(\omega^2 - q^2\right)\psi(r) = 0$$

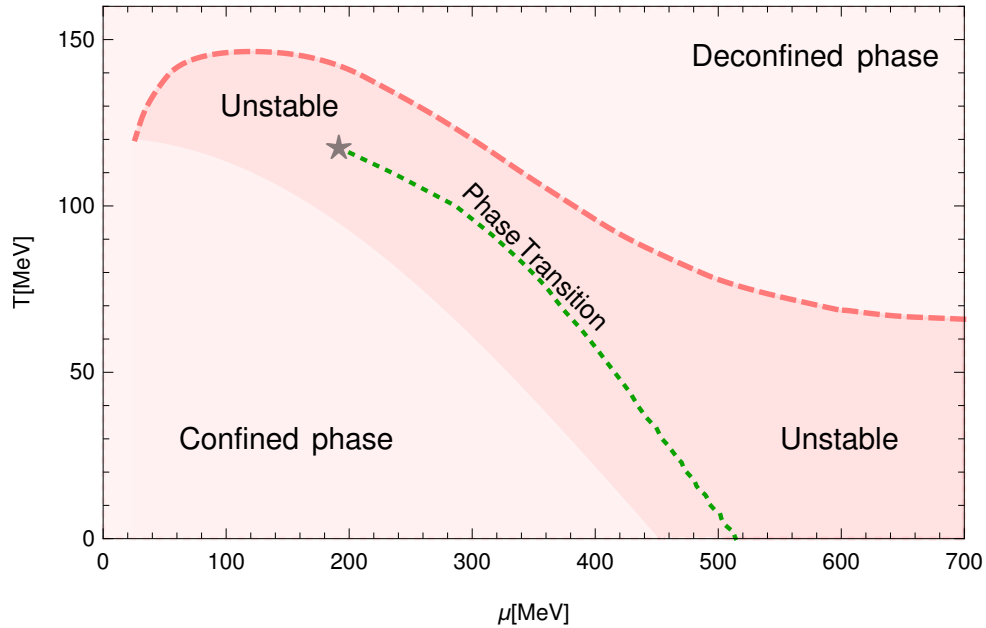
$\psi = \delta A^{x,a}_{L/R} \pm i\delta A^{y,a}_{L/R}$ $\qquad$ $r =$ holographic coord.

# Instability in V-QCD



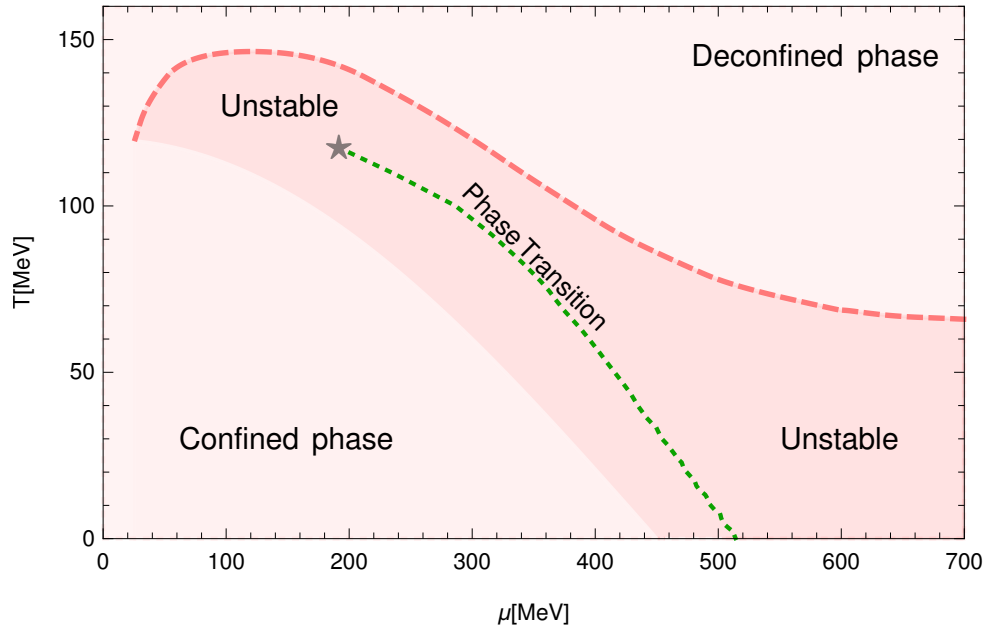

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- Instability is also found at higher $T$, near the regime with critical point?! (a big surprise) [Cruz Rojas,Demircik,MJ 2405.02399]
- Does not affect only V-QCD: essentially same results in Einstein-Maxwell-dilaton [Demircik,Jokela,MJ,Piispa 2405.02392]
- Effect of strange quark mass currently unknown

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Summary:

- Flavor effects dramatically change the picture of the holographic QCD critical point
  - $\beta$-equilibrium: critical point at unexpectedly low $\mu_B$
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Outlook:

1. Locate the inhomogeneous instability for $N_f = 2 + 1$
2. Address the off-diagonal quark susceptibilities
3. Apply the model to analyze further observables (bulk viscosity, finite magnetic field, neutron stars, ...)
4. Put improvements together, publish a new version of a general purpose V-QCD equation of state

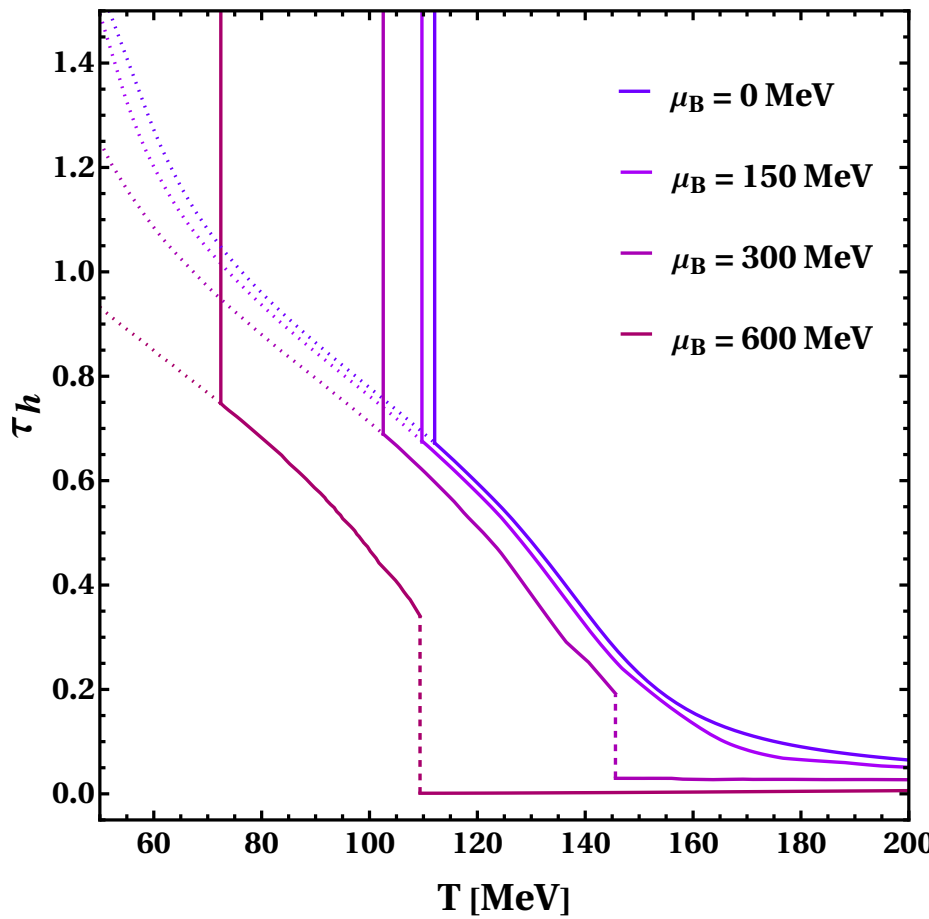
Thank you!

# Results: chiral symmetry

We only study chiral symmetry in the strange quark sector

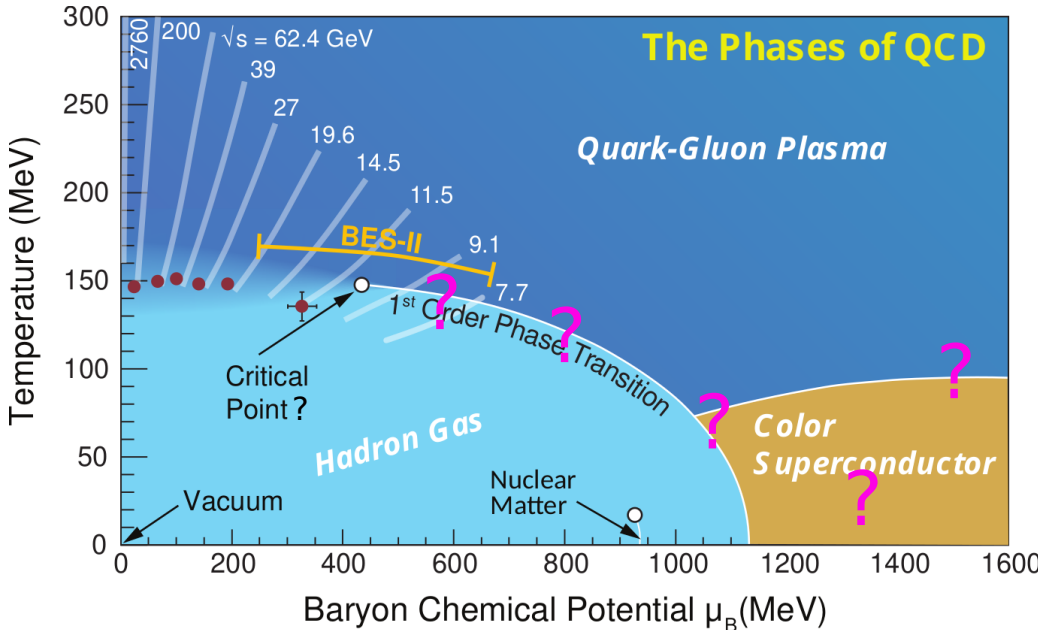
- Asymptotically, schematically $\tau_s \sim m_s r + \langle \bar{s}s \rangle r^3 \quad (r \to 0)$
- Extracting the condensate numerically annoying
- Since $m_s$ rather small, $\tau_s$ at the horizon is a proxy for $\langle \bar{s}s \rangle$ — plot it instead of the condensate

- Clear jump of the horizon value at the transition
- Interpret as chiral restoration transition (in the strange sector)

