

Non-equilibrium corrections to the Wigner function at freeze-out

F.B., D. Roselli, X.L. Sheng arXiv:2512.24994

Deeply revised version to appear including Z. H. Zhang as co-author

CONTENT

An ambitious attempt to calculate non-equilibrium corrections with an *ab initio* calculation in a statistical QFT framework

RESULT

A critical revisitation of Kubo formulae: appearance of memory terms

Motivations and outline

Search for a first-principle derivation of dissipative corrections to the spin polarization of particles emitted in relativistic heavy ion physics. Especially needed for spin alignment.

RECENT STUDIES:

D. Wagner, N. Weickgenannt and E. Speranza, Phys. Rev. Res. 5 (2023) no.1, 013187 **Relativistic kinetic theory**

Y. L. Yin, W. B. Dong, J. Y. Pang, S. Pu and Q. Wang, Phys. Rev. C 110 (2024) no.2, 024905

S. Z. Yang, X. Q. Xie, S. Pu, J. H. Gao and Q. Wang, Phys. Rev. D 112 (2025) no.9, 094040

M. Buzzegoli, JHEP 07 (2025), 255

Y. Li, S.Y.F. Liu, Phys. Rev. C 113 (2026), 034911

Statistical QFT

J. R. Wang, S. Fang, D. L. Yang and S. Pu, *Is the shear induced spin polarization non-dissipative?* [arXiv:2507.15238 [hep-ph]].

PROGRAMME: An *ab initio* calculation within a statistical QFT framework

EXECUTION: Spin fields too complicated. Scalar field first.

OUTPUT: Surprising conclusions

Rigorous formulation within QFT

FREE SCALAR FIELD

$$\hat{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\varepsilon_p} (\hat{a}(p)e^{-ip \cdot x} + \hat{b}^\dagger(p)e^{ip \cdot x})$$

$$\widehat{W}(x, k) = \frac{2}{(2\pi)^4} \int d^4s e^{-is \cdot k} : \hat{\psi}^\dagger\left(x + \frac{s}{2}\right) \hat{\psi}\left(x - \frac{s}{2}\right) :$$

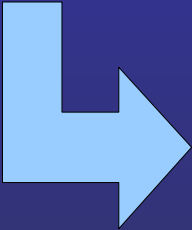
$$W(x, k) \equiv \text{Tr} \left[\hat{\rho} \widehat{W}(x, k) \right] ,$$

$$\widehat{W}(x, k) \equiv \widehat{W}^+(x, k) + \widehat{W}^-(x, k) + \widehat{W}_S(x, k) ,$$

$$\widehat{W}^\pm(x, k) \equiv \widehat{W}(x, k) \theta(k^2) \theta(\pm k^0) , \quad \widehat{W}_S \equiv \widehat{W}(x, k) \theta(-k^2)$$

Particle,
antiparticle and
mixed
components

SPECTRUM


$$\frac{dN_k}{d^3\mathbf{k}} = \int dk^0 \int_\Sigma d\Sigma_\mu k^\mu W^+(x, k) = \frac{1}{2\varepsilon_{\mathbf{k}}} \langle \hat{a}^\dagger(k) \hat{a}(k) \rangle ,$$

Rigorous formulation within QFT (2)

In a scattering experiment the free out-fields are (weak) limits of interacting fields

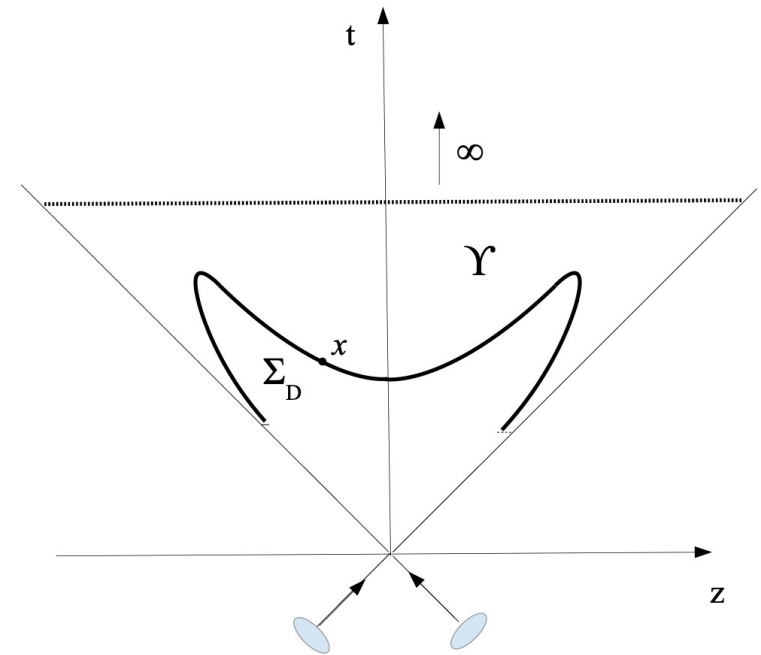
Yang-Feldman equation connecting interacting and out-field

$$\hat{\phi}(x) = \hat{\phi}_{\text{out}}(x) - \int d^4y \Delta_{\text{adv}}(x-y) \hat{J}(y),$$

$$\lim_{t \rightarrow +\infty} \hat{\phi}(x) = \hat{\phi}_{\text{out}}(x). \quad (\square + m^2) \hat{\phi}(x) = \hat{J}(x).$$

Wigner function with interacting field

$$\widehat{W}(x, k) = \frac{2}{(2\pi)^4} \int d^4s e^{-is \cdot k} : \hat{\phi}^\dagger \left(x + \frac{s}{2} \right) \hat{\phi} \left(x - \frac{s}{2} \right) :$$



$$\frac{dN_k}{d^3k} = \lim_{t \rightarrow +\infty} \int dk^0 \int_{\Sigma} d\Sigma_{\mu} k^{\mu} W^+(x, k) = \frac{1}{2\varepsilon} \langle \hat{a}_{\text{out}}^\dagger(k) \hat{a}_{\text{out}}(k) \rangle$$

By using Gauss theorem

$$\frac{dN_k}{d^3k} = \int dk^0 \int_{\Sigma_D} d\Sigma_{\mu} k^{\mu} W^+(x, k) + \int dk^0 \int_{\Gamma} d^4x k^{\mu} \partial_{\mu} W^+(x, k),$$

Fluid-decoupling hypersurface

Residual interaction (RKT)

Rigorous formulation within QFT (3)

