Spin Hydrodynamics

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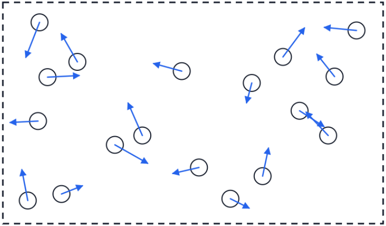
Perfect spin hydrodynamics

Standard relativistic hydrodynamics: perfect fluid

- Gas of spinless particles

$$\partial_\mu T^{\mu\nu}_{\rm eq} = 0, \qquad (1)$$
$$T^{\mu\nu}_{\rm eq} = (\varepsilon + P)u^\mu u^\nu - P g^{\mu\nu},$$
$$\partial_\mu N^\mu_{\rm eq} = 0$$

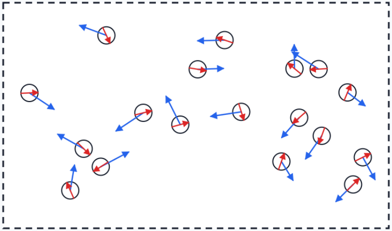


- 5 equations for 5 unknown functions
  $T,\ \mu,\ \vec{u} = \gamma \vec{v}$
  $u^\lambda = \gamma(1, \vec{v})$

Perfect spin hydrodynamics

- Gas of particles with spin

$$\partial_\mu T^{\mu\nu}_{\rm eq} = 0, \qquad (2)$$
$$\partial_\mu N^{\mu}_{\rm eq} = 0,$$
$$J^{\lambda\mu\nu} = L^{\lambda\mu\nu} + S^{\lambda\mu\nu}$$



$$\partial_\lambda J^{\lambda\mu\nu}_{\rm eq} = 0 \Leftrightarrow \partial_\lambda S^{\lambda\mu\nu}_{\rm eq} = T^{\nu\mu}_{\rm eq} - T^{\mu\nu}_{\rm eq}$$
$$= 0 \text{ for symmetric } T^{\mu\nu}$$

- Nontrivial forms of energy–momentum and spin tensors

For spin 1/2

$$\begin{cases} T^{\mu\nu}_{\rm eq} = T^{\mu\nu}_{\rm GLW} \\ S^{\lambda\mu\nu}_{\rm eq} = S^{\lambda\mu\nu}_{\rm GLW} \end{cases}$$

For spin 1 ($m \neq 0$)

$$\begin{cases} T^{\mu\nu}_{\rm eq} = T^{\mu\nu}_{\rm KG} \\ S^{\lambda\mu\nu}_{\rm eq} = S^{\lambda\mu\nu}_{\rm KG} \end{cases}$$

- In general, six new degrees of freedom in addition to T, μ, $\vec{u} = \gamma\vec{v}$. We can parametrize the spin polarization tensor $\omega_{\alpha\beta}$ as

$$\omega_{\alpha\beta} = k_\alpha u_\beta - k_\beta u_\alpha + \epsilon_{\alpha\beta\gamma\delta} u^\gamma \omega^\delta, \qquad (3)$$

with $k \cdot u = \omega \cdot u = 0$.

Classical spin description

- Spin 1/2

$$N^{\mu}_{\rm eq} = \int \mathrm{d}P\,\mathrm{d}S\; p^{\mu}\; \left[f^{+}_{\rm eq}(x,p,s) - f^{-}_{\rm eq}(x,p,s)\right], \qquad (4)$$

$$T^{\mu\nu}_{\rm eq} = \int \mathrm{d}P\,\mathrm{d}S\; p^{\mu} p^{\nu}\; \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s)\right], \qquad (5)$$

$$S^{\lambda\mu\nu}_{\rm eq} = \int \mathrm{d}P\,\mathrm{d}S\; p^{\lambda}\, s^{\mu\nu} \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s)\right]. \qquad (6)$$

- The equilibrium distribution functions are
  Fermi–Dirac $f^{\pm}_{\rm eq}(x,p,s) = \left[\exp\left(\mp\xi(x) + p_{\mu}\beta^{\mu}(x) - \frac{1}{2}\omega_{\mu\nu}(x)s^{\mu\nu}\right) + 1\right]^{-1}$
  or Boltzmann $f^{\pm}_{0,\rm eq}(x,p,s) = \exp\left(\pm\xi(x) - p_{\mu}\beta^{\mu}(x) + \frac{1}{2}\omega_{\mu\nu}(x)s^{\mu\nu}\right)$,

- $\mathrm{d}P = \frac{\mathrm{d}^3 p}{(2\pi)^3 E_p}$, $\mathrm{d}S = \frac{m}{\pi\mathfrak{s}}\mathrm{d}^4 s\,\delta(s\cdot s + \mathfrak{s}^2)\delta(p\cdot s)$,
  $\xi \equiv \mu/T$ and $\beta^{\mu} \equiv u^{\mu}/T$.

