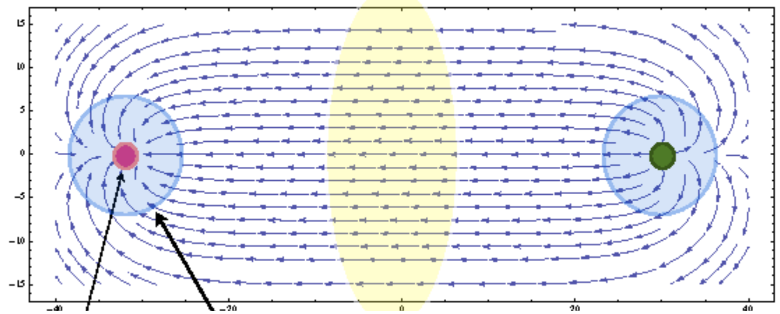
Xiong-Hui Cao 曹雄辉

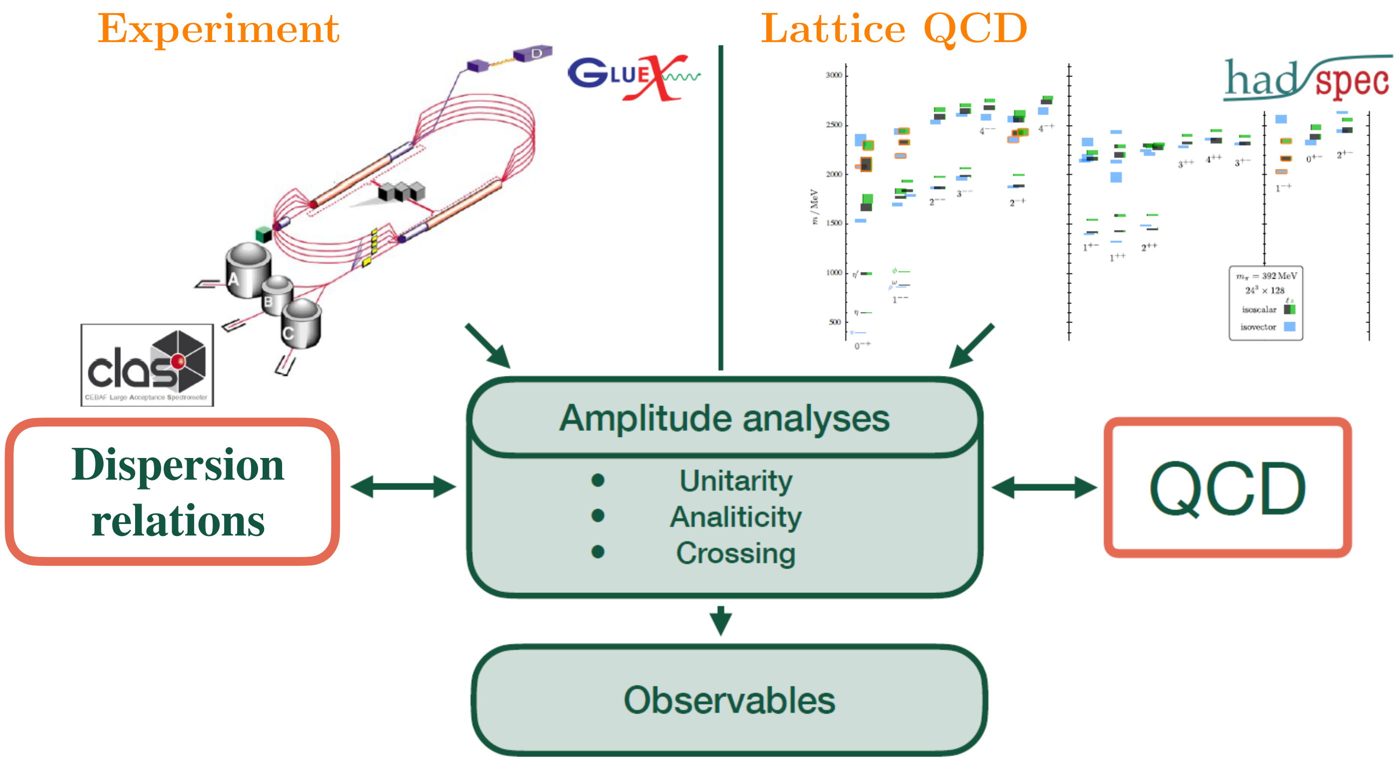
第一届中国电子离子对撞机相关物理年会

2026年4月22日，青岛，山东大学

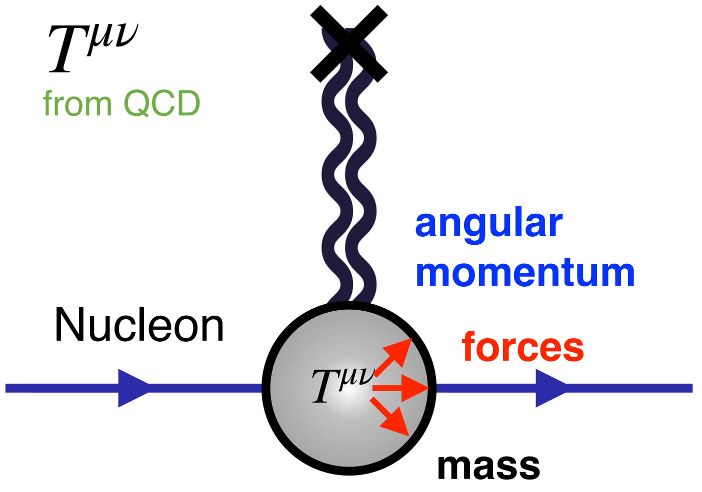
QCD, hadron physics and the analytic S-matrix

- **Asymptotic freedom**:
  only perturbative at high energies
- **Confinement**:
  at low energies the gluons bind the quarks together to form the hadrons
- Quark models give rough structure of **hadron spectrum**
- Need **non-perturbative & model-independent approaches** to describe these hadrons rigorously:





Gravitational structure of nucleons



- Gravity couples to matter through energy-momentum tensor (EMT) $T^{\mu\nu}$

$$T^{\mu\nu}_{\mathrm{QCD}} = T^{\mu\nu}_q + T^{\mu\nu}_g$$

study gravitational forces within hadrons

study **mechanical properties** of hadrons

- Definition:

$$\left\langle N(p') \middle| T^{\mu\nu} \middle| N(p) \right\rangle = \frac{1}{4m_N} \bar{u}(p') \left[A(t) P^\mu P^\nu + J(t) \left(i P^{\{\mu} \sigma^{\nu\}\rho} \Delta_\rho \right) + D(t) \left(\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2 \right) \right] u(p)$$

Kobzarev, Okun (1962)
Pagels (1966)