General field expansion (formal)

$$\widehat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{p} \int_{-|\mathbf{p}|^2}^{+\infty} dM^2 \frac{1}{2\sqrt{M^2 + |\mathbf{p}|^2}} \frac{\varrho(p)}{2\pi} \left(e^{-ip \cdot x} \widehat{A}(M^2, \mathbf{p}) + e^{ip \cdot x} \widehat{B}^\dagger(M^2, \mathbf{p}) \right),$$

$$\widehat{A}(p) \equiv \frac{1}{(2\pi)^{3/2} \varrho(p)} \theta(p^0) \widehat{\phi}_F(p), \quad \widehat{B}^\dagger(p) \equiv \frac{1}{(2\pi)^{3/2} \varrho(p)} \theta(p^0) \widehat{\phi}_F(-p),$$

Spectral function in some state

$$\varrho(p) \equiv \int d^4x e^{ip \cdot x} \langle [\widehat{\phi}(x), \widehat{\phi}^\dagger(0)] \rangle.$$

$$\varrho_{\text{free}}(p) = 2\pi \text{sign}(p^0) \delta(p^2 - m^2) = 2\pi \text{sign}(p^0) \delta(M^2 - m^2).$$

BEWARE! A and B do not fulfill free field commutation relations!

Wigner operator (particle part)

$$\begin{aligned} \widehat{W}^+(x, k) : &= \frac{2}{(2\pi)^5} \int d^4q e^{ix \cdot q} \left\{ \varrho(k_+) \varrho(k_-) \widehat{A}^\dagger(k_+) \widehat{A}(k_-) \theta(k_+^0) \theta(k_-^0) \right. \\ &\quad \left. + \varrho(k_+) \varrho(-k_-) \left[\widehat{A}^\dagger(k_+) \widehat{B}^\dagger(-k_-) + \text{h.c.} \right] \theta(k_+^0) \theta(-k_-^0) \right\}. \end{aligned}$$

$$k_\pm = k \pm q/2$$

Density operator

Quantum state (assumption): local equilibrium at the initial 3D Cauchy hypersurface

$$\hat{\rho} = \hat{\rho}_{\text{LE}}(\tau_0) = \frac{1}{Z} \exp \left[- \int_{\Sigma_0} d\Sigma_\mu(y) \left(\hat{T}^{\mu\nu}(y) \beta_\nu(y) - \hat{j}^\mu(y) \zeta(y) \right) \right],$$

In traditional calculations, the density operator is rewritten using the Gauss theorem:

$$\hat{\rho} = \frac{1}{Z} \exp \left[- \int_{\Sigma_D} d\Sigma_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu \right) + \int_{\Omega} d^4y \left(\hat{T}^{\mu\nu}(y) \partial_\mu \beta_\nu(y) - \partial_\mu \zeta(y) \hat{j}^\mu(y) \right) \right]$$

local equilibrium

dissipation

Reworking the Zubarev's approach to non-equilibrium quantum statistical mechanics

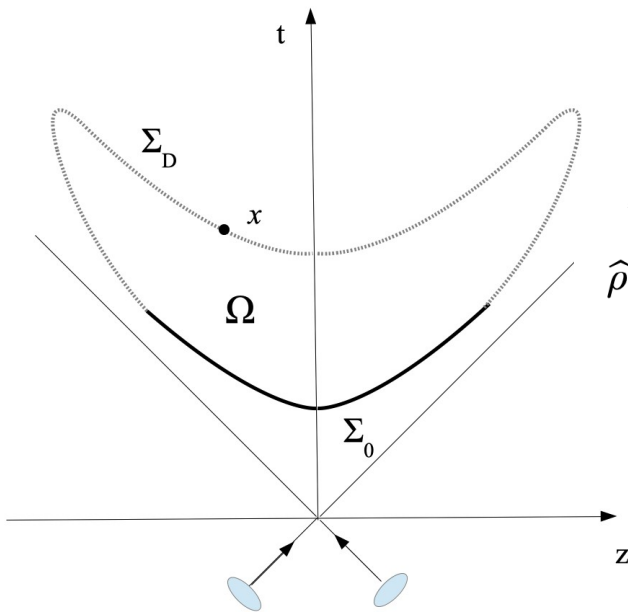
F. Becattini (Florence U. and INFN, Florence), M. Buzzegoli (Florence U. and INFN, Florence), E. Grossi (Heidelberg U.) (Feb 4, 2019)

Published in: *Particles* 2 (2019) 2, 197-207 • e-Print: [1902.01089](https://arxiv.org/abs/1902.01089) [cond-mat.stat-mech]

$$\langle \hat{O}(x) \rangle = \text{Tr}(\hat{\rho} \hat{O}(x)) = \text{Tr}(\hat{\rho}_{\text{LE}} \hat{O}(x)) + \Delta O(x) = \langle \hat{O}(x) \rangle_{\text{LE}} + \Delta O(x)$$

Local equilibrium part

Dissipative part



A new method to calculate the off-equilibrium corrections

X. L. Sheng, F. B. and D. Roselli, arXiv:2509.14301

$$\hat{\rho} = \hat{\rho}_{\text{LE}}(\tau_0) = \frac{1}{Z} \exp \left[- \int_{\Sigma_0} d\Sigma_\mu(y) \left(\hat{T}^{\mu\nu}(y) \beta_\nu(y) - \hat{j}^\mu(y) \zeta(y) \right) \right],$$

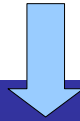
Rewrite:

$$\beta_\nu(y) = \beta_\nu(x) + (\beta_\nu(y) - \beta_\nu(x)) = \beta_\nu(x) + \Delta\beta_\nu(y, x)$$

$$\zeta(y) = \zeta(x) + (\zeta(y) - \zeta(x)) = \zeta(x) + \Delta\zeta(y, x),$$



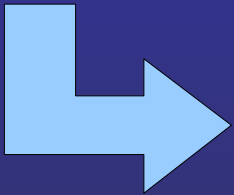
$$\hat{\rho} = \frac{1}{Z} \exp \left[-\beta(x) \cdot P + \zeta(x) \hat{Q} - \int_{\Sigma_0} d\Sigma_\mu(y) \left(\hat{T}^{\mu\nu}(y) \Delta\beta_\nu(y, x) - \hat{j}^\mu(y) \Delta\zeta(y, x) \right) \right],$$



Main term: global equilibrium at $\beta(x), \zeta(x)$



Perturbation: $\Delta\beta, \Delta\zeta$, assumed to be small enough



$$\langle \hat{O}(x) \rangle \simeq \langle \hat{O}(x) \rangle_{\text{GE}} + \Delta O(x),$$

$$\hat{\rho}_{\text{GE}} = \frac{1}{Z_{\text{GE}}} \exp \left[-\beta(x) \cdot \hat{P} + \zeta(x) \hat{Q} \right]$$

