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Thermodynamics of rotating fermions

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Outline

Introduction

Rigid rotation

- States under rigid rotation are described by the following Killing vector:

$$\beta^\mu \partial_\mu = \beta_0(\partial_t + \Omega_0 \partial_\varphi). \tag{1}$$

- u^μ has unit 4-norm and is parallel to β^μ:

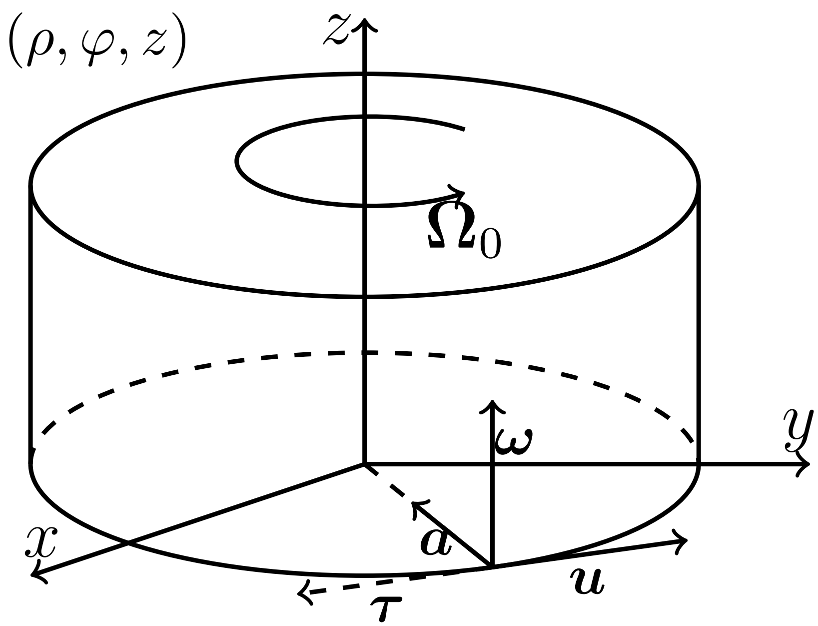
$$u^\mu \partial_\mu = \frac{\beta^\mu \partial_\mu}{\sqrt{\beta_\nu \beta^\nu}} = \Gamma(\partial_t + \Omega_0 \partial_\varphi), \qquad \Gamma = (1 - \rho^2 \Omega_0^2)^{-1/2}, \tag{2}$$

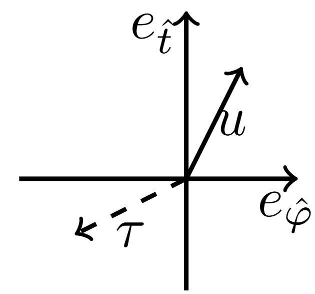
  with $\Gamma \equiv$ Lorentz factor and $\rho \equiv$ distance to the rotation axis.
- The local temperature T represents the inverse norm of β^μ:

$$T = 1/\sqrt{\beta_\mu \beta^\mu} = \Gamma T_0, \tag{3}$$

  with $T_0 \equiv$ the temperature on the rotation axis.
- In global equilibrium, $\mu/T = \text{const.} \Rightarrow \mu = \Gamma \mu_0$.
- The state diverges at the light cylinder, where $\rho \Omega_0 = 1$.

Kinematic frame for rigid rotation





A "kinematic" orthogonal tetrad is given by: [Becattini, Grossi, PRD 2015]

$$\text{Velocity}: \quad u^\mu \partial_\mu = \Gamma(\partial_t + \Omega_0 \partial_\varphi), \qquad \Gamma = (1 - \rho^2 \Omega_0^2)^{-1/2},$$

$$\text{Acceleration}: \quad a^\mu = u^\nu \partial_\nu u^\mu = -\rho \Omega_0^2 \Gamma^2 \delta^\mu_\rho, \qquad a^2 = -\mathbf{a}^2 = -\Omega_0^2 \Gamma^2 (\Gamma^2 - 1),$$

$$\text{Vorticity}: \quad \omega^\mu = \tfrac{1}{2} \varepsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta = \Gamma^2 \Omega \delta^\mu_z, \qquad \omega^2 = -\boldsymbol{\omega}^2 = -\Omega_0^2 \Gamma^4,$$

$$\text{Fourth vector:}^{1} \quad \tau^\mu \partial_\mu = \varepsilon^{\mu\nu\alpha\beta} u_\nu a_\alpha \omega_\beta \partial_\mu = -\Omega_0^3 \Gamma^5 (\rho^2 \Omega_0 \partial_t + \partial_\varphi), \qquad \tau^2 = -\boldsymbol{\tau}^2 = -\Omega_0^4 \Gamma^6 (\Gamma^2 - 1).$$

