# Effective field theory of quasi-hydro from kinetic theory

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# What is quasi-hydro?

Conventions:
$c = \hbar = 1$
$(-,+,+,+)$

Take a hydro theory. Weakly break some of its conservation laws. You have quasi-hydro.

Example: Start from ideal fluid at zero chemical potential (in linear regime).

$$\partial_t T^{00} + \partial_x T^{01} = 0$$
$$\partial_t T^{01} + c_s^2 \partial_x T^{00} = 0$$

Weakly break momentum conservation. You get Maxwell-Cattaneo heat transfer.

$$\partial_t T^{00} + \partial_x T^{01} = 0$$
$$\partial_t T^{01} + c_s^2 \partial_x T^{00} = -\frac{1}{\tau} T^{01}$$



$$\partial_t T^{00} + \partial_x T^{01} = 0$$
$$\tau \partial_t T^{01} + T^{01} = -D \partial_x T^{00}$$

Remark: For this to be quasi-hydro, you need $\tau \gg \tau_{\text{microscopic}}$. One can justify the existence of a PDE for the non-conserved quantity only in this limit.

# Israel-Stewart as quasi-hydro

Example 2: Take the theory of linear elasticity.

$$\partial_t T^{0y} + \partial_x T^{xy} = 0$$
$$\partial_t T^{xy} + w^2 \partial_x T^{0y} = 0$$ (time derivative of Hooke's law)

Weakly break elastic memory. You get Israel-Stewart theory of viscosity.

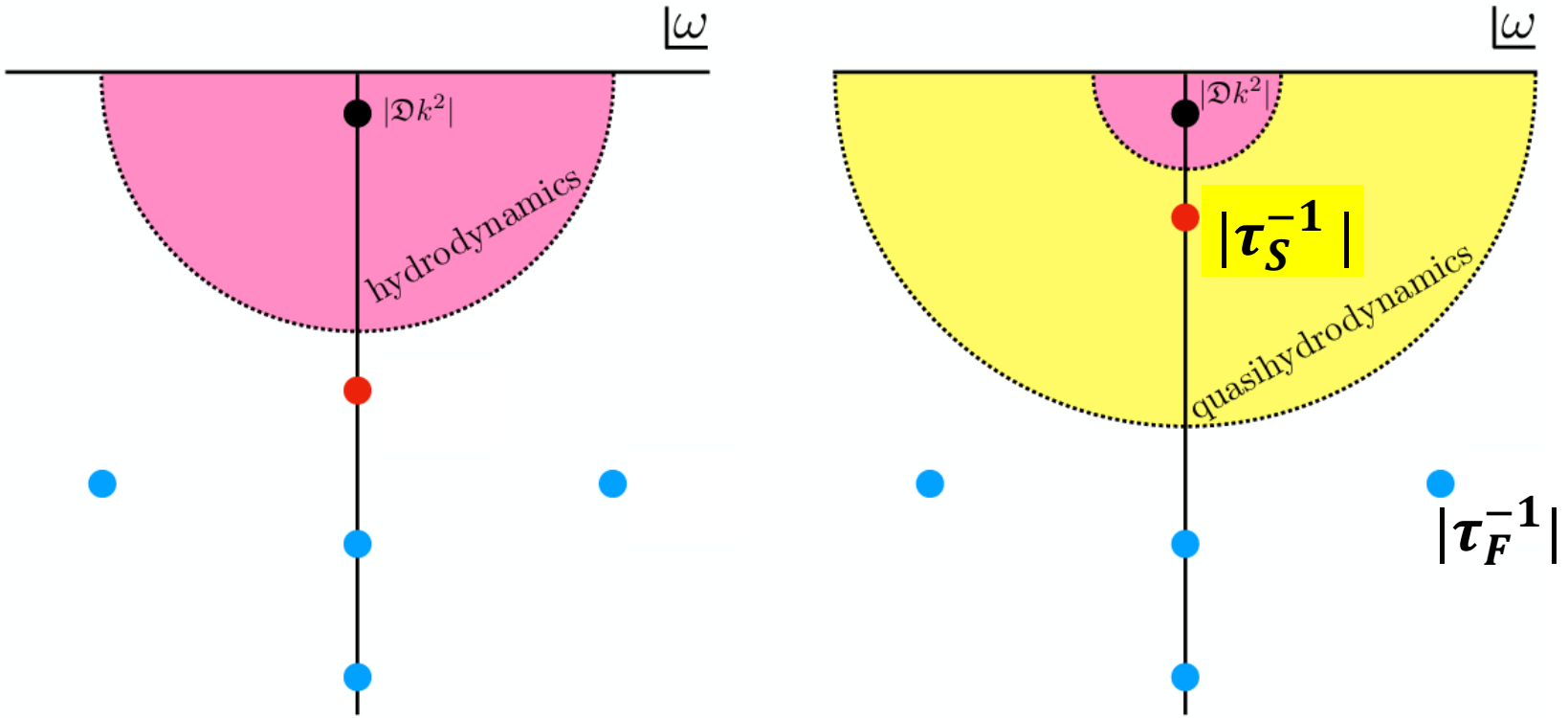
$$\partial_t T^{0y} + \partial_x T^{xy} = 0$$
$$\partial_t T^{xy} + w^2 \partial_x T^{0y} = -\frac{1}{\tau} T^{xy}$$



$$\partial_t T^{0y} + \partial_x T^{xy} = 0$$
$$\tau \partial_t T^{xy} + T^{xy} = -D \partial_x T^{0y}$$

Remark: For this to be quasi-hydro, you need $\tau \gg \tau_{\text{microscopic}}$. In this regime, Israel-Stewart theory must be interpreted as model of viscoelasticity, where the elastic stresses slowly relax to zero.