W. Florkowski, M. Hontarenko, Phys.Rev.Lett. 134 (2025) 8, 082302.
ZD, W. Florkowski, M. Hontarenko, (2024), Phys.Rev.D, **110**(9), 096018.

- The distribution functions can be expanded to a given order in ω, and the integrals defining the hydrodynamic tensors can be computed.
- Up to the second order in ω,

$$N^\mu_{\rm eq} = (n_0 + n_2^k + n_2^\omega) u^\mu + n_t t^\mu, \tag{7}$$

$$\begin{aligned} T^{\mu\nu}_{\rm eq} &= (\varepsilon_0 + \varepsilon_2^k + \varepsilon_2^\omega) u^\mu u^\nu - (P_0 + P_2^k + P_2^\omega)\Delta^{\mu\nu} \\ &\quad + P_{k\omega}(k^\mu k^\nu + \omega^\mu \omega^\nu) + P_t(t^\mu u^\nu + t^\nu u^\mu), \end{aligned} \tag{8}$$

Those tensors contain only even powers of ω.
The scalar coefficients multiplying the vectors and tensors on the RHS are functions of T, μ, k^2, ω^2.

$$\begin{aligned} S^{\lambda\mu\nu}_{\rm eq} &= u^\lambda \left[A\,(k^\mu u^\nu - k^\nu u^\mu) + A_1 t^{\mu\nu}\right] \\ &\quad + \frac{A}{2}\left(t^{\lambda\mu} u^\nu - t^{\lambda\nu} u^\mu + \Delta^{\lambda\mu} k^\nu - \Delta^{\lambda\nu} k^\mu\right), \end{aligned} \tag{9}$$

The spin tensor contains only odd powers of ω.
The coefficients A, A_1 are functions of T, μ.
$t^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} u_\alpha \omega_\beta$.
$t^\mu = t^{\mu\nu} k_\nu = \epsilon^{\mu\nu\alpha\beta} k_\nu U_\alpha \omega_\beta$.

Quantum spin description

- Start from the Wigner function

$$W^{\pm}(x,k)=\frac{1}{4m}\int \mathrm{d}P\,\delta^{(4)}(k\mp p)(\not{p}\pm m)X_s^{\pm}(\not{p}\pm m), \tag{10}$$

- Use the new equilibrium spin density

$$X_s^{\pm}(x,p) = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu + \gamma_5 \not{a}\right]. \tag{11}$$

- The conserved currents and tensors are

$$N^\mu(x) = \sum_{r=1}^{2}\int \mathrm{d}P\,p^\mu\left[f_{rr}^{+}(x,p) - f_{rr}^{-}(x,p)\right], \tag{12}$$

$$T^{\mu\nu}(x) = \sum_{r=1}^{2}\int \mathrm{d}P\,p^\mu p^\nu\left[f_{rr}^{+}(x,p) + f_{rr}^{-}(x,p)\right], \tag{13}$$

$$S^{\lambda\mu\nu}(x) = \frac{1}{2}\sum_{r,s=1}^{2}\int \mathrm{d}P\,p^\lambda \sigma_{sr}^{+\mu\nu}(p)f_{rs}^{+}(x,p) + \sigma_{sr}^{-\mu\nu}(p)f_{rs}^{-}(x,p), \tag{14}$$

where
$\sigma_{sr}^{+\mu\nu}(p) = 1/(2m)\,\bar{u}_s(p)\sigma^{\mu\nu}u_r(p)$ and $\sigma_{sr}^{-\mu\nu}(p) = 1/(2m)\,\bar{v}_r(p)\sigma^{\mu\nu}v_s(p)$

S. Bhadury, ZD, W. Florkowski, S.K. Kar, V. Mykhaylova, arXiv:2505.02657 [hep-ph].
S.K. Kar, V. Mykhaylova, arXiv:2511.09580 [quant-ph].