The vorticity tensor is then

$$\omega_{\alpha\beta} = a_\alpha u_\beta - a_\beta u_\alpha - \varepsilon_{\alpha\beta\mu\nu} u^\mu \omega^\nu = \Omega_0 \Gamma (g_{\alpha x} g_{\beta y} - g_{\alpha y} g_{\beta x}). \tag{4}$$

[1] Aka *another vector*.

[Buzzegoli, Becattini, JHEP**12**(2018)002]

Grand canonical ensemble

Thermodynamics of static systems

- In the case of a static system ($\Omega_0 = 0$), we have

$$\hat{\rho} = e^{-\beta_0(\widehat{H} - \mu_0 \widehat{Q})}, \tag{5}$$

where $\widehat{H} \equiv$ Hamiltonian and $\widehat{Q} \equiv$ charge operators.

- The total thermodynamic potential Φ, defined as

$$\Phi = -T_0 \ln Z = \phi V, \qquad Z = \mathrm{tr}(\hat{\rho}), \tag{6}$$

satisfies the Gibbs-Duhem relation $d\Phi = -\mathcal{S} dT_0 - \mathcal{Q} d\mu_0 - P dV$, where

$$P = -\frac{d\Phi}{dV} = -\phi, \quad \mathcal{Q} = -\frac{\partial \Phi}{\partial \mu_0} = QV, \quad \mathcal{S} = -\frac{\partial \Phi}{\partial T_0} = sV. \tag{7}$$

- Imposing $\mathcal{E} = \langle \widehat{H} \rangle = Z^{-1} \mathrm{tr}(\hat{\rho} \widehat{H})$ gives Euler relation

$$\mathcal{E} = T_0 \mathcal{S} + \Phi + \mu_0 \mathcal{Q}. \tag{8}$$

- Since the static system is homogeneous, all relations scale as V, hence:

$$P = -\frac{\Phi}{V} = -\phi, \quad Q = -\frac{\partial \phi}{\partial \mu_0}, \quad s = -\frac{\partial \phi}{\partial T_0}, \quad \epsilon = T_0 s + \phi + \mu_0 Q. \tag{9}$$

Rotating system: Grand potential

- The rotating system is constructed using $\widehat{M}^z \equiv$ total angular momentum:

$$\hat{\rho} = e^{-\beta_0(\widehat{H} - \mu_0 \widehat{Q} - \Omega_0 \widehat{M}^z)}. \tag{10}$$

- The rotating system is inhomogeneous and ill defined as $\rho \to \Omega^{-1} \Rightarrow P \neq -\Phi/V$!
- In the rotating case, we introduce the grand canonical potential as

$$\Phi = \int_V d^3x \phi(x) = \mathcal{E} - T_0 \mathcal{S} - \mu_0 \mathcal{Q} - \Omega_0 \mathcal{M}^z, \tag{11}$$

where we demand

$$\mathcal{S} = -\frac{\partial \Phi}{\partial T_0}, \quad \mathcal{Q} = -\frac{\partial \Phi}{\partial \mu_0}, \quad \mathcal{M}^z = -\frac{\partial \Phi}{\partial \Omega_0}, \tag{12}$$

thus ensuring the *extended* Gibbs-Duhem relation:

$$d\Phi = -\mathcal{S} dT_0 - \mathcal{Q} d\mu_0 - P dV - \boldsymbol{\mathcal{M}} \cdot d\boldsymbol{\Omega}_0. \tag{13}$$

Rotating system: Grand potential

- We can express $\mathcal{E} = \Phi + T_0 \mathcal{S} + \mu_0 \mathcal{Q} + \Omega_0 \mathcal{M}^z$ in terms of Φ as

$$\mathcal{E} = \Phi + \beta_0 \left(\frac{\partial \Phi}{\partial \beta_0}\right)_{\mu_0,\Omega_0} - \mu_0 \left(\frac{\partial \Phi}{\partial \mu_0}\right)_{\beta_0,\Omega_0} - \Omega_0 \left(\frac{\partial \Phi}{\partial \Omega_0}\right)_{\beta_0,\mu_0}$$
$$= \left(\frac{\partial \Phi}{\partial \beta_0}\right)_{\beta_0\mu_0,\beta_0\Omega_0}, \qquad \Phi(\beta_0,\mu_0,\Omega_0) \to \Phi\left[\beta_0, \frac{\beta_0\mu_0}{\beta_0}, \frac{\beta_0\Omega_0}{\beta_0}\right]. \tag{14}$$