# Spectral viewpoint




Two timescales:
$\tau_F$: microscopic fast timescale;
$\tau_S$: relaxation timescale of the slow quasi-hydro modes.

$$\tau_S \gg \tau_F$$

Figure from Grozdanov, Lucas, Poovuttikul (2019)

**Our goal:**
**Build a classical EFT using $\tau_F^{-1}$ as the UV-cutoff momentum scale**

# More explicitly

1) Given the low-energy degrees of freedom (i.e. the conserved and quasi-conserved modes), build the most general PDEs as series expansions in the fast timescale.

*Cattaneo example:*

$$\partial_t T^{00} + \partial_x T^{01} + O(\tau_F) + O\left(\tau_F^2\right) + O\left(\tau_F^3\right) + \cdots = 0$$
$$\tau \partial_t T^{01} + T^{01} + D \partial_x T^{00} + O(\tau_F) + O\left(\tau_F^2\right) + O\left(\tau_F^3\right) + \cdots = 0$$

2) Provide microscopic formulae for all the EFT coefficients, and use them to place thermodynamic and causality constraints on transport order-by-order.

3) Prove that, within the regime of applicability of the EFT, the series converges to the exact result, and quantitatively estimate the truncation error.

Main result: In the linear regime, this can be done 100% rigorously, provided that the microscopic theory is kinetic-like and causal.

# Kinetic-like microscopic theories

$$\partial_t \Psi = -\left(R + V^j \partial_j\right)\Psi \quad \xrightarrow{\text{Read}} \quad \partial_t \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \vdots \end{bmatrix} = -\left( \begin{bmatrix} R_{11} & R_{12} & R_{13} & \cdots \\ R_{21} & R_{22} & R_{23} & \cdots \\ R_{31} & R_{32} & R_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} V^j_{11} & V^j_{12} & V^j_{13} & \cdots \\ V^j_{21} & V^j_{22} & V^j_{23} & \cdots \\ V^j_{31} & V^j_{32} & V^j_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \partial_j \right) \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \vdots \end{bmatrix}$$

Relaxation

Transport

If

a) We normalized $\Psi_n$ such that the free-energy density is $\Delta F = \frac{1}{2}\sum_n |\Psi_n|^2$

b) All $\Psi_n$ are even under PT, i.e. $\Psi(x^\mu) \to +\Psi(-x^\mu)$

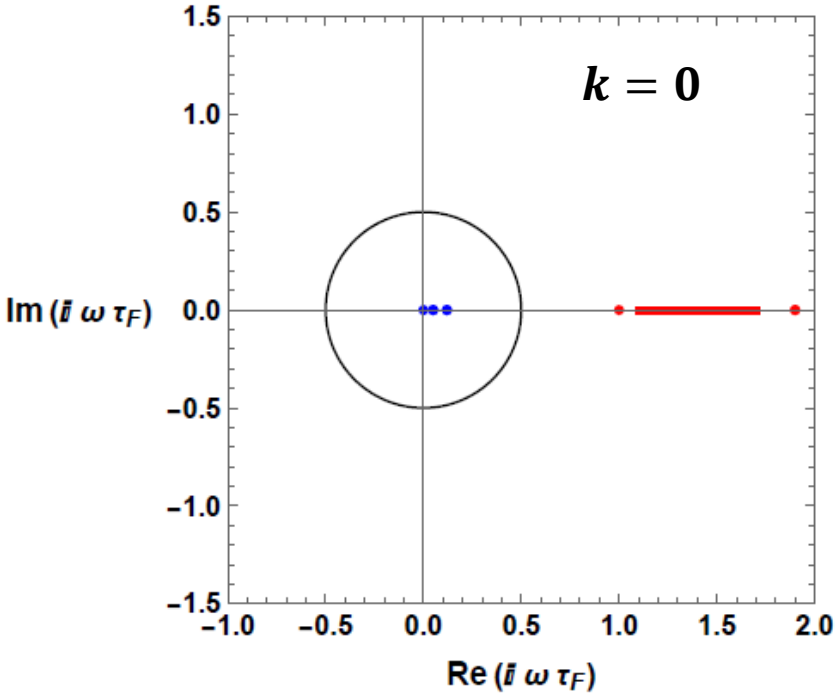
Then $R$ and $V$ are real-symmetric (due to Onsager).
The second law of thermodynamics is equivalent to $R \geq 0$.
Causality is equivalent to $\left\|V^j n_j\right\| \leq 1$

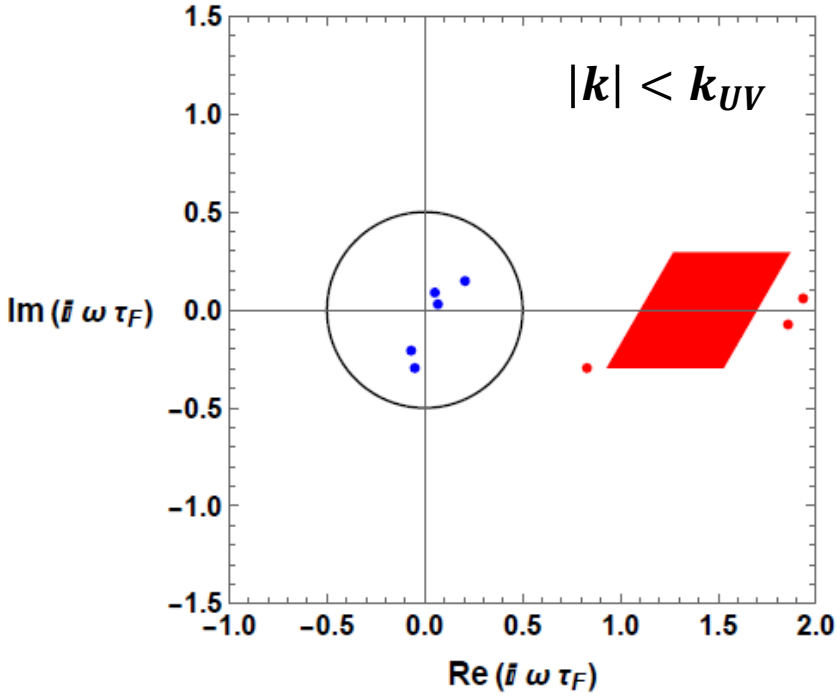
# Spectral separation

Recall: $\partial_t \Psi = -\left(R + V^j \partial_j\right)\Psi$

Assume spectral separation at $k = 0$, i.e. assume $\mathbf{Spectrum}(R) \subset \left[0, \tau_S^{-1}\right] \cup \left[\tau_F^{-1}, \infty\right)$

