Quantum Metric Corrected Kinetic Theory

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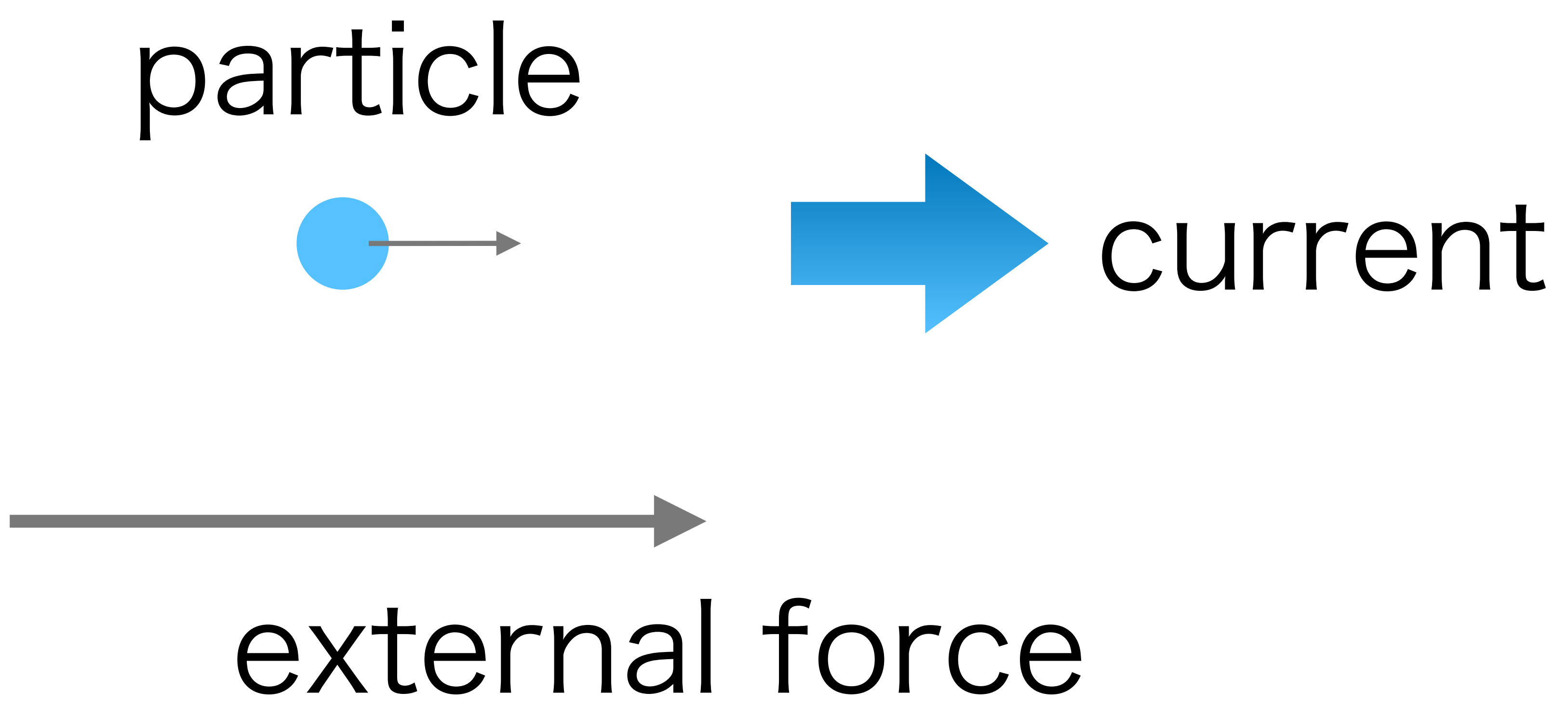
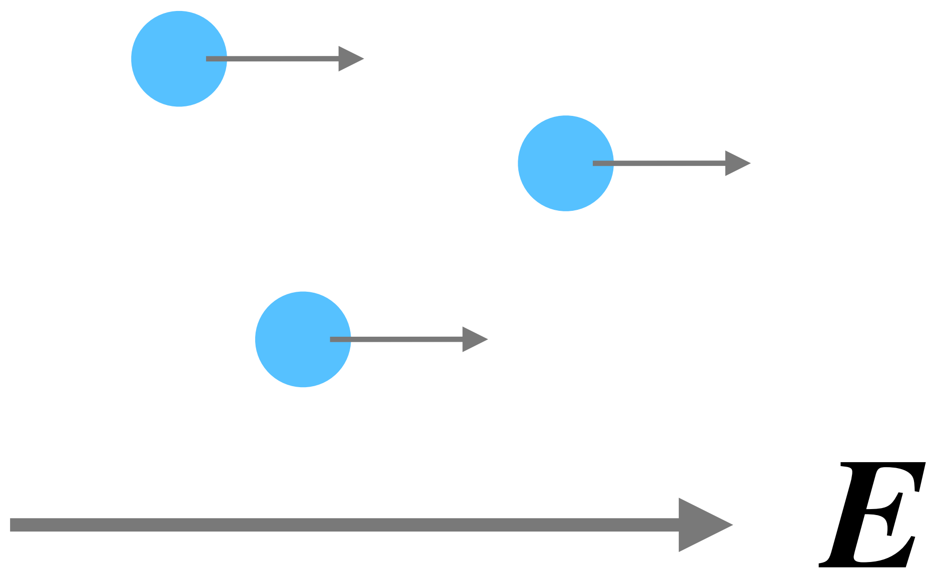
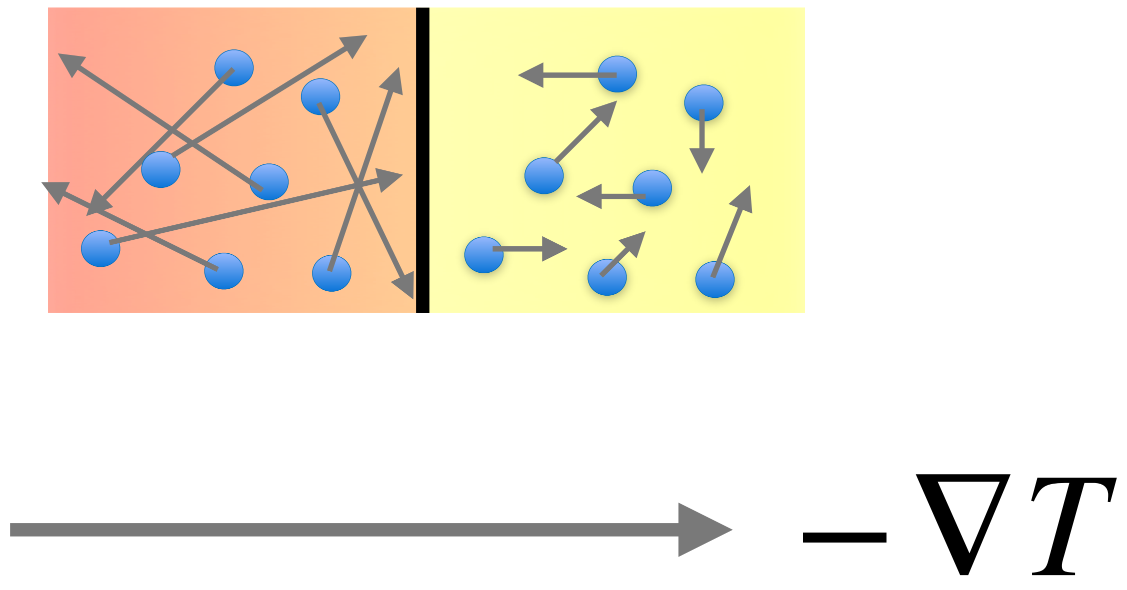
with Naoki Yamamoto, arXiv: 2509.15731 (2025)