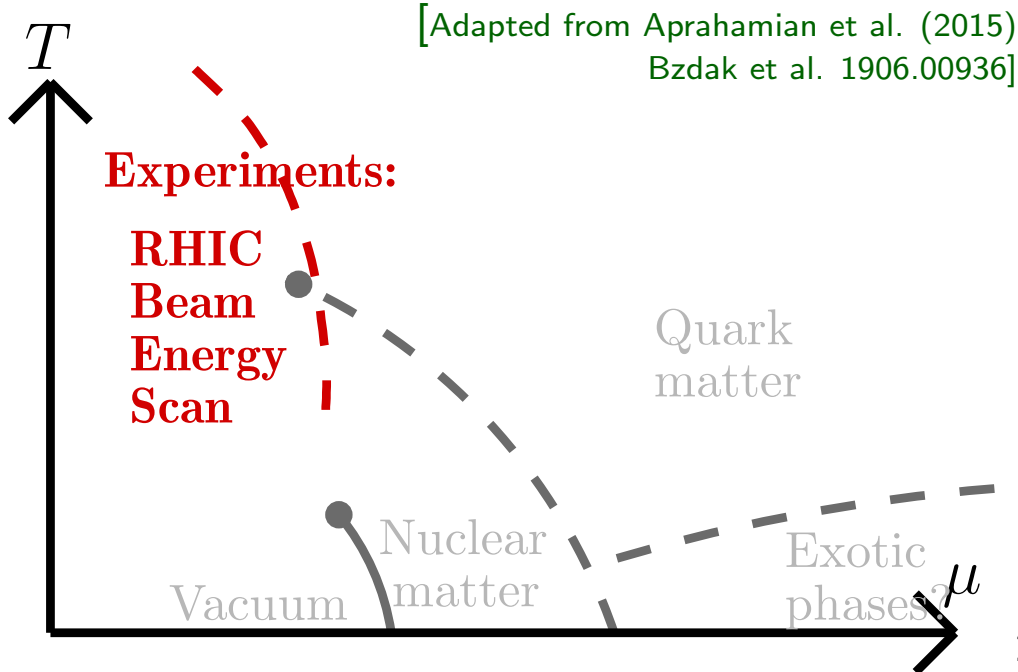


# QCD phase diagram and the critical point

Search for the critical point: ongoing analysis of Beam Energy Scan data at RHIC



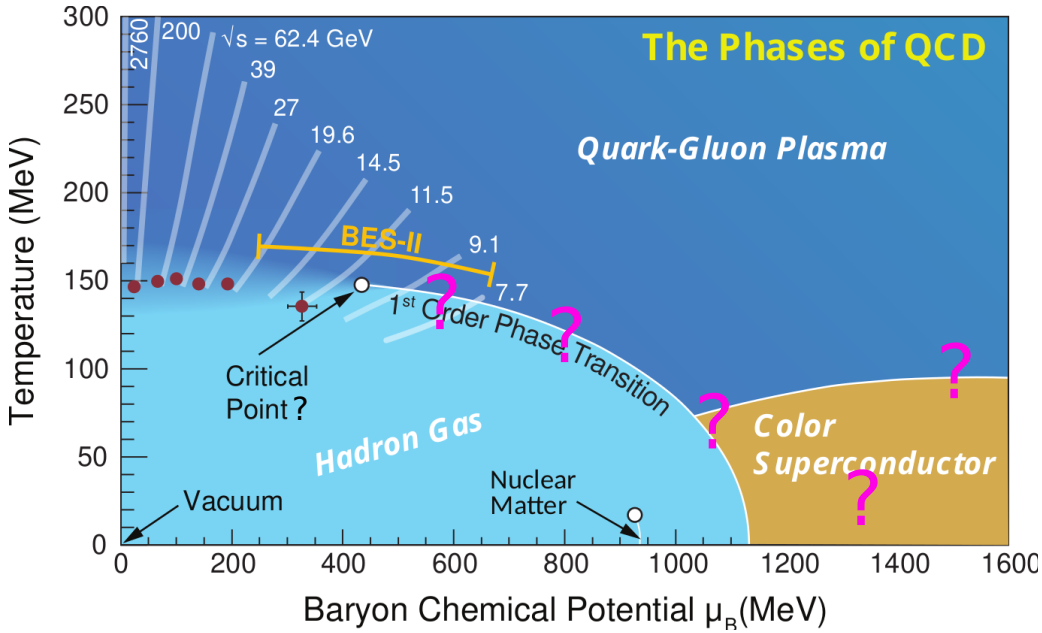

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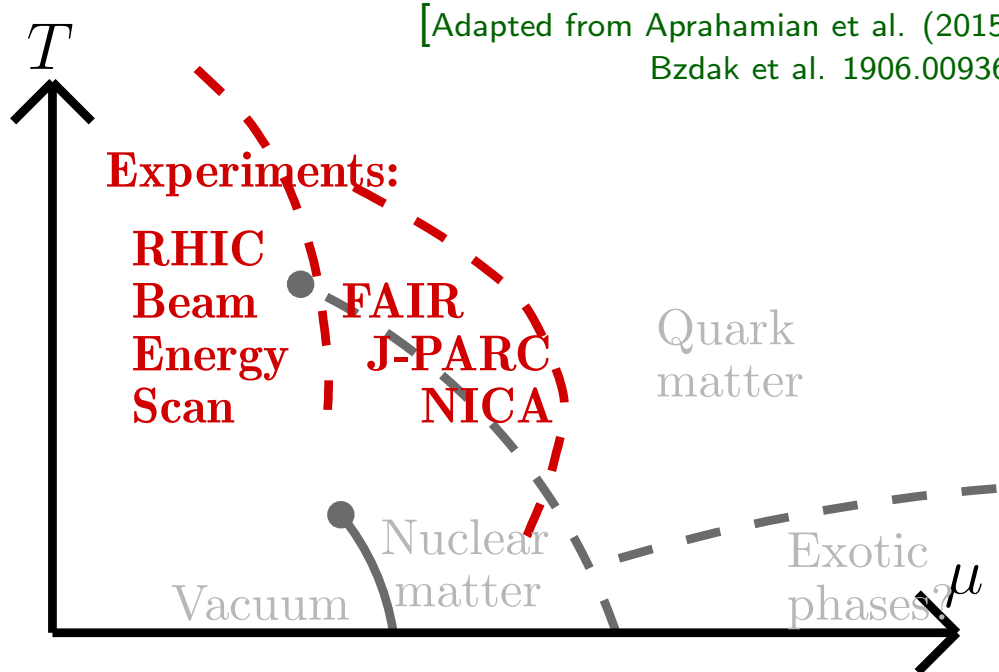
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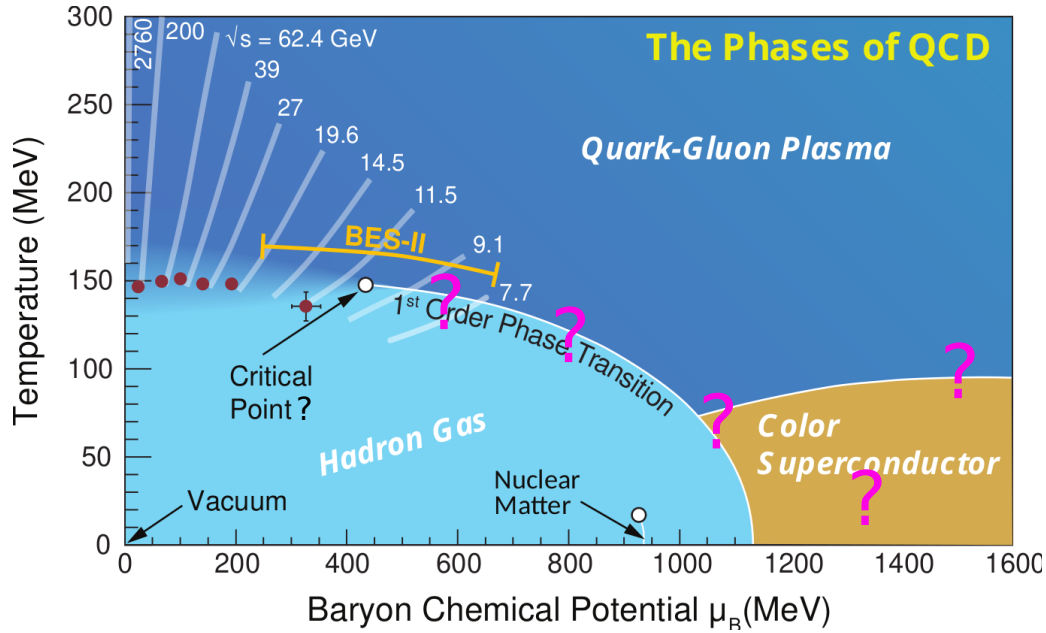


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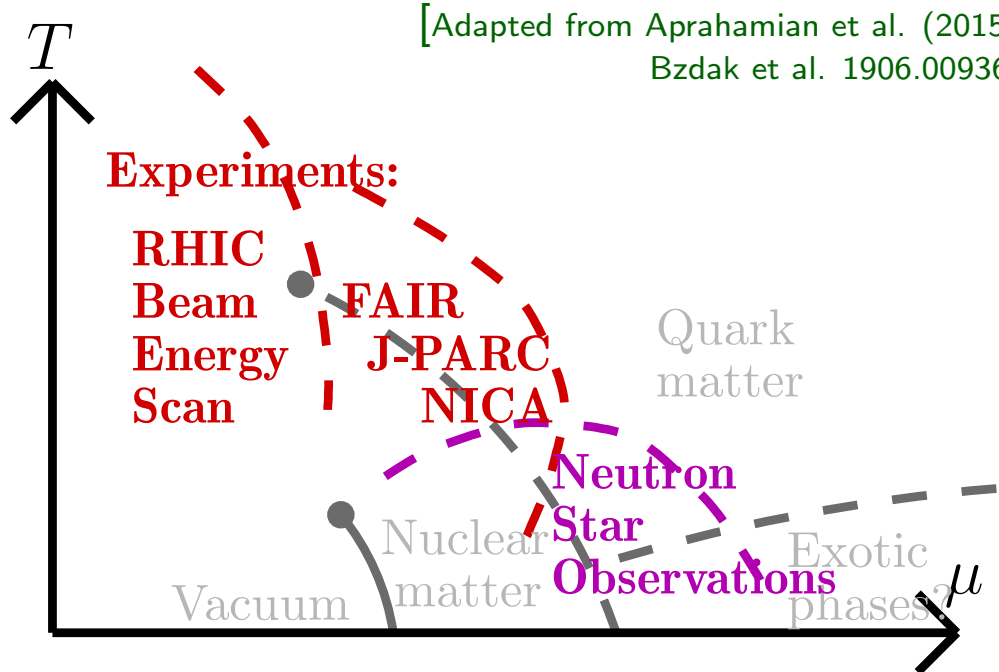
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Neutron star observations give complementary information at high density

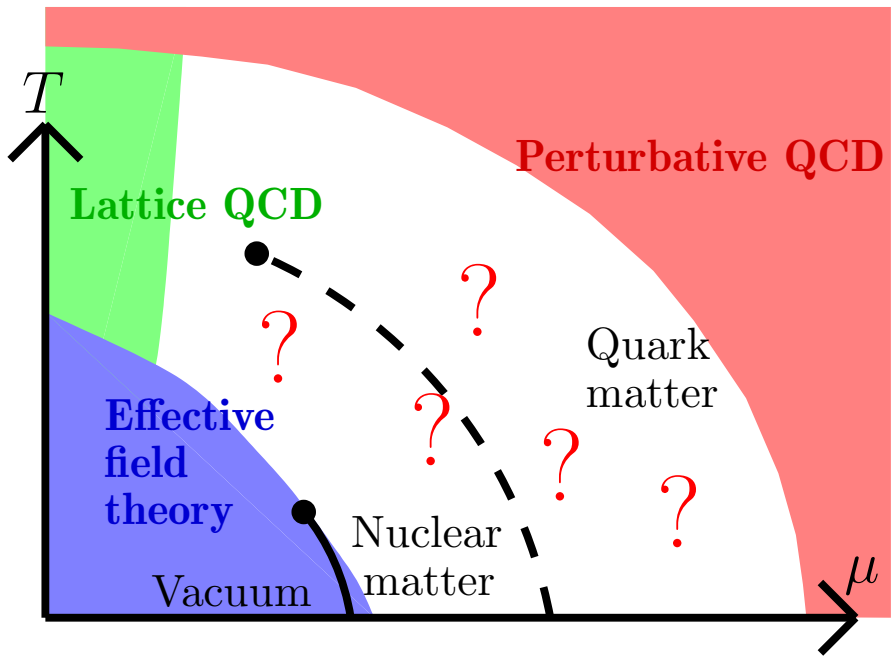



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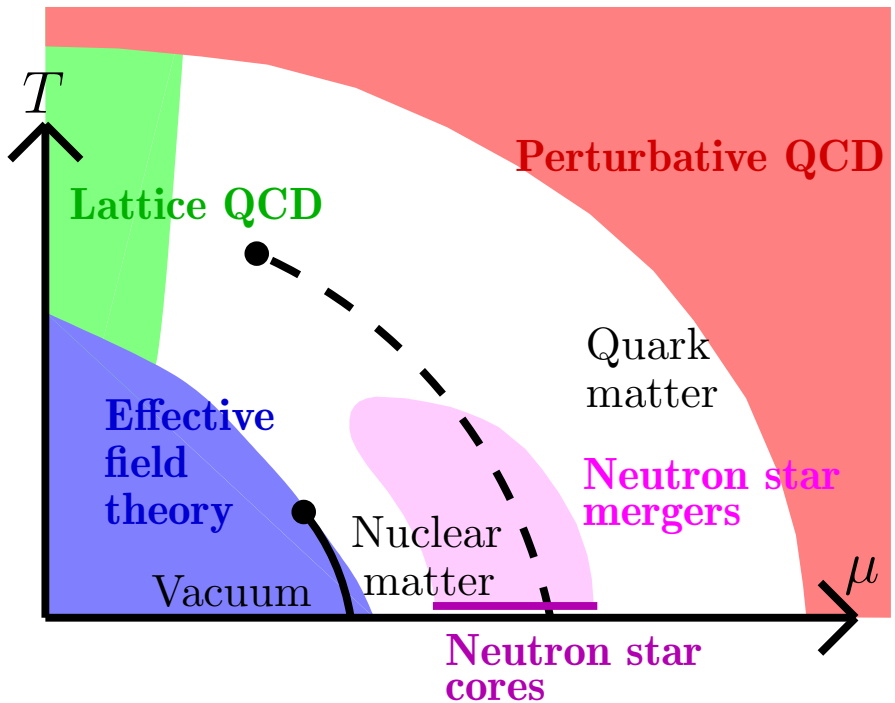
# QCD phase diagram: theory status

- Lattice data only available at zero/small chemical potentials
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- This region is highly relevant for neutron star physics!
- Improving theoretical predictions important!
- Strongly coupled physics – use the gauge/gravity duality?

# Why use holography for dense matter?

Already various models available in the literature – perhaps the gauge/gravity duality is just another uncontrolled approximation?