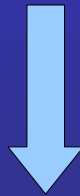
Linear response

By expanding the density operator from GE, we can calculate the non-equilibrium correction of local operators at the linear order

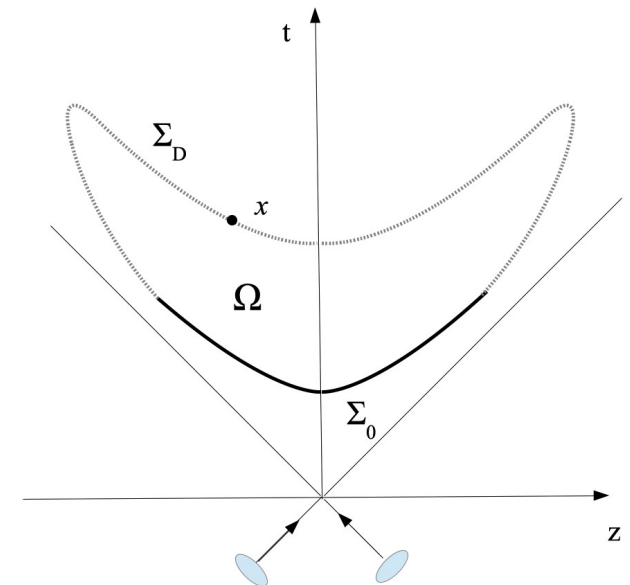
$$\Delta O(x) = - \int_{\Sigma_0} d\Sigma_\mu(y) \int_0^1 dz \langle \hat{O}(x), e^{z\hat{\mathcal{E}}_{\text{GE}}} \left(\hat{T}^{\mu\nu}(y) \Delta\beta_\nu(y, x) - \hat{j}^\mu \Delta\zeta(y, x) \right) e^{-z\hat{\mathcal{E}}_{\text{GE}}} \rangle_{c, \text{GE}} ,$$

This expression is totally equivalent to the one obtained by separating local equilibrium and dissipative terms

$$\begin{aligned} \Delta O(x) = & - \int_{\Sigma} d\Sigma_\mu(y) \int_0^1 dz \langle \hat{O}(x), e^{z\hat{\mathcal{E}}_{\text{GE}}} \left(\hat{T}^{\mu\nu}(y) \Delta\beta_\nu(y, x) - \hat{j}^\mu \Delta\zeta(y, x) \right) e^{-z\hat{\mathcal{E}}_{\text{GE}}} \rangle_{c, \text{GE}} \\ & + \int_{\Omega} d^4y \int_0^1 dz \langle \hat{O}(x), e^{z\hat{\mathcal{E}}_{\text{GE}}} \left(\hat{T}^{\mu\nu}(y) \partial_\mu \beta_\nu(y) - \hat{j}^\mu \partial_\mu \zeta(y) \right) e^{-z\hat{\mathcal{E}}_{\text{GE}}} \rangle_{c, \text{GE}} . \end{aligned}$$



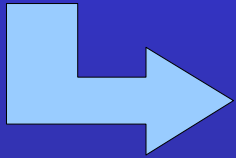
Linear response commutes with Gauss theorem



The Wigner function: leading term

$$\langle \widehat{W}^+(x, k) \rangle \simeq \langle \widehat{W}^+(x) \rangle_{\text{GE}} + \Delta W^+(x, k) ,$$

$$\langle \widehat{W}^+(x, k) \rangle_{\text{GE}} = \frac{2}{(2\pi)^4} n_B(\beta(x) \cdot k) \varrho(k) ,$$

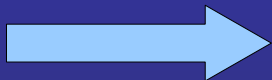


$$\frac{dN_k}{d^3k} = \frac{2}{(2\pi)^4} \int_0^{+\infty} dk^0 \varrho(k) \int_{\Sigma_D} d\Sigma \cdot k n_B(\beta(x) \cdot k) ,$$

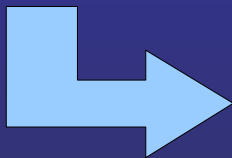
Cooper-Frye formula

NOTE: because

$$e^{iy \cdot \widehat{P}} \widehat{\phi}(x) e^{-iy \cdot \widehat{P}} = \widehat{\phi}(x + y) ,$$



$$e^{iy \cdot \widehat{P}} \widehat{A}(p) e^{-iy \cdot \widehat{P}} = e^{-ip \cdot y} \widehat{A}(p) , \quad e^{iy \cdot \widehat{P}} \widehat{B}^\dagger(p) e^{-iy \cdot \widehat{P}} = e^{ip \cdot y} \widehat{B}^\dagger(p) ,$$



$$\langle \widehat{A}^\dagger(p) \widehat{A}(p') \rangle_{\text{GE}} = \frac{2\pi}{\varrho^2(p)} \theta(p^0) \theta(p'^0) \delta^4(p - p') n_B(p) \varrho_{\text{GE}}(p)$$

$$\langle \widehat{B}(p) \widehat{A}(p') \rangle_{\text{GE}} = \langle \widehat{A}^\dagger(p') \widehat{B}^\dagger(p) \rangle_{\text{GE}}^* = 0 .$$

The Wigner function: leading non-equilibrium correction

$$\langle \widehat{W}^+(x, k) \rangle \simeq \langle \widehat{W}^+(x) \rangle_{\text{GE}} + \Delta W^+(x, k) ,$$

$$\begin{aligned} \Delta W^+(x, k) = & \frac{2}{(2\pi)^5} \int_{\Sigma_0} d\Sigma_\mu(y) \int d^4q \left\{ \varrho(k_+) \varrho(k_-) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} e^{iq \cdot (x-y)} \theta(k_+^0) \theta(k_-^0) \right. \\ & \times \left(\underbrace{\langle \widehat{A}^\dagger(k_+) \widehat{A}(k_-), \widehat{T}^{\mu\nu}(0) \rangle_{c, \text{GE}}}_{\substack{\text{red line} \\ \text{red bracket}}} \Delta\beta_\nu(y, x) - \underbrace{\langle \widehat{A}^\dagger(k_+) \widehat{A}(k_-), \widehat{j}^\mu(0) \rangle_{c, \text{GE}}}_{\substack{\text{red line} \\ \text{red bracket}}} \Delta\zeta(y, x) \right) \\ & + \varrho(k_+) \varrho(-k_-) \theta(k_+^0) \theta(-k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \\ & \times \left. \left(e^{iq \cdot (x-y)} \langle \widehat{A}^\dagger(k_+) \widehat{B}^\dagger(-k_-), \widehat{T}^{\mu\nu}(0) \rangle_{c, \text{GE}} \Delta\beta_\nu(y, x) - \langle \widehat{A}^\dagger(k_+) \widehat{B}^\dagger(-k_-), \widehat{j}^\mu(0) \rangle_{c, \text{GE}} \Delta\zeta(y, x) + \text{c.c.} \right) \right\} \end{aligned}$$