- Mass normalization:

$$m_N = \int d^3 r T_{00}(r)$$

$$A(0) = 1$$

- Spin normalization:

$$J^i = \epsilon_{ijk} \int d^3 r r_j T_{0k}(r)$$

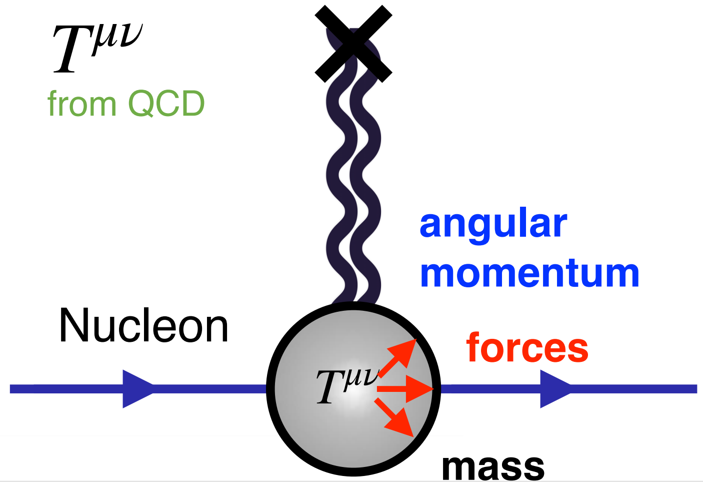
$$J(0) = 1/2$$

- D-term: $D \equiv D(0)$

$$D = -\frac{m_N}{2} \int d^3 r \left(r_i r_j - \frac{1}{3} \delta_{ij} \right) T_{ij}(r)$$

$$D = ?$$

Gravitational structure of nucleons



D-term as the "last unknown global property"

Polyakov, and Schweitzer, (2018)

em:	$\partial_\mu J^\mu_{\rm em} = 0$	$\langle N'\vert J^\mu_{\rm em}\vert N\rangle$	$\longrightarrow$	$Q = 1.602176487(40)\times 10^{-19}\mathrm{C}$ $\mu = 2.792847356(23)\mu_N$
weak:	PCAC	$\langle N'\vert J^\mu_{\rm weak}\vert N\rangle$	$\longrightarrow$	$g_A = 1.2694(28)$ $g_p = 8.06(55)$
gravity:	$\partial_\mu T^{\mu\nu}_{\rm grav} = 0$	$\langle N'\vert T^{\mu\nu}_{\rm grav}\vert N\rangle$	$\longrightarrow$	$m = 938.272013(23)\,\mathrm{MeV}/c^2$ $J = \frac{1}{2}$ $D = ?$

Scalar radius puzzle

- Trace FF:

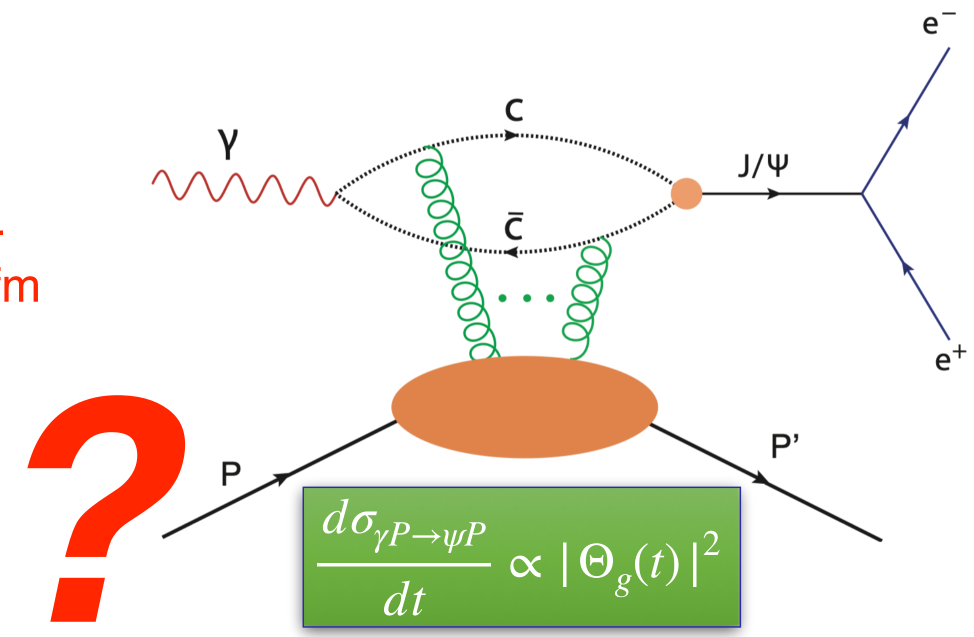
$$\left\langle N(p')\middle| T^{\mu}_{\ \mu} \middle| N(p)\right\rangle = m_N \bar{u}(p')\left[A(t) - \frac{t}{4m_N^2}[A(t) - 2J(t) + 3D(t)]\right] u(p) \equiv \bar{u}(p')\Theta(t)u(p)$$

- Trace anomaly in QCD: $T^{\mu}_{\ \mu} = \frac{\beta(g)}{2g} F^{a,\mu\nu} F^{a,}{}_{\mu\nu} + (1+\gamma_m)\sum_q m_q \bar{\psi}_q \psi_q$

- Scalar radius: $\Theta(t) = m_N\left(1 + t\langle r_\Theta^2\rangle/6 + \ldots\right)$

  - Kharzeev proposed it can be extracted from the threshold photoproduction of the vector-meson, e.g., J/ψ, and the fit result is ~0.55 fm
    Kharzeev, PRD (2021)……

  - Recently, two LQCD calculations at near physical quark mass result in a large scalar radius, ~1 fm
    Hackett et al., PRL (2024); Wang et al. [χQCD], PRD (2024)



$$\frac{d\sigma_{\gamma P\to\psi P}}{dt} \propto |\Theta_g(t)|^2$$

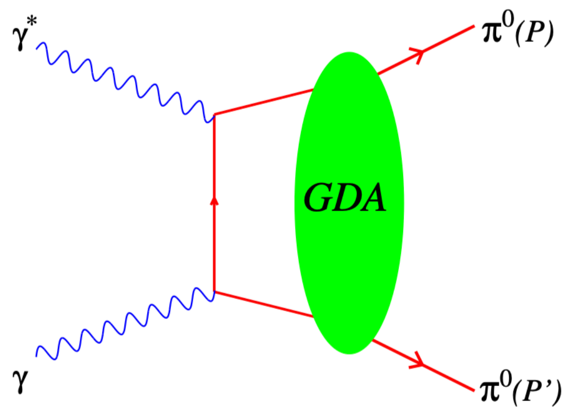
D-term from experiments

- All GFFs can be measured from DVCS through sum rules derived for GPDs Ji, PRL (1997); Ji, Melnitchouk and Song, PRD (1997)
- π^0: $\gamma^*\gamma \to \pi^0\pi^0$ in e^+e^- Belle data: PRD 93, 032003 (2016)