Then, spectral separation survives up until $k_{UV} = (2\tau_F)^{-1} - \tau_S^{-1} \approx (2\tau_F)^{-1}$.



# Integrate out the fast sector

$$\partial_t \Psi = -\left(R + V^j \partial_j\right)\Psi \xrightarrow[\Psi \propto e^{ik_j x^j - i\omega t}]{\text{Assume}} \left(R + ik_j V^j\right)\Psi = \mathrm{i}\omega\Psi$$

Orthogonal projectors:
$P_S$: projects onto the slow invariant space of $R$
$P_F$: projects onto the fast invariant space of $R$

$$P_S\left(R - i\omega + ik_j V^j\right)P_S\Psi + ik_j P_S V^j P_F \Psi = 0$$
$$ik_j P_F V^j P_S \Psi + P_F\left(R - i\omega + ik_j V^j\right)P_F\Psi = 0$$

Solving for $\Psi_F = P_F\Psi$, we obtain an equation for $\Psi_S = P_S\Psi$

$$\longrightarrow \quad \left(R_S - i\omega + ik_j V_S^j\right)\Psi_S - ik_j ik_k P_S V^j P_F \left(R_F - i\omega + ik_l V_F^l\right)^{-1} P_F V^k P_S \Psi_S = 0$$

# Gradient expansion

$$\left(R_S - i\omega + ik_j V_S^j\right)\Psi_S - ik_j ik_k P_S V^j P_F \left(R_F - i\omega + ik_l V_F^l\right)^{-1} P_F V^k P_S \Psi_S = 0$$

For quasi-hydro modes with $|k| < k_{UV}$, we can expand in Neumann series:

$$\left(R_F - i\omega + ik_l V_F^l\right)^{-1} = \sum_a \left[R_F^{-1}\left(i\omega - ik_l V_F^l\right)\right]^a R_F^{-1}$$

This converges to the exact operator for $|k| < k_{UV}$ thanks to causality!

Final result:

$$\left(\partial_t + R_S + V_S^j \partial_j\right)\Psi_S = P_S V^j P_F R_F^{-1} P_F V^k P_S \partial_j \partial_k \Psi_S \quad O(\tau_F)$$
$$+ P_S V^j P_F R_F^{-1}\left(-\partial_t - V_F^l \partial_l\right) R_F^{-1} P_F V^k P_S \partial_j \partial_k \Psi_S \quad O(\tau_F^2)$$
$$+ P_S V^j P_F R_F^{-1}\left(-\partial_t - V_F^l \partial_l\right) R_F^{-1}\left(-\partial_t - V_F^l \partial_l\right) R_F^{-1} P_F V^k P_S \partial_j \partial_k \Psi_S \quad O(\tau_F^3)$$
$$+ \cdots$$

# Truncated theory

For clarity, let's stop at second order:

$$\left(\partial_t + R_S + V_S^j \partial_j\right) \Psi_S = D^{jk}\partial_j\partial_k\Psi_S + B^{jk}\partial_j\partial_k\partial_t\Psi_S + C^{jkl}\partial_j\partial_k\partial_l\Psi_S + \mathrm{O}(\tau_F^3)$$

$O(\tau_F)$ $O(\tau_F^2)$ $O(\tau_F^3)$

| EFT coefficient | Microscopic formula | Thermodynamic constraints | Causality bound |
|---|---|---|---|
| $R_S$ | $P_S R P_S$ | $R_S \geq 0, \;\; R_S = R_S^T$ | None |
| $V_S^j$ | $P_S V^j P_S$ | $V_S^j = \left(V_S^j\right)^T$ | $\left\|V_S^j n_j\right\| \leq 1$ |
| $D^{jk}$ | $P_S V^{(j} P_F R_F^{-1} P_F V^{k)} P_S$ | $\Phi_j D^{jk} \Phi_k \geq 0, \;\; D^{jk} = \left(D^{jk}\right)^T$ | $\left\|n_j D^{jk} n_k\right\| \leq \tau_F$ |
| $B^{jk}$ | $P_S V^{(j} P_F R_F^{-2} P_F V^{k)} P_S$ | $\Phi_j B^{jk} \Phi_k \geq 0, \;\; B^{jk} = \left(B^{jk}\right)^T$ | $\left\|n_j B^{jk} n_k\right\| \leq \tau_F^2$ |
| $C^{jkl}$ | $P_S V^{(j} P_F R_F^{-1} V_F^l R_F^{-1} P_F V^{k)} P_S$ | $C^{jkl} = \left(C^{jkl}\right)^T$ | $\left\|C^{jkl} n_j n_k n_l\right\| \leq \tau_F^2$ |