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There is however strong motivation for this approach:

- Strongly coupled physics: holography may work better than many other approaches
- Different phases (quark, nuclear, color superconducting, quarkyonic . . . ) on the same footing or even in a single model
  - Typically not achieved in the literature
  - Gives rise to predictions for phase transitions
- As it turns out, predictions do make sense!
  - Highly nontrivial – as the precise holographic dual for QCD is not known, these models cannot be derived
  - In particular, many details work out of the box at finite density

# Quark flavors in V-QCD

- Goal: construct a model for $N_f = 2 + 1$ — two massless and one massive (strange) quark
- The tachyonic brane action readily implements $N_f$ flavors and full chiral symmetry
- Optimally, one would just take the "old" model and turn on quark masses or chemical potentials for each flavor
- We actually tried this in order to analyze the bulk viscosity

  [Cruz Rojas, Gorda, Hoyos, Jokela, MJ, Kurkela, Paatelainen, Säppi, Vuorinen 2402.00621 (PRL)]
- However, the details of the flavor dependence turn out to be in conflict with lattice data $\Rightarrow$ model needs adjusting

  [MJ, Mitra 2507.08087]

# Background geometries

Three classes of geometries are relevant

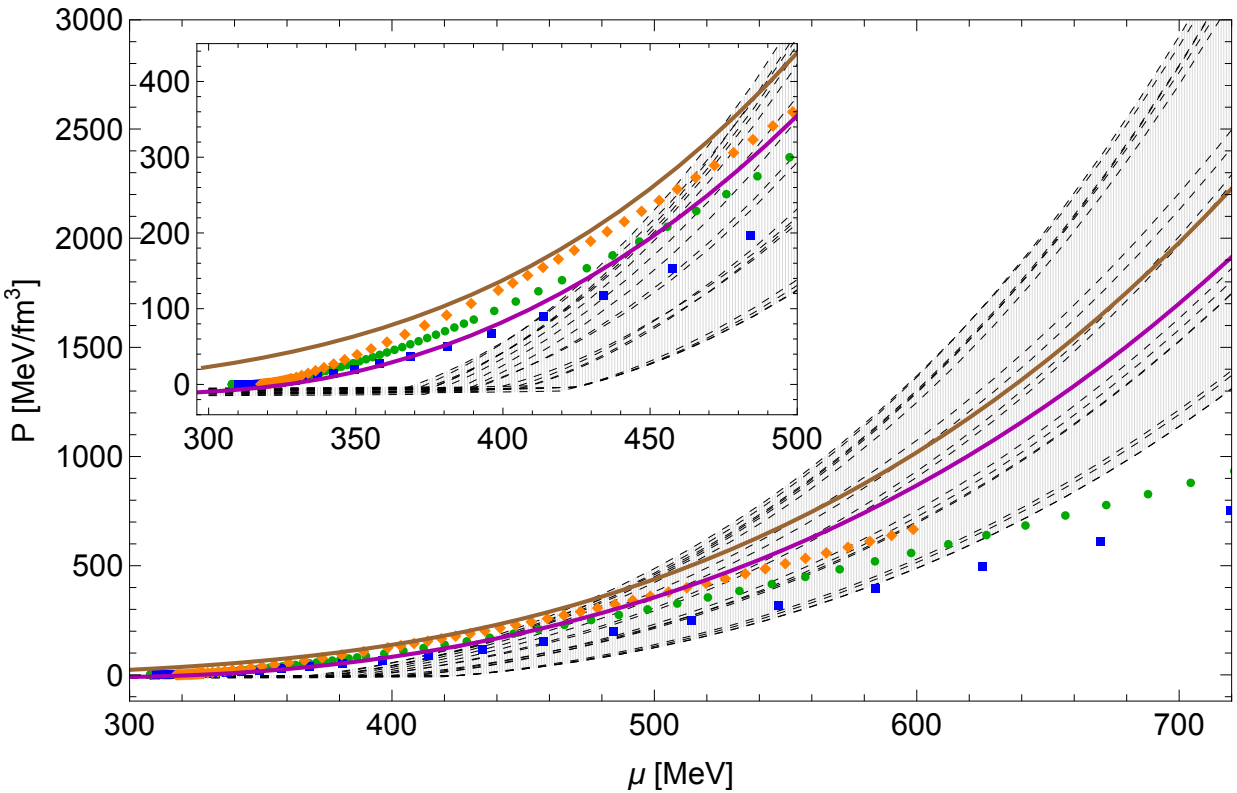
$$ds^2 = e^{2A(r)}\left(\frac{dr^2}{f(r)} - f(r)dt^2 + d\mathbf{x}^2\right)$$

1. "Chirally symmetric" black holes — meaning that light quark tachyons vanish $\tau_u = 0 = \tau_d$, while $\tau_s \neq 0$
   [main focus of this talk]
2. "Chirally broken" black holes — meaning that all tachyons nonzero, $\tau_i \neq 0$
3. Chirally broken thermal gas vacuum with $f(r) = 1$

We construct such solutions numerically

- Obtain thermodynamics through holographic dictionary, including black hole thermodynamics, and integration of the 1st law
- No single numerically expensive task, but still rather tedious

# Implications for neutron star EOS




Comparison of

1. Our models (solid curves)
2. Nuclear theory models (colored dots)
3. Old unflavored V-QCD EOSs (gray bands and dashed curves)

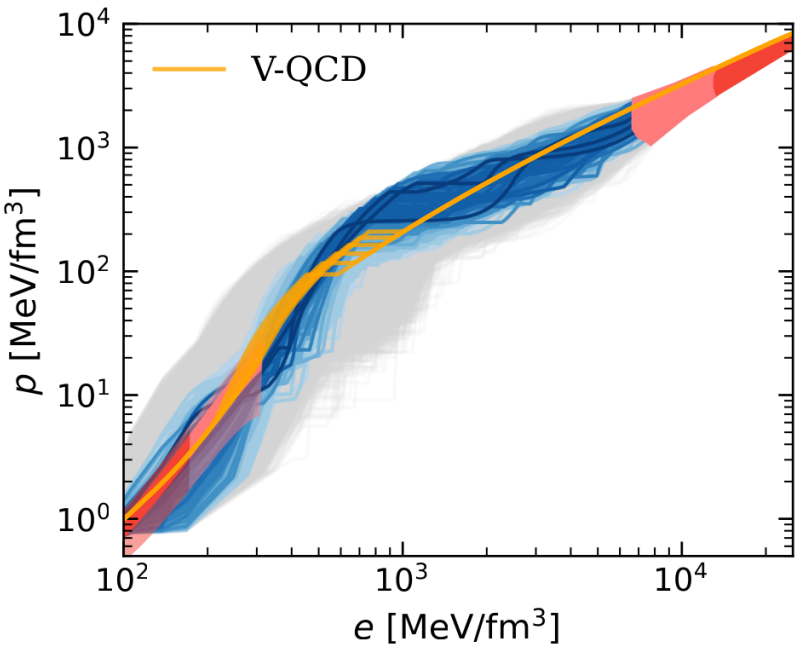
[Jokela, MJ, Remes 1809.07770]

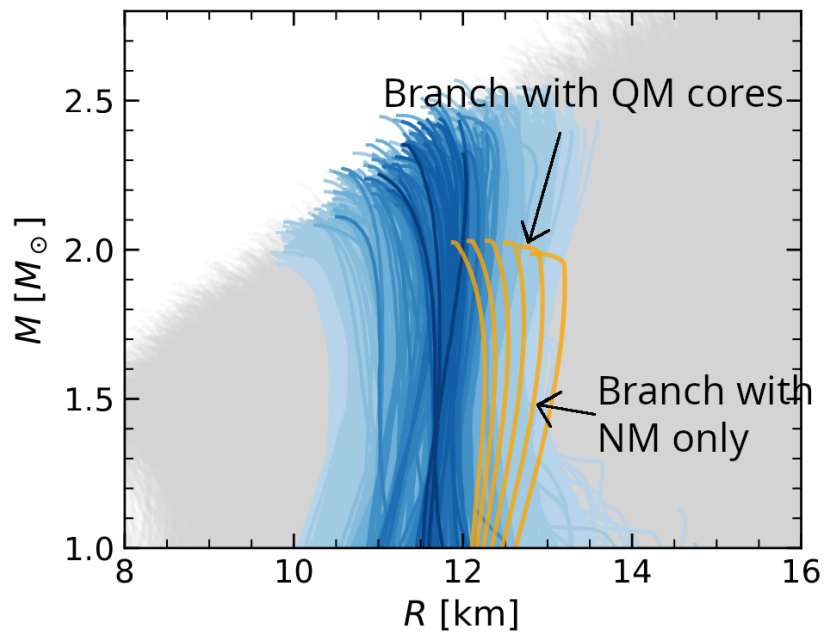
- New V-QCD close enough to nuclear theory to allow construction of feasible neutron star equations of state

# Work in progress: neutron star properties

Some preliminary results for neutron stars

- Adjusted parameters of the model to achieve realistic maximum neutron star masses and sizable quark cores
- Also added holographic nuclear matter
- Impose theoretical and observational constraints
- Compare with model-independent ensemble



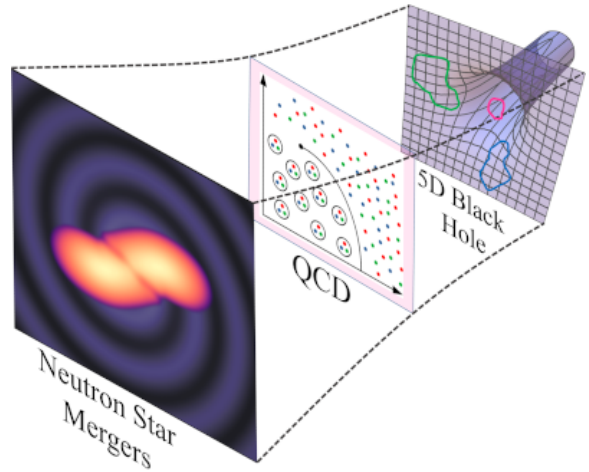




[with Christian Ecker and Toshali Mitra]

# The starting point: our earlier EOS

A state-of-the-art EOS, to be used

1. to describe (isolated) neutron stars
2. in simulations of neutron star mergers
3. in simulations of core collapse supernovae
4. when analyzing heavy-ion collisions (?)

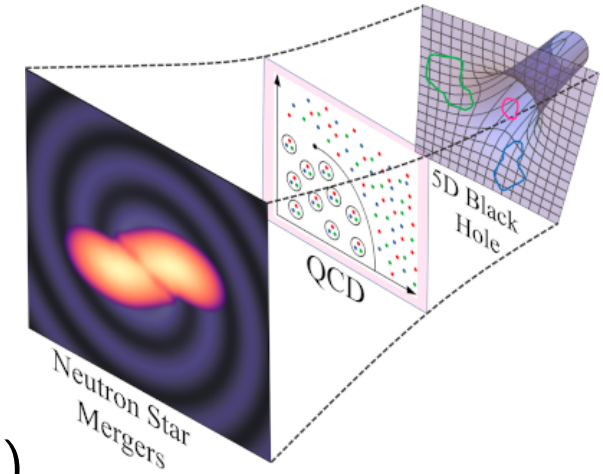



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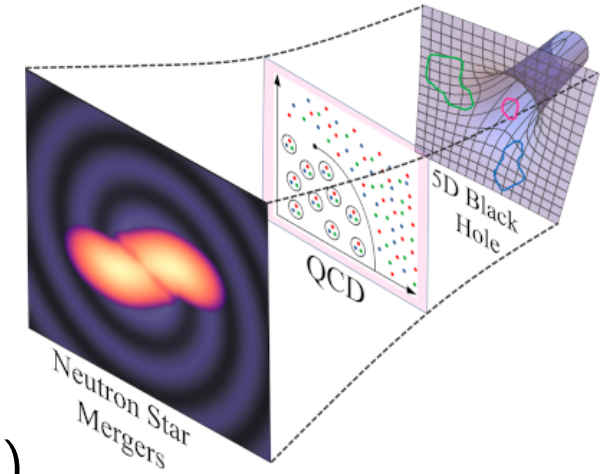
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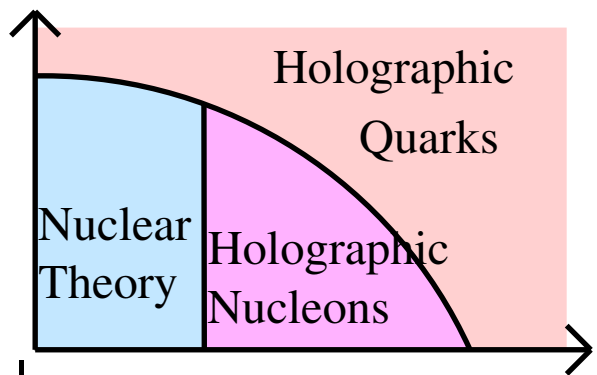



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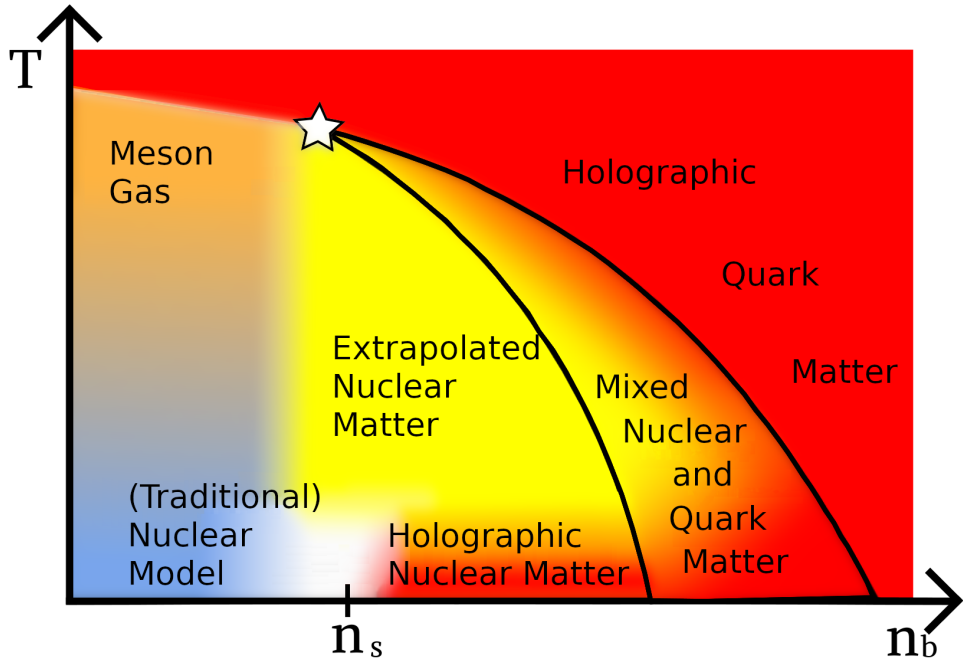
Main ingredients are

1. Holographic model for quark matter
2. (Slightly adjusted) holographic model for nuclear matter
3. Nuclear theory model for hadronic phase – at low density holography not very useful