Unknown correlators: they play a crucial role

We can write the most general expression by using available vectors k, q, β

Correlators (with stress-energy tensor)

$$\Theta^{\mu\nu}(k, q, \beta) \equiv \varrho(k) \langle \hat{A}^\dagger(k_+) \hat{A}(k_-), \hat{T}^{\mu\nu}(0) \rangle_{c, \text{GE}},$$

Constraints from energy-momentum conservation:

$$\Theta^{0\nu}(k, q^0, \mathbf{q} = 0, \beta) = -\frac{\theta(k^0)}{(2\pi)^2 \varrho(k)} \delta(q^0) \left[\frac{\partial}{\partial \beta_\nu(x)} (n_B(k) \varrho(k)) \right],$$

$$q_\mu \Theta^{\mu\nu}(k, q, \beta) = 0, \quad \forall k, q, \beta.$$

Ward identity

Most general form:

$$\Theta^{\mu\nu}(k, q, \beta) = \int_{-\pi/2}^{\pi/2} d\vartheta \sum_f \delta(q \cdot w(\vartheta) - f(S)) \Gamma_{\vartheta, f}^{\mu\nu}(k, q, \beta) + \delta(q^2) \Gamma_{q^2}^{\mu\nu}(k, q, \beta) + \Xi^{\mu\nu}(k, q, \beta).$$

S stands for scalars formed out of vectors k, q, β except linear combinations of kq and $q\beta$

$$w^\mu(\vartheta) = \cos \vartheta \frac{k^\mu}{\sqrt{k^2}} + \sin \vartheta \frac{\beta^\mu}{\sqrt{\beta^2}}$$

Analitycity in $q=0$, delta distributions and Ward identities constrain the form of the correlators:

Example:

$$\begin{aligned} \Gamma_{\vartheta, f}^{\mu\nu}(k, q, \beta) = & \Gamma_{1, f}(\vartheta; S) \left\{ (q \cdot k)(q \cdot \beta) w^\mu(\vartheta) w^\nu(\vartheta) - f(S) \left[(q \cdot \beta) \frac{\cos \vartheta}{\sqrt{k^2}} k^\mu k^\nu + (q \cdot k) \frac{\sin \vartheta}{\sqrt{\beta^2}} \beta^\mu \beta^\nu \right] \right\} \\ & + \Gamma_{2, f}(\vartheta; S) (q^\mu q^\nu - q^2 g^{\mu\nu}) + \Gamma_{3, f}(\vartheta; S) [(q \cdot k) (k^\mu q^\nu + k^\nu q^\mu) - q^2 k^\mu k^\nu - (q \cdot k)^2 g^{\mu\nu}] \\ & + \Gamma_{4, f}(\vartheta; S) [(q \cdot \beta) (q^\mu \beta^\nu + q^\nu \beta^\mu) - q^2 \beta^\mu \beta^\nu - (q \cdot \beta)^2 g^{\mu\nu}], \end{aligned}$$

$\Gamma_i \equiv$ thermal-gravitational form factors

Correlators (2)

Analitycity in $q=0$, delta distributions and Ward identities imply:

$$\Gamma_{\vartheta, f}^{\mu\nu} = \mathcal{O}(q^2) \text{ if } f \neq 0 \quad \Gamma_{q^2}^{\mu\nu} = \mathcal{O}(q^2) \quad \Xi^{\mu\nu} = \mathcal{O}(q^2)$$

Special case if $f(S)=0$

$$\begin{aligned} \Gamma_{\vartheta}^{\mu\nu}(k, q, \beta) = & \Gamma_1(\vartheta; S) w^\mu(\vartheta) w^\nu(\vartheta) + \Gamma_2(\vartheta; S) (q^\mu q^\nu - q^2 g^{\mu\nu}) \\ & + \Gamma_3(\vartheta; S) [(q \cdot k) (k^\mu q^\nu + k^\nu q^\mu) - q^2 k^\mu k^\nu - (q \cdot k)^2 g^{\mu\nu}] \\ & + \Gamma_4(\vartheta; S) [(q \cdot \beta) (q^\mu \beta^\nu + q^\nu \beta^\mu) - q^2 \beta^\mu \beta^\nu - (q \cdot \beta)^2 g^{\mu\nu}] \end{aligned}$$

The only correlator surviving in the $q=0$ limit

Energy-momentum conservation implies:

$$\begin{aligned} \frac{1}{\sqrt{k^2}} \int_{-\pi/2}^{\pi/2} d\vartheta \cos \vartheta \Gamma_1(\vartheta; k, 0, \beta) &= \frac{\theta(k^0)}{(2\pi)^2} n_B(k) \left[1 + n_B(k) - \frac{\partial \log \varrho(k)}{\partial(k \cdot \beta)} \right], \\ \frac{1}{\sqrt{\beta^2}} \int_{-\pi/2}^{\pi/2} d\vartheta \sin \vartheta \Gamma_1(\vartheta; k, 0, \beta) &= -\frac{2\theta(k^0)}{(2\pi)^2} n_B(k) \frac{\partial \log \varrho(k)}{\partial \beta^2}. \end{aligned}$$

Correlation function

Quantifies the correlation between Wigner operator and SET

$$\begin{aligned} \Delta W^+(x, k) \Big|_{WT} &= - \int_{\Sigma_0} d\Sigma_\mu(y) \Delta\beta_\nu(y, x) \int_0^1 dz \langle \widehat{W}^+(x, k), e^{z\widehat{\mathcal{E}}_{GE}} \widehat{T}^{\mu\nu}(y) e^{-z\widehat{\mathcal{E}}_{GE}} \rangle_{c, GE} \\ &\equiv \int_{\Sigma_0} d\Sigma_\mu(y) \Delta\beta_\nu(y, x) C_{WT}^{\mu\nu}(x - y), \end{aligned}$$

Apply Gauss theorem

$$\Delta W^+(x, k) = \int_{\Sigma_D} d\Sigma_\mu(y) \Delta\beta_\nu(y, x) C_{WT}^{\mu\nu}(x - y, k) + \int_{\Omega} d^4y C_{WT}^{\mu\nu}(y - x, k) \frac{\partial}{\partial y^\mu} \Delta\beta_\nu(y, x).$$