# Example: Cattaneo EFT

Including first-order corrections to Cattaneo, we have

$$\partial_t \rho + c_S \partial_j J^j = D_1 \partial_j \partial^j \rho + O\left(\tau_F^2\right)$$
$$\partial_t J_l + \tau_S^{-1} J_l + c_S \partial_l \rho = D_2 \partial^j \left(\partial_j J_l + \partial_l J_j - \frac{2}{3}\delta_{jl}\partial_k J^k\right) + D_3 \partial_l \partial_k J^k + O\left(\tau_F^2\right)$$

with

$$|c_S| \leq 1 \qquad 0 \leq D_n \leq \tau_F \qquad \frac{4}{3}D_2 + D_3 \leq \tau_F$$

# 3 microscopic models with a quasi-hydro behaviour

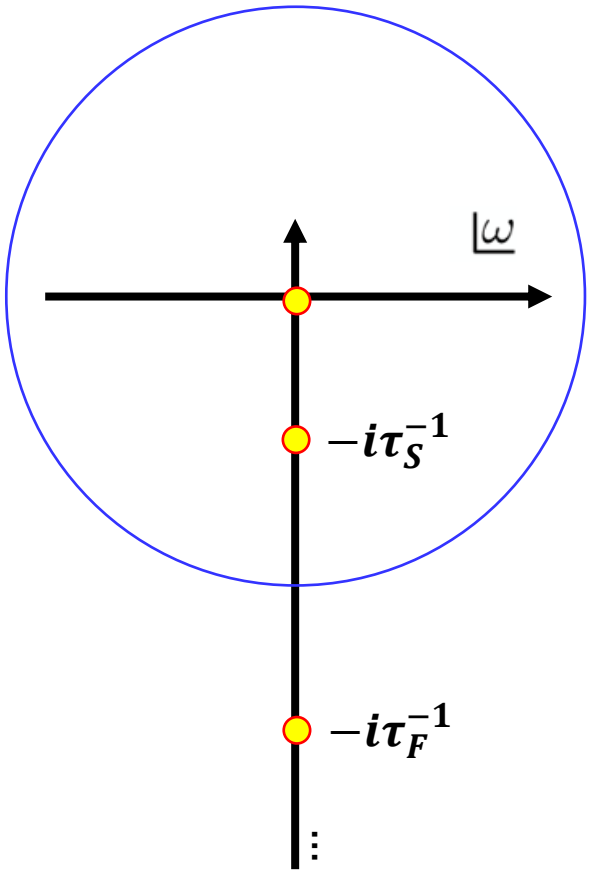
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# Ex1: Cattaneo EFT of small-angle scattering

Monoenergetic ultra-relativistic particles (photons or neutrinos) scatter-off a background medium.
The relevant kinetic distribution function is the angular density $\boldsymbol{I(t, x^j, n^j)}$, with $\boldsymbol{n^j \in S^2}$.
In the small-angle scattering limit, the Boltzmann equation reads

$$\boldsymbol{(\partial_t + n^j \partial_j) I = \frac{1}{2\tau} \Delta_{S^2} I} \qquad (\boldsymbol{\Delta_{S^2}} = \text{Laplacian on } \boldsymbol{S^2})$$

At $\boldsymbol{k = 0}$, the excitation modes are spherical harmonics.
The corresponding relaxation rates are $\boldsymbol{\frac{l(l+1)}{2\tau} = 0, \frac{1}{\tau}, \frac{3}{\tau}, \frac{6}{\tau}, \frac{10}{\tau}}$ ...




Set $\boldsymbol{\tau_S = \tau}$ and $\boldsymbol{\tau_F = \frac{\tau}{3}}$.

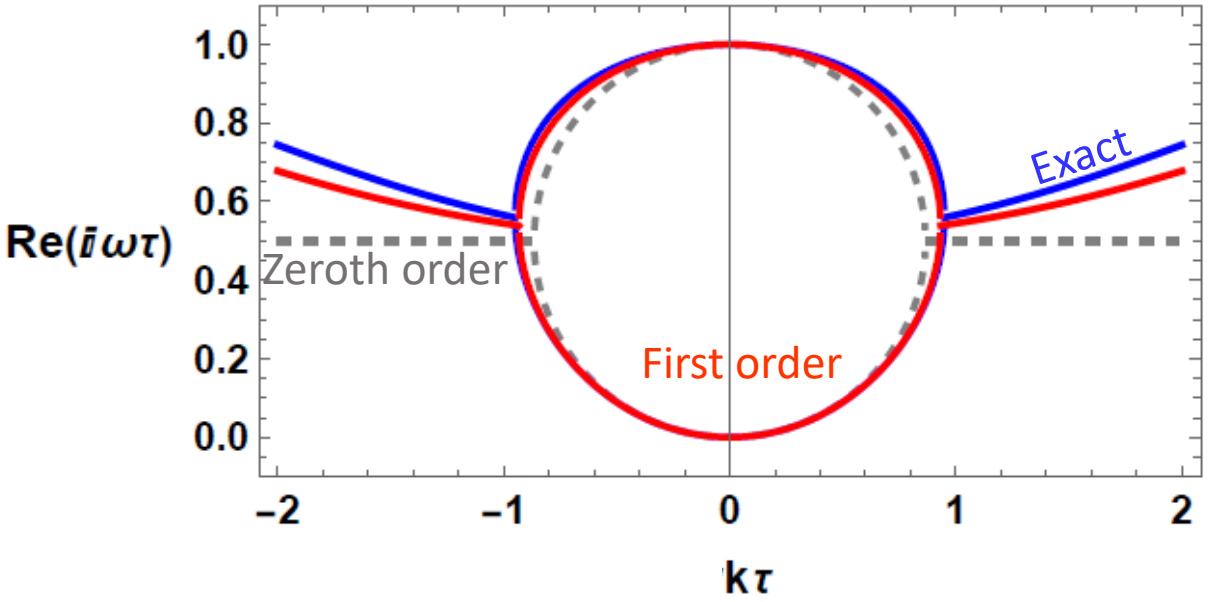
This gives an EFT with a conserved scalar density and a quasi-conserved vector current, i.e. a Cattaneo EFT.