- Then, Φ can be obtained from $\mathcal{E}$ by ensemble integration:

$$\Phi = \frac{1}{\beta_0}\int d\beta_0 (\mathcal{E})_{\beta_0\mu_0,\beta_0\Omega_0} = \int d^3x\, \phi(x). \tag{15}$$

- The most natural candidate for the total energy is

$$\mathcal{E} = \int d^3x\, \Theta^{tt} \quad \Rightarrow \quad \phi(x) = \frac{1}{\beta_0}\int d\beta_0 (\Theta^{tt})_{\beta_0\mu_0,\beta_0\Omega_0}, \tag{16}$$

with $\Theta^{\mu\nu} = \frac{i}{2}\langle \widehat{\overline{\psi}} \gamma^\mu \overleftrightarrow{\partial^\nu} \widehat{\psi}\rangle \equiv$ energy-momentum tensor.

Rotating system: Grand potential   VEA, JHEP**08**(2020)016

- Θ^{tt} is given by $\left[\sum_j = \sum_{\sigma_j=\pm 1} \sum_{\lambda_j=\pm\frac{1}{2}} \sum_{m_j=-\infty}^{\infty} \int_M^\infty dE_j \int_{-p_j}^{p_j} dk_j\right]$

$$\Theta^{tt} = \frac{1}{8\pi^2} \sum_j f_j \Theta_j^{tt}, \quad f_j = [e^{\beta_0(E_j - \Omega_0 m_j - \sigma_j \mu_0)} + 1]^{-1}. \tag{17}$$

- Performing the β_0 integration leads to $F_j = \ln\left[1 + e^{-\beta_0(E_j - \Omega_0 m_j - \sigma_j \mu_0)}\right]$

$$\phi(x) = -\frac{1}{8\pi^2 \beta_0} \sum_j E_j^{-1} \Theta_j^{tt} F_j \to -\Theta^{zz}, \tag{18}$$

with $F_j = -\ln(1 - f_j) = \ln[1 + e^{-\beta(E_j \cdots)}]$.

- The (relativistically non-covariant) thermodynamic pressure is $P_{\rm GP} = \Theta^{zz}$.

$$\Theta_j^{tt} = E_j \left(J_j^+ + \frac{2\lambda_j k_j}{p_j} J_j^-\right), \quad \Theta^{zz} = \frac{k_j^2}{E_j} J_j^+ + \frac{2\lambda_j p_j k_j}{E_j} J^-,$$
$$J_j^\pm \equiv J^2_{m_j - \frac{1}{2}}(q_j \rho) \pm J^2_{m_j + \frac{1}{2}}(q_j \rho). \tag{19}$$

Rotating system: Thermodynamic consistency

- The Grand Canonical pressure $P_{\rm GP}$ is related to the time components of the relevant currents:

$$\frac{\partial P_{\rm GP}}{\partial \mu_0} = J_V^t, \quad \frac{\partial P_{\rm GP}}{\partial \Omega_0} = \rho^2 \Theta^{t\varphi} + \frac{1}{2} J_A^z = M_C^{t,xy},$$
$$\frac{\partial P_{\rm GP}}{\partial T_0} = \frac{1}{T_0}\left(\Theta^{tt} + P_{\rm GP} - \rho^2 \Omega_0 \Theta_j^{t\varphi} - \mu_0 J_V^t - \frac{1}{2}\Omega_0 J_A^z\right), \qquad (20)$$

with $M_C^{t,xy} = x\Theta^{ty} - y\Theta^{tx} + S_C^{t,xy}$ and $S_C^{t,xy} = -\frac{1}{2}\varepsilon^{txyz} J_{A;z}$ chosen automatically in the canonical pseudogauge.