Hydrodynamic limit: separation of scales

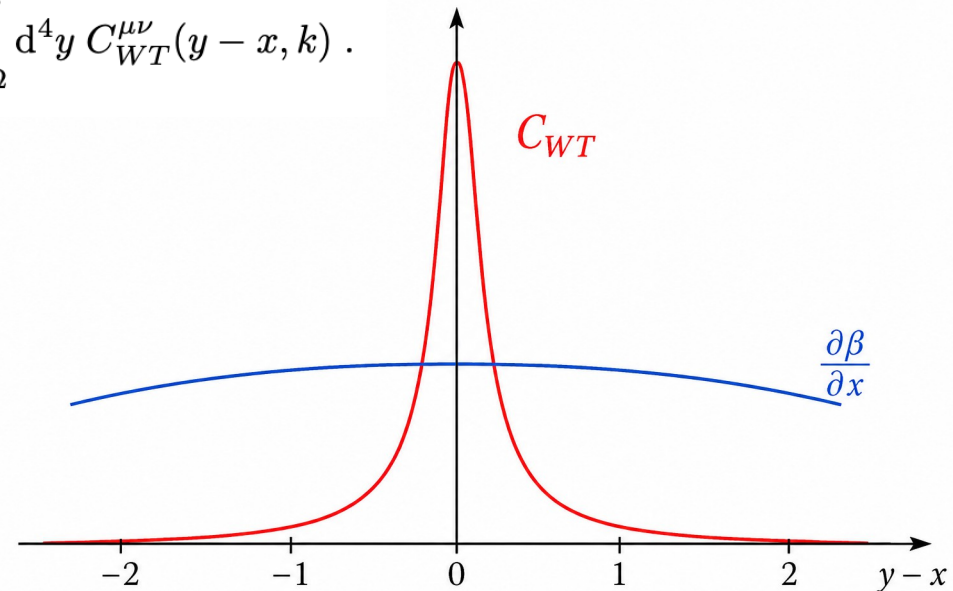
$$\Delta W^+(x, k) \simeq \frac{\partial\beta_\nu}{\partial x^\lambda} \int_{\Sigma_D} d\Sigma_\mu(y) (y - x)^\lambda C_{WT}^{\mu\nu}(x - y, k) + \frac{\partial\beta_\nu}{\partial x^\mu} \int_{\Omega} d^4y C_{WT}^{\mu\nu}(y - x, k).$$

Local equilibrium correction

Dissipative part

“Transport” coefficients encoded in

$$\int_{\Omega} d^4y C_{WT}^{\mu\nu}(y - x, k)$$



Correlation function (2)

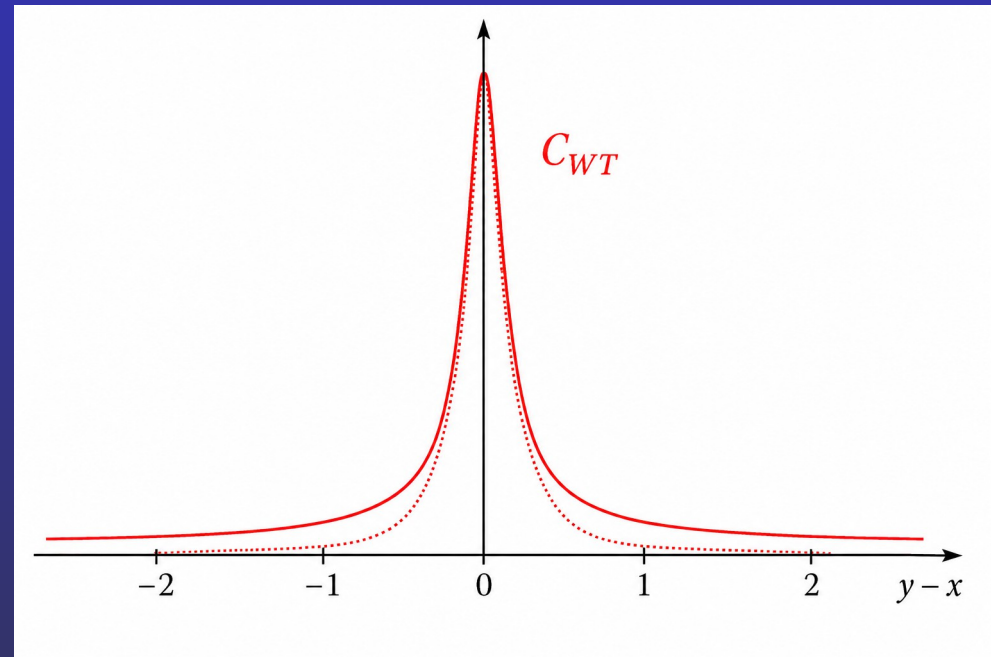
$$C_{WT}^{\mu\nu}(x-y, k) = \frac{2}{(2\pi)^5} \int d^4q \frac{\varrho(k_+) \varrho(k_-)}{\varrho(k)} \theta(k_+^0) \theta(k_-^0) e^{iq \cdot (x-y)} \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \Theta^{\mu\nu}(k, q, \beta)$$

if $\Omega = \mathbb{R}^3 \times [t_0, t]$

$$\int_{\Omega} d^4y C_{WT}^{\mu\nu}(y-x, k) = (t-t_0) (2\pi)^3 \varrho(k) \theta(k^0) \int_{-\pi/2}^{\pi/2} d\vartheta \frac{1}{|w^0(\vartheta)|} \Gamma_{\vartheta}^{\mu\nu}(k, 0, \beta)$$

Surprising result! The correlation function is not LOCAL enough

It is NOT possible to express the dissipative part of the Wigner function as a traditional gradient expansion with gradients evaluated at current time



Correlation function (3)

$$C_{WT1}^{\mu\nu}(x-y, k) = \frac{2}{(2\pi)^5} \int d^4q e^{iq \cdot (x-y)} \frac{\varrho(k_+) \varrho(k_-)}{\varrho(k)} \theta(k_+^0) \theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \int_{-\pi/2}^{\pi/2} d\vartheta \delta(q \cdot w(\vartheta)) \Gamma_{\vartheta}^{\mu\nu}(k, q, \beta)$$

$$C_{WT2}^{\mu\nu}(x-y, k) = \frac{2}{(2\pi)^5} \int d^4q e^{iq \cdot (x-y)} \frac{\varrho(k_+) \varrho(k_-)}{\varrho(k)} \theta(k_+^0) \theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \\ \times \int_{-\pi/2}^{\pi/2} d\vartheta \sum_{f \neq 0} \delta(q \cdot w(\vartheta) - f(S)) \Gamma_{\vartheta, f}^{\mu\nu}(k, q, \beta)$$