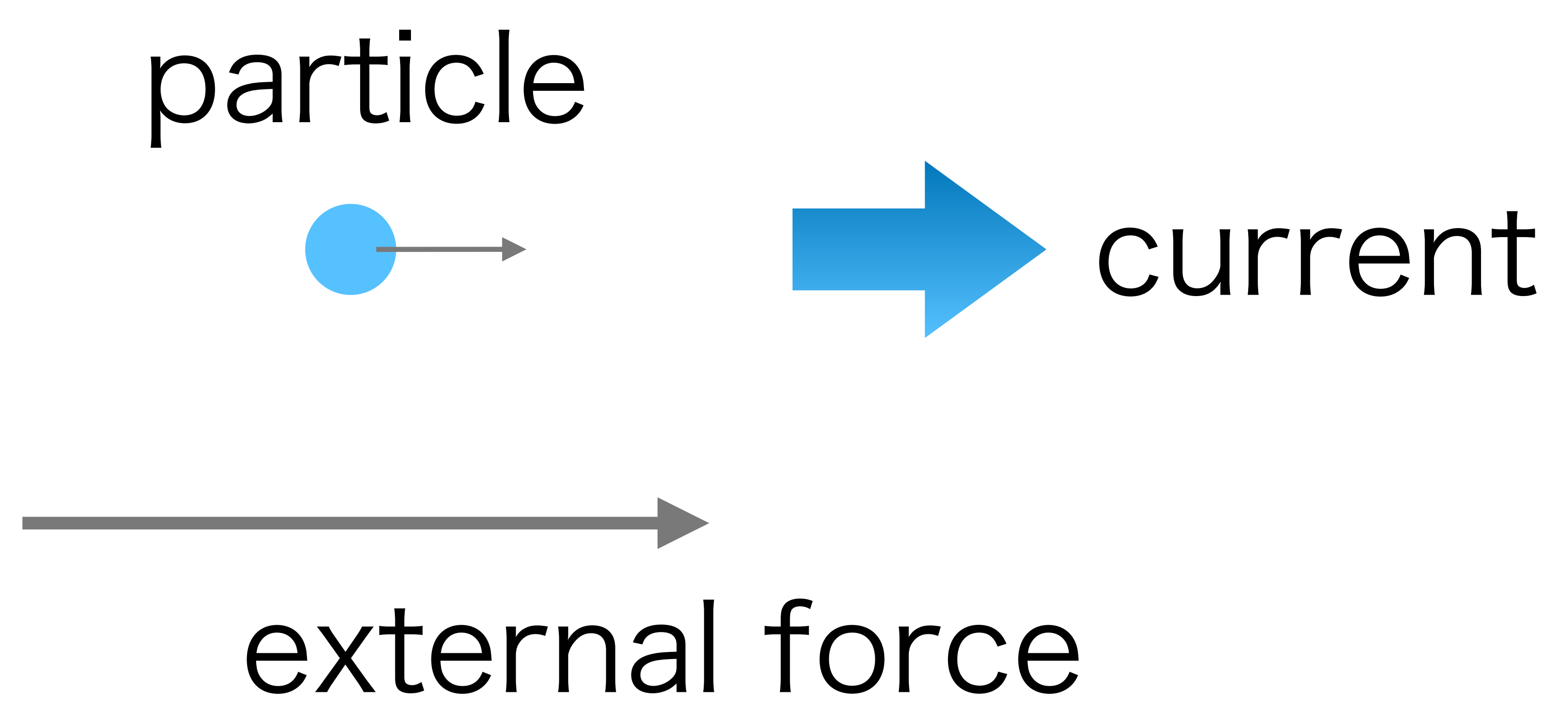
Classical Transport Phenomena

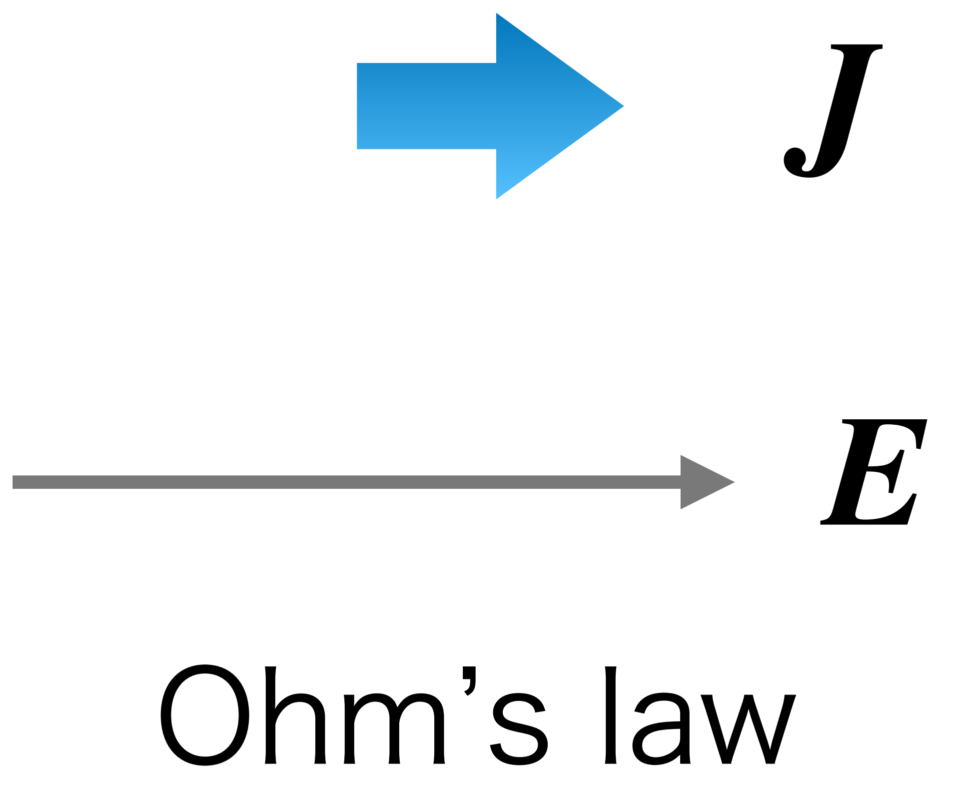


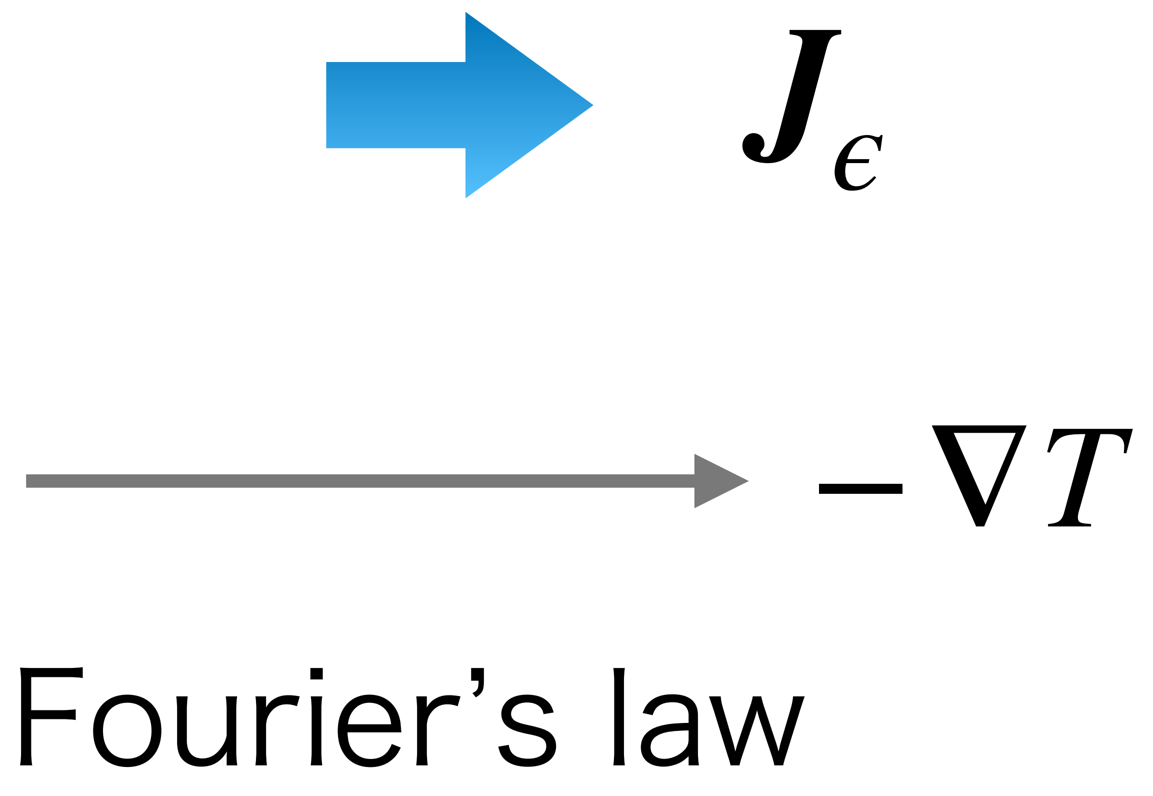




Classical Transport Phenomena







Quantum Transport Phenomena

$\boldsymbol{E}$ → $\boldsymbol{J}$ anomalous Hall effect

Karplus, Luttinger (1954)

$\boldsymbol{J}$

$\boldsymbol{B}$

chiral magnetic effect

Vilenkin (1980)

Fukushima, Kharzeev, Warringa (2008)

$\boldsymbol{J}$

$\boldsymbol{\omega}$

chiral vortical effect

Vilenkin (1979), Son, Surowka (2009)