- Then, the energy density is recovered as expected:

$$\Theta^{tt} = T_0 \frac{\partial P_{\rm GP}}{\partial T_0} - P_{\rm GP} + \mu_0 \frac{\partial P_{\rm GP}}{\partial \mu_0} + \Omega_0 \frac{\partial P_{\rm GP}}{\partial \Omega_0}. \qquad (21)$$

Rotating system: Restoring Lorentz covariance

- The grand canonical pressure is not a Lorentz scalar: $P_{\rm GP} = \Theta^{zz}$.
- The thermodynamic quantities arising from $P_{\rm GP}$ are also not Lorentz scalars (Θ^{tt}, J_V^t, $M_C^{t,xy}$,...).
- A first step towards restoring Lorentz covariance is to write

$$\Theta^{tt} - \rho^2 \Omega_0 \Theta_j^{t\varphi} = \frac{1}{\Gamma} \Theta^{t\mu} u_\mu, \quad u^\mu \partial_\mu = \Gamma(\partial_t + \Omega_0 \partial_\varphi). \tag{22}$$

- Using $\frac{1}{2}\Omega_0 J_A^z = \Omega_0 S_C^{t,xy}$, and the vorticity tensor

$$\omega_{\alpha\beta} = a_\alpha u_\beta - a_\beta u_\alpha - \varepsilon_{\alpha\beta\mu\nu} u^\mu \omega^\nu = \Omega_0 \Gamma (g_{\alpha x} g_{\beta y} - g_{\alpha y} g_{\beta x}), \tag{23}$$

we can write

$$\frac{\partial P_{\rm GP}}{\partial T_0} = \frac{1}{T} \left(\Theta^{t\mu} u_\mu - \tilde{\phi}^t - \mu J_V^t - \frac{1}{2} S_C^{t,\alpha\beta} \omega_{\alpha\beta} \right) \equiv s^t, \tag{24}$$

with $T = \Gamma T_0$, $\mu = \Gamma \mu_0$ and $\tilde{\phi}^t = \Gamma \phi$.

Thermodynamic potential current

Grand potential current

- It is clear that $P_{\rm GP} = -\phi^t$ is just the time component of a four vector, which we construct below:

$$\phi^\mu = \frac{1}{\beta_0}\int d\beta_0 (\Theta^{\mu t}_C)_{\beta_0\mu_0,\beta_0\Omega_0} = -\frac{1}{8\pi^2\beta_0}\sum_j E_j^{-1}\Theta_j^{\mu t} F_j. \tag{25}$$

- This current can be shown to satisfy

$$-\frac{\partial\phi^\mu}{\partial\mu_0} = J^\mu_V, \quad -\frac{\partial\phi^\mu}{\partial\Omega_0} = M^{\mu,xy}_C = \rho^2\Theta^{\mu\varphi}_C + S^{\mu,xy}_C,$$
$$-\frac{\partial\phi^\mu}{\partial T_0} = \frac{1}{T_0}(\Theta^{\mu t}_C - \phi^\mu - \mu_0 J^\mu_V - \rho^2\Omega\Theta^{\mu\varphi}_C + \tfrac{1}{2}\Omega_0\varepsilon^{\mu xy\nu}J_{A;\nu}). \tag{26}$$

- Using $\Theta^{\mu\nu}_C u_\nu = \Gamma(\Theta^{\mu t} - \rho^2\Omega_0\Theta^{\mu\varphi}_C)$ and $\omega_{\alpha\beta}S^{\mu,\alpha\beta}_C = -\Gamma\Omega_0\varepsilon^{\mu xy\nu}J_{A;\nu}$ gives

$$s^\mu = \frac{1}{T}\left(\Theta^{\mu\nu}_C u_\nu - \tilde\phi^\mu - \mu J^\mu_V - \frac{1}{2}S^{\mu,\alpha\beta}_C\omega_{\alpha\beta}\right), \tag{27}$$

with $\tilde\phi^\mu = \Gamma\phi^\mu$.