- Whereas the spin-1 case is

$$T^{\mu\nu}_{\mathrm{KG}} = \int \mathrm{d}P\, f_0\, p^\mu p^\nu \left[2\cosh(2a_*) + 1\right] \tag{15}$$

$$S^{\lambda\mu\nu}_{\mathrm{KG}} = 2\int \mathrm{d}P\, f_0\, p^\lambda \frac{\sinh\left(2\sqrt{-a^2}\right)}{m\sqrt{-a^2}} \epsilon^{\mu\nu\alpha\beta} a_\alpha p_\beta \tag{16}$$

- The Boltzmann distribution function

$$f_0(x,p) = \exp\left(\xi(p) - p_\mu \beta^\mu(x)\right), \tag{17}$$

$$a^\mu = -\frac{1}{4m}\epsilon^{\mu\nu\rho\sigma} p_\nu \omega_{\rho\sigma} \tag{18}$$

W. Florkowski, S. K. Kar, V. Mykhaylova, arXiv:2602.00819 [nucl-th]

Generating functions

- Spin 1/2, classical approach

$$\chi_{\rm cl} = 2 \int \mathrm{d}P \cosh\xi \exp\left(-p_\mu \beta^\mu\right) \int \mathrm{d}S \exp\left(\frac{1}{2}\omega_{\mu\nu} s^{\mu\nu}\right) \tag{19}$$

- Spin 1/2, quantum approach

$$\chi_{\rm qt}(x) = 4 \int \mathrm{d}P \exp\left(-p\cdot\beta\right) \cosh\xi \, \cosh\sqrt{-a^2}, \tag{20}$$

$$N^\mu_{\rm eq} = -\frac{\partial^2 \chi}{\partial\beta_\mu \partial\xi}, \tag{21}$$

$$T^{\mu\nu}_{\rm eq} = \frac{\partial^2 \chi}{\partial\beta_\mu \partial\beta_\nu}, \tag{22}$$

$$S^{\lambda\mu\nu}_{\rm eq} = -\frac{\partial^2 \chi}{\partial\beta_\lambda \partial\omega_{\mu\nu}}. \tag{23}$$

- The integrals converge for physical values of spin polarization ω.
  ZD, W. Florkowski, V. Mykhaylova, Phys.Rev.D 112 (2025) 5, L051901
- The hydrodynamic tensors of classical and quantum spin frameworks exactly agree in an expansion up to the second order in ω. Higher-order corrections differ only by multiplicative numerical factors but their structure is the same.
  ZD, Phys.Lett.B 873 (2026) 140205

Causality and stability

- We can define spin hydrodynamics as a divergence-type theory (DTT).
- We use a multi-index notation $N^{\lambda A} \equiv (N^{\lambda} T^{\lambda\mu}, S^{\lambda\mu\nu})$ to write the conservation laws in a compact form,

$$\partial_\lambda N^{\lambda A} = 0 \quad \text{or} \quad M^{\lambda AB}\partial_\lambda \zeta_B = 0, \tag{24}$$

where

$$\zeta_A = (\zeta, \zeta_\mu, \zeta_{\mu\nu}) = (\xi, -\beta_\mu, \omega_{\mu\nu}), \quad M^{\lambda AB} = \left(\frac{\partial N^{\lambda A}}{\partial \zeta_B}\right) = -\left(\frac{\partial^3 \chi}{\partial \zeta_B \partial \zeta_A \partial \beta_\lambda}\right). \tag{25}$$

- Equations are symmetric hyperbolic if the four-vector

$$M^{\lambda} \equiv M^{\lambda AB} Z_A Z_B \tag{26}$$

for any nonvanishing real elements $Z_A = (Z, Z_\mu, Z_{\mu\nu})$ is future oriented and timelike. Then, spin hydrodynamics is nonlinearly causal and stable.

- Analogous as in Enrico Speranza's talk.
- We write M^λ as

$$M^\lambda = \left(Z\frac{\partial}{\partial \xi} - Z_\mu \frac{\partial}{\partial \beta_\mu} + \frac{1}{2} Z_{\mu\nu}\frac{\partial}{\partial \omega_{\mu\nu}}\right)^2 \mathcal{N}^\lambda \quad \text{where} \quad \mathcal{N}^\lambda = \frac{\partial \chi}{\partial \zeta_\lambda} = -\frac{\partial \chi}{\partial \beta_\lambda} \tag{27}$$

and prove those properties for each of the four cases: underlying particle statistics Fermi–Dirac vs. Boltzmann, classical vs. quantum spin.

R. P. Geroch and L. Lindblom, Phys. Rev. D 41, 1855 (1990).
R. P. Geroch and L. Lindblom, Annals of Physics 207, 394 (1991).
L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. D 106, 056010 (2022).
N. Abboud, L. Gavassino, R. Singh, and E. Speranza, Phys. Rev. D 112, 094043 (2025).
S. Bhadury, ZD, W. Florkowski, S.K. Kar, V. Mykhaylova, Phys.Rev.D 113 (2026) 3, 036017.