Landsteiner, Megias, Pena-Benitez (2011)

kinetically described by **Berry curvature** Berry (1984)

$$\boldsymbol{b}(\boldsymbol{p}) = \boldsymbol{\nabla}_{\boldsymbol{p}} \times \boldsymbol{a} \qquad \boldsymbol{a}(\boldsymbol{p}) = -\mathrm{i}\langle u_{\boldsymbol{p}} | \boldsymbol{\nabla}_{\boldsymbol{p}} u_{\boldsymbol{p}} \rangle$$

Berry Curvature Corrected Kinetic Theory

Xiao, Shi, Niu (2005)

$$(1 + \hbar \boldsymbol{B} \cdot \boldsymbol{b})\partial_t f + [\boldsymbol{v} + \hbar \boldsymbol{E} \times \boldsymbol{b} + \hbar\, (\boldsymbol{v} \cdot \boldsymbol{b})\, \boldsymbol{B}] \cdot \boldsymbol{\nabla} f$$

$$+ [\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} + \hbar(\boldsymbol{E} \cdot \boldsymbol{B})\boldsymbol{b}] \cdot \boldsymbol{\nabla}_{\boldsymbol{p}} f = (1 + \hbar \boldsymbol{B} \cdot \boldsymbol{b}) c\,[f]$$

universal formulation up to Berry curvature (BC)

e.g. chiral kinetic theory

Son, Yamamoto (2012) Stephanov, Yin (2012)
Chen, Pu, Wang, Wang (2012)

$$\boldsymbol{b} = \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}$$

Modified Phase Space

Boltzmann kinetic theory

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

solution of Boltzmann eq.

Modified Phase Space

BC corrected kinetic theory (without anomaly)

$$(n, \boldsymbol{j}) \neq \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} \neq 0$$

Modified Phase Space

BC corrected kinetic theory (without anomaly)

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

$$= 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B}$$

Modified Phase Space

BC corrected kinetic theory (without anomaly)

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

$$= 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B}$$

modified phase space measure $\sqrt{\omega}\, \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{p}$

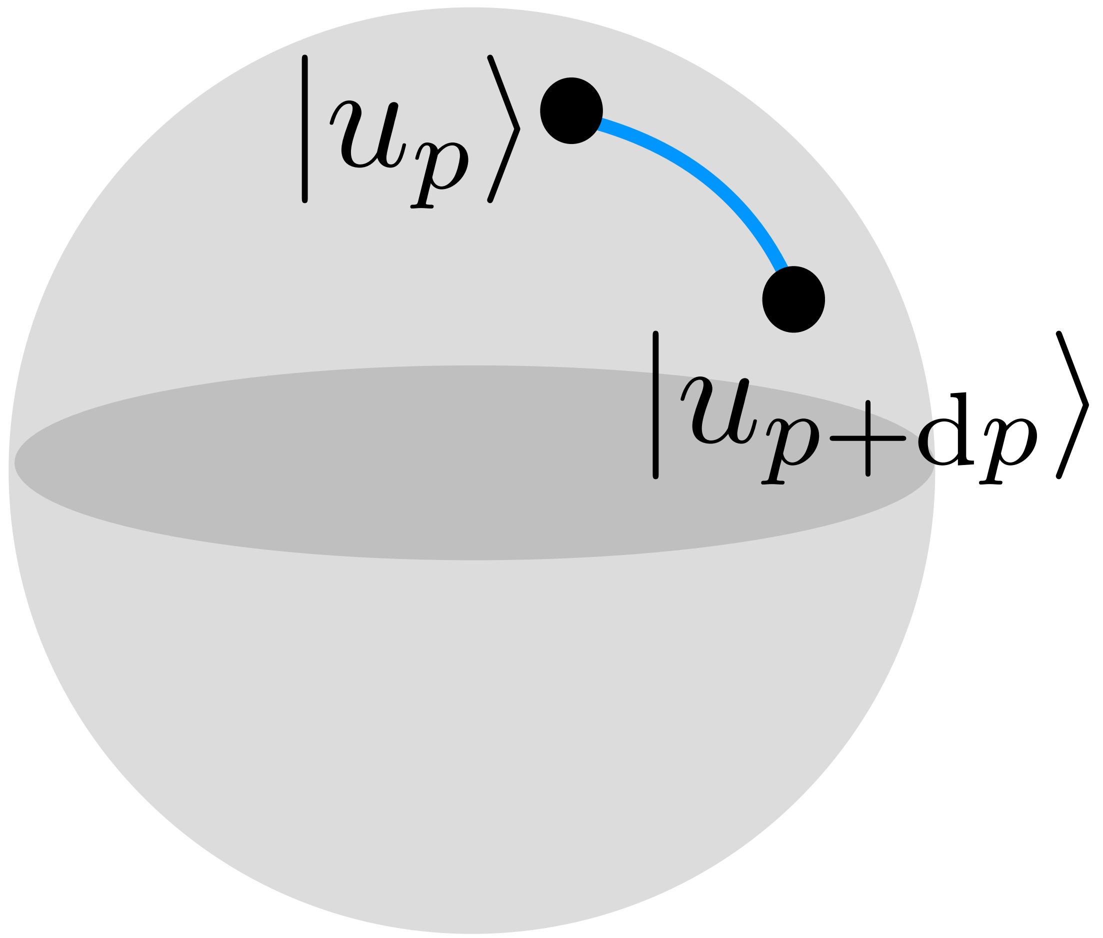
Xiao, Shi, Niu (2005)
Duval, Horvath, Horvathy, Martina, Stichel (2005)

⇔ The Poisson bracket forms a nontrivial algebra

$$\omega = \det \begin{pmatrix} \{p_i, p_j\} & \{p_i, x_j\} \\ \{x_i, p_j\} & \{x_i, x_j\} \end{pmatrix}^{-1}$$

Is this a special structure for the BC effect?