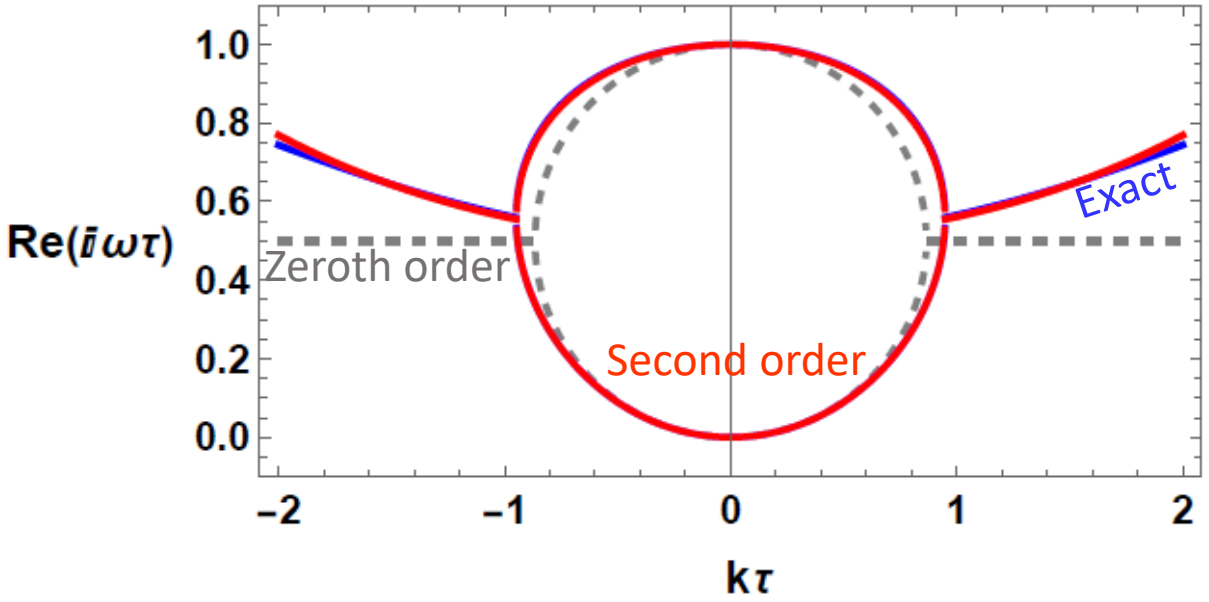
# Ex1: Cattaneo EFT of small-angle scattering

At second order

$$\partial_t \rho + \partial_j J^j = 0$$

$$\tau \partial_t J^k + J^k + D\partial^k \rho - (1 - \tau_F \partial_t)\frac{6}{5}\tau_F D \partial_j \left(\partial^{(j} J^{k)} - \frac{1}{3}\delta^{jk}\partial_l J^l\right) + O\left(\tau_F^3\right) = 0$$



# Ex2: Two-temperature EFT of radiative transfer

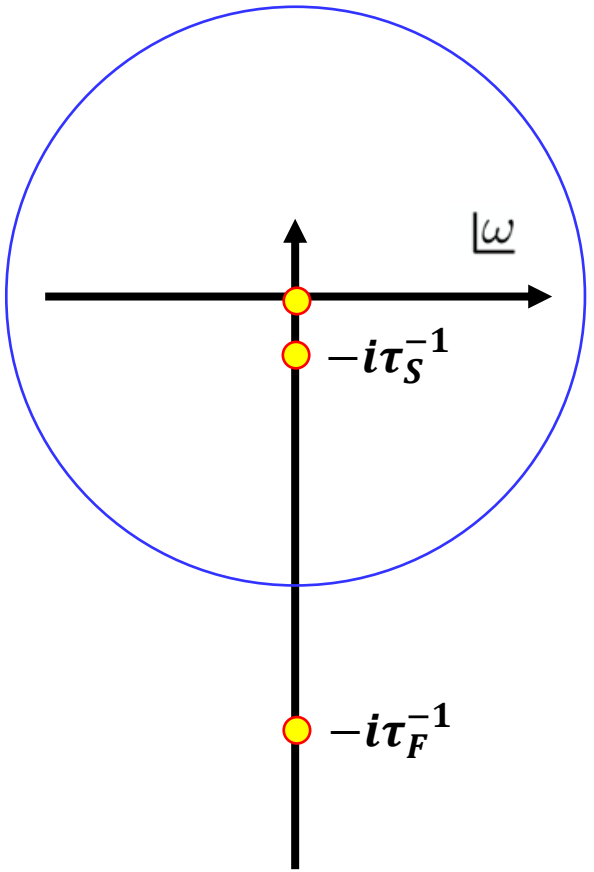
Photons travel through a grey medium in local thermal equilibrium.
The relevant degrees of freedom are $\boldsymbol{\delta T_M}$ and $\boldsymbol{I(t, x^j, n^j)}$, with $\boldsymbol{n^j \in S^2}$.
The equations of motion are

$$\boldsymbol{c_M \partial_t \delta T_M = \frac{c_R(\langle I\rangle - \delta T_M)}{\tau_{abs}}}$$

$$\boldsymbol{(\partial_t + n^j \partial_j) I = \frac{\delta T_M - I}{\tau_{abs}} + \frac{\langle I\rangle - I}{\tau_{sc}}}$$

Assume that scattering is much faster than absorption,
$\boldsymbol{\tau_S \sim \tau_{abs} \gg \tau_{sc} \sim \tau_F}$

Then there is one conserved scalar, $\boldsymbol{c_M \delta T_M + c_R \langle I\rangle}$, and one quasi-conserved scalar, $\boldsymbol{\langle I\rangle - \delta T_M}$. The fast part is $\boldsymbol{I - \langle I\rangle}$.
This gives rise to a two-temperature EFT, with $\boldsymbol{\delta T_R = \langle I\rangle}$.