Pressure as a Lorentz scalar

- Contracting s^μ with u_μ leads to

$$s = \frac{1}{T}\left(\epsilon + P - \mu Q_V - \frac{1}{2} S_C^{\alpha\beta}\omega_{\alpha\beta}\right), \tag{28}$$

  with the usual definitions:

$$\epsilon = u_\mu \Theta_C^{\mu\nu} u_\nu, \quad Q_V = u_\mu J_V^\mu, \quad S_C^{\alpha\beta} = u_\mu S_C^{\mu,\alpha\beta}, \tag{29}$$

  while the thermodynamic pressure reads now $P = -\tilde{\phi}^\mu u_\mu \neq \Theta_C^{zz}$.
- Differentiating P w.r.t. T_0, μ_0 yields standard relations:

$$\frac{1}{\Gamma}\frac{\partial P}{\partial T_0} = \frac{\partial P}{\partial T} = s, \quad \frac{1}{\Gamma}\frac{\partial P}{\partial \mu_0} = \frac{\partial P}{\partial \mu} = Q_V. \tag{30}$$

- The derivative w.r.t. Ω_0 is subtle, as both Γ and u^μ depend on Ω_0:

$$\frac{\partial P}{\partial \Omega_0} = -\frac{\partial(\Gamma u_\mu)}{\partial \Omega_0}\phi^\mu - \Gamma u_\mu \frac{\partial \phi^\mu}{\partial \Omega_0}. \tag{31}$$

- To achieve thermodynamic consistency, $\partial P/\partial \omega_{\alpha\beta} = S_C^{\alpha\beta}$ and for this, one must reexpress P w.r.t. the local parameters: T, μ, $\omega_{\alpha\beta}$.

From global to local state

- So far, we considered thermodynamic derivatives with respect to the global ensemble parameters: T_0, μ_0 and Ω_0.
- We aim to regard $\tilde{\phi}^\mu = \Gamma \phi^\mu \equiv \tilde{\phi}^\mu(T, \mu, \omega_{xy})$, instead of $\tilde{\phi}^\mu(T_0, \mu_0, \Omega_0)$.
- The transition $(T_0, \mu_0) \to (T, \mu)$ is straightforward, since:

$$\frac{\partial \tilde{\phi}^\alpha}{\partial T} = \frac{\partial \phi^\alpha}{\partial T_0} = -s^\alpha, \quad \frac{\partial \tilde{\phi}^\alpha}{\partial \mu} = \frac{\partial \phi^\alpha}{\partial \mu_0} = -J_V^\alpha. \tag{32}$$

- However, $\Omega_0 \to \omega_{xy}$ is less clear, since

$$\frac{\partial \phi^\mu}{\partial \Omega_0} = \Gamma^2 \frac{\partial \tilde{\phi}^\mu}{\partial \omega_{xy}} - \rho^2 \Omega_0 \Gamma (\tilde{\phi}^\mu - T s^\mu - \mu J_V^\mu). \tag{33}$$

- A little rearrangement reveals that

$$-\frac{\partial \tilde{\phi}^\mu}{\partial \omega_{xy}} = S_C^{\mu,xy} + \Theta_C^{\mu\nu} \tilde{\tau}_\nu, \quad \tilde{\tau}^\mu \partial_\mu = -\rho^2 \Omega \partial_t - \partial_\varphi. \tag{34}$$

- The RHS displays the desired spin term, plus an undesirable orbital term.

Local state thermodynamics

Spin potential / vorticity duality

- **Idea:** In equilibrium, $\Omega_{\alpha\beta} = \omega_{\alpha\beta}$. [Becattini, PRL **108** (2012) 244502; Weickgenannt et al, PRL **127** (2021) 052301]
- $\omega_{\alpha\beta}$ is kinematic in nature, while $\Omega_{\alpha\beta}$ is thermodynamic.
- We wish to disentangle vortical = kinematic + thermodynamic, s.t.:

$$-\frac{\partial\tilde{\phi}^\mu}{\partial\omega_{xy}} = \Theta^{\mu\nu}\tilde{\tau}_\nu, \quad -\frac{\partial\tilde{\phi}^\mu}{\partial\Omega_{xy}} = S_C^{\mu,xy}. \tag{35}$$

- The kinematic part is comprised of u^μ, a^μ, ω^μ and $\tau^\mu = \varepsilon^{\mu\nu\alpha\beta}u_\nu a_\alpha \omega_\beta$ and

$$\omega_{\alpha\beta} = a_\alpha u_\beta - a_\beta u_\alpha - \varepsilon_{\alpha\beta\mu\nu}u^\mu\omega^\nu. \tag{36}$$

- Similarly, we introduce

$$\Omega_{\alpha\beta} = \kappa_\alpha u_\beta - \kappa_\beta u_\alpha - \varepsilon_{\alpha\beta\mu\nu}u^\mu\Omega^\nu,$$
$$\kappa^\mu = \Omega^{\mu\nu}u_\nu, \quad \Omega^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}u_\nu\Omega_{\alpha\beta}. \tag{37}$$

- To implement this programme, we consider massless (free) fermions.