Another Quantum-Geometric Quantity



"distance" between nearby states

$$\mathrm{d}s^2 = 1 - |\langle u_p | u_{p+\mathrm{d}p} \rangle|^2 \approx g_{ij}\mathrm{d}p_i \mathrm{d}p_j$$

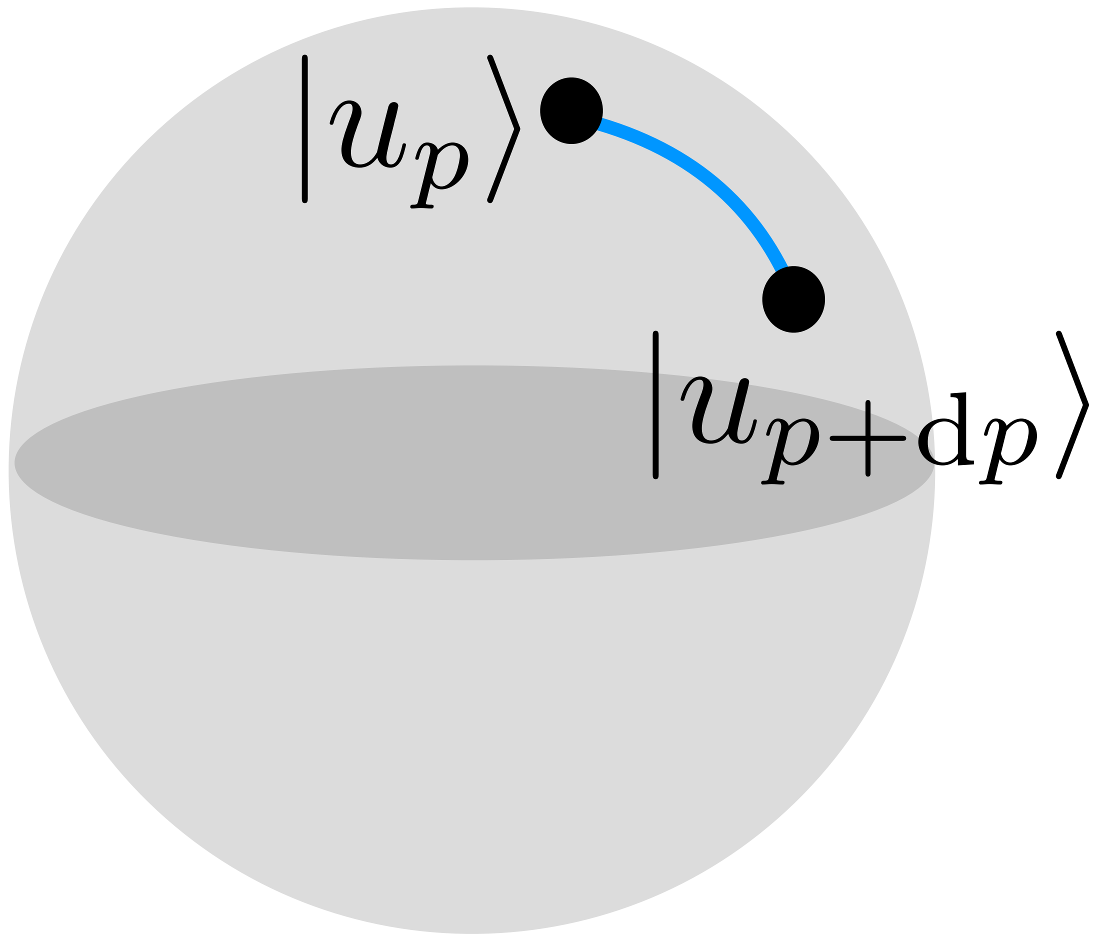
quantum metric

Provost, Vallee (1980)

many recent reviews of
"quantum geometry"

Liu, Qiang, Lu, Xie (2024) Gao, Nagaosa, Ni, Xu (2025)
Verma, Moll, Holder, Queiroz (2025) Jiang, Holder, Yan (2025)
Yu, Bernevig, Queiroz, Rossi, Törmä, Yang (2025)

Another Quantum-Geometric Quantity



"distance" between nearby states

$$\mathrm{d}s^2 = 1 - |\langle u_p|u_{p+\mathrm{d}p}\rangle|^2 \approx g_{ij}\mathrm{d}p_i\mathrm{d}p_j$$

quantum metric

Provost, Vallee (1980)

ex.) nonlinear anomalous Hall effect (nondissipative current)

Gao, Yang, Niu (2014)

$$j_i = \hbar^2 E_j E_k \int_{\boldsymbol{p}} (\partial_i^p G_{jk} - \partial_j^p G_{ik}) f \qquad G_{ij} = g_{ij}/\epsilon_{\boldsymbol{p}}$$

The Foundation Is Missing

	Berry curvature	Quantum metric
transport phenomena	AHE/CME/CVE etc.	nonlinear AHE Gao, Yang, Niu (2014)
definition of currents	$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f$ $\sqrt{\omega} = 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B}$	?
conservation (up to anomaly)	$\partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$	?

It is hard to extract QM from the Wigner function

Berry Curvature Corrected Kinetic Theory

Dynamical System with Berry Connection

Sundaram, Niu (1999)   Stephanov, Yin (2012)

$$L = \boldsymbol{p} \cdot \dot{\boldsymbol{x}} + \boldsymbol{A} \cdot \dot{\boldsymbol{x}} - \hbar \boldsymbol{a} \cdot \dot{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}} - \Phi \qquad \xi^a = (p_i, x_i)$$

universal form up to $\boldsymbol{a}$ (not restricted in CKT)