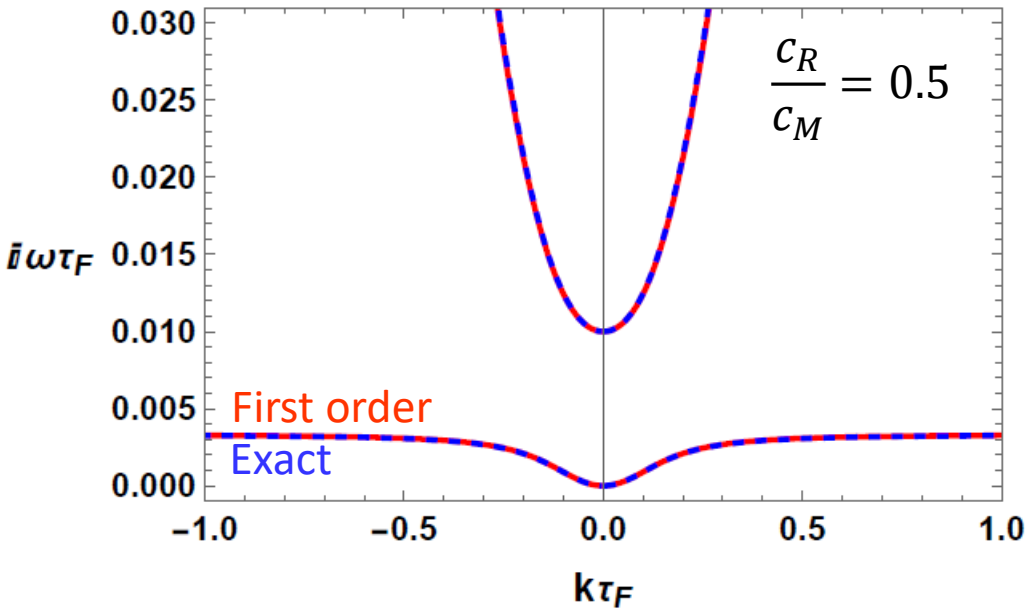


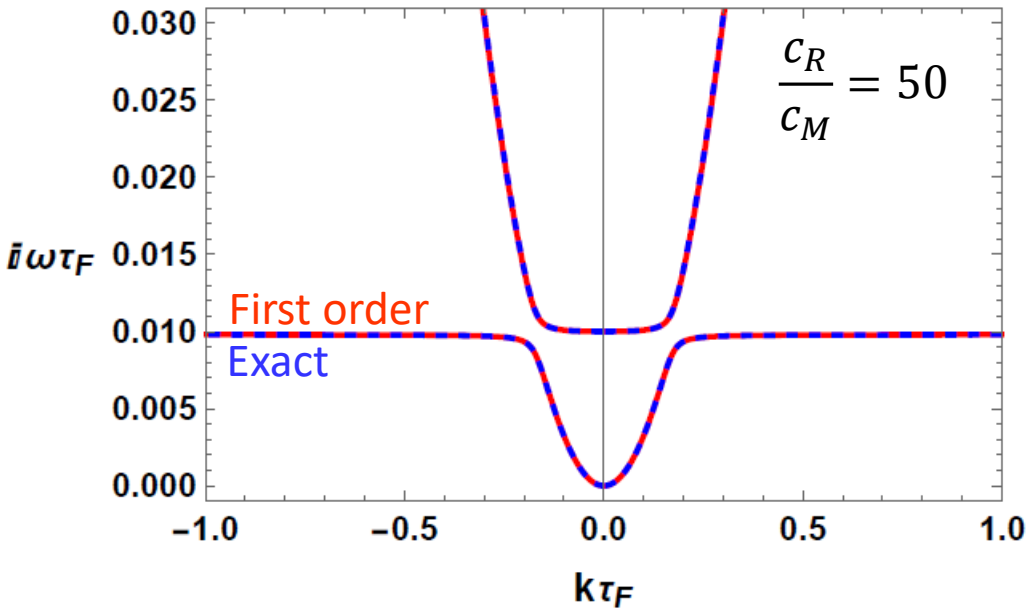
# Ex2: Two-temperature EFT of radiative transfer

At first order

$$\partial_t \delta T_M = \frac{c_R(\delta T_R - \delta T_M)}{c_M \, \tau_{abs}}$$

$$\partial_t \delta T_R = \frac{\delta T_M - \delta T_R}{\tau_{abs}} + \frac{\tau_F}{3} \partial_j \partial^j \delta T_R + O(\tau_F^2)$$



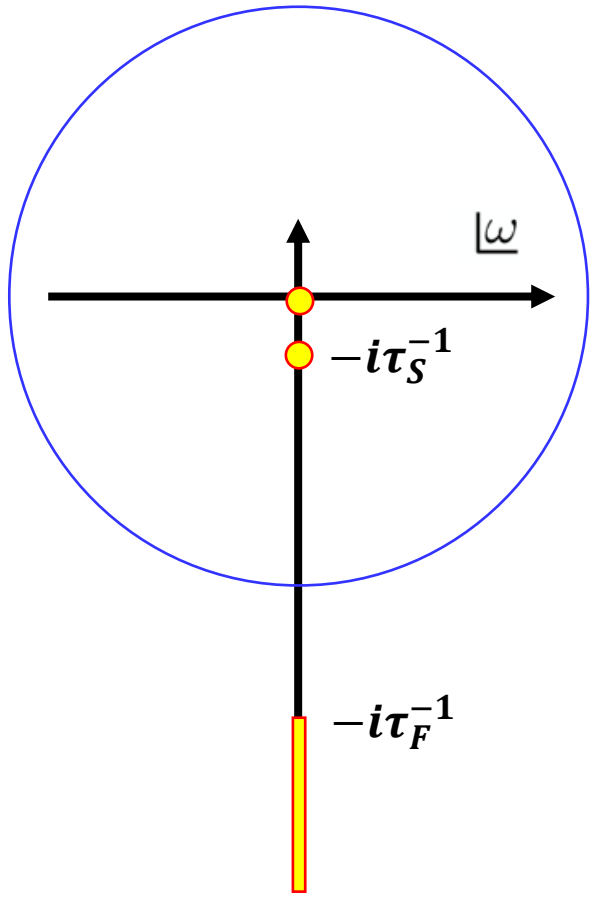
# Ex3: Cattaneo EFT of bottlenecked momentum diffusion in 1D