Massless fermions: exact results

VEA, Chernodub, EPJC **83** (2023) 111

- In the case of massless fermions, we have the following beta-frame decomposition of the Belinfante energy-momentum tensor $T_B^{\mu\nu} = \Theta_C^{(\mu\nu)}$:

$$T_B^{\mu\nu} = \epsilon u^\mu u^\nu - P_{\rm eff} \Delta^{\mu\nu} + \pi^{\mu\nu} + \sigma_\varepsilon^\tau (\tau^\mu u^\nu + \tau^\nu u^\mu),$$
$$J_V^\mu = Q_V u^\mu + \sigma_V^\tau \tau^\mu, \quad J_A^\mu = \sigma_A^\omega \omega^\mu, \tag{38}$$

- We'll focus just on the blue terms, given as:

$$\frac{1}{3}\epsilon = P_{\rm eff} = P_{\rm cl} - \frac{3\omega^2 + a^2}{12}\sigma^\omega_{A;{\rm cl}} + \frac{\omega^4 + \frac{46}{45}\omega^2 a^2 - \frac{17}{15}a^4}{192\pi^2},$$
$$\sigma_A^\omega = \sigma^\omega_{A;{\rm cl}} - \frac{\omega^2 + 3a^2}{24\pi^2}, \quad Q_V = Q_{V;{\rm cl}} - \frac{\mu(\omega^2 + a^2)}{4\pi^2}, \tag{39}$$

with $\omega^2 = \omega_\mu \omega^\mu = -\Omega^2 \Gamma^4$ and $a^2 = a_\mu a^\mu = -\Omega^2\Gamma^2(\Gamma^2 - 1)$ and the classical contributions:

$$P_{\rm cl} = \frac{7\pi^2 T^4}{180} + \frac{\mu_V^2 T^2}{6} + \frac{\mu_V^4}{12\pi^2}, \tag{40}$$
$$Q_{V;{\rm cl}} = \frac{\mu_V T^2}{3} + \frac{\mu_V^3}{3\pi^2}, \quad \sigma^\omega_{A;{\rm cl}} = \frac{T^2}{6} + \frac{\mu_V^2}{2\pi^2}, \tag{41}$$

Thermodynamic potential current

- To compute $\phi^\mu = \beta_0^{-1} \int d\beta_0 (\Theta_C^{\mu t})_{\beta_0\mu_0,\beta_0\Omega_0}$, we require $\Theta_C^{\mu\nu}$, satisfying

$$\Theta_C^{\mu\nu} = T_B^{\mu\nu} - \frac{1}{2}\partial_\lambda (S_C^{\lambda,\mu\nu} + S_C^{\mu,\nu\lambda} - S_C^{\nu,\lambda\mu}), \tag{42}$$

  which leads to $\Theta_C^{t\varphi} = T_B^{t\varphi} + \frac{1}{4\rho}\partial_\rho J_A^z$ and $\Theta_C^{\varphi t} = T_B^{\varphi t} - \frac{1}{4\rho}\partial_\rho J_A^z$.

- Performing the integration $\tilde{\phi}^\mu = \beta_0^{-1}\Gamma \int d\beta_0 (\Theta^{\mu t})_{\beta_0\mu_0,\beta_0\Omega_0}$, we arrive at

$$\tilde{\phi}^t = -\Gamma\left[P_{\rm cl} - \frac{3\omega^2 + a^2}{12}\sigma^\omega_{A;{\rm cl}} + \frac{\omega^4 + \frac{122}{15}\omega^2 a^2 - \frac{17}{15}a^4}{192\pi^2}\right],$$
$$\tilde{\phi}^\varphi = -\Omega\Gamma\left[P_{\rm cl} - \frac{\omega^2 + 3a^2}{12}\sigma^\omega_{A;{\rm cl}} + \frac{\omega^4 + 22\omega^2 a^2 + 17a^4}{960\pi^2}\right]. \tag{43}$$