Dynamical System with Berry Connection

Sundaram, Niu (1999)   Stephanov, Yin (2012)

$$L = \underbrace{\boldsymbol{p}\cdot\dot{\boldsymbol{x}} + \boldsymbol{A}\cdot\dot{\boldsymbol{x}} - \hbar\boldsymbol{a}\cdot\dot{\boldsymbol{p}}}_{\pi_a(\xi)\dot{\xi}^a} \underbrace{- \epsilon_{\boldsymbol{p}} - \Phi}_{-H} \qquad \xi^a = (p_i, x_i)$$

canonical eq. $\dot{\xi}^a = \{H, \xi^a\} = -\{\xi^a, \xi^b\}\dfrac{\partial H}{\partial \xi^b}$

from the matching with EL eq. $\delta \int L\,\mathrm{d}t = 0$

Son, Yamamoto (2012)

Berry Curvature Corrections

Modified Poisson bracket   Duval, Horvath, Horvathy, Martina, Stichel (2005)

$$\{p_i, p_j\} = -(1 - \hbar \boldsymbol{b} \cdot \boldsymbol{B})\epsilon_{ijk} B_k$$

$$\{x_i, x_j\} = \hbar \epsilon_{ijk} b_k$$

$$\{p_i, x_j\} = (1 - \hbar \boldsymbol{b} \cdot \boldsymbol{B})\delta_{ij} + \hbar b_i B_j$$

Phase space factor

$$\sqrt{\omega} := \det\{\xi^a, \xi^b\}^{-1} = 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B}$$

Xiao, Shi, Niu (2005)

Quantum Metric Corrected Kinetic Theory

Dynamical System with Quantum Metric

w/o mag. field

Ren (2025) Yoshida, Yokoyama (2025)

$$L = \boldsymbol{p} \cdot \dot{\boldsymbol{x}} - \hbar \boldsymbol{a} \cdot \dot{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}} - \Phi - \frac{\hbar^2}{2} \dot{p}_i \dot{p}_j G_{ij}$$

universal form up to $\boldsymbol{a}$ and $G_{ij} = g_{ij} / \epsilon_{\boldsymbol{p}}$

KM, Yamamoto (2025)

We have derived it from the extension of the path integral formalism

Stephanov, Yin (2012)

Dynamical System with Quantum Metric

Ren (2025) Yoshida, Yokoyama (2025)

w/o mag. field

$$L = \underbrace{\boldsymbol{p} \cdot \dot{\boldsymbol{x}} - \hbar \boldsymbol{a} \cdot \dot{\boldsymbol{p}}}_{\pi_a(\xi)\dot{\xi}^a} \underbrace{- \epsilon_{\boldsymbol{p}} - \Phi}_{-H} - \frac{\hbar^2}{2} \dot{p}_i \dot{p}_j G_{ij}$$

canonical eq. $\dot{\xi}^a = \{H, \xi^a\} = -\{\xi^a, \xi^b\} \frac{\partial H}{\partial \xi^b}$

from the matching with EL eq. $\delta \int L \, \mathrm{d}t = 0$

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Dynamical System with Quantum Metric

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$$L = \boldsymbol{p} \cdot \dot{\boldsymbol{x}} - \hbar \boldsymbol{a} \cdot \dot{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}} - \Phi - \frac{\hbar^2}{2} \dot{p}_i \dot{p}_j G_{ij}$$

$\pi_a(\xi)\dot{\xi}^a$ $-H$ different form $\sim \dot{\xi}^a \dot{\xi}^b$

canonical eq. $\dot{\xi}^a = \{H, \xi^a\} = -\{\xi^a, \xi^b\} \dfrac{\partial H}{\partial \xi^b}$

from the matching with EL eq. $\delta \int L \, \mathrm{d}t = 0$

Son, Yamamoto (2012)