  $D^Q_{\pi^0} \simeq -0.7$ at $\langle Q^2 \rangle = 16.6$ GeV^2

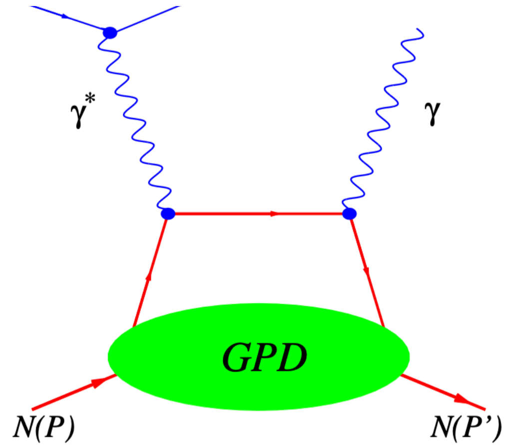
  Kumano, Song, Teryaev, PRD (2018)

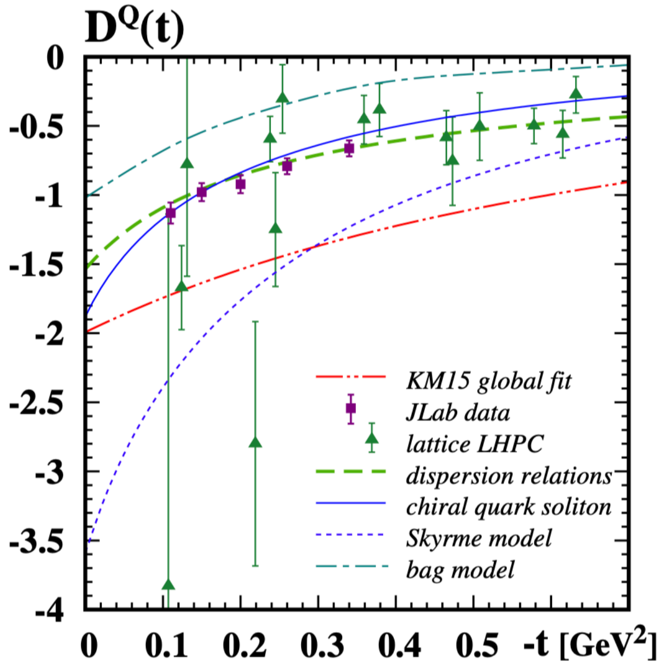
  Chiral symmetry: total $D_{\pi^0} \simeq -1$ (gluons contribute the rest)



- Proton: Burkert, Elouadrhiri, Girod, Nature 557, 396 (2018)

  JLab data: PRL 100, 162002 (2008) & PRL 115, 212003 (2015)





Subtraction function in dispersion relations for DVCS amplitudes

- $\Delta(t, \mu) \to D^Q(t, \mu)$ scale-dependent

  model-dependent

- Explore t range at EIC & EicC

Why data-driven dispersion theory?

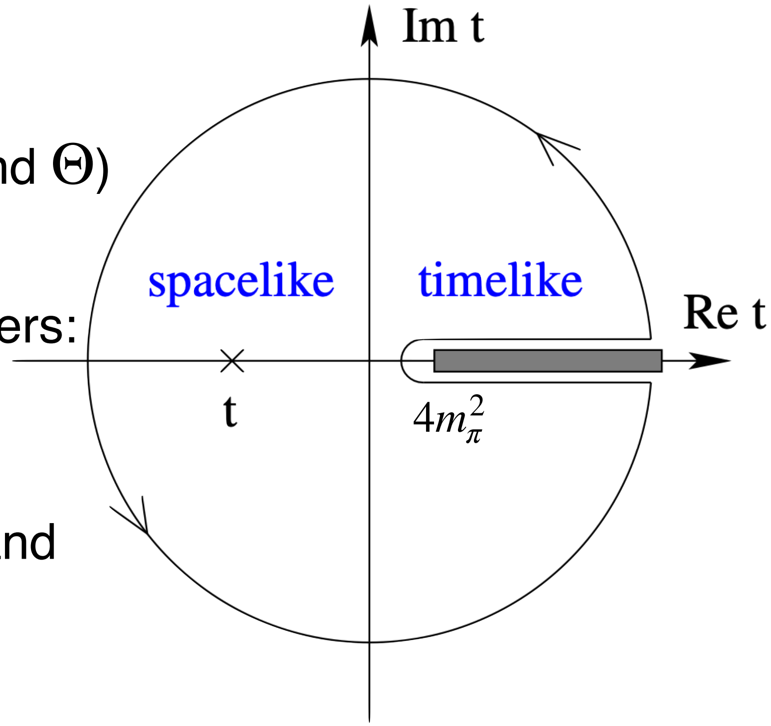
- Based on fundamental principles: unitarity, analyticity and crossing symmetry

- Simultaneous analysis of all four GFFs (A, J, D and Θ)

- Connects FFs over full range of momentum transfers: timelike and spacelike data

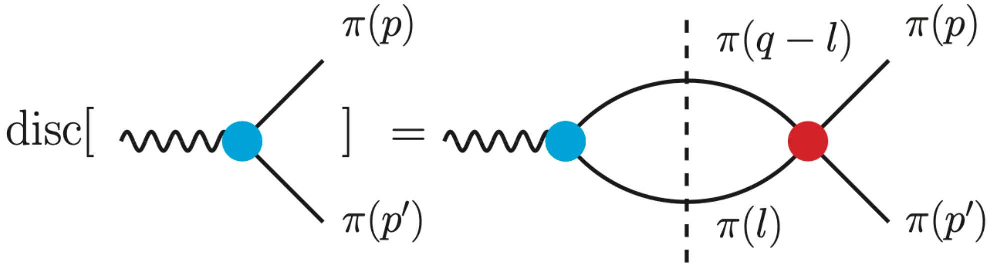
- Connects to data from other processes ($\pi\pi, K\bar{K}$ and $\pi N, KN$ scatterings…)

- Constraints from ChPT, pQCD…



Example: the pion vector form factor

- Unitarity: (only $\ell = 1$ is projected out)



$$\mathrm{disc}F_\pi(s) \;=\; 2i\mathrm{Im}F_\pi(s) = 2i\sigma_\pi(s)F_\pi(s)t_1^{1*}(s) = 2iF_\pi(s)\sin\delta_1^1(s)e^{-i\delta_1^1(s)}\,,$$

- Watson's theorem: Watson, (1954) $\mathrm{Im}F_\pi(s) = |F_\pi(s)|e^{i\delta_{F_\pi}(s)}\sin\delta_1^1(s)e^{-i\delta_1^1(s)}\theta(s-4m_\pi^2)\,,$