Entropy, charge and dynamic pressure

- Since $\tilde{\phi}^\rho = \tilde{\phi}^z = 0$, we can decompose $\tilde{\phi}^\mu$ w.r.t. u^μ and τ^μ:

$$\tilde{\phi}^\mu = -P u^\mu - \sigma^\tau_\phi \tau^\mu, \quad \tau^\mu = \Omega_0^3 \Gamma^5 \tilde{\tau}^\mu, \tag{44}$$

where

$$P = -u_\mu \tilde{\phi}^\mu = P_{\rm cl} - \frac{\omega^2 + a^2}{4} \sigma^\omega_{A;{\rm cl}} + \frac{\omega^4 + \frac{134}{15}\omega^2 a^2 + \frac{17}{5} a^4}{192\pi^2},$$
$$\sigma^\tau_\phi = \frac{1}{6}\sigma^\omega_{A;{\rm cl}} - \frac{3\omega^2 + 17a^2}{720\pi^2}. \tag{45}$$

- The entropy and charge densities read

$$s = \frac{\partial P}{\partial T} = s_{\rm cl} - \frac{T(\omega^2 + a^2)}{12}, \quad Q_V = \frac{\partial P}{\partial \mu} = Q_{V;{\rm cl}} - \frac{\mu(\omega^2 + a^2)}{4\pi^2}. \tag{46}$$

- Since $P \neq P_{\rm eff}$, the system develops a dynamic pressure:

$$\Pi = P_{\rm eff} - P = \frac{a^2}{6}\sigma^\omega_{A;{\rm cl}} - \frac{a^2(89\omega^2 + 51a^2)}{2160\pi^2}. \tag{47}$$

Spin tensor

P. Singha et al, arXiv:2508.20237 [Accepted in PRD]

- The spin tensor is known for systems under rotation ($\Omega_{\mu\nu} = \omega_{\mu\nu}$):

$$\Omega_{\mu\nu} = \omega_{\mu\nu} : \qquad S_C^{\alpha\beta} = -\frac{1}{2} u_\mu \varepsilon^{\mu\alpha\beta\nu} J_{A;\nu} = \frac{1}{2}(\omega^{\alpha\beta} + u^\alpha a^\beta - u^\beta a^\alpha)\sigma_A^\omega. \quad (48)$$

- In the absence of rotation but finite spin potential ($\omega_{\mu\nu} = 0$, $\Omega_{\mu\nu} = -\frac{1}{2}\varepsilon_{0\mu\nu\lambda}\mu^\lambda$), the energy dispersion reads:

$$E_{\mathbf{p}}^{(s)} = \sqrt{E_p^2 + s^2\mu_\Sigma^2 - 2s\mu_\Sigma E_{p_z}}, \quad (49)$$

with $s = \pm 1/2$, $E_p = \sqrt{\mathbf{p}^2 + m^2}$ and $E_{p_z} = \sqrt{p_z^2 + m^2}$.

- Using $\mathcal{L}_q = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \Omega_{\mu\nu} S_C^{0,\mu\nu}$ gives $\phi = \phi_{\rm ZP} + \phi_\beta$, with

$$\phi_{\rm ZP} = -\sum_s \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}^{(s)}, \quad \phi_\beta = -\frac{2}{\beta}\sum_s \frac{d^3p}{(2\pi)^3} \ln(1 + e^{-\beta E_{\mathbf{p}}^{(s)}}). \quad (50)$$

- For massless fermions, $\phi_\beta = -7\pi^2 T_0^4/180$ is spin-independent $\Rightarrow S^{\alpha\beta} = -\partial\phi/\partial\Omega_{\alpha\beta} = 0$.

Spin tensor

- Since $S^{\alpha\beta} = 0$ at finite $\Omega^{\mu\nu}$ but vanishing $\omega^{\mu\nu}$, the tensor structure of $S^{\alpha\beta}$ is still given by the vorticity tensor: [Singha et al, arXiv:2508.20237 [Accepted in PRD]]

$$S_C^{\alpha\beta} = \frac{1}{2}(\omega^{\alpha\beta} + u^\alpha a^\beta - u^\beta a^\alpha)\sigma_A^\omega \equiv \frac{\partial P_\Omega}{\partial \Omega_{\alpha\beta}}, \tag{51}$$

which can be integrated to yield

$$P_\Omega = P_{\rm cl} - \frac{\omega^2 + a^2}{4}\sigma^\omega_{A;{\rm cl}} + \frac{\omega^4 + \frac{134}{15}\omega^2 a^2 + \frac{17}{15}a^4}{192\pi^2} - \frac{1}{2}\sigma_A^\omega(\omega \cdot \delta\Omega), \tag{52}$$

where we introduced $\delta\Omega^\mu = \Omega^\mu - \omega^\mu$.