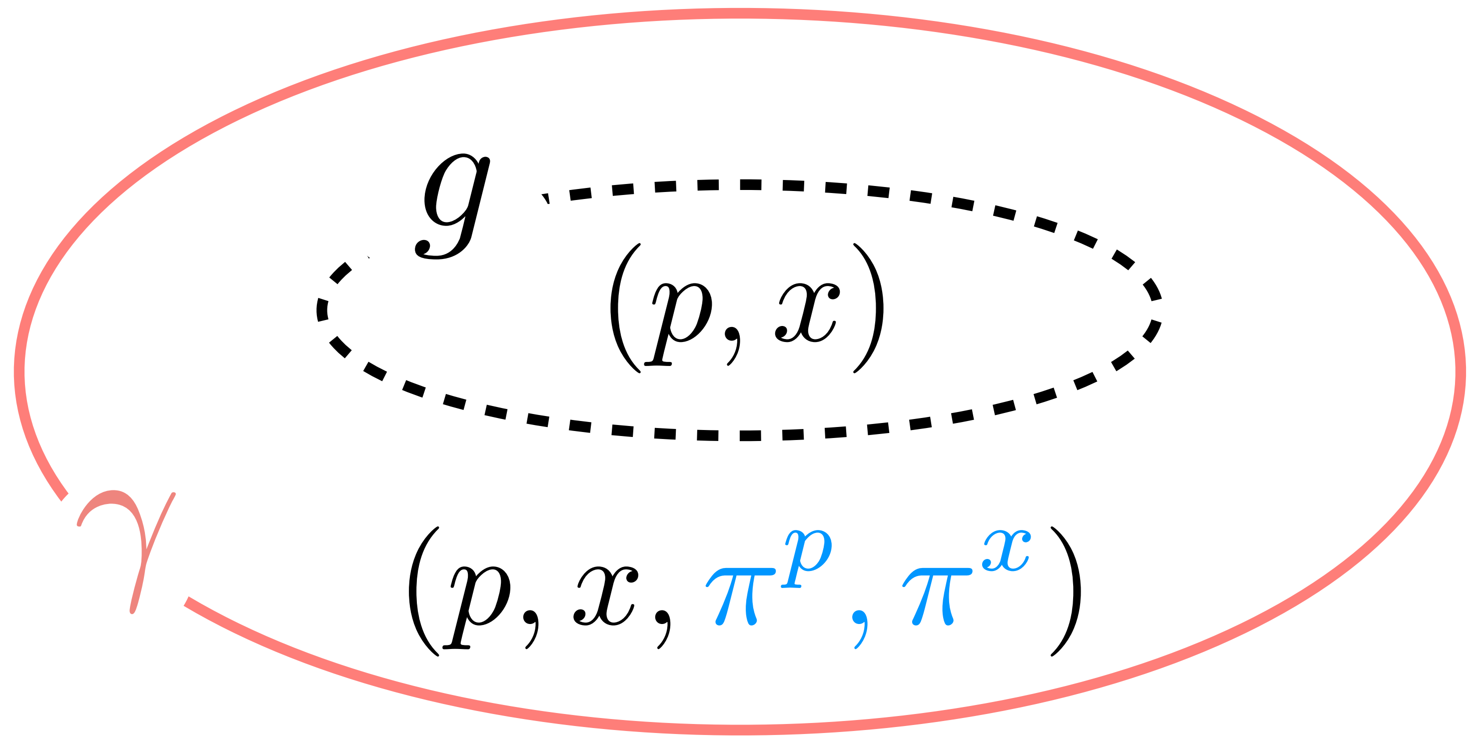
Unusual Dynamical System

Lagrangian $$L = \boldsymbol{p} \cdot \dot{\boldsymbol{x}} - \hbar \boldsymbol{a} \cdot \dot{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}} - \Phi - \frac{\hbar^2}{2} \dot{p}_i \dot{p}_j G_{ij}$$