Numerical results

- To perform numerical simulations of perfect spin hydrodynamics, we write down the conservation equations in a selected geometry – eleven first-order differential equations for eleven unknowns in general.
- We may solve the equations from $\partial_\mu N^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$ without spin first, and then evolve spin on this background, as an approximation; or we may include spin feedback and solve all the equations at once.
- Boost-invariant, transversely homogeneous
  ZD, W. Florkowski, N. Łygan, R. Ryblewski, Phys.Rev.C 111 (2025) 2, 024909
  ZD, N. Łygan, 2604.17392 [hep-ph]
- Boost-invariant, cylindrically symmetric
  ZD, W. Florkowski, J. Witkowski arXiv:2605.01857 [hep-ph]
- This showed the feasibility of the approach, although the geometries are too simple for comparison with experiment yet.
- Simulations in more general geometries and with dissipation (see the later part of the talk) come next!

Symmetric hyperbolicity vs. global existence of solutions

- The theory truncated at a finite order in ω is well posed, causal and stable too!
- Caveat: these properties do not guarantee that singularities can never form.
- We observed singularities in the boost-invariant transversely homogeneous case in the 2nd-order theory for extremely high initial longitudinal spin polarizations.
- There are additional conditions for divergence-type theories to guarantee global existence of solutions, e.g., should be "totally dissipative"
  H.-O. Kreiss, G. B. Nagy, O. E. Ortiz, O. A. Reula, J. Math. Phys. 38 (1997) 5272
- Singularities form in finite time from smooth initial data in BDNK hydrodynamics (Bemfica, Disconzi, Noronha, Kovtun), which is also causal, stable and well posed
  L. S. Keeble, F. Pretorius, PRD 112 (2025) 124034
- Singularities form in DTT for conformal fluids
  P. E. Montes, M. E. Rubio, O. A. Reula, PRD 107 (2023) 103041
- Singularities form in Israel-Stewart F.S. Bemfica, PRE 112 (2025) 065105
- What helps:
  - less restrictive geometries: no singularities formed in the case of spin polarization transverse to the flow for *any* initial values ZD, N. Łygan, 2604.17392 [hep-ph]
  - expansion to the fourth order in ω [upcoming]
  - dissipation.

Thermodynamics with spin

Generalized thermodynamic relations

- Old approach, using the spin density tensor $S^{\alpha\beta}$ of J. Weyssenhoff, A. Raabe, Acta Phys.Polon. 9 (1947) 7

$$\varepsilon + P = T\sigma + \mu n + \frac{1}{2}\Omega_{\alpha\beta} S^{\alpha\beta}, \tag{28}$$

$$\mathrm{d}\varepsilon = T\mathrm{d}\sigma + \mu \mathrm{d}n + \frac{1}{2}\Omega_{\alpha\beta} S^{\alpha\beta}, \quad \mathrm{d}P = \sigma \mathrm{d}T + n\mathrm{d}\mu + \frac{1}{2}\Omega_{\alpha\beta} S^{\alpha\beta} \tag{29}$$

Multiplication by the flow vector u^μ gave

$$S^\mu_{\rm eq} = P\beta^\mu - \xi N^\mu_{\rm eq} + \beta_\lambda T^{\lambda\mu}_{\rm eq} - \frac{1}{2}\omega_{\alpha\beta} S^{\mu\alpha\beta}_{\rm eq}, \tag{30}$$

$$\mathrm{d}S^\mu_{\rm eq} = -\xi \mathrm{d}N^\mu_{\rm eq} + \beta_\lambda \mathrm{d}T^{\lambda\mu}_{\rm eq} - \frac{1}{2}\omega_{\alpha\beta} \mathrm{d}S^{\mu\alpha\beta}_{\rm eq}, \tag{31}$$

$$\mathrm{d}(P\beta^\mu) = N^\mu_{\rm eq}\mathrm{d}\xi - T^{\lambda\mu}_{\rm eq}\mathrm{d}\beta_\lambda + \frac{1}{2} S^{\mu\alpha\beta}_{\rm eq}\mathrm{d}\omega_{\alpha\beta}, \tag{32}$$

with the "phenomenological spin tensor"

$$S^{\mu\alpha\beta}_{\rm eq} = u^\mu S^{\alpha\beta}_{\rm eq}. \tag{33}$$

- However, both through kinetic theory and the Wigner function, we find, up to the second order in ω,

$$N^\mu_{\rm eq} = \tilde{n} u^\mu + n_t t^\mu, \tag{34}$$

$$T^{\mu\nu}_{\rm eq} = \tilde{\varepsilon} u^\mu u^\nu - \tilde{P}\Delta^{\mu\nu} + P_{k\omega}(k^\mu k^\nu + \omega^\mu \omega^\nu) + P_t(t^\mu u^\nu + t^\nu u^\mu),$$

$$S^{\lambda\mu\nu}_{\rm eq} = u^\lambda \left(A\left(k^\mu u^\nu - k^\nu u^\mu\right) + A_1 t^{\mu\nu}\right) + \frac{A}{2}\left(t^{\lambda\mu} u^\nu - t^{\lambda\nu} u^\mu + \Delta^{\lambda\mu} k^\nu - \Delta^{\lambda\nu} k^\mu\right).$$