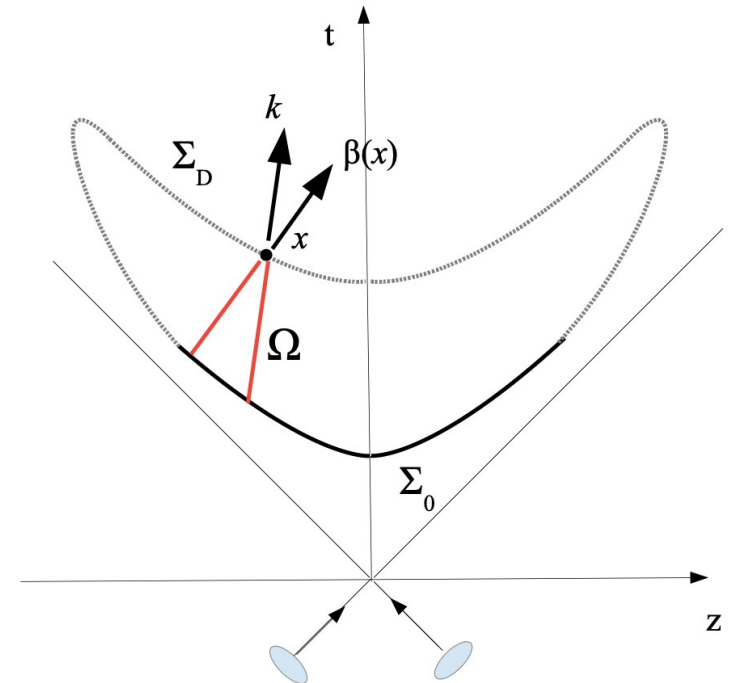
$$C_{WT3}^{\mu\nu}(x-y, k) = \frac{2}{(2\pi)^5} \int d^4q e^{iq \cdot (x-y)} \frac{\varrho(k_+) \varrho(k_-)}{\varrho(k)} \theta(k_+^0) \theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \delta(q^2) \Gamma_{q^2}^{\mu\nu}(k, q, \beta)$$

$$C_{WT4}^{\mu\nu}(x-y, k) = \frac{2}{(2\pi)^5} \int d^4q e^{iq \cdot (x-y)} \frac{\varrho(k_+) \varrho(k_-)}{\varrho(k)} \theta(k_+^0) \theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \Xi^{\mu\nu}(k, q, \beta)$$

Constant over the world-lines

$$(x-y)^\mu = w^\mu(\vartheta) \tau$$

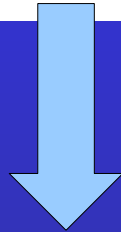
A long-distance contribution to the correlation function



Free field case

In the free field case, the correlation function can be calculated exactly

$$C_{WT}(x-y) = \int d^4q \delta(k \cdot q) e^{iq \cdot (x-y)} G^{\mu\nu}(q) = -(2\pi)^3 \int_0^1 dz \langle \widehat{W}^+(x, k), e^{z\widehat{A}} \widehat{T}^{\mu\nu}(y) e^{-z\widehat{A}} \rangle_{c,GE} .$$



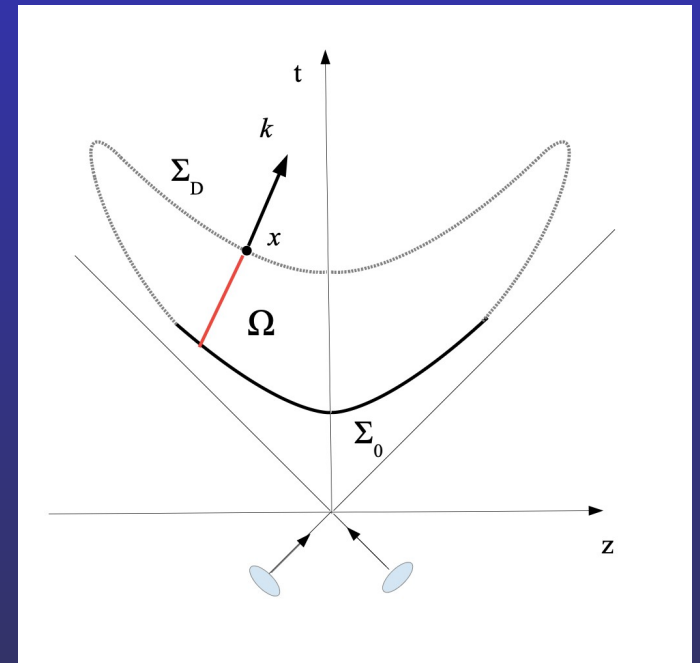
$$G_{\text{free}}^{\mu\nu}(k, q, \beta) = \theta(k_+^0) \theta(k_-^0) \delta\left(k^2 + \frac{q^2}{4} - m^2\right) \frac{n_B(k_+) + n_B(k_-)}{(2\pi)^3 \beta(x) \cdot q} \left[k^\mu k^\nu - \frac{1}{4} (q^\mu q^\nu - q^2 g^{\mu\nu}) \right] ,$$

The correlation function is constant over the worldline

$$\mathbf{y} = \mathbf{x} - \frac{\mathbf{k}}{k^0} (y^0 - x^0) ,$$

consistently with the *free streaming* from Σ_0 to Σ_D

It should not be surprising that for an interacting system a similar effect partially survives



Gradient expansion

$$\underline{C_{WT1}^{\mu\nu}(x-y, k)} = \frac{2}{(2\pi)^5} \int d^4q e^{iq \cdot (x-y)} \frac{\varrho(k_+) \varrho(k_-)}{\varrho(k)} \theta(k_+^0) \theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \int_{-\pi/2}^{\pi/2} d\vartheta \delta(q \cdot w(\vartheta)) \Gamma_{\vartheta}^{\mu\nu}(k, q, \beta)$$

$$\underline{C_{WT2}^{\mu\nu}(x-y, k)} = \frac{2}{(2\pi)^5} \int d^4q e^{iq \cdot (x-y)} \frac{\varrho(k_+) \varrho(k_-)}{\varrho(k)} \theta(k_+^0) \theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q}$$

$$\times \int_{-\pi/2}^{\pi/2} d\vartheta \sum_{f \neq 0} \delta(q \cdot w(\vartheta) - f(S)) \Gamma_{\vartheta, f}^{\mu\nu}(k, q, \beta)$$