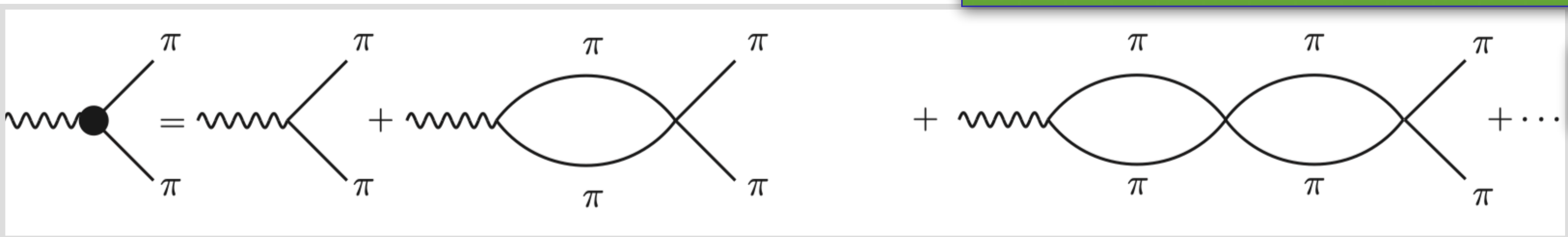
$$\Rightarrow \quad \delta_{F_\pi}(s) = \delta_1^1(s)\,,$$

- Analytic solution (only in single channel), Omnes equation: Omnes, (1958)

$$F_\pi(s) = P(s)\Omega_1^1(s)\,, \quad \Omega_1^1(s) = \frac{f(s)}{f(0)} = \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\},$$

depends solely on the P-wave phase shift of $\pi\pi$

- Diagrammatic interpretation



Data-driven!

Unitarity relation for pion GFFs

- Gravitational form factors (GFFs) for spin-0 particles, e.g., for pion:

$$\left\langle \pi^a(p') \left| \hat{T}^{\mu\nu}(0) \right| \pi^b(p) \right\rangle = \frac{\delta^{ab}}{2}\left[A^\pi(t)P^\mu P^\nu + D^\pi(t)\left(\Delta^\mu\Delta^\nu - \mathrm{t}g^{\mu\nu}\right)\right]$$

$$P^\mu = p'^\mu + p^\mu, \quad \Delta^\mu = p'^\mu - p^\mu$$

Crossing symmetry
for constructing dispersion relations

$$\left\langle \pi^a(p')\,\pi^b(p) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle = \frac{\delta^{ab}}{2}\left[A^\pi(t)\Delta^\mu\Delta^\nu + D^\pi(t)\left(P^\mu P^\nu - \mathrm{t}g^{\mu\nu}\right)\right]$$

- Unitarity for full amplitude ⇒discontinuity (imaginary part) of the pion GFFs

$$\mathrm{Disc}\left\langle \pi^a(p')\,\pi^b(p) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle$$

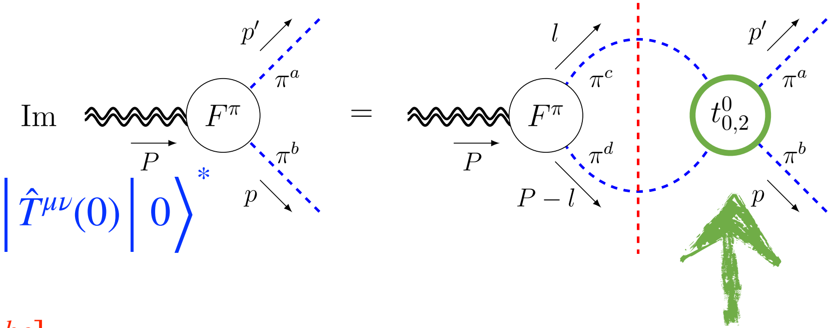
$$= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int d\Omega_l \left\langle \pi^a(p')\,\pi^b(p) \mid \pi^c(l)\pi^d(P-l)\right\rangle \left\langle \pi^c(l)\pi^d(P-l) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle^*$$

$$= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int d\Omega_l \left[A(t,s,u)\delta^{ab}\delta^{cd} + A(s,t,u)\delta^{ac}\delta^{bd} + A(u,s,t)\delta^{ad}\delta^{bc}\right]$$

$$\times \frac{\delta^{cd}}{2}\left[\left(A^\pi(t)\right)^*(2l-P)^\mu(2l-P)^\nu + \left(D^\pi(t)\right)^*\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]$$

$$= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int d\Omega_l \frac{\delta^{ab}}{2}A^{I=0}(t,s,u)\left[\left(A^\pi(t)\right)^*(2l-P)^\mu(2l-P)^\nu + \left(D^\pi(t)\right)^*\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]$$

S-,*D*-waves ... *S*-wave



$\pi\pi$ scattering

Unitarity relation for pion GFFs

- Discontinuity of A^π and D^π: associated only with t_0^0, t_2^0 partial waves.

$$\text{Disc}\left\langle \pi^a(p')\,\pi^b(p)\right|\hat{T}^{\mu\nu}(0)\,|\,0\rangle = \frac{\delta^{ab}}{2}\left[\text{Disc}\,A^\pi(t)\Delta^\mu\Delta^\nu + \text{Disc}\,D^\pi(t)\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]$$

$$= 2i\frac{2p_\pi}{\sqrt{t}}\frac{\delta^{ab}}{2}\left[\left(A^\pi(t)\right)^*\left(\frac{4}{3t}p_\pi^2\boxed{\left(t_0^0(t)-t_2^0(t)\right)}\left(P^\mu P^\nu - tg^{\mu\nu}\right)+\boxed{t_2^0(t)}\Delta^\mu\Delta^\nu\right)+\left(D^\pi(t)\right)^*\boxed{t_0^0(t)}\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]$$

Partial-wave amplitudes of $\pi\pi$ scattering

- Partial-wave unitarity

$$\text{Im}\,A^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_2^0(t)\right)^* A^\pi(t)$$

$$\text{Im}\,D^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left[\frac{4}{3}\frac{p_\pi^2}{t}\left(t_0^0(t)-t_2^0(t)\right)^* A^\pi(t) + \left(t_0^0(t)\right)^* D^\pi(t)\right] \;\Rightarrow\; \text{Im}\,\Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_0^0(t)\right)^*\Theta^\pi(t)$$

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$, matrix elements (conserved separately):

$$\left\langle \pi^a(p')\,\pi^b(p)\right|\hat{T}^{\mu\nu}(0)\,|\,0\rangle = \delta^{ab}\left\{\frac{1}{3}\left(g^{\mu\nu}-\frac{P^\mu P^\nu}{P^2}\right)\Theta^\pi(t) + \left[\Delta^\mu\Delta^\nu + \frac{\Delta^2}{3t}\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]A^\pi(t)\right\}$$

Trace part: $\left\langle \pi^a(p')\,\pi^b(p)\right|\hat{T}^\mu{}_\mu(0)\left|\,0\right\rangle = \delta^{ab}\Theta^\pi(t)$, $\quad \Theta^\pi(t) = -\frac{1}{2}\left(4p_\pi^2 A^\pi(t) + 3tD^\pi(t)\right)$

A^π: D-wave single channel

- **Single-channel**: Watson's theorem ⇒ phase of FF = scattering phase shift

$$\text{disc}\, A^\pi(t) = 2iA^\pi(t)\sin\delta_2^0(t)e^{-i\delta_2^0(t)}\theta\left(t-t_\pi\right)$$

$$\left|t_2^0\right| e^{i\phi_2^0} = \frac{\eta_2^0 e^{2i\delta_2^0}-1}{2i\sigma_\pi}$$