$S^{\lambda\mu\nu}_{\rm eq}$ contains terms that are not proportional to u^λ!

- The solution is obtained from Boltzmann's definition of the entropy (H-function)

$$S^\mu = -\int \mathrm{d}P\,\mathrm{d}S\, p^\mu \left[f^+\left(\ln f^+ - 1\right) + f^-\left(\ln f^- - 1\right)\right] \quad \text{classical spin} \qquad (35)$$

$$S^\mu = -\frac{1}{2}\int \mathrm{d}P\, p^\mu \left\{\mathrm{tr}_4\left[X^+\left(\ln X^+ - 1\right)\right] + \mathrm{tr}_4\left[X^-\left(\ln X^- - 1\right)\right]\right\} \quad \text{quantum spin} \qquad (36)$$

- Using those together with other kinetic-theory expressions, one obtains the tensor form of thermodynamic relations,

$$S^\mu_{\mathrm{eq}} = T^{\mu\alpha}_{\mathrm{eq}}\beta_\alpha - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\mathrm{eq}} - \xi N^\mu_{\mathrm{eq}} + \mathcal{N}^\mu_{\mathrm{eq}}, \qquad \mathcal{N}^\mu = \coth\xi N^\mu_{\mathrm{eq}} \neq P u^\mu \qquad (37)$$

$$dS^\mu_{\mathrm{eq}} = -\xi dN^\mu_{\mathrm{eq}} + \beta_\lambda dT^{\lambda\mu}_{\mathrm{eq}} - \frac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}_{\mathrm{eq}}, \qquad (38)$$

$$d\mathcal{N}^\mu_{\mathrm{eq}} = N^\mu_{\mathrm{eq}}d\xi - T^{\lambda\mu}_{\mathrm{eq}}d\beta_\lambda + \frac{1}{2}S^{\mu,\alpha\beta}_{\mathrm{eq}}d\omega_{\alpha\beta}. \qquad (39)$$

W. Florkowski, M. Hontarenko, Phys.Rev.Lett. 134 (2025) 8, 082302,
ZD, W. Florkowski, M. Hontarenko, (2024), Phys.Rev.D, 110 (2024) 9, 096018.

Pseudo-gauge freedom

Pseudo-gauge freedom

- The currents $T^{\mu\nu}$ and $J^{\lambda\mu\nu}$ are still conserved and the total charges are the same after a *pseudo-gauge transformation*

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda G^{\lambda\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda(\Phi^{\lambda\mu\nu} + \Phi^{\nu\mu\lambda} + \Phi^{\mu\nu\lambda}), \qquad (40)$$

$$S'^{\lambda\mu\nu} = S^{\lambda\mu\nu} - \Phi^{\lambda\mu\nu} + \partial_\rho Z^{\mu\nu\lambda\rho}, \qquad (41)$$

defined by any differentiable tensors

$$\Phi^{\lambda\mu\nu} = -\Phi^{\lambda\nu\mu}, \quad (\text{this implies } G^{\lambda\mu\nu} = -G^{\mu\lambda\nu}), \qquad (42)$$

$$Z^{\mu\nu\lambda\rho} = -Z^{\nu\mu\lambda\rho} = -Z^{\mu\nu\rho\lambda}. \qquad (43)$$

- For example, the Belinfante symmetric $T^{\mu\nu}$ is obtained through

$$\Phi^{\lambda\mu\nu} = S^{\lambda\mu\nu}, \qquad Z^{\mu\nu\lambda\rho} = 0. \qquad (44)$$

- Various pseudo-gauge choices have their specific advantages
  $T^{\mu\nu}_{\rm Bel}$ (Belinfante-Rosenfeld) $\rightarrow$ coupling to gravity
  $S^{\lambda\mu\nu}_{\rm can}$ (canonical) $\rightarrow$ gauge invariant, related to spin operator