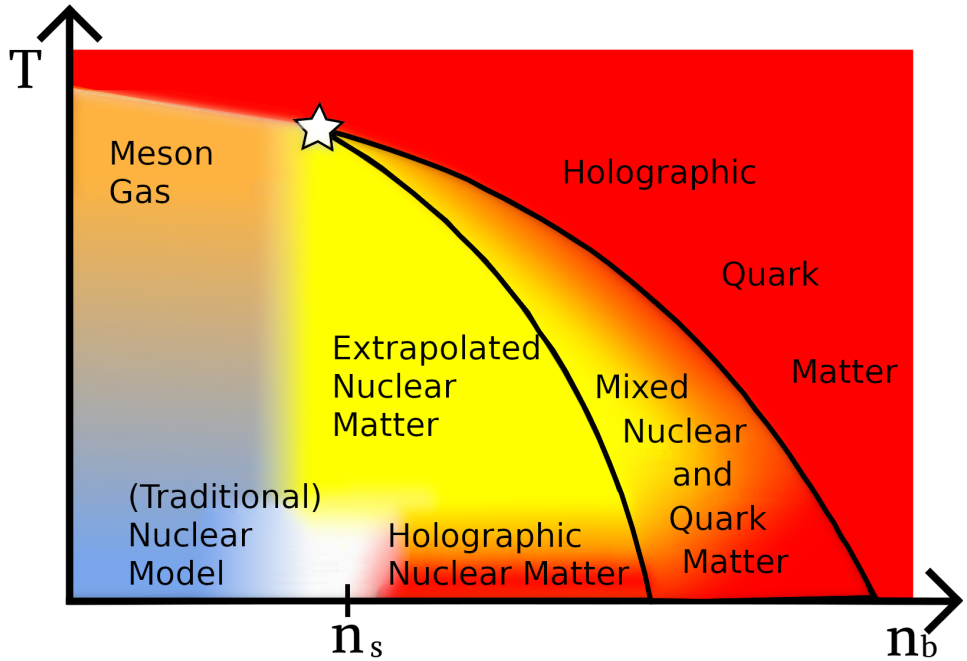



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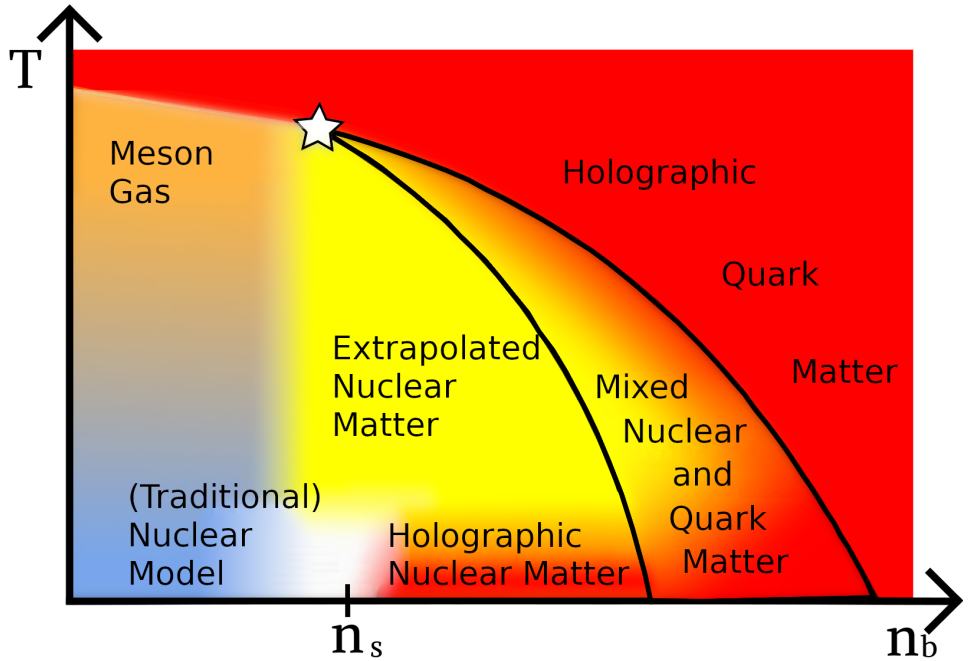


# Combining the building blocks: the hybrid model




Covers regions relevant for neutron stars and heavy-ion collisions!

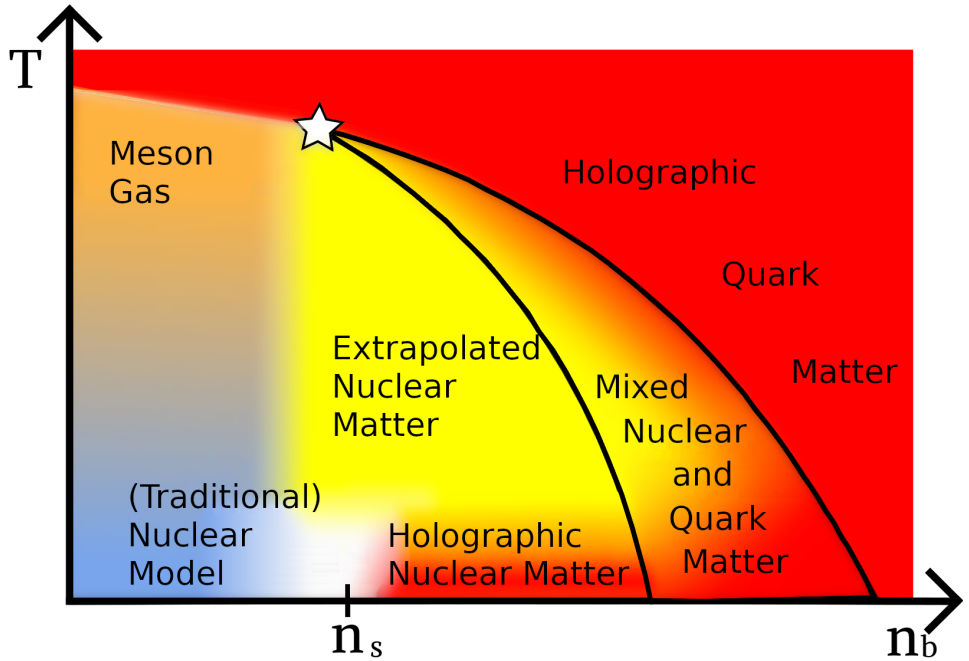
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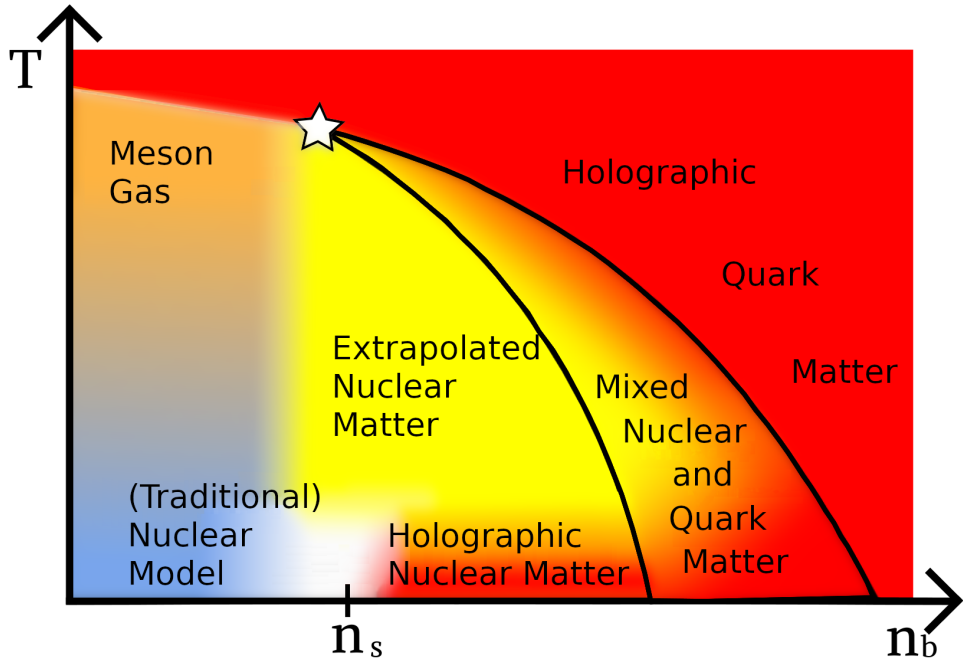
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- Probabilistic approach using these EOSs: see Christian's talk

# Shortcomings of the current EOSs

However our model has some obvious shortcomings:

I Deviation from $\beta$-equilibrated or isospin symmetric matter not implemented through holography
- Neutron star merger EOSs depend on temperature $T$, baryon number density $n_b$, and charge fraction $Y_q$
- $Y_q$ dependence not coming from holography so far in any phase

II No dependence on quark masses
- So far matching $N_f = 2 + 1$ data with a massless model

III No model for "exotic" color superconducting or color-flavor locked phases

IV Holographic nuclear matter model overly simple (homogeneous field instead of instantons)

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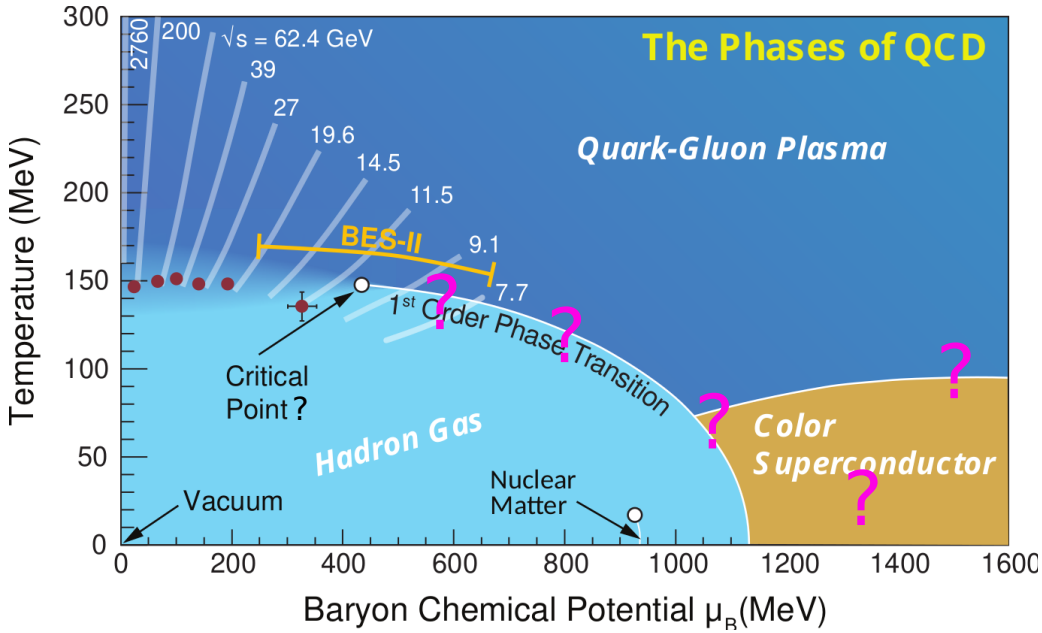
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This talk: developments towards fixing these problems

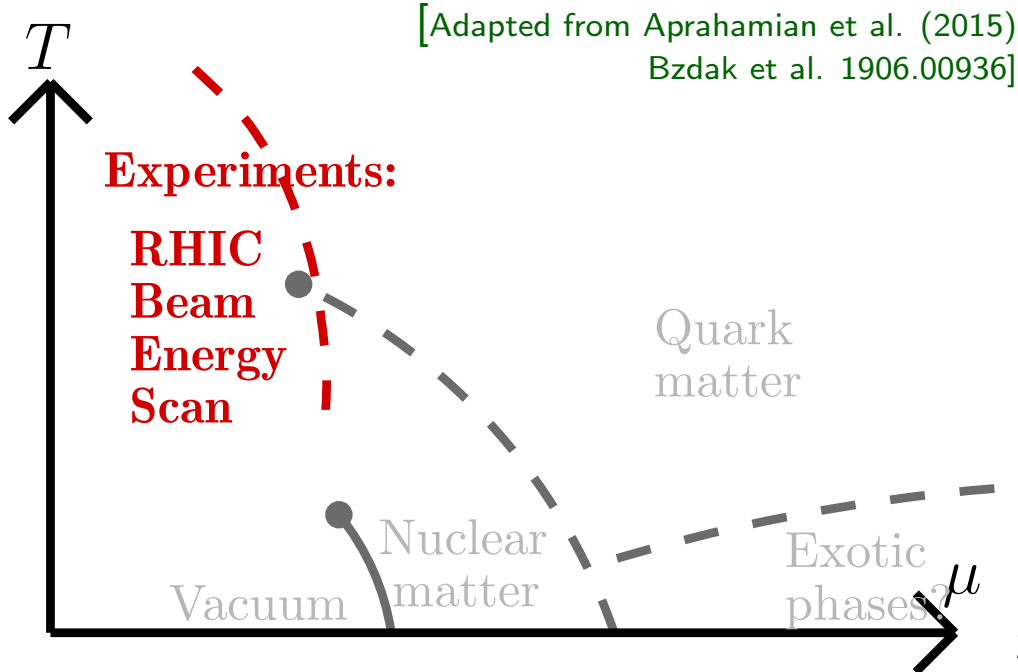
1. Quark flavor dependence in V-QCD (issues I and II)
2. Simple model for color superconductor (issue III)
3. Simple model for isospin asymmetric nuclear matter (issue I)

# QCD phase diagram and the critical point

Search for the critical point: ongoing effort at RHIC (Beam Energy Scan)