Ultrarelativistic particles are confined on a line, and scatter off a medium at fixed temperature $\beta^{-1}$ in local thermal equilibrium. Collisions are infinitely soft but infinitely frequent (Fokker-Planck limit).
The Boltzmann equation for $\boldsymbol{\delta f(t,x,p)}$ reads

$$\left(\partial_t + n^j \partial_j\right)\delta f = \partial_p\left[B(p)\left(\partial_p \delta f + \beta v \delta f\right)\right]$$

Suppose collisions are effective at high energy but suppressed at low energy:

$$B(p) = \frac{4}{\beta^2 \tau_F}\left[1 - \left(1 - \sqrt{\frac{\tau_F}{\tau_0}}\right)e^{-\beta\varepsilon}\right]^2 \qquad (\tau_0 \gg \tau_F)$$

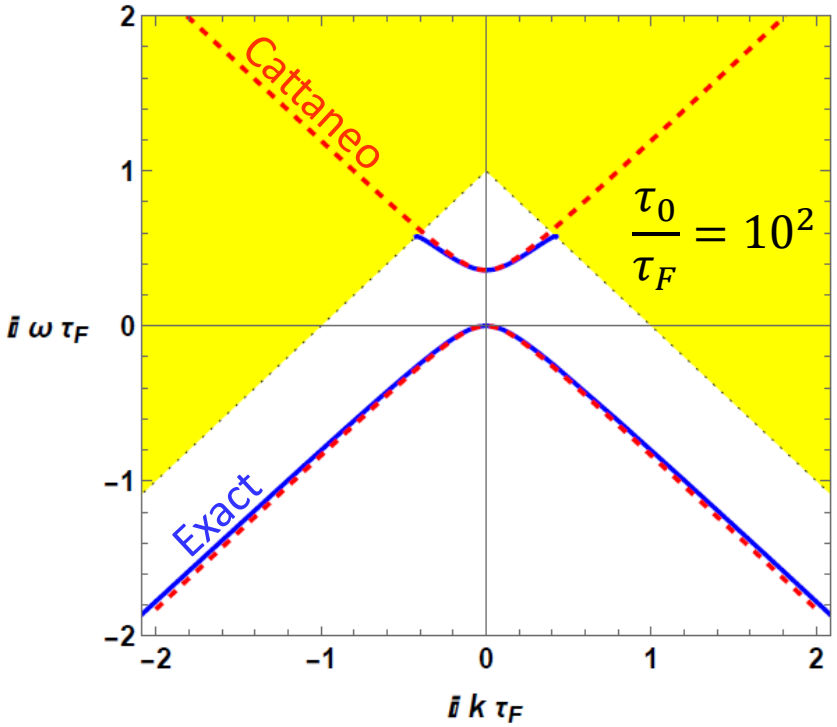


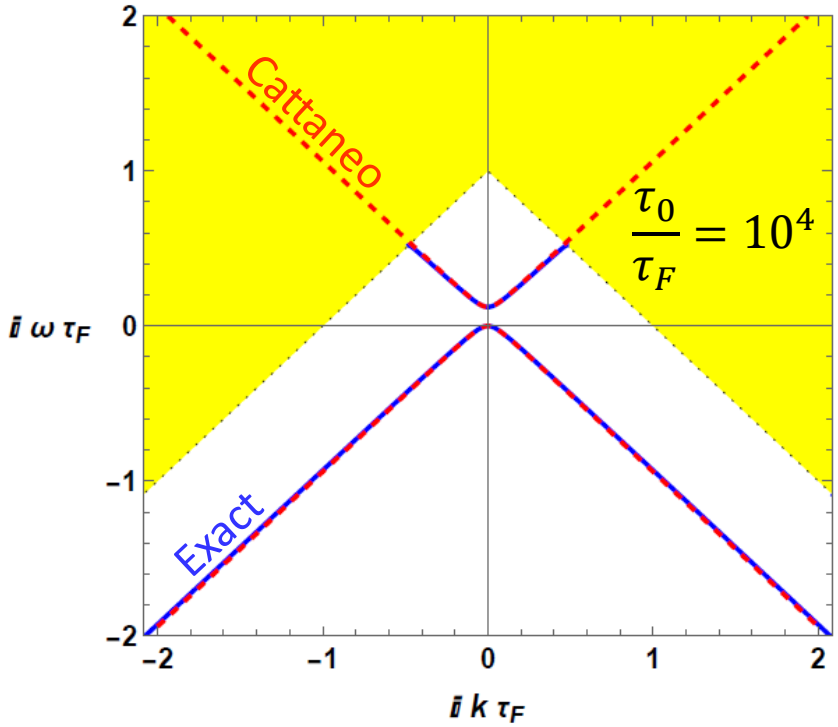

Then we find that, while the total density is conserved, the current is quasi-conserved, and relaxes over a timescale $\tau_S \approx \sqrt{\tau_F \tau_0/4}$.
The result is again a Cattaneo EFT.