$S^{\lambda\mu\nu}_{\rm GLW}$ (de Groot, van Leuveen, van Weert)
$S^{\lambda\mu\nu}_{\rm HW}$ (Hilgevoord and Wouthuysen)
$\rightarrow$ good for thermodynamics

- Sidney Coleman:

> *...we have an infinite family of possible definitions of the local current [...] and the right answer is, of course, there's nothing to natter about, there's nothing to be disturbed about...it is something to be pleased about. If we have many objects that satisfy desirable general criteria, then that's better than having just one... the more freedom you have, the better. It's like being passed a plate of cookies and someone starts arguing about which is the best cookie. They're all edible!*

Quantum Field Theory Lectures of Sidney Coleman, World Scientific 2019

We can go from the GLW pseudo-gauge to canonical and to Belinfante:

- GLW → canonical: $\Phi^{\lambda\mu\nu}_{\rm can} = S^{\mu\lambda\nu}_{\rm GLW} - S^{\nu\lambda\mu}_{\rm GLW}$

$$S^{\lambda\mu\nu}_{\rm can} = S^{\lambda\mu\nu}_{\rm GLW} - \Phi^{\lambda\mu\nu}_{\rm can} \tag{45}$$

$$T^{\mu\nu}_{\rm can} = T^{\mu\nu}_{\rm GLW} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda\mu\nu}_{\rm can} + \Phi^{\mu,\nu\lambda}_{\rm can} + \Phi^{\nu,\mu\lambda}_{\rm can}\right) \tag{46}$$

- canonical → Belinfante: $\Phi_{\rm Bel} = S^{\lambda\mu\nu}_{\rm can}$

$$S^{\lambda\mu\nu}_{\rm Bel} = 0 \tag{47}$$

$$T^{\mu\nu}_{\rm Bel} = T^{\mu\nu}_{\rm can} + \frac{1}{2}\partial_\lambda \left(S^{\lambda\mu\nu}_{\rm can} + S^{\mu,\nu\lambda}_{\rm can} + S^{\nu,\mu\lambda}_{\rm can}\right) \tag{48}$$

Spin–orbit coupling

Spin–orbit coupling

- Spin–orbit coupling in atomic physics is mediated in coherent processes by effective magnetic field, which can be included in spin magneto-hydrodynamics (spin-MHD)
  S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, R. Ryblewski, Phys.Rev.Lett. 129 (2022) 192301
- Conservation of the total angular momentum of a particle implies conservation of spin if collisions are local (spacetime coordinate x^μ can always be set equal to 0)

$$j^{\alpha\beta} = l^{\alpha\beta} + s^{\alpha\beta} = x^\alpha p^\beta - x^\beta p^\alpha + s^{\alpha\beta} \tag{49}$$

- Transfer between orbital and spin part is possible if collisions are nonlocal, which leads to dissipation and entropy production.
  N. Weickgenannt, E. Speranza, X.-L. Sheng, Q. Wang, D. Rischke, Phys.Rev.Lett.127 (2021) 052301
- Perfect spin hydrodynamics = spin conservation.
- *Dissipative spin hydrodynamics* = transfer between **S** and **L** is possible.

Close-to-equilibrium dynamics

- Israel–Stewart approach to dissipative terms as an alternative to the nonlocal collision formalism
- We write general nonequilibrium expressions as the equilibrium terms plus corrections

$$N^\mu = N^\mu_{eq} + \delta N^\mu, \quad T^{\mu\nu} = T^{\mu\nu}_{eq} + \delta T^{\mu\nu}, \quad S^{\mu,\alpha\beta} = S^{\mu,\alpha\beta}_{eq} + \delta S^{\mu,\alpha\beta}. \tag{50}$$

- We get

$$\partial_\mu S^\mu = \delta N^\mu \partial_\mu \xi + \delta T^{\mu\nu}_s \partial_\mu \beta_\nu + \delta T^{\mu\nu}_a (\partial_\mu \beta_\nu - \omega_{\nu\mu}) - \frac{1}{2} \delta S^{\mu,\alpha\beta} \partial_\mu \omega_{\alpha\beta}. \tag{51}$$

(Agrees with, e.g., K. Hattori et al., Phys.Lett.B **795** (2019) 100-106,
F. Becattini, A. Daher, X.-L. Sheng, Phys.Lett.B **850** (2024), 138533.)