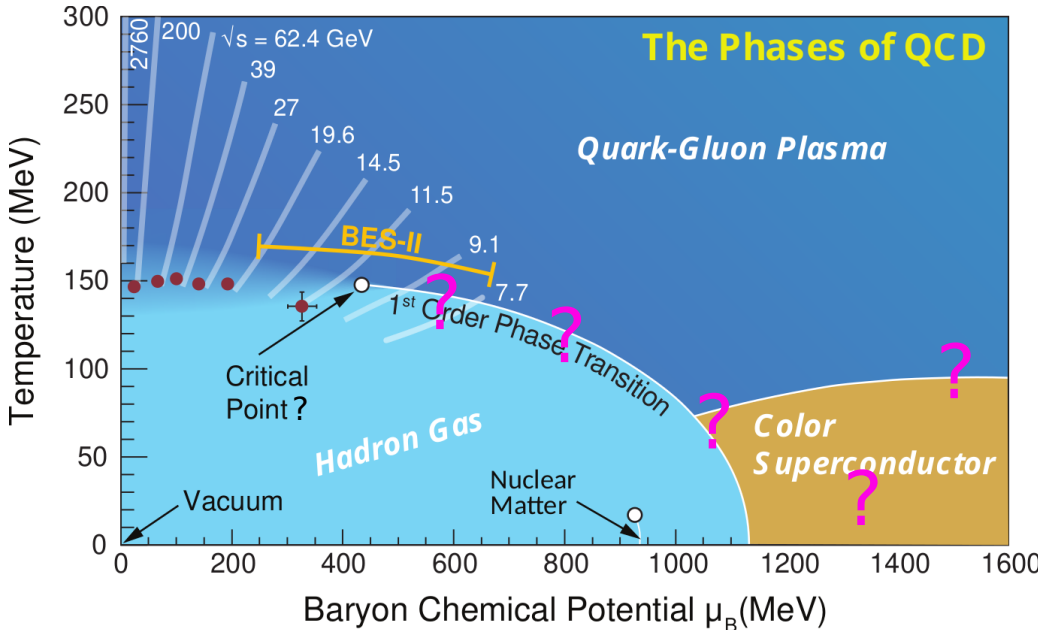

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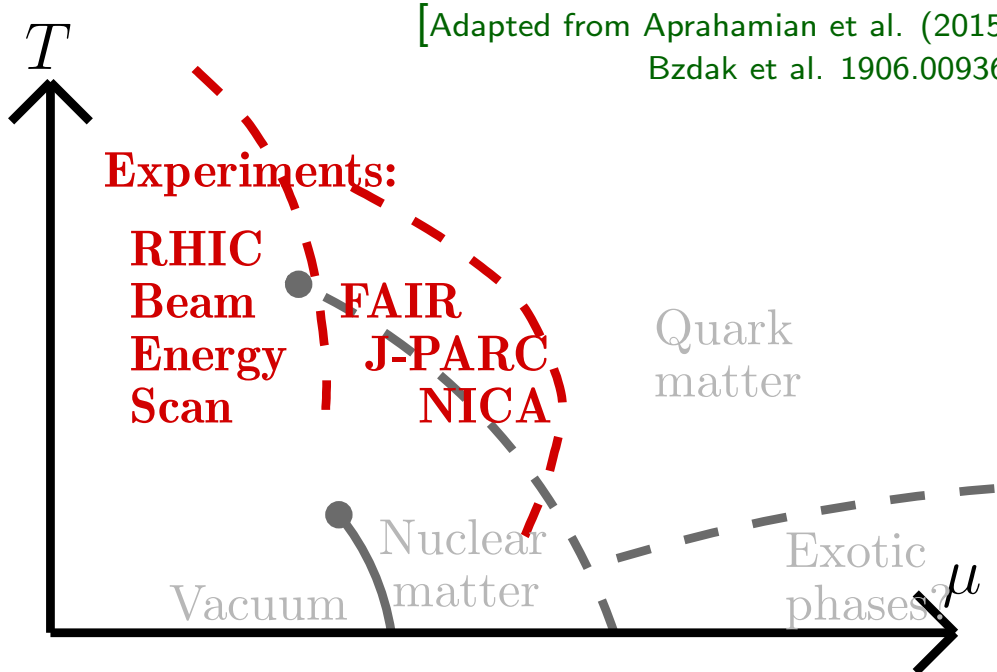
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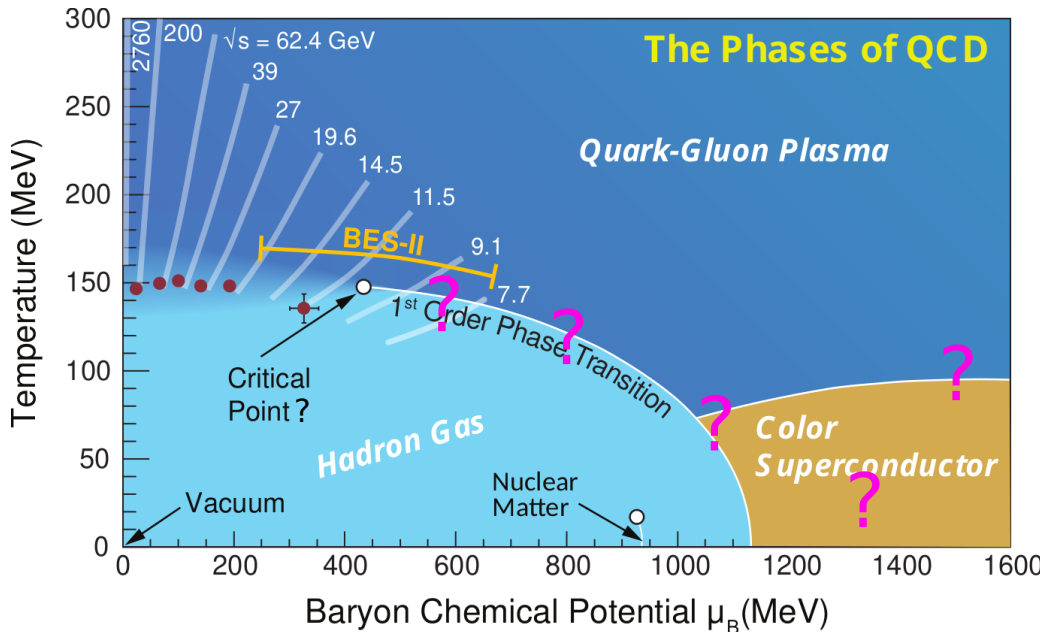


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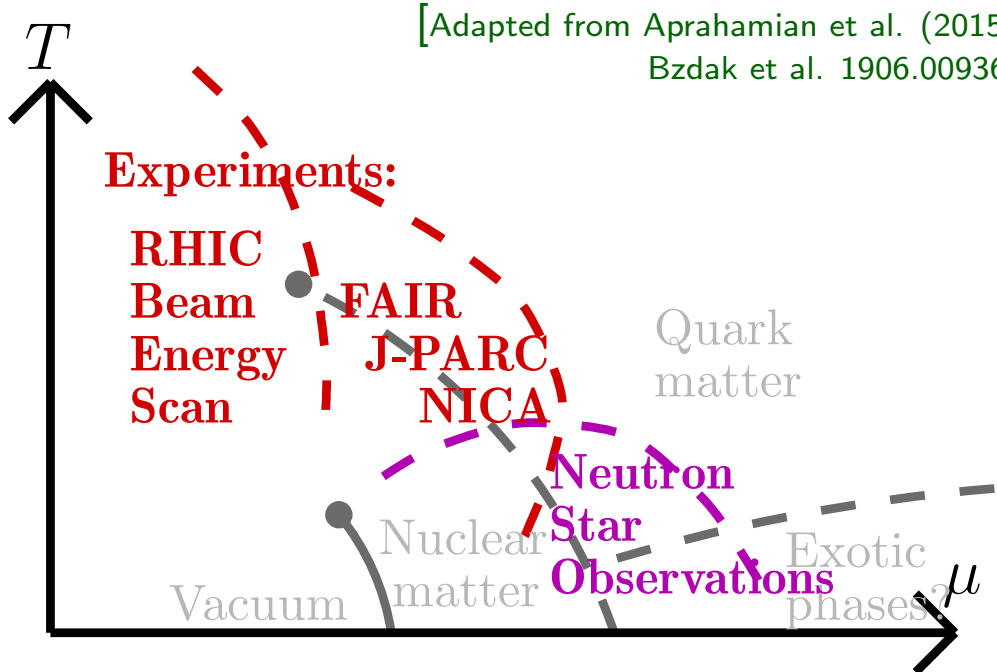
Search for the critical point: ongoing effort at RHIC (Beam Energy Scan)

Will be extended by future experiments at FAIR, J-PARC, NICA

Neutron star observations give complementary information at high density




[Adapted from Aprahamian et al. (2015) Bzdak et al. 1906.00936]



# Homogeneous nuclear matter in V-QCD

Nuclear matter in the probe limit: consider full brane action $S = S_{\rm DBI} + S_{\rm CS}$ where

[Bigazzi, Casero, Cotrone, Kiritsis, Paredes; Casero, Kiritsis, Paredes]

$$S_{\rm DBI} = -\frac{1}{2} M^3 N_c \, \mathbb{T}r \int d^5x \, V_{f0}(\lambda) e^{-\tau^2} \left( \sqrt{-\det \mathbf{A}^{(L)}} + \sqrt{-\det \mathbf{A}^{(R)}} \right)$$

$$\mathbf{A}^{(L/R)}_{MN} = g_{MN} + \delta^r_M \delta^r_N \kappa(\lambda) \tau'(r)^2 + \delta^{rt}_{MN} w(\lambda) \Phi'(r) + w(\lambda) F^{(L/R)}_{MN}$$

gives the dynamics of the solitons (will be expanded in $F^{(L/R)}$) and

$$S_{\rm CS} = \frac{N_c}{8\pi^2} \int \Phi(r) e^{-b\tau^2} dt \wedge \left( F^{(L)} \wedge F^{(L)} - F^{(R)} \wedge F^{(R)} + \cdots \right)$$

sources the baryon number for the solitons

- Extra parameter, $b > 1$, to ensure regularity of solutions

Set $N_f = 2$ and consider the homogeneous SU(2) Ansatz

[Rozali, Shieh, Van Raamsdonk, Wu 0708.1322]

$$A^i_L = -A^i_R = h(r) \sigma^i$$

[Ishii, MJ, Nijs, 1903.06169]

# Discontinuity and smeared instantons

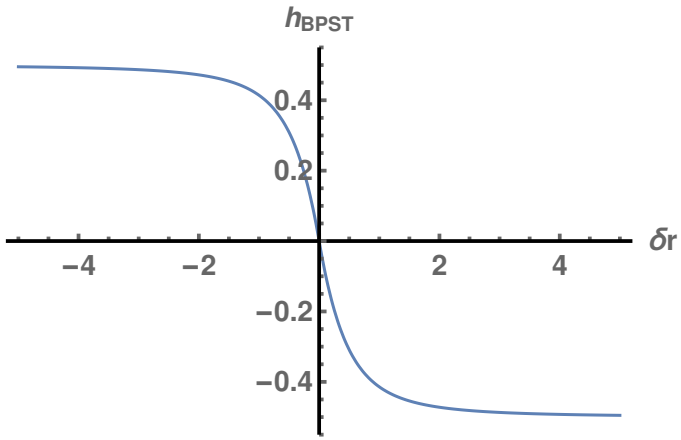
With the homogeneous Ansatz $A_i^a(r) = h(r)\delta_i^a$ baryon number vanishes for any smooth $h(r)$:

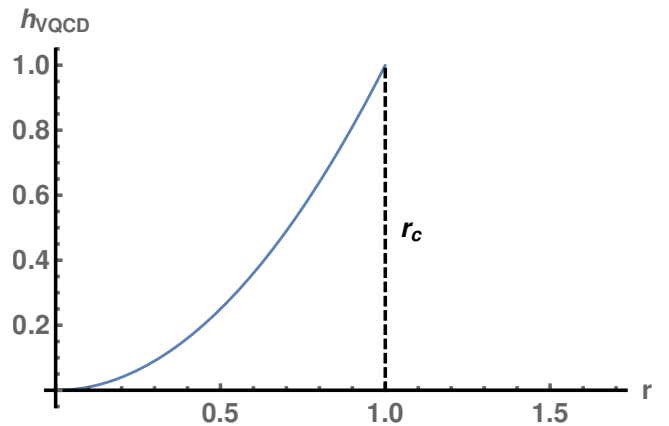
$$N_b \propto \int dr \frac{d}{dr}[\mathrm{CS-term}] = 0$$

How can this issue be avoided?