δ_2^0 replaced by the phase of $\pi\pi$ partial wave ϕ_2^0 to account for inelasticity

Hoferichter et al., Phys. Rept. (2016)

Omnes solution

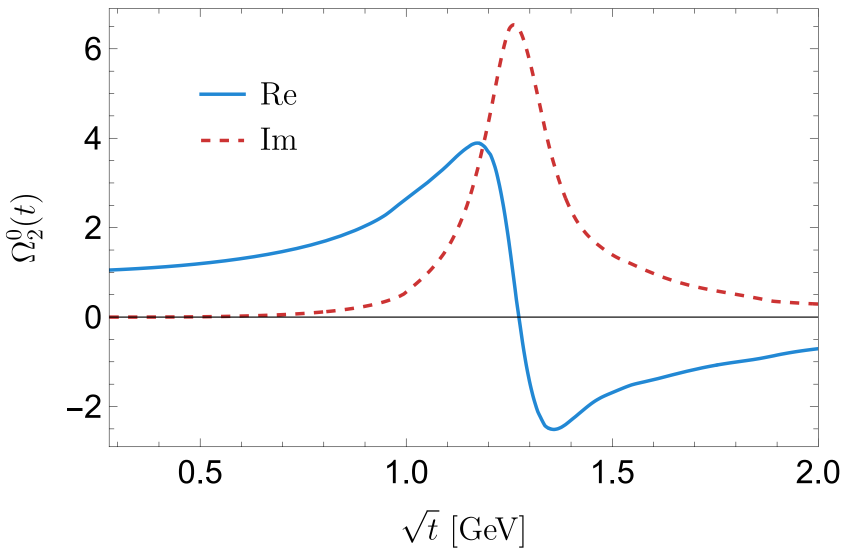
$$A^\pi(t) = P_2^\pi(t)\Omega_2^0(t), \quad \Omega_2^0(t) \equiv \exp\left\{\frac{t}{\pi}\int_{t_\pi}^\infty \frac{dt'}{t'}\frac{\phi_2^0(t')}{t'-t}\right\}$$

- δ_2^0, η_2^0 up to $E_0 \simeq 2\ \text{GeV}$ from dispersive analysis Bydžovský et al., PRD (2016)
- Polynomial: $P_2^\pi(t) = 1+\alpha t$ matched to NLO ChPT

$$\Rightarrow A^\pi(t) = 1 - \frac{2L_{12}^r}{F_\pi^2}t$$

Tensor-meson dominance estimate: $L_{12}^r = -\frac{F_\pi^2}{2m_{f_2}^2}$

$$P_2^\pi(t) = 1 + \left[\frac{1}{m_{f_2}^2} - \dot{\Omega}_2^0(0)\right] t \simeq 1 - \left(0.01\text{GeV}^{-2}\right) t$$



Θ^π: S-wave $\pi\pi - K\bar{K}$ coupled channel

- $\pi\pi$ S-wave phase shifts known precisely from Roy(-like) equation analyses

  Bern group; Madrid-Krakow group

- Generalisation to coupled channels: isoscalar-scalar $f_0(500)$, $f_0(980)$ mesons

  - Unitarity relation for Θ^π $\Rightarrow$

$$\mathrm{Im}\,\Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}} \left(t_0^0(t)\right)^* \Theta^\pi(t)$$

Matrix relation for coupled channels (both pion and kaon trace GFFs):

$$\mathrm{Im}\boldsymbol{\Theta}(t) = [\mathbf{T}_0^0(t)]^* \, \boldsymbol{\Sigma}_0^0(t) \, \boldsymbol{\Theta}(t)$$

$$\boldsymbol{\Theta}(t) = \begin{pmatrix} \Theta^\pi(t) \\ \frac{2}{\sqrt{3}}\Theta^K(t) \end{pmatrix}$$

Phase-space factor

$$\boldsymbol{\Sigma}_0^0(t) \equiv \mathrm{diag}(\sigma_\pi \theta(t - t_\pi), \sigma_K \theta(t - t_K))$$

With $\sigma_i(t) = \sqrt{1 - 4m_i^2/t}$ $(i = \pi, K)$

$\pi\pi$-$K\bar{K}$ T-matrix

$$\mathbf{T}_0^0(t) = \begin{pmatrix} \frac{\eta_0^0(t) e^{2i\delta_0^0(t)} - 1}{2i\sigma_\pi} & |g_0^0(t)| e^{i\Psi_0^0(t)} \\ |g_0^0(t)| e^{i\Psi_0^0(t)} & \frac{\eta_0^0(t) e^{2i(\Psi_0^0(t) - \delta_0^0(t))} - 1}{2i\sigma_K} \end{pmatrix}.$$

$$\eta_0^0(t) = \sqrt{1 - 4\sigma_\pi\sigma_K |g_0^0(t)|^2 \theta\left(t - t_K\right)}$$

Muskhelishvili-Omnes problem

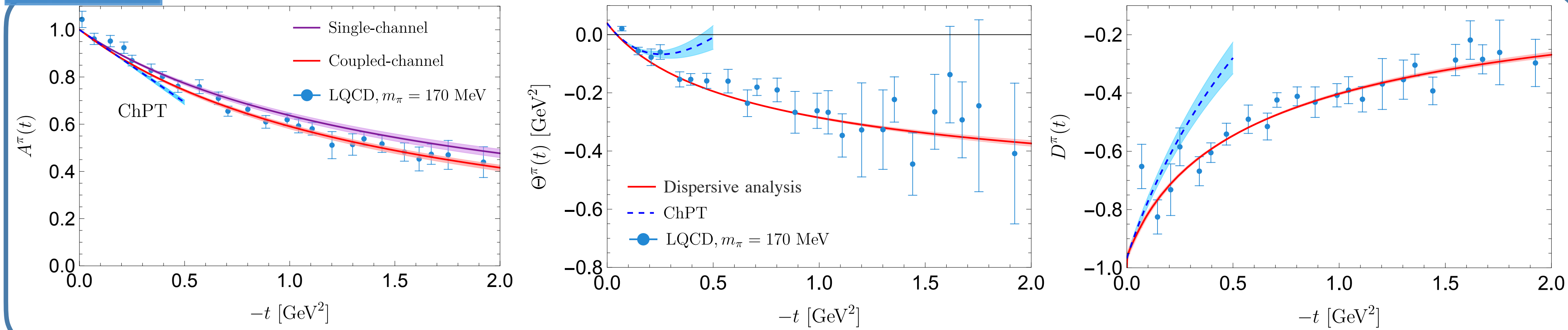
Donoghue, Gasser, Leutwyler, NPB (1991)

Pion and Kaon GFFs

XHC, F.-K. Guo, Q.-Z. Li and D.-L. Yao, Nature Commun. (2025)