$$C_{WT3}^{\mu\nu}(x-y, k) = \frac{2}{(2\pi)^5} \int d^4q e^{iq \cdot (x-y)} \frac{\varrho(k_+) \varrho(k_-)}{\varrho(k)} \theta(k_+^0) \theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \delta(q^2) \Gamma_{q^2}^{\mu\nu}(k, q, \beta)$$

$$\underline{C_{WT4}^{\mu\nu}(x-y, k)} = \frac{2}{(2\pi)^5} \int d^4q e^{iq \cdot (x-y)} \frac{\varrho(k_+) \varrho(k_-)}{\varrho(k)} \theta(k_+^0) \theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \Xi^{\mu\nu}(k, q, \beta)$$

$$\begin{aligned} \Delta W_i^+(x, k) &= \int_{\Sigma_0} d\Sigma_\mu(y) \left[C_{WTi}^{\mu\nu}(x-y, k) \Delta\beta_\nu(y, x) - C_{Wji}^{(\Xi)\mu\nu}(x-y, k) \Delta\zeta(y, x) \right] \\ &= \int_{\Sigma_D} d\Sigma_\mu(y) \left[C_{WTi}^{\mu\nu}(x-y, k) \Delta\beta_\nu(y, x) - C_{Wji}^{(\Xi)\mu\nu}(x-y, k) \Delta\zeta(y, x) \right] \\ &+ \int_{\Omega} d^4y \left[C_{WTi}^{\mu\nu}(x-y, k) \frac{\partial\beta_\nu}{\partial y^\mu} - C_{Wji}^{(\Xi)\mu\nu}(x-y, k) \frac{\partial\zeta}{\partial y^\mu} \right] \quad i = 1, \dots, 4 \end{aligned}$$

 Gradients at point x (traditional)

 Gradients at point on Σ_0 (new)

Gradient expansion (2)

$$G_{\Xi}^{\mu\nu}(k, q, \beta) \equiv \theta(k_+^0)\theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)} \Xi^{\mu\nu}(k, q, \beta),$$

$$H_{\Psi}^{\mu}(k, q, \beta) \equiv \theta(k_+^0)\theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)} \Psi^{\mu}(k, q, \beta)$$

$$\Delta W_4^+(x, k)_{\text{diss}} = \frac{2}{(2\pi)} \sum_{N=0}^{\infty} \frac{i^N}{N!} \left[\partial_{\nu_1}^q \dots \partial_{\nu_N}^q G_{\Xi}^{\mu\nu}(k, q, \beta) \Big|_{q^\mu=0} \partial_x^{\nu_1} \dots \partial_x^{\nu_N} \partial_{x\mu} \beta_\nu \right. \\ \left. - \partial_{\nu_1}^q \dots \partial_{\nu_N}^q H_{\Psi}^{\mu}(k, q, \beta) \Big|_{q^\mu=0} \partial_x^{\nu_1} \dots \partial_x^{\nu_N} \partial_{x\mu} \zeta \right]$$

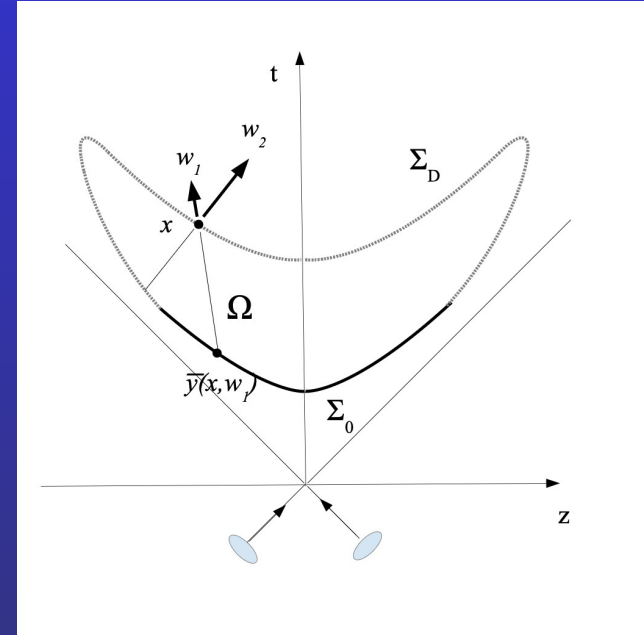
Using the same method as in:

X.L. Sheng, F.B., D. Roselli, arXiv:2509.14301

$$G_{\vartheta, f}^{\mu\nu}(k, q, \beta) \equiv \theta(k_+^0)\theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)} \Gamma_{\vartheta, f}^{\mu\nu}(k, q, \beta)$$

$$H_{\vartheta, f}^{\mu}(k, q, \beta) \equiv \theta(k_+^0)\theta(k_-^0) \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)} \Upsilon_{\vartheta, f}^{\mu}(k, q, \beta)$$

$$\Delta W_1^+(x, k) = \int_{-\pi/2}^{\pi/2} d\vartheta \sum_{N=0}^{\infty} \frac{2\theta_\vartheta(x)}{(2\pi)^2} \frac{(-i)^N}{N!} [D_y(\bar{y}_k(x))]^N \\ \times \left\{ \frac{\hat{n}_\mu(y)}{|w(\vartheta) \cdot n(y)|} [G_{\vartheta}^{\mu\nu}(k, q, \beta) \Delta\beta_\nu(y, x) - H^\mu(\vartheta; k, q, \beta) \Delta\zeta(y, x)] \right\} \Big|_{q=0, y=\bar{y}_\vartheta(x)}$$



Leading contributions

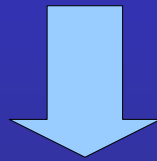
$$\Delta W(x, k)_4^+ = \mathcal{O}(\partial^3 \beta(x)) \quad \longrightarrow \text{usual gradient correction}$$