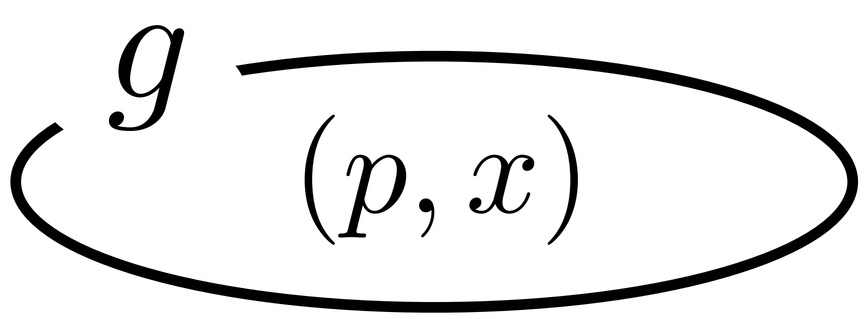
$$= L(p, x, \dot{p}, \dot{x})$$

Unusual Dynamical System

Hamiltonian $H = H(p, x, \pi^p, \pi^x)$



$\neq$

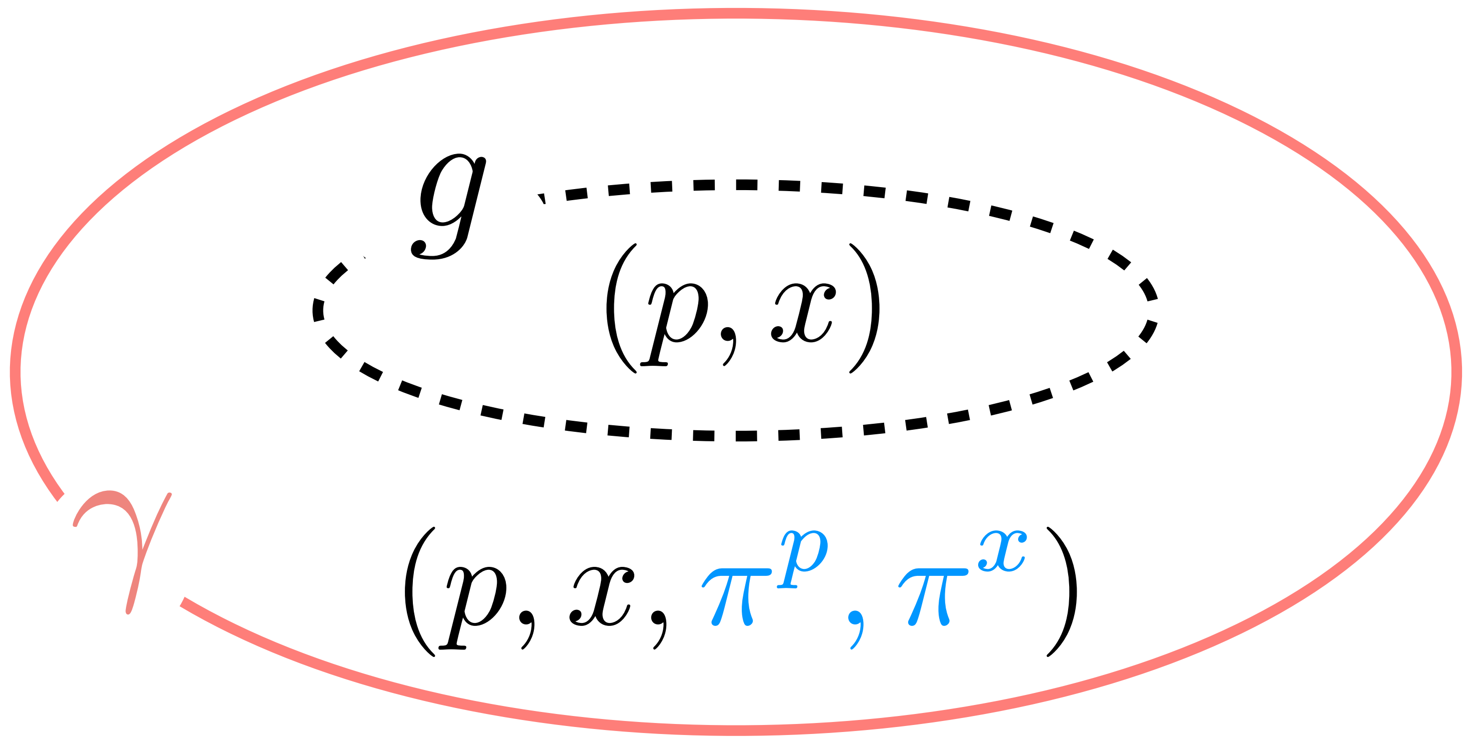


Whole phase space of H

Kinetic phase space

Unusual Dynamical System

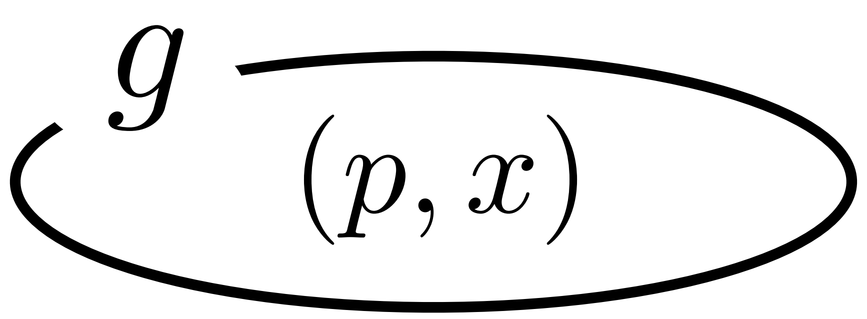
Hamiltonian $H = H(p, x, \pi^p, \pi^x)$



+ 6 constraints

$\phi_n = \phi_n(p, x, \pi^p, \pi^x)$

=



Dirac (1950) Bergmann (1949)

We need **canonical formalism in constrained systems**

using Dirac bracket

$$\{X, Y\}_{\mathrm{D}} = \{X, Y\} - \{X, \phi_n\}\{\phi_n, \phi_m\}^{-1}\{\phi_m, Y\}$$

Quantum Metric Corrections

KM, Yamamoto (2025)

Modified Poisson bracket (w/o mag. field)

$$\{p_i, p_j\}_{\mathrm{D}} = 0$$

$$\{x_i, x_j\}_{\mathrm{D}} = \hbar\epsilon_{ijk}b_k - \hbar^2(\partial_i^p G_{jk} - \partial_j^p G_{ik})E_k$$

$$\{p_i, x_j\}_{\mathrm{D}} = \delta_{ij} - \hbar^2 G_{ik}\partial_j E_k$$

Phase space factor

$$\sqrt{\omega} = \det\{\xi^a, \xi^b\}_{\mathrm{D}}^{-1} = 1 + \hbar^2 G_{ij}\partial_i E_j$$

QM Corrected Kinetic Equation

Modified Liouville's theorem

KM, Yamamoto (2025)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \partial_t \rho + \{H, \rho\}_{\mathrm{D}} = 0 \qquad \rho = \sqrt{\omega} f$$

w/o mag. field and collisions

$$\left[\partial_t + \left(\boldsymbol{v}' + \hbar \boldsymbol{E} \times \boldsymbol{b}' + \hbar^2 \boldsymbol{V}\right) \cdot \boldsymbol{\nabla} + \boldsymbol{E} \cdot \boldsymbol{\nabla}_{\boldsymbol{p}}\right] f = 0$$

$$\boldsymbol{v}' = \boldsymbol{\nabla}_{\boldsymbol{p}} \left(\epsilon_{\boldsymbol{p}} - \frac{\hbar^2}{2} G_{ij} E_i E_j\right) \qquad b_i' = \epsilon_{ijk} \partial_j^p (a_k + \hbar G_{kl} E_l) \qquad V_k = -G_{kj} v_i \partial_i E_j$$

valid for chiral fermions: we have derived the $\mathcal{O}(\hbar^2)$ chiral kinetic eq.

Conserved Charge/Energy Currents