# Ex3: Cattaneo EFT of bottlenecked momentum diffusion in 1D

At zeroth order

$$\partial_t \rho + \partial_x J = 0$$
$$\tau_S \partial_t J + J + \tau_S \partial_x \rho + O(\tau_F) = 0$$



# Universality of quasi-hydro EFTs

# Key question

When we are first introduced to field theory, we learn that Lorentz invariance can be used to classify field theories based on the geometric character of their degrees of freedom (scalar field, vector field, Dirac field,…).

Can we do something similar here?

# The free-energy current

Recall that, at zeroth order, we have

$$\partial_t \Psi_S + V_S^j \partial_j \Psi_S = -R_S \Psi_S$$

Multiplying by $\Psi_S^\dagger$ on both sides, and recalling that $V_S^j$ and $R_S$ are Hermitian, we obtain

$$\partial_\mu F^\mu = -Q,$$

With $F^\mu$ the free-energy current, and $\mathbf{Q} \geq \mathbf{0}$ the energy dissipation rate:

$$F^0 = \frac{\Psi_S^\dagger \Psi_S}{2} \qquad F^j = \frac{\Psi_S^\dagger V_S^j \Psi_S}{2} \qquad Q = \Psi_S^\dagger R_S \Psi_S$$

# Consequences of rotational invariance

Suppose that the background state is isotropic.
Then, for every rotation $\boldsymbol{O}$, there is a transformation $\boldsymbol{U(O)}$ that implements its action in $\boldsymbol{\Psi}$ −space:

$$\boldsymbol{\Psi_S(t,x) \rightarrow U(O)\Psi_S\left(t, O^{-1}x\right)}$$

Since $\boldsymbol{2F^0 = \Psi_S^\dagger \Psi_S}$ is a rotational scalar, $\boldsymbol{U(O)}$ is unitary

⟹ $\boldsymbol{\Psi}$ −space decomposes into unitary irreducible representations of $\boldsymbol{SO(3)}$

Moreover, rotational invariance implies that, if $\boldsymbol{\Psi_S(t,x)}$ is a solution of the equations of motion, then so is $\boldsymbol{U(O)\Psi_S\left(t, O^{-1}x\right)}$. Therefore, $\boldsymbol{U(O)^\dagger R_S U(O) = R_S}$ and $\boldsymbol{U(O)^\dagger V_S^j U(O) = O_k^j V_S^k}$. Therefore:

a) $\boldsymbol{R_S} \propto$ Identity within each irreducible sector;

b) $\boldsymbol{F^j}$ transforms like a vector and $\boldsymbol{Q}$ transforms like a scalar (as expected).

# Classification

State which irreducible representation enter the model, and state whether each of them is conserved or quasi-conserved. Then, write down the most general scalar $Q$ and vector $F^j$. This fully specifies the zeroth-order theory (and identifies the universality class of the full EFT).

Example: $\mathbf{1}_C \oplus \mathbf{3}_C \oplus \mathbf{5}_Q$

$$F^0 = \frac{1}{2}\left(\rho^2 + J_j J^j + \pi^{jk}\pi_{jk}\right)$$

$$F^k = c_1 \rho J^k + c_2 J_j \pi^{jk}$$

$$Q = c_3 \pi^{jk}\pi_{jk}$$

$$\partial_\mu \frac{\partial F^\mu}{\partial \Psi_I} = -\frac{1}{2}\frac{\partial Q}{\partial \Psi_I}$$

$$\partial_t \rho + c_1 \partial_k J^k = 0$$

$$\partial_t J^l + c_1 \partial^l \rho + c_2 \partial_j \pi^{jl} = 0$$

$$\partial_t \pi^{jk} + c_3 \pi^{jk} + c_2 \langle \partial^j J^k \rangle = 0$$

This is just Israel-Stewart theory of shear viscosity at zero chemical potential!

# Universality classes

In general, many different systems fall within the same EFT classification in the linear regime (i.e. have the same IR degrees of freedom). Hence, we can pick one physical representative for an entire universality class of models.
Examples:

| EFT | Representative |
|---|---|
| $(\mathbf{1}_C \oplus \mathbf{1}_C \oplus \cdots) \oplus (\mathbf{1}_Q \oplus \mathbf{1}_Q \oplus \cdots)$ | Chemistry |
| $(\mathbf{1}_C \oplus \mathbf{1}_C \oplus \cdots) \oplus (\mathbf{1}_Q \oplus \mathbf{1}_Q \oplus \cdots) \oplus \mathbf{3}_C$ | Chemically active ideal fluid |
| $\mathbf{1}_C \oplus \mathbf{3}_Q$ | Cattaneo |
| $\mathbf{1}_C \oplus \mathbf{3}_C \oplus \mathbf{5}_Q$ | Israel-Stewart shear viscosity at zero chemical potential |
| $\mathbf{1}_C \oplus \mathbf{3}_C \oplus \mathbf{5}_Q \oplus \mathbf{5}_Q$ | MIS$^\star$ at zero chemical potential |
| $\mathbf{1}_C \oplus \mathbf{1}_C \oplus \mathbf{1}_Q \oplus \mathbf{3}_C \oplus \mathbf{3}_Q \oplus \mathbf{5}_Q$ | Full Israel-Stewart at finite chemical potential |

# Summary

Strengths:

- Rigorous and systematic;
- Applicable in a variety of contexts at weak and intermediate coupling;
- Zeroth-order theories are always symmetric hyperbolic, causal, and covariantly stable;
- Keeps track of microcausality order by order (despite truncated theories not being causal themselves);
- Reveals a universality of the linearized dynamics.

Limitations:

- Linear;
- Assumes kinetic-like microscopic dynamics.

Main references:


- LG, arXiv:2608.13542
- LG, Disconzi, Noronha, Phys. Rev. Lett. 132, 222302 (May 2024)