- To find the form of the deviations δN^μ, $\delta T^{\mu\nu}$, $\delta S^{\mu,\alpha\beta}$, we use a decomposition of general tensors of the given symmetry via projections along u^μ and separation into symmetric and antisymmetric parts

$$\begin{aligned} N^\mu &= a u^\mu + b^\mu, \\ T^{\mu\nu} &= c u^\mu u^\nu + d^\mu_s u^\nu + d^\nu_s u^\mu + d^\mu_a u^\nu - d^\nu_a u^\mu + e^{\mu\nu}_a + e^{\mu\nu}_s, \\ S^{\lambda,\mu\nu} &= u^\lambda \left[(f^\mu u^\nu - f^\nu u^\mu) + \epsilon^{\mu\nu\rho\sigma} u_\rho w_\sigma \right] + i^{\lambda\mu} u^\nu - i^{\lambda\nu} u^\mu + j^{\lambda\mu\nu}, \end{aligned} \tag{52}$$

with the constraints $b^\mu u_\mu = 0$, $d^\mu_s u_\mu = d^\mu_a u_\mu = e^{\mu\nu}_a u_\mu = e^{\mu\nu}_s u_\mu = 0$,
$f^\mu u_\mu = 0$, $h^{\mu\nu} = -h^{\nu\mu}$, $h^{\mu\nu} u_\mu = 0$, $i^{\lambda\mu} u_\lambda = i^{\lambda\mu} u_\mu = 0$,
$j^{\lambda\mu\nu} = -j^{\lambda\nu\mu}$, $j^{\lambda\mu\nu} u_\lambda = j^{\lambda\mu\nu} u_\mu = j^{\lambda\mu\nu} u_\nu = 0$, $w \cdot u = 0$.

The dissipative corrections

Some parts of the general tensors have the same form as the equilibrium ones. When we enforce the Landau matching conditions

$$
\begin{aligned}
N^\mu u_\mu &= N^\mu_{\rm eq} u_\mu, \\
T^{\mu\nu} u_\mu u_\nu &= T^{\mu\nu}_{\rm eq} u_\mu u_\nu, \\
S^{\lambda,\mu\nu} u_\lambda &= S^{\lambda,\mu\nu}_{\rm eq} u_\lambda,
\end{aligned} \tag{53}
$$

we obtain

$$
\begin{aligned}
a = \bar{n}(T,\xi,k^2,\omega^2), \quad & c = \bar{\varepsilon}(T,\xi,k^2,\omega^2), \\
f^\mu = A(T,\xi)k^\mu, \quad & w^\mu = A_1(T,\xi)\omega^\mu.
\end{aligned} \tag{54}
$$

The remaining terms are

$$
\begin{aligned}
\delta N^\mu &= V^\mu, \\
\delta T^{\mu\nu}_s &= -\Pi\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}, \\
\delta T^{\mu\nu}_a &= d^\mu_a u^\nu - d^\nu_a u^\mu + e^{\mu\nu}_a, \\
\delta S^{\lambda,\mu\nu} &= \Sigma^{\lambda\mu} u^\nu - \Sigma^{\lambda\nu} u^\mu + \phi^{\lambda\mu\nu},
\end{aligned} \tag{55}
$$

with

$$
\begin{aligned}
V^\mu &= b^\mu - n_t t^\mu, \quad \Pi = e - \bar{P}_{k\omega}, \quad W^\mu = d^\mu_s - P_t t^\mu, \\
\pi^{\mu\nu} &= e^{\langle\mu\nu\rangle}_s - P_{k\omega}(k^{\langle\mu}k^{\nu\rangle} + \omega^{\langle\mu}\omega^{\nu\rangle}), \quad \Sigma^{\lambda\mu} = i^{\lambda\mu} - \frac{A}{2}t^{\lambda\mu}, \\
\phi^{\lambda\mu\nu} &= j^{\lambda\mu\nu} - \frac{A}{2}(\Delta^{\lambda\mu}k^\nu - \Delta^{\lambda\nu}k^\mu).
\end{aligned} \tag{56}
$$

The equations have the same form as in (Biswas 2023) and can be expressed in terms of gradients of hydrodynamic variables multiplied by kinetic coefficients.