$$\Delta W(x, k)_2^+ = \mathcal{O}(\partial^2 \beta(y(x)))$$

$$\Delta W(x, k)_3^+ = ?$$

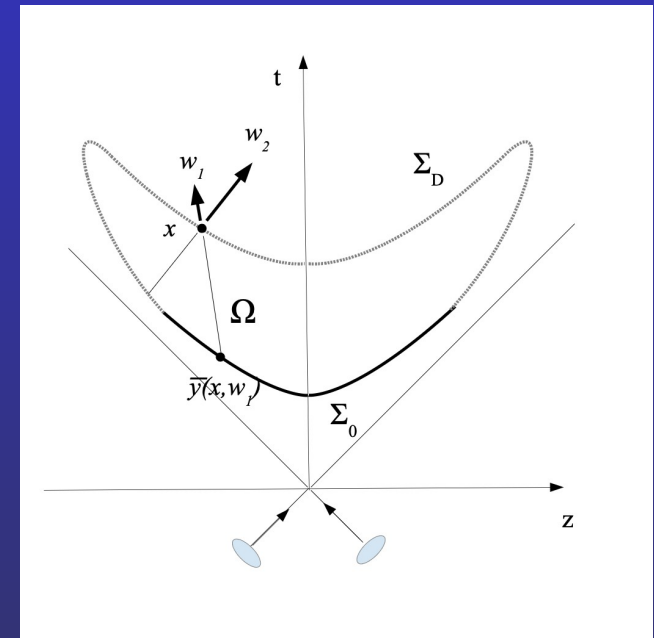
$$\Delta W(x, k)_1^+ = \mathcal{O}(\beta(\bar{y}(x)) - \beta(x))$$

long distance
terms or
memory terms



$$\Delta^{(0)} W^+(x, k) \simeq \frac{2\varrho(k)}{(2\pi)^2} \int_{-\pi/2}^{\pi/2} d\vartheta \theta_\vartheta(x) \left\{ \Gamma_1(\vartheta; S_0) [w_\vartheta \cdot \beta(x) - w_\vartheta \cdot \beta(\bar{y}_\vartheta(x))] \right. \\ \left. - \Upsilon_1(\vartheta; S_0) [\zeta(x) - \zeta(\bar{y}_\vartheta(x))] \right\}.$$

$$\Delta \frac{dN}{d^3k} \simeq \int dk^0 \int_{\Sigma_D} d\Sigma \cdot k \Delta^{(0)} W^+(x, k)$$



Is this the main dissipative correction?

Summary

- ★ The non-equilibrium corrections to the Wigner function cannot be expressed solely by means of an expansion in gradients evaluated at the current space-time point
- ★ A memory effect survives which cannot be captured by traditional relativistic kinetic theory
- ★ Corrections depend on unknown dynamical coefficients, thermo-gravitational form factors
- ★ One, perhaps two, non-equilibrium terms can be expressed with an expansion in gradients evaluated at the initial hypersurface; one non-equilibrium term with a gradient expansion at the decoupling hypersurface (leading term at least 2nd order derivative)
- ★ The leading term is proportional to the difference between the hydro-thermodynamic fields at decoupling and initial hypersurface

Outlook

- ★ Extension to fields with spin to find all possible corrections to spin polarization

Comparison with stress-energy tensor

$$\Delta T^{\mu\nu}(x)_{\text{diss}} = \int_{\Omega} d^4y C_{TT}^{\mu\nu\rho\sigma}(x-y) \partial_{\rho} \beta_{\sigma}(y) \simeq \partial_{\rho} \beta_{\sigma}(x) \int_{\Omega} d^4y C_{TT}^{\mu\nu\rho\sigma}(x-y)$$

$$C_{TT}^{\mu\nu\rho\sigma}(x-y) = \int_0^1 dz \langle \hat{T}^{\mu\nu}(x), e^{z\hat{\mathcal{E}}_{\text{GE}}} \hat{T}^{\rho\sigma}(y) e^{-z\hat{\mathcal{E}}_{\text{GE}}} \rangle_{c, \text{GE}}$$

Encodes transport coefficients:
Kubo formula of shear viscosity

Write the correlation function as a Fourier transform

$$C_{TT}^{\mu\nu\rho\sigma}(x-y) = \int d^4q e^{iq \cdot (x-y)} \tilde{C}_{TT}^{\mu\nu\rho\sigma}(q)$$

Constraints from energy-momentum conservation:

$$\int d^3y C_{TT}^{\mu\nu 0\sigma}(x-y) = \langle \hat{T}^{\mu\nu}(x), \hat{P}^{\sigma} \rangle_{c, \text{GE}} = -\frac{\partial}{\partial \beta_{\sigma}} \langle \hat{T}^{\mu\nu}(x) \rangle_{\text{GE}}$$

$$(2\pi)^3 \int dq^0 e^{iq^0(x^0-y^0)} \tilde{C}_{TT}^{\mu\nu 0\sigma}(q^0, \mathbf{q}=0) = -\frac{\partial}{\partial \beta_{\sigma}} \langle \hat{T}^{\mu\nu}(x) \rangle_{\text{GE}}$$

Ward identities

$$q_{\mu_i} \tilde{C}_{TT}^{\mu_1 \mu_2 \mu_3 \mu_4} = 0$$

Comparison with stress-energy tensor (2)

$$C_{WT}^{\mu\nu}(x-y, k) = \frac{2}{(2\pi)^5} \int d^4q \frac{\varrho(k_+) \varrho(k_-)}{\varrho(k)} \theta(k_+) \theta(k_-) e^{iq \cdot (x-y)} \frac{1 - e^{\beta(x) \cdot q}}{\beta(x) \cdot q} \Theta^{\mu\nu}(k, q, \beta)$$

$$\Theta^{0\nu}(k, q^0, \mathbf{q} = 0, \beta) = -\frac{\theta(k^0)}{(2\pi)^2 \varrho(k)} \delta(q^0) \left[\frac{\partial}{\partial \beta_\nu(x)} (n_B(k) \varrho(k)) \right],$$

$$(2\pi)^3 \int dq^0 e^{iq^0(x^0-y^0)} \tilde{C}_{TT}^{\mu\nu 0\sigma}(q^0, \mathbf{q} = 0) = -\frac{\partial}{\partial \beta_\sigma} \langle \hat{T}^{\mu\nu}(x) \rangle_{\text{GE}}$$



Fourier transform of C_{WT}
is much more constrained
Why?

Mathematically, the above constraint on Θ stems from:

$$\varrho(k_+) \varrho(k_-) \langle \hat{A}^\dagger(k_+) \hat{A}(k_-), \hat{P}^\nu \rangle_{c, \text{GE}} = -\frac{\partial}{\partial \beta_\nu(x)} \left(\varrho(k_+) \varrho(k_-) \langle \hat{A}^\dagger(k_+) \hat{A}(k_-) \rangle_{\text{GE}} \right)$$

$$\langle \hat{A}^\dagger(p) \hat{A}(p') \rangle_{\text{GE}} = \frac{2\pi}{\varrho^2(p)} \theta(p^0) \theta(p'^0) \delta^4(p - p') n_B(p) \varrho_{\text{GE}}(p)$$

More physically, the stress-energy tensor of an interacting system is not just an integral of Wigner functions of separated fields, but it includes 3-field contributions