- Smearing the BPST soliton in singular Landau gauge:

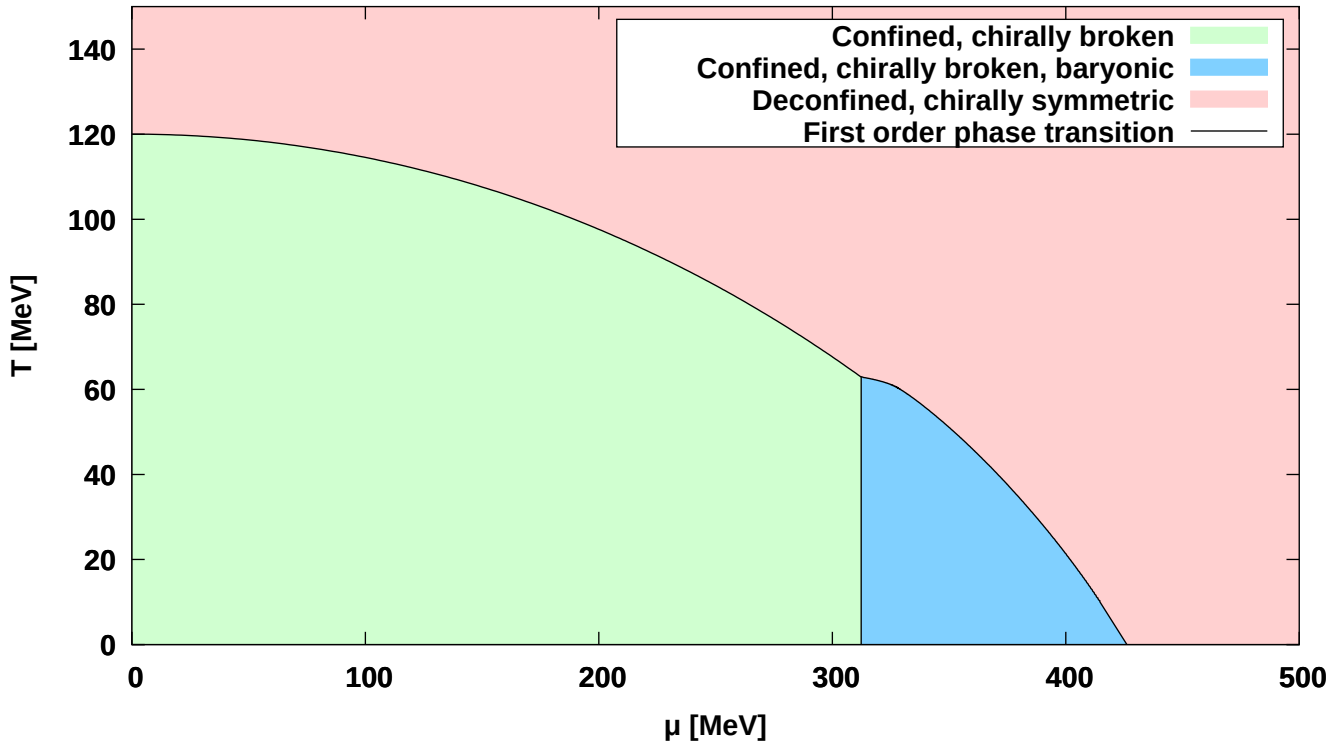
$$\langle A_i^a \rangle \sim \int \frac{d^3\mathbf{x}\ \ \eta_{i4}^a\ \delta r}{(\delta r^2 + \mathbf{x}^2 + \rho^2)(\delta r^2 + \mathbf{x}^2)} \sim -\frac{\delta_i^a\ \delta r}{\sqrt{\delta r^2 + \rho^2} + |\delta r|}$$




- This suggests a solution: introduce a discontinuity in $h(r)$ at $r = r_c$
- The discontinuity sources nonzero baryon charge!

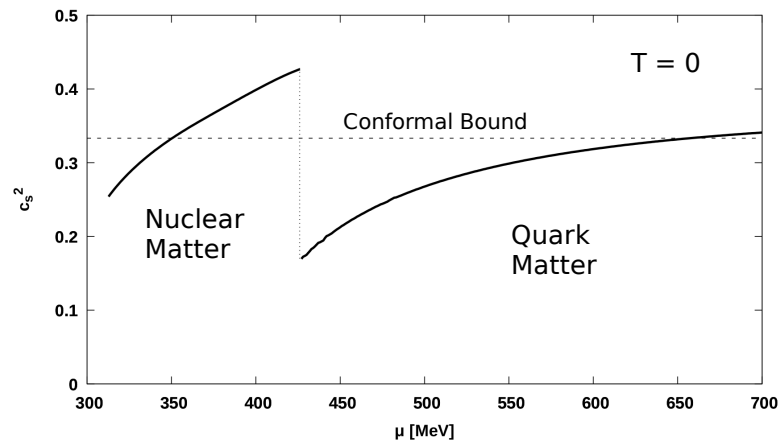
# Phase diagram after including nuclear matter




Stiff EOS (high $c_s^2$) in the nuclear matter phase $\Rightarrow$ helps to pass the bounds from neutron star observations!

Such a stiff EOS also found in WSS and hard wall models

[Kovensky, Poole, Schmitt 2111.03374; Bartolini, Gudnason, Leutgeb, Rebhan 2202.12845]



# Adjusting the nuclear matter model

The V-QCD nuclear matter EOS as such is
however not fully satisfactory:

- Temperature dependence is absent in the confined phases, and therefore also for holographic nuclear matter
- This is likely to be a good first approximation, but not enough for a state-of-the-art model

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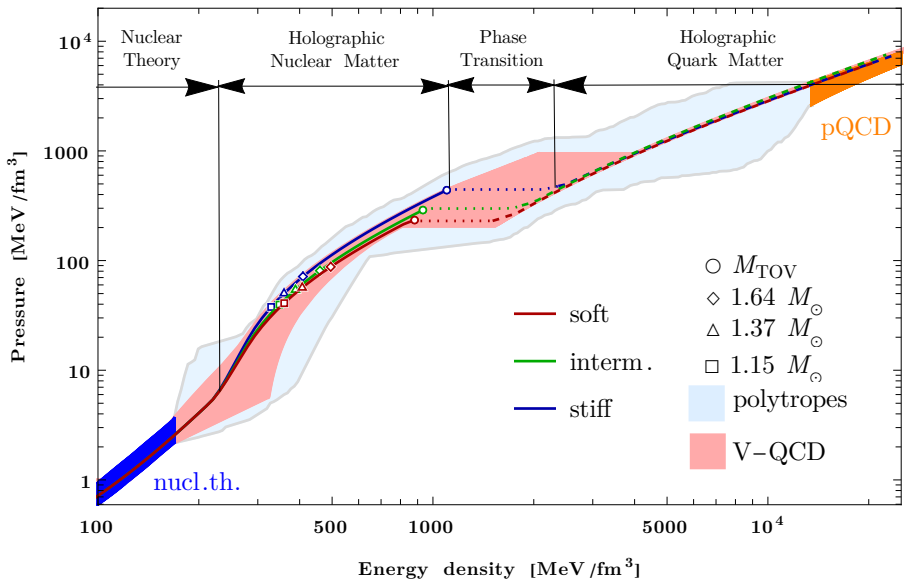
Our solution: we extrapolate the holographic nuclear matter EOS to nonzero $T$ by a using a van der Waals approach

- Gas of protons, neutrons and electrons with an excluded volume correction and a potential term

[Demircik, Ecker, MJ 2112.12157]

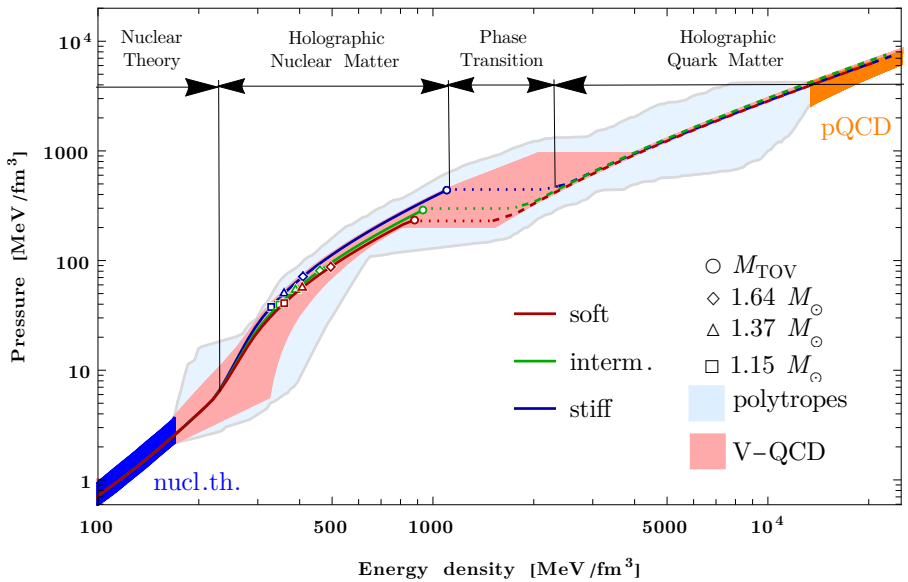
# Cold EoS and known constraints

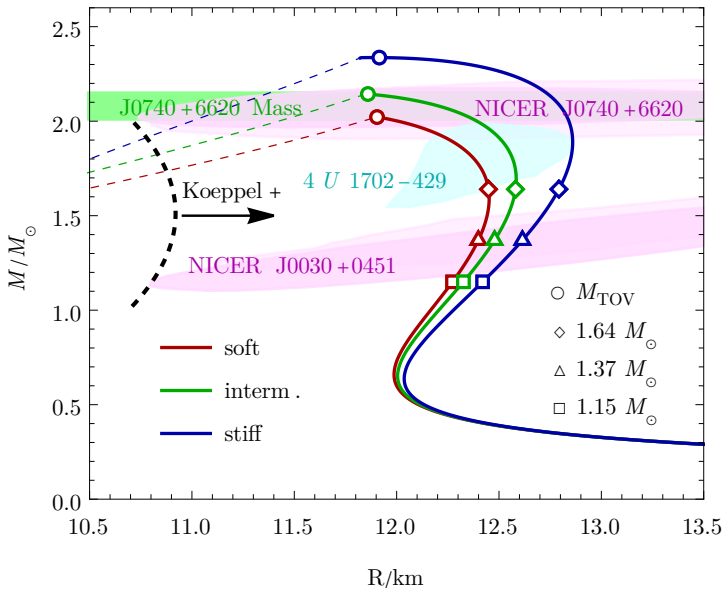
- Three choices of EoSs: soft, intermediate, and stiff $\leftrightarrow$ the degrees of freedom of V-QCD left free by fit to lattice data
- Compared to bands of all feasible cold matter EoS: Without and with holography



# Cold EoS and known constraints

- Three choices of EoSs: soft, intermediate, and stiff $\leftrightarrow$ the degrees of freedom of V-QCD left free by fit to lattice data
- Compared to bands of all feasible cold matter EoS: Without and with holography




- Plug EoSs in TOV: neutron star $M(R)$ curves (left plot)
- Compares well with mass/radius observations
- No stable quark cores inside neutron stars

[Ecker, MJ, Nijs, van der Schee 1908.03213]
[Jokela, MJ, Nijs, Remes 2006:01141]
[Demircik, Ecker, MJ 2112.12157]

# Advantages of the model

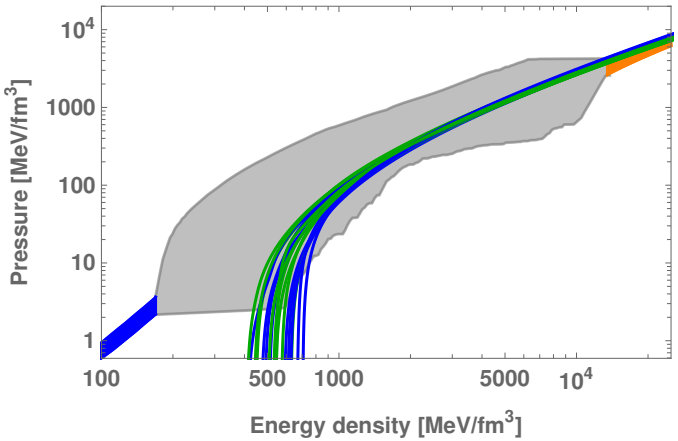
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1. Precise fit of lattice data possible
   - Mild parameter dependence, works with simple potentials

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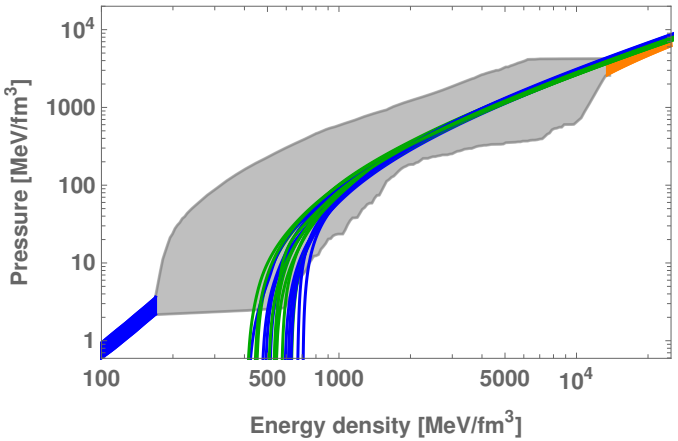
1. Precise fit of lattice data possible
   - Mild parameter dependence, works with simple potentials
2. Extrapolated quark equation of state to high density reasonable
   - Quite nontrivial since there are strict bounds

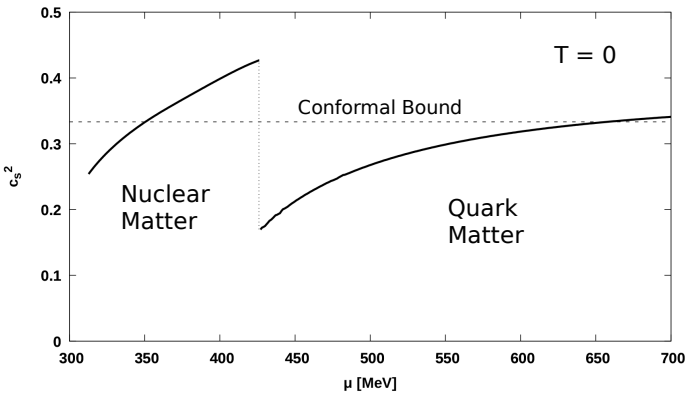


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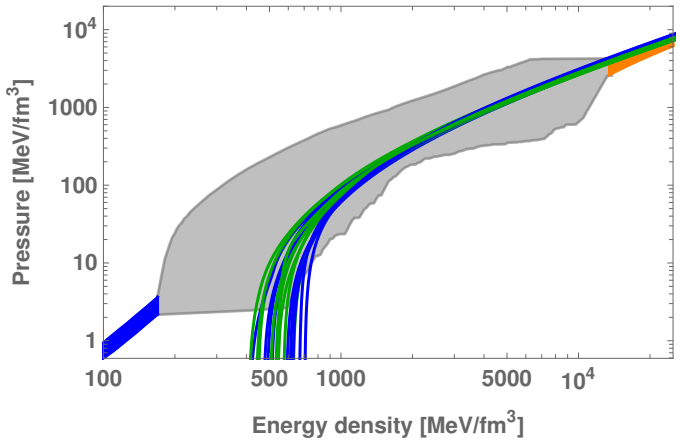