- The local quantities read

$$s_\Omega = \frac{\partial P_\Omega}{\partial T} = s_{\rm cl} - \frac{T(\omega^2 + a^2)}{12} - \frac{T}{6}\omega \cdot \delta\Omega,$$

$$Q_V^\Omega = \frac{\partial P_\Omega}{\partial \mu} = Q_{V;{\rm cl}} - \frac{\mu(\omega^2 + a^2)}{4\pi^2} - \frac{\mu}{2\pi^2}\omega \cdot \delta\Omega. \tag{53}$$

Local energy density

- The energy density $\epsilon_\Omega = Ts_\Omega - P_\Omega + \mu Q_V^\Omega + \frac{1}{2}\Omega_{\alpha\beta} S_C^{\alpha\beta}$ reads

$$\epsilon_\Omega = \epsilon_{\rm cl} - \frac{3\omega^2 + a^2 + 4\omega\cdot\delta\Omega}{4}\sigma^\omega_{A;\rm cl} + \frac{\omega^4 + \frac{46}{45}\omega^2 a^2 - \frac{17}{45}a^4}{64\pi^2}. \quad (54)$$

- The dynamic pressure $\Pi_\Omega = \frac{1}{3}\epsilon_\Omega - P_\Omega$ gets modified to

$$\Pi_\Omega = \frac{a^2}{6}\sigma^\omega_{A;\rm cl} - \frac{a^2(89\omega^2 + 17a^2)}{2160\pi^2} + \left(\frac{1}{2}\sigma^\omega_A - \frac{1}{3}\sigma^\omega_{A;\rm cl}\right)(\omega\cdot\delta\Omega). \quad (55)$$

- Using $S_C^{\mu,\alpha\beta} = -\frac{1}{2}\sigma^\omega_A \varepsilon^{\mu\alpha\beta\lambda}\omega_\lambda = -\partial\tilde{\phi}^\mu_\Omega/\partial\Omega_{\alpha\beta}$, the thermodynamic four-current can be obtained as:

$$\tilde{\phi}^\mu_\Omega = -(P + \delta P)u^\mu - \sigma^\tau_\phi \tau^\mu - \delta\sigma^\tau_\phi \delta\tau^\mu, \quad (56)$$

where

$$\delta P = -\frac{1}{2}\sigma^\omega_A \omega\cdot\delta\Omega, \quad \delta\tau^\mu = \varepsilon^{\mu\nu\alpha\beta}u_\nu(\kappa_\alpha - a_\alpha)\omega_\beta, \quad \delta\sigma^\tau_\phi = \frac{1}{2}\sigma^\omega_A. \quad (57)$$

Bonus: Unruh effect

- Consider now a fluid undergoing accelerated motion (no vorticity):

$$\epsilon = \epsilon_{\rm cl} - \frac{a^2}{4}\sigma^{\omega}_{A;\rm cl} - \frac{17a^4}{960\pi^2}. \tag{58}$$

- The solution $\epsilon = 0$ gives

$$\mathbf{a}^2 = \frac{120\pi^2}{17}\left(\sigma^{\omega}_{A;\rm cl} + \sqrt{(\sigma^{\omega}_{A;\rm cl})^2 + \frac{17}{15\pi^2}\epsilon_{\rm cl}}\right). \tag{59}$$

- When $\mu = 0$, we have $\sigma^{\omega}_{A;\rm cl} = T^2/6$ and $\epsilon_{\rm cl} = 7\pi^2 T^4/60$, such that

$$\mathbf{a}^2 = (2\pi T)^2. \tag{60}$$

- The above relation confirms our thermodynamic relations are compatible with the Unruh effect:

$$T > T_U = \frac{|\mathbf{a}|}{2\pi}. \tag{61}$$

- When $T < T_U$, we have $\epsilon < 0$ and consequently, $P_{\rm eff} = P + \Pi < 0 \Rightarrow$ instability.

Conclusion

Conclusion

- We explored the thermodynamics of a quantum fluid exhibiting vorticity and acceleration.
- In the rigidly rotating ensemble, we constructed the thermodynamic potential and its four-current within a fictitious cylinder of radius $R < \Omega^{-1}$.
- Locally, kinematic (vorticity tensor) and thermodynamic (spin potential) vortical contributions must be disentangled.
- In the frame of massless fermions, we obtained the thermodynamic pressure of a fluid at finite spin potential and under finite vorticity.

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