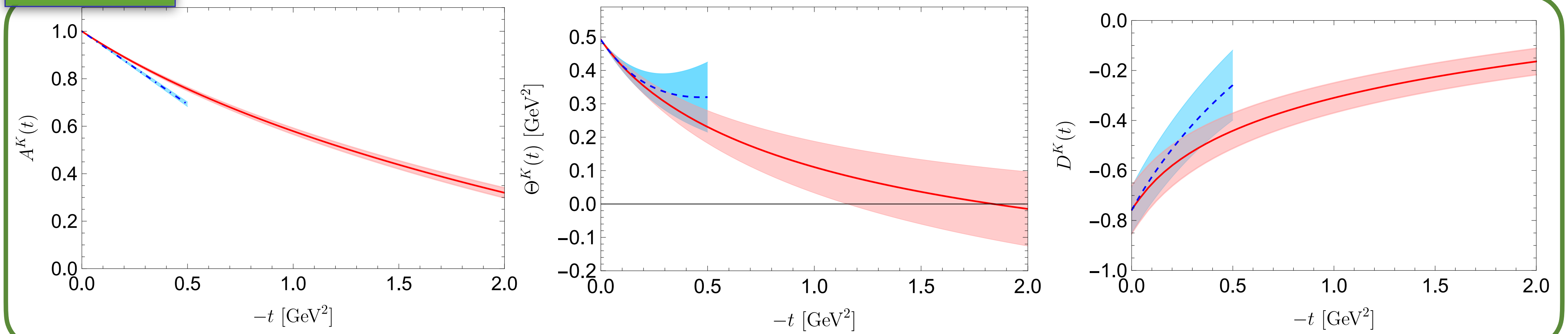
Prediction, Not fit

Pion



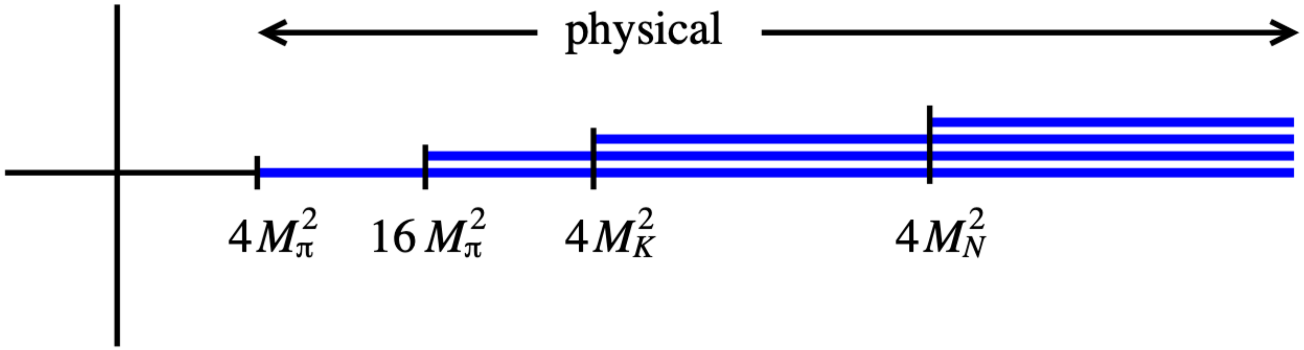
Kaon

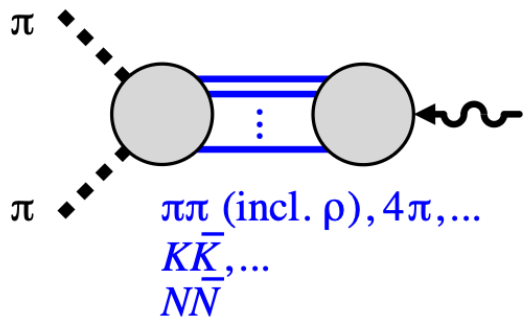
LQCD ($m_\pi \simeq 170$ MeV): Hackett et al., PRD (2023)



Unitarity relation for nucleon GFFs

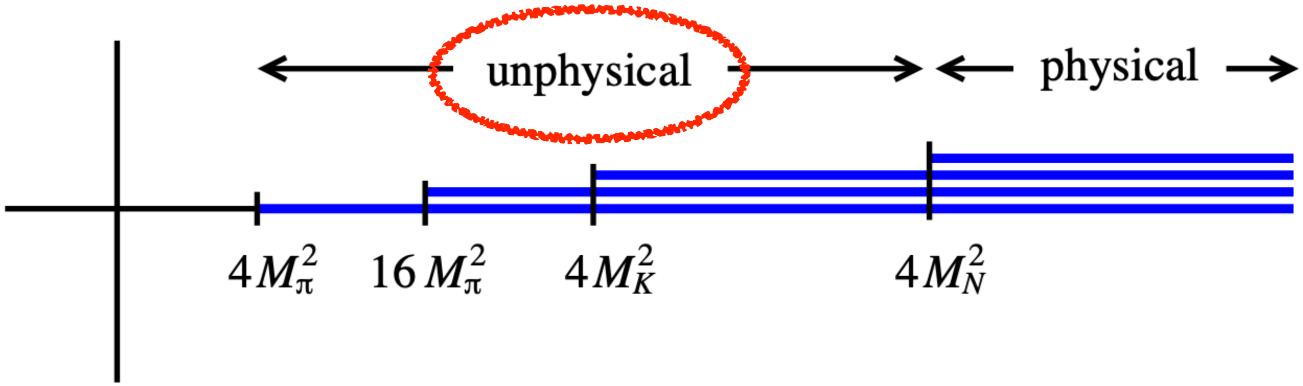
- Pion GFFs

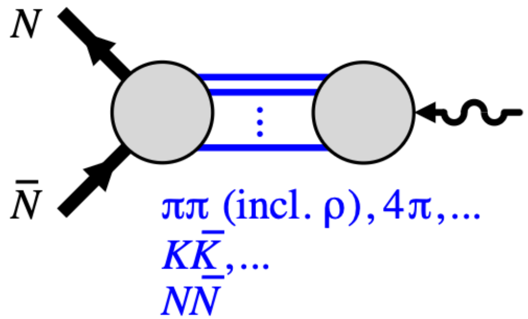




- Nucleon GFFs

Analytic continuation!