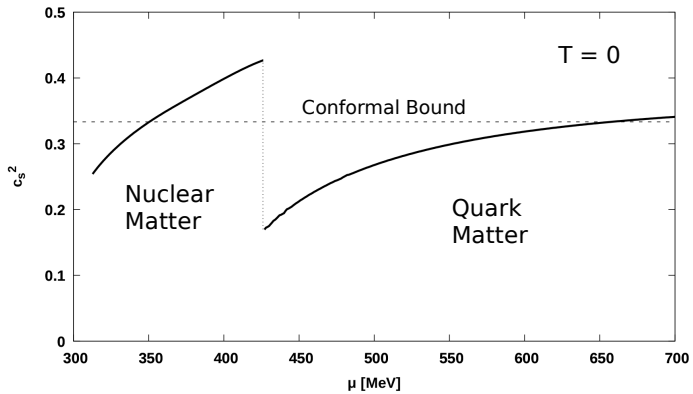



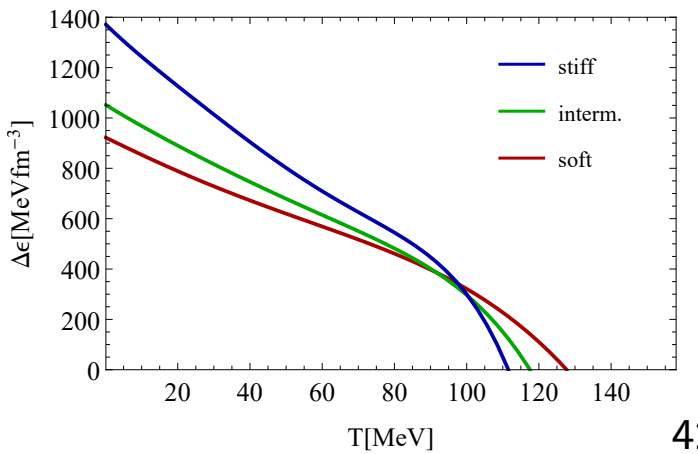
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3. Nuclear matter EOS is stiff (high speed of sound)
   - Easier to pass constraints from neutron star observations
4. Simultaneous modeling of nuclear and quark matter phases
   - Predictions for the phase transition



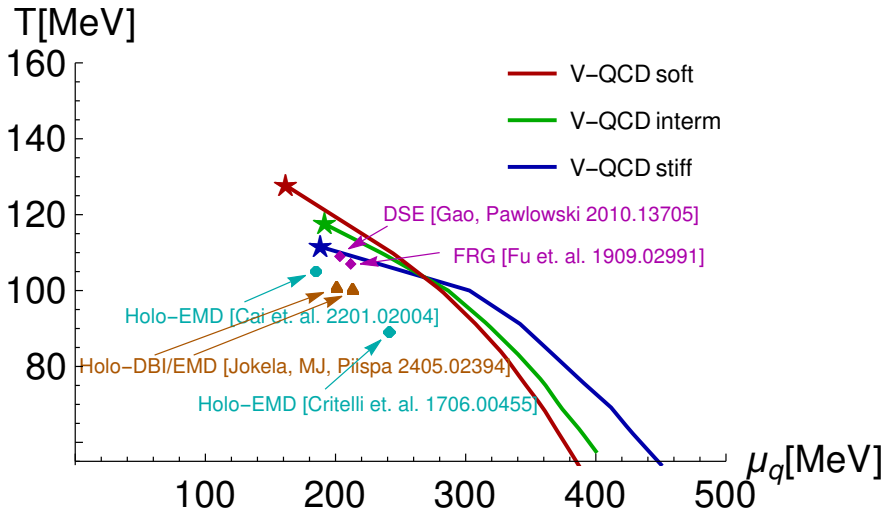



# Agreement with FRG

Close agreement with functional renormalization group (FRG)

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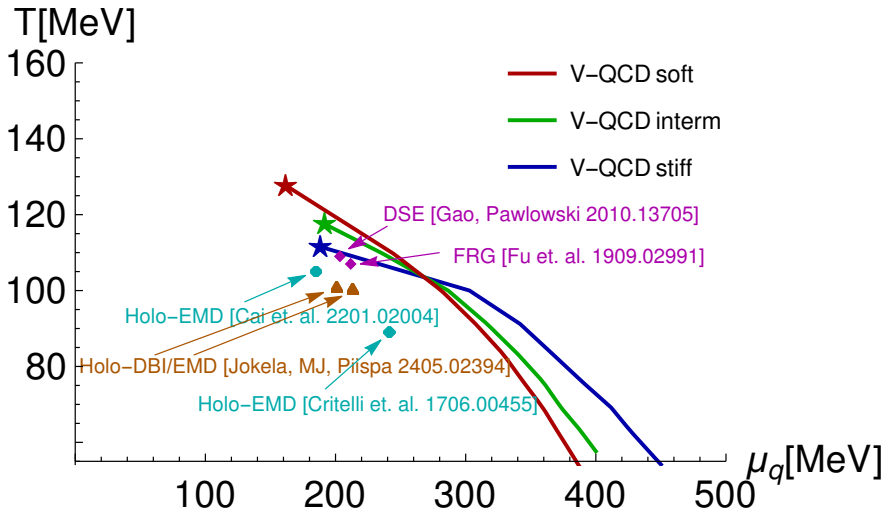
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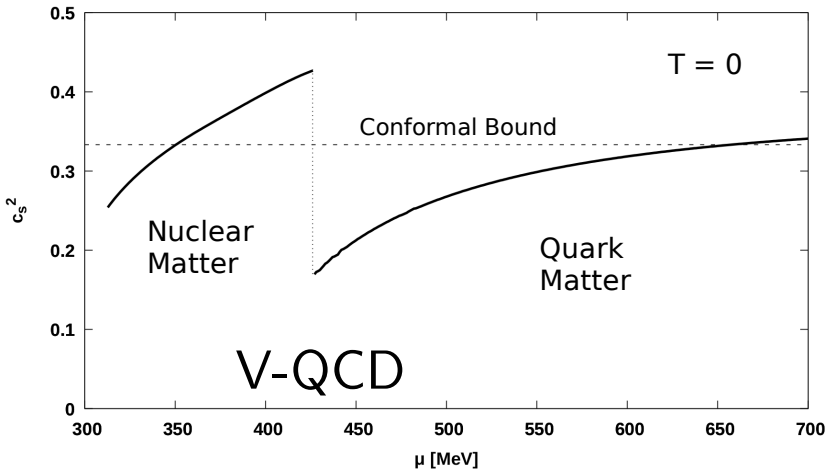

- Critical point

# Agreement with FRG

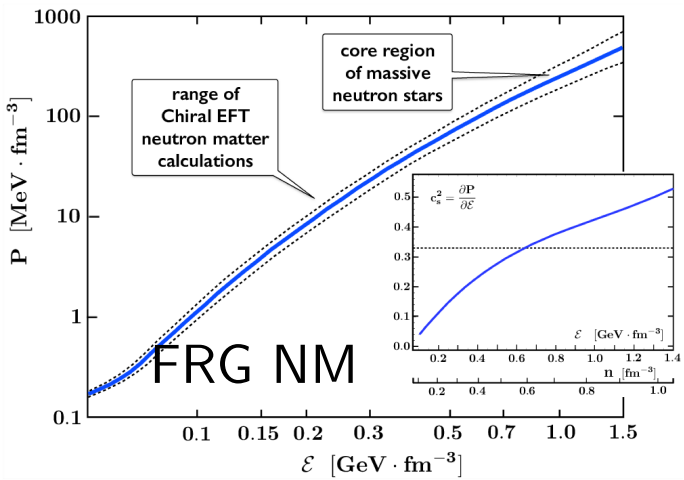
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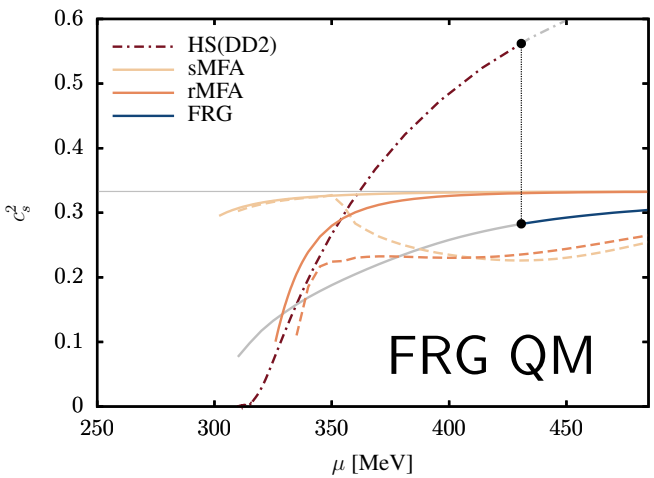



- Critical point




- Speed of sound at $T = 0$




[Drews, Weise 1610.07568; Otto, Oertel, Schaefer 1910.11929]

# Ansatz for potentials

$$V_{\rm g}(\lambda) = 12\left[1 + V_{g,1}\lambda + \frac{V_{g,2}\lambda^2}{1 + c_\lambda\lambda/\lambda_0} + V_{\rm IR}\, e^{-\lambda_0/(c_\lambda\lambda)}\left(\frac{c_\lambda\lambda}{\lambda_0}\right)^{4/3}\sqrt{\log(1 + c_\lambda\lambda/\lambda_0)}\right]$$

$$V_{\rm f}(\lambda,\tau_i) = V_{{\rm f}\lambda}(\lambda)\left(1+\tau_i^4\right)^{\tau_p}\exp\left(-\tau_i^2\right)$$

$$V_{{\rm f}\lambda}(\lambda) = W_0 + W_1\lambda + \frac{W_2\lambda^2}{1 + c_\lambda\lambda/\lambda_0} + W_{\rm IR}\, e^{-\lambda_0/(c_\lambda\lambda)}(c_\lambda\lambda/\lambda_0)^2$$

$$\kappa(\lambda) = \kappa_0\left[1 + \bar\kappa_0\left(1 + \frac{\bar\kappa_1\lambda_0}{c_\lambda\lambda}\right)e^{-\lambda_0/(c_\lambda\lambda)}\frac{(c_\lambda\lambda/\lambda_0)^{4/3}}{\sqrt{\log(1 + c_\lambda\lambda/\lambda_0)}}\right]^{-1}$$

$$w(\lambda,\tau) = \frac{w_0}{1+\beta_s\tanh(\gamma_s^2\tau_i^2)}\left[1 + \frac{w_1 c_\lambda\lambda/\lambda_0}{1 + c_\lambda\lambda/\lambda_0} + \bar w_0\, e^{-\lambda_0/(c_w\lambda)}\frac{(c_w\lambda/\lambda_0)^{4/3}}{\log(1 + c_w\lambda/\lambda_0)}\right]^{-1}.$$

# Paired phases in QCD

Model computations, and analysis at large $\mu$, indicate that at low $T$ deconfined quarks become paired

- Color–flavor locking (CFL)
  - Known to be present at asymptotically high $\mu$
  - $\mathrm{SU}(3)_c \times \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_B \to \mathrm{SU}(3)_{c+L+R}$
- Two-flavor color superconductivity (2SC)

Symmetry breaking patterns involve the gauge group — tricky in holography, where observables are gauge singlets . . .