R. Biswas, A. Daher, A. Das, W. Florkowski, R. Ryblewski, Phys.Rev.D **108**, 014024 (2023).

Thus, we get the form of the tensors

$$\begin{aligned}
b^{\mu} &= \lambda \nabla^{\mu} \xi + n_t t^{\mu}. \quad c = \bar{\varepsilon}(T, \xi, k^2, \omega^2), \quad d_s^{\mu} = -\kappa(Du^{\mu} - \beta \nabla^{\mu} T) + P_t t^{\mu}, \\
d_a^{\mu} &= \lambda_a \beta^{-1}(\beta D u^{\mu} + \beta^2 \nabla^{\mu} T - 2k^{\mu}), \quad e = \bar{P} - \zeta\theta - (1/3)P_{k\omega}(k^2 + \omega^2), \\
e_s^{\langle\mu\nu\rangle} &= 2\eta\sigma^{\mu\nu} + P_{k\omega}(k^{\langle\mu} k^{\nu\rangle} + \omega^{\langle\mu}\omega^{\nu\rangle}), \quad e_a^{\mu\nu} = \gamma\beta\nabla^{[\mu} u^{\nu]}, \\
i^{\lambda\mu} &= -\chi_1 \Delta^{\lambda\mu} u^{\beta} \nabla^{\alpha} \omega_{\alpha\beta} - \chi_2 u_{\nu} \nabla^{\langle\lambda} \omega^{\mu\rangle\nu} - \chi_3 u_{\nu} \Delta^{[\mu}_{\rho} \nabla^{\lambda]} \omega^{\rho\nu} + \frac{A}{2} t^{\lambda\mu}, \\
j^{\lambda\mu\nu} &= \frac{\chi_4}{2} \nabla^{\lambda} \omega^{\langle\mu\rangle\langle\nu\rangle} + \frac{A}{2}(\Delta^{\lambda\mu} k^{\nu} - \Delta^{\lambda\nu} k^{\mu}),
\end{aligned} \tag{57}$$

with $D = u^{\mu}\partial_{\mu}$, $\theta = p_{\mu} u^{\mu}$, $\sigma^{\mu\nu} = p^{\langle\mu} u^{\nu\rangle}$, η, ζ – shear and bulk viscosity, $\lambda =$ the diffusion coefficient, κ – thermal conductivity, λ_a, γ – coefficients introduced in Hattori 2019, $\chi_1, \chi_2, \chi_3, \chi_4$ – coefficients from Biswas 2023.

K. Hattori et al., Phys.Lett.B **795** (2019) 100-106.

R. Biswas, A. Daher, A. Das, W. Florkowski, R. Ryblewski, Phys.Rev.D **108**, 014024 (2023).

- This, together with the conservation laws, forms a framework of dissipative spin hydrodynamics.
- We include second-order terms in ω in a consistent way.
- Including second-order terms in gradients is straightforward.

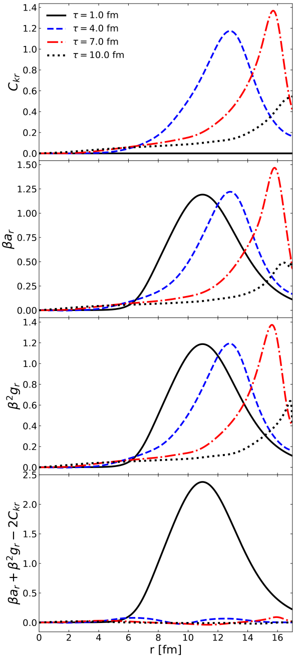
- Preliminary: results with dissipation in the boost-invariant cylindrically symmetric gemetry.
- ZD, W. Florkowski, J. Witkowski arXiv:2605.01857 [hep-ph] + dissipation
- Initial temperature: 0.3 GeV (Woods–Saxon profile in r), particle mass: 1 GeV
- Nonzero dissipative λ_a^μ coefficient.
- Decomposition of $\omega_{\mu\nu}$:

$$\omega_{\alpha\beta} = k_\alpha u_\beta - k_\beta u_\alpha + t_{\alpha\beta},$$

$$k^\mu = C_{kr} R^\mu + C_{k\phi} \Phi^\mu + C_{kz} Z^\mu$$

$$\omega^\mu = C_{\omega r} R^\mu + C_{\omega\phi} \Phi^\mu + C_{\omega z} Z^\mu$$

- The system reaches global equilibrium in which the spin polarization tensor is given by thermal vorticity, $\omega_{\mu\nu} = \partial_{[\nu}\beta_{\mu]}$.
- The difference to thermal vorticity is proportional to the quantity in the last panel and decreases quickly.



Summary

Summary

- Perfect spin hydrodynamics = spin conservation.
- Classical and quantum spin approaches to spin hydrodynamics agree at the second order in spin polarization, and higher-order corrections differ by multiplicative numerical factors.
- Spin hydrodynamics is a divergence-type theory, well posed, causal and stable.
- It leads to consistent thermodynamic relations.
- Perhaps pseudo-gauge freedom is useful, not problematic.
- Dissipation can be introduced without the nonlocal collisions framework.
- Numerical simulations in simple geometries have already been performed.
- We have preliminary results showing dissipation and the spin–orbit interaction, and they look promising.
- Simulations in more general geometries and with a greater variety of dissipative terms come next.

Thank you for your attention!

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