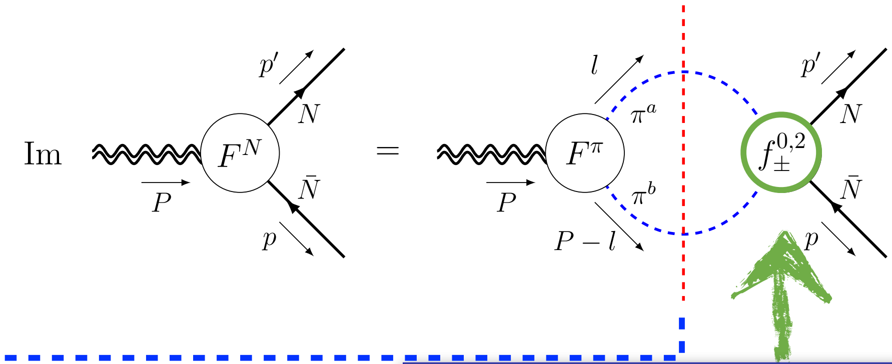


Unitarity relation for nucleon GFFs

- Discontinuity

$$\text{Disc}\left\langle N(p')\bar{N}(p)\middle|\hat{T}^{\mu\nu}(0)\middle|0\right\rangle$$

$$\propto \sum_n \langle N(p')\bar{N}(p)|n\rangle\langle n|\hat{T}^{\mu\nu}(0)|0\rangle^*\delta^4(p+p'-p_n)$$



$\pi\pi(K\bar{K}) \to N\bar{N}$ scattering

$$\text{Im}\,A(t) = \frac{3p_\pi^5}{\sqrt{6t}}\left[f_-^2(t) + \sqrt{\frac{3}{2}}\frac{m_N}{p_N^2}\Gamma^2(t)\right]^* A^\pi(t)$$

$$\text{Im}\,J(t) = \frac{3p_\pi^5}{2\sqrt{6t}}\left[f_-^2(t)\right]^* A^\pi(t)$$

$$\text{Im}\,D(t) = -\frac{3m_N p_\pi}{2p_N^2\sqrt{t}}\left[\frac{4p_\pi^2}{3t}\left(\left(f_+^0(t)\right)^* - \left(p_\pi p_N\right)^2\left(f_+^2(t)\right)^*\right)A^\pi(t) + \left(f_+^0(t)\right)^* D^\pi(t)\right]$$

$$\Theta(t) = \frac{1}{4m_N}\left[-4p_N^2 A(t) + 2tJ(t) - 3tD(t)\right]$$

$$\text{Im}\,\Theta(t) = -\frac{3p_\pi}{4p_N^2\sqrt{t}}\left[f_+^0(t)\right]^*\Theta^\pi(t)$$

$f_\pm^J$: $\pi\pi \to K\bar{K}$ partial wave amp. With $+/-$ for parallel/anti-parallel $N\bar{N}$ helicities

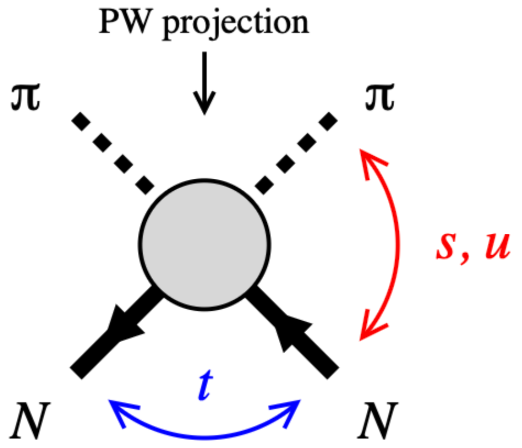
$$\Gamma^2 = m_N^2\sqrt{\frac{2}{3}}f_-^2 - f_+^2$$

Frazer & Fulco (1960); Höhler (1983)

Coupled-channel

$$\text{Im}\,\Theta(t) = -\frac{3}{4p_N^2\sqrt{t}}\left\{p_\pi\left[f_+^0(t)\right]^*\Theta^\pi(t)\theta\left(t-t_\pi\right) + \frac{4}{3}p_K\left[h_+^0(t)\right]^*\Theta^K(t)\theta\left(t-t_K\right)\right\}$$

Analysis: $\pi\pi \to N\bar{N}$ partial-wave amplitudes



- Idea: Construct $\pi\pi \to N\bar{N}$ PWAs from $\pi N \to \pi N$ scattering data using amplitude analysis techniques

Two major challenges:

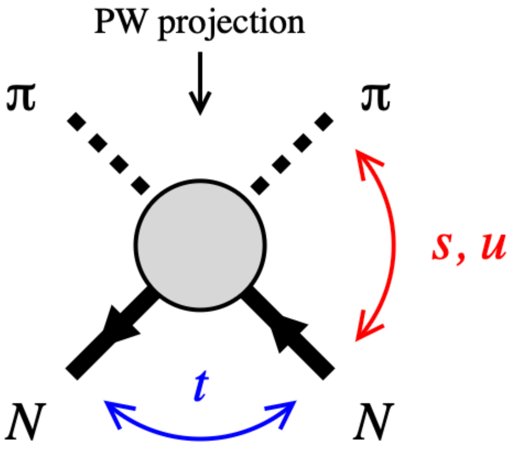
- t–channel partial wave projection requires knowledge of amplitude in unphysical region of s, u–channel processes $s, u \; < \; (M_\pi + M_N)^2$

  $\to$ analytic continuation in s, u at fixed t

- For spectral function we need t-channel PWAs at $t > 4\ M_\pi^2 > 0$, while $\pi N \to \pi N$ data are at $t < 0$