Despite this, many studies already exist (see also talk by Edwan Préau) . . . we did the first attempt in V-QCD

# Simple model for paired phases in V-QCD

We model the quark condensate through an additional charged scalar field $\psi$

$$S_\psi = -M^3 N_c N_f \int d^5x \, \sqrt{-\det g}\, Z(\lambda) \left[ |D_M \psi|^2 + M(\lambda)^2 \, |\psi|^2 \right]$$

$$D_M \psi = \partial_M \psi - i \hat{A}_M \psi \ , \qquad \hat{A}_M = Tr[A_M]/N_f$$

- Trivial (massless) flavors so far
- Unit charge: allows generalization to a matrix field
- Allow for dilaton dependent potential functions $Z$, $M$
- Here restrict to probe field $\psi$: no backreaction to metric
- Similar to "holographic superconductor"

[Cruz Rojas, Demircik, Ecker, MJ 2505.06338]

# The holographic mechanism

Mechanism leading to instabilities at low $T$

- Deconfined quark matter phase $\leftrightarrow$ black hole geometry
- As $T \to 0$, black hole (nearly) extremal: IR geometry $\sim AdS_2 \times \mathbb{R}^3$
- This geometry easily leads to instabilities due to its low Breitenlohner-Freedman (BF) bound for fluctuations
  - Violation at zero momentum: instability towards a homogeneous condensate
  - Violation at nonzero momentum: instability towards an inhomogeneous condensate
- Both kind of instabilities possible in V-QCD
  - The charge scalar $\psi$ may condense homogeneously (focus in this talk)
  - Gauge fields may condense inhomogeneously: Nakamura–Ooguri–Park instability driven by the Chern–Simons term

[Cruz Rojas, Demircik, MJ 2405.02399; Demircik, Jokela, MJ, Piispa 2405.02392 ]

- $AdS_2 \times \mathbb{R}^3$ geometry implies nonzero entropy at zero $T$
  - Condensing charged $\psi$ expected to absorp the charge supporting the $AdS_2$, leading to zero entropy at zero $T$

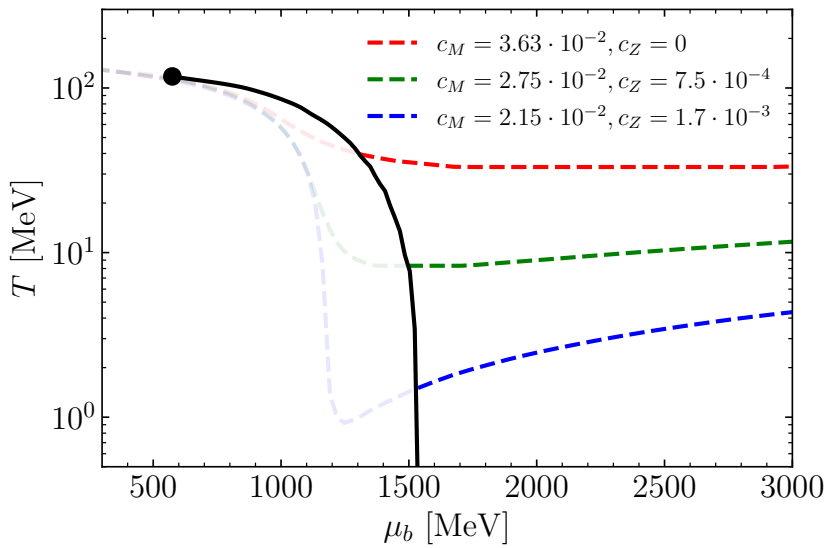
# Choice of potentials and extent of instability

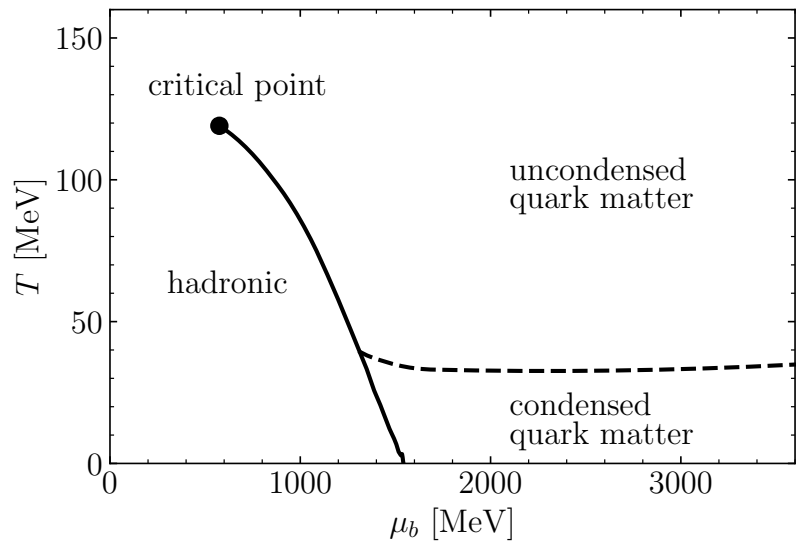
$$\mathcal{L}_\psi \propto Z(\lambda)\left[|D_M\psi|^2 + M(\lambda)^2\,|\psi|^2\right]$$

How to choose the potentials $M$ and $Z$ in the Lagrangian?

- Earlier observation: working potentials obey asymptotics such that they become flat in the string frame, $g_s = \lambda^{4/3} g$
- Require the same as $\lambda \to \infty$:

$$Z(\lambda) = 1 + c_Z\lambda^2\,, \qquad M(\lambda) = -3(1 + c_M\lambda^{4/3})$$



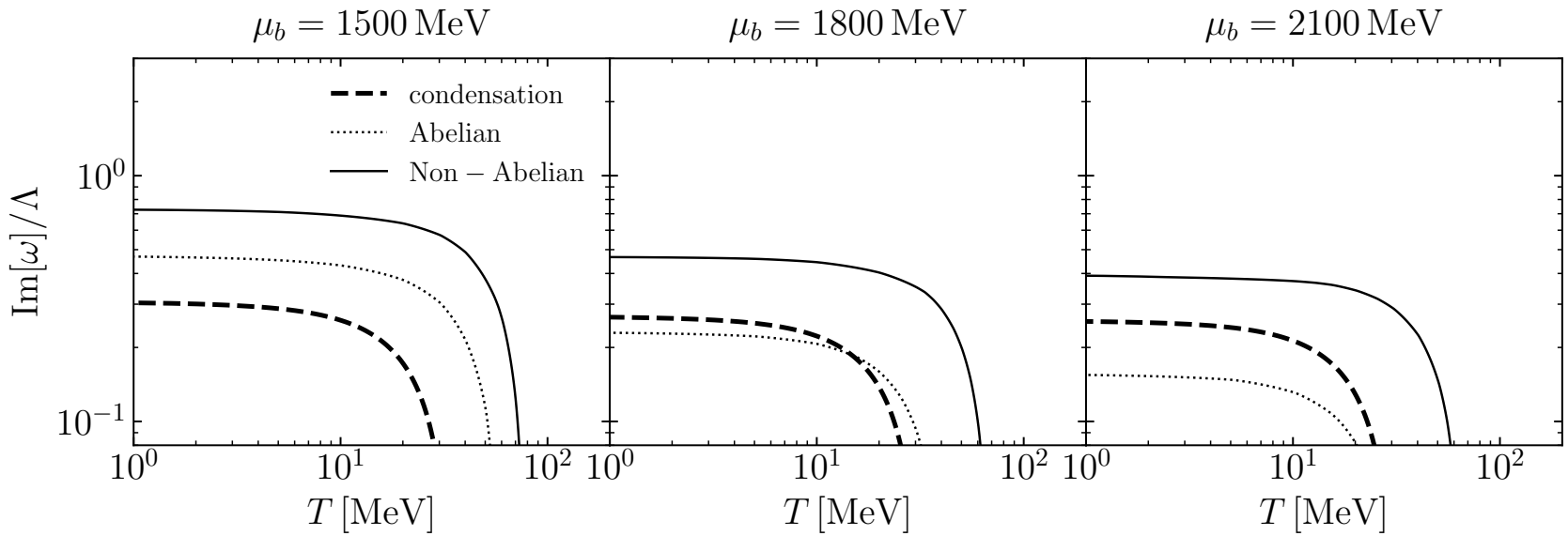




- Vary other parameters such that no instability in deconfined region at $\mu = 0 \Rightarrow$ maximal phenomenologically viable effect
- N.B. constant $M$ does not "work": essentially no instability

# Competition with modulated instability

Compare the (maximal) frequency of the unstable mode to the frequency of the modulated Nakamura–Ooguri–Park instability

- Both singlet (Abelian) and non-singlet (non-Abelian) gauge fields unstable




The color superconducting instability (thick dashed)

- appears only at lower temperature
- is "weaker" than the modulated instability

But the final ground state could include both effects . . . merits further study

# Model for nuclear matter

Standard method for baryons in holographic models: Each baryon maps to a solitonic 5D "instanton" of gauge fields

- Such instantons have been studied in many models, including V-QCD
  [MJ, Kiritsis, Nitti, Préau 2209.05868; 2212.06747]
- Already constructing an isolated instanton solution is nontrivial
- Dense nuclear matter requires studying many-instanton solutions . . . extremely challenging!

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- Already constructing an isolated instanton solution is nontrivial
- Dense nuclear matter requires studying many-instanton solutions . . . extremely challenging!
- Simple approach: V-QCD with two flavors and a homogeneous gauge field, mimicking dense solitons: $A^i = h(r)\sigma^i$
  [Rozali, Shieh, Van Raamsdonk, Wu 0708.1322; Li, Schmitt, Wang 1505.04886]
  [Ishii, MJ, Nijs, 1903.06169]
  - Note: spatial and flavor indices linked
- This is what we used so far . . . however by construction isospin symmetric: no distinction between protons and neutrons

# Going away from symmetric case

Introduce asymmetric gauge field and isospin chemical potential

$$A_t(r) = \widehat{a}_0(r)\mathbb{1} + a_0(r)\sigma^3 \ , \qquad a_0(0) = \mu_I$$

- Here $r$ is the holographic coordinate, bdry at $r = 0$

Then consistency requires a more general Ansatz

$$A_i = H(r)\sigma^i, \qquad (i = 1, 2)$$
$$A_3 = H_3(r)\sigma^3 + L_3(r)\mathbb{1}$$

[Kovensky, Schmitt 2105.03218; Bartolini, Gudnason 2209.14309; Kovensky, Poole, Schmitt 2302.10675]

- A discontinuity at some $r = r_c$ required for nonzero $n_b$
- Beyond the discontinuity, $r > r_c$, gauge fields vanish
- Solution found by minimizing the action, both wrt functions and $r_c$

See also talk by Andreas Schmitt

# Model parameters and the Chern–Simons term

We follow the approach from isospin symmetric construction

[Ishii, MJ, Nijs, 1903.06169; Demircik, Ecker, MJ 2112.12157]

- Match with a nuclear theory model at $\sim 1.5 n_s \Rightarrow$ determine remaining model parameters
- These include the parameters from the Chern–Simons action, in our case (including L/R components) [MJ, Kiritsis, Nitti, Préau ]

$$S_{\rm TCS} \sim iN_c \int F_1(\tau)\,{\rm tr}\left[A\wedge A\wedge A\wedge\left(F^{(L)}+F^{(R)}\right)\right]$$
$$-iF_3(\tau)\,{\rm tr}\left[A\wedge\left(F^{(L)}-F^{(R)}\right)\wedge\left(F^{(L)}-F^{(R)}\right)\right]$$

with $A = A^{(L)} - A^{(R)}$ and taking

$$F_1(\tau) = e^{-b_1\tau^2}\,, \qquad F_3(\tau) = e^{-b_3\tau^2}$$

- $b_3$ does not appear in iso-symmetric setup, we set it to $b_3 = 1$
- $b_1$ and normalization of the action $c_f$ determined by matching with nuclear models

# Symmetry energy

A key characteristic of EOS away from isospin symmetry: symmetry energy $S_N$

$$\frac{\mathcal{E}}{n_b} = \frac{\mathcal{E}}{n_b}\bigg|_{\delta=0} + S_N(n_b)\delta^2 + \cdots \qquad \delta = \frac{n_p - n_n}{n_p + n_n} = \frac{2n_I}{n_b}$$

Two ways to compute in holography:

- "Directly" by turning on small $\mu_I$ and analyzing the energy
- Computing the isospin density by quantizing the iso–rotation of the homogeneous nuclear matter in analogy to what one does with instanton solutions

Both give the same answer

[Bartolini, Gudnason 2209.14309]

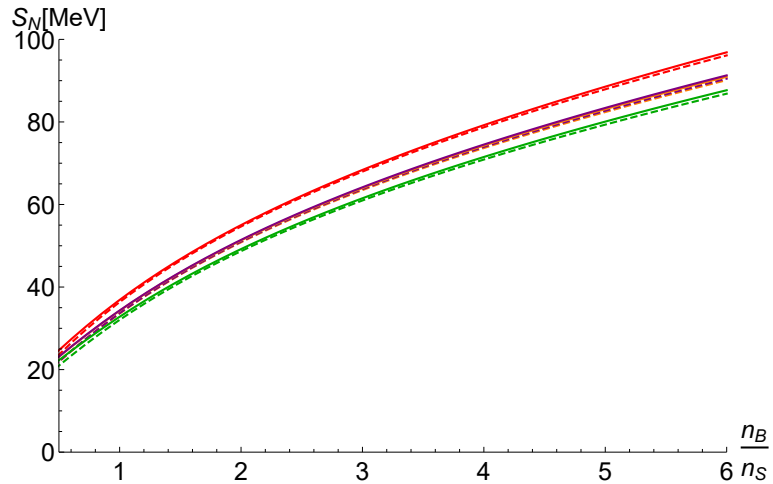
Computation done in the literature for WSS and hard-wall models, giving too large $S_N(n_b = n_s)$ as compared to experiments

# Symmetry energy: results

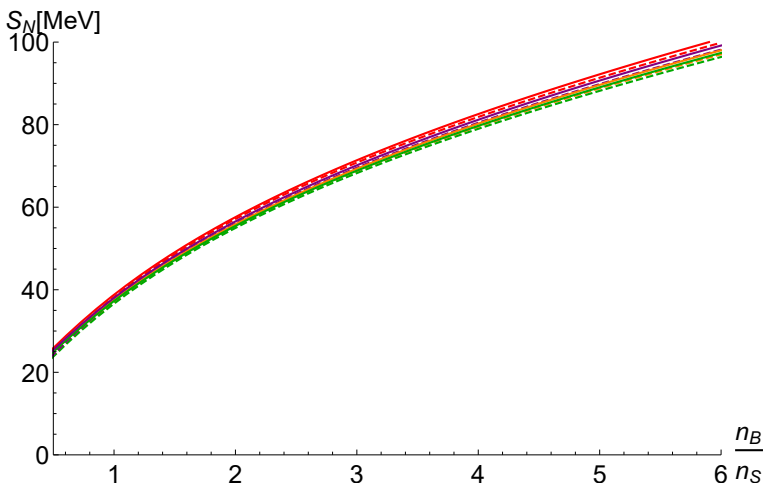
Our results for various nuclear models and matching points

- Value $\sim 30\,\mathrm{MeV}$ at saturation, agreement with experiment!
- Comes at the expense of playing with parameters (we set $b_3 = 1$ and matched $b_1$)

Matching at $1.5 n_s$




Matching at $1.2 n_s$




Matching at $1.8 n_s$