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f$$

$$(\varepsilon, \boldsymbol{J}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \left[\epsilon_{\boldsymbol{p}} - \frac{\hbar^2}{2} E_i E_j G_{ij} \right] \sqrt{\omega} f$$

$$\partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

$$\partial_t \varepsilon + \boldsymbol{\nabla} \cdot \boldsymbol{J} = \boldsymbol{E} \cdot \boldsymbol{j}$$

ex.) nonlinear AHE is correctly reproduced

$$j_i = \hbar^2 E_j E_k \int_{\boldsymbol{p}} (\partial_i^p G_{jk} - \partial_j^p G_{ik}) f$$

Gao, Yang, Niu (2014)

First derivation meeting the conservation law

→ resolution of the tensions between many literatures

Application to Chiral/Dirac Fermions

chiral fermion : $G_{ij} = \frac{1}{4|\boldsymbol{p}|^3}(\delta_{ij} - \hat{p}_i\hat{p}_j)$

- equilibrium current: consistent with QFT

Yang, Gao, Liang, Wang (2020)
Mameda (2023) Yang, Gao, Pu (2025)

$$n^{(EE)} = -\frac{\hbar^2 C}{24\pi^2}|\boldsymbol{E}|^2 \qquad \boldsymbol{j}^{(EE)} = \frac{\hbar^2 C}{12\pi^2}\left[-\boldsymbol{E}(\boldsymbol{E}\cdot\boldsymbol{u}) + |\boldsymbol{E}|^2\boldsymbol{u}\right] \qquad \varepsilon^{(\partial E)} = \frac{\hbar^2 \mu}{12\pi^2}\boldsymbol{\nabla}\cdot\boldsymbol{E}$$

- first application of QM to high-energy physics

- Application to Dirac (massive) fermions is straightforward

  QM does not require chirality imbalance, unlike Berry curvature

Summary

- Derived quantum metric (QM) correction to quantum kinetic theory
- QM is essential not only for nondissipative transport but also the conservation law through an anomalous algebra
- The first finding of the QM corrections in high-energy physics
  QM effect on EoSs
- The interplay between QM and magnetic field is also an important direction, which is related to the Lorentz invariance problem

Hayata, Hidaka, Mameda (2020)
Mameda (2023)