  $\to$ analytic continuation in t

Analysis: $\pi\pi \to N\bar{N}$ partial-wave amplitudes

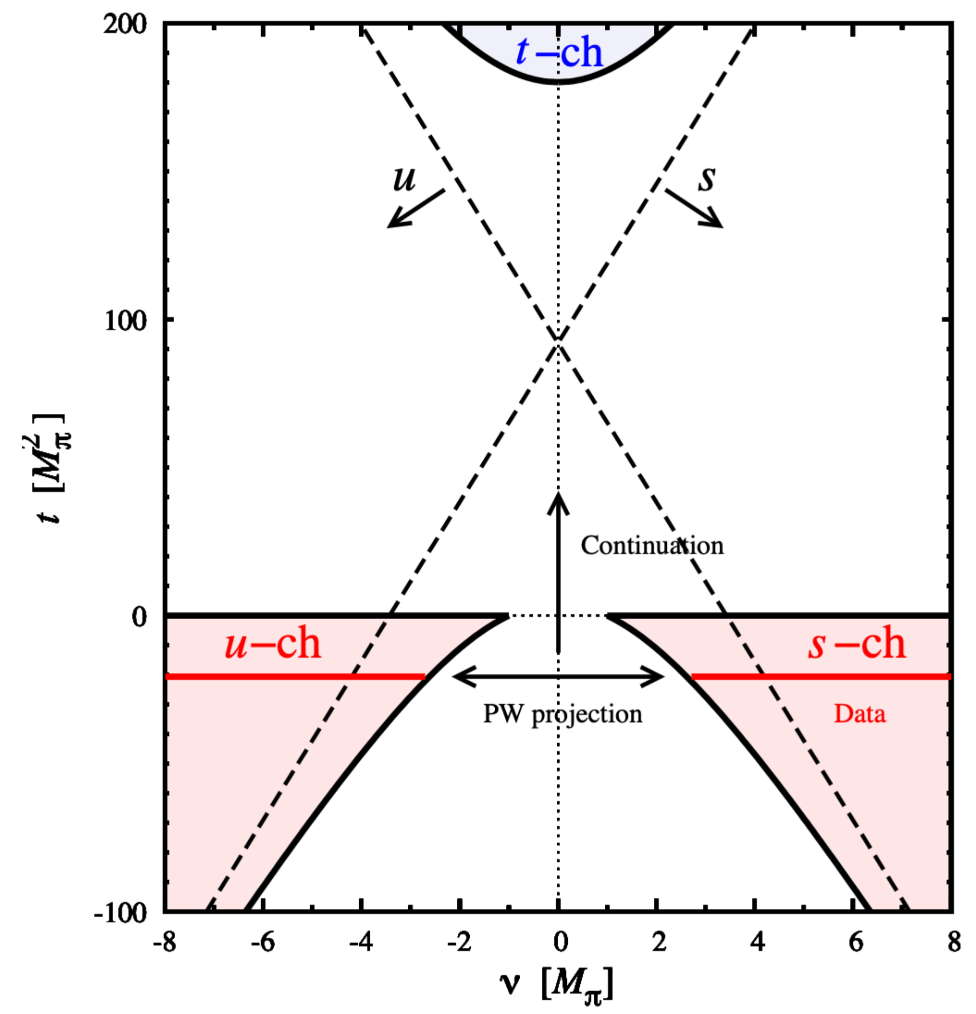


- t–channel partial wave
  in unphysical region of

  $\to$ analytic continua

- For spectral function w
  while $\pi N \to \pi$

  $\to$ analytic continuation in t

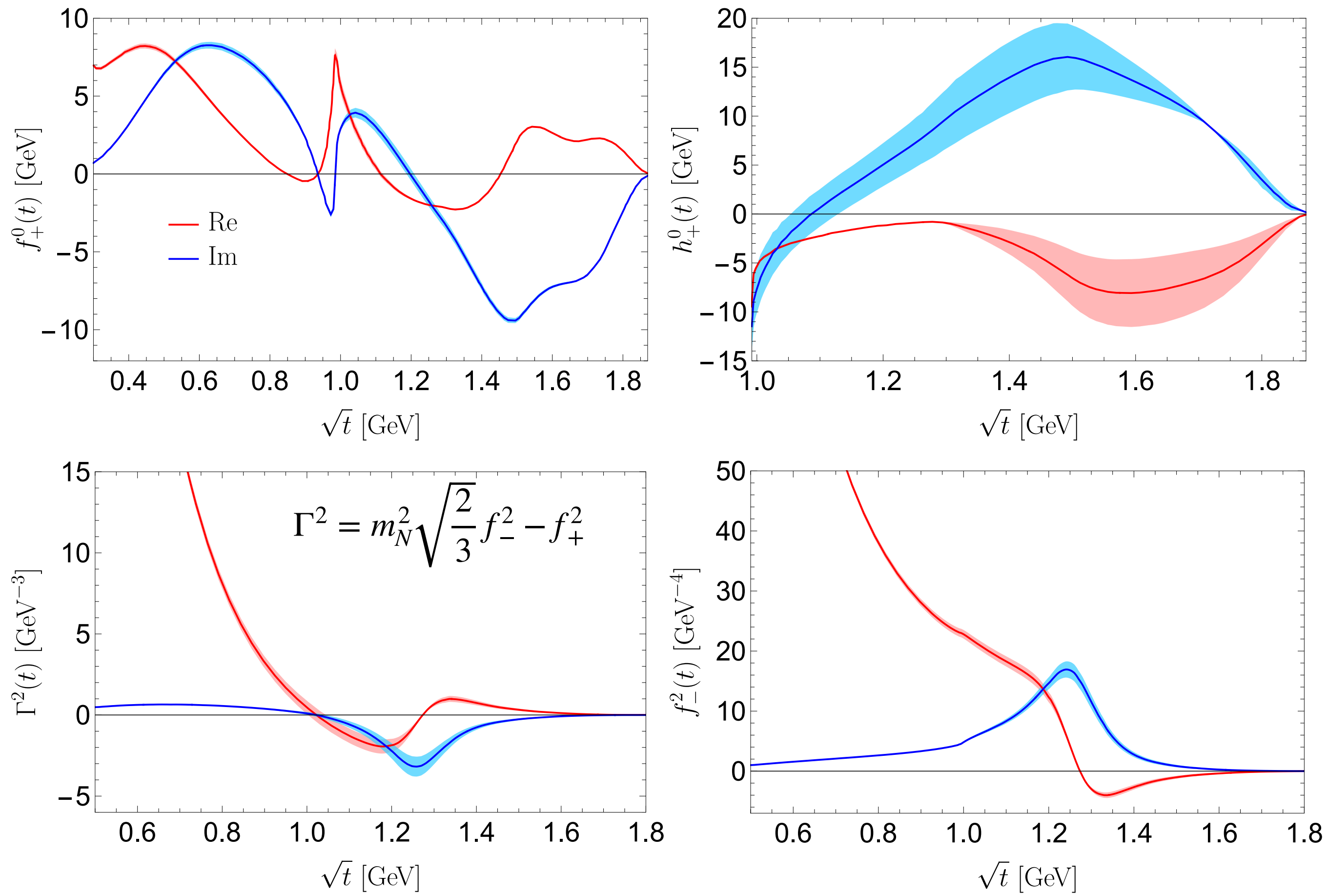


$$\nu = \frac{s-u}{4m_N}$$

Analytically continue PWAs to region $t > 4M_\pi^2$ via **Roy-Steiner eq.**

$\pi\pi/K\bar{K} \to N\bar{N}$ partial wave amplitudes

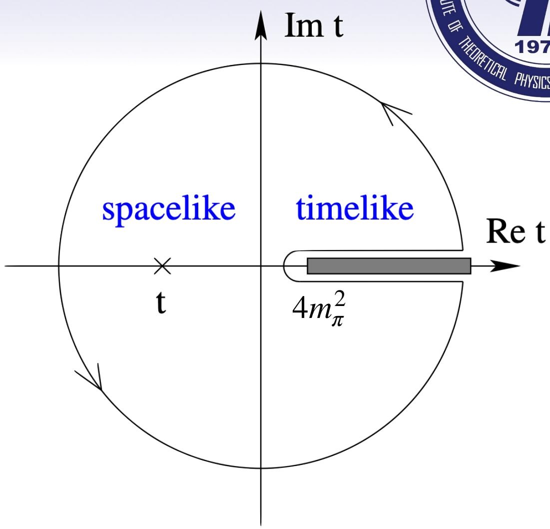
- Inputs: $\pi\pi/K\bar{K} \to N\bar{N}$ partial wave amp. $f_{\pm}^{0,2}, h_{+}^{0}$ from Roy-Steiner eq. analyses



πN amplitudes from Roy-Steiner equation analyses

Ditsche, et al., JHEP (2012); Hoferichter, et al., PRL 115, 092301(2015); PRL 115, 192301 (2015); Phys. Rept. (2016); **XHC, Q.-Z. Li & H.-Q. Zheng, JHEP (2022)**; Hoferichter, et al., PLB (2024)

Spectral functions of nucleon GFFs



- Dispersive relations (DRs) for the nucleon GFFs :

$$(A,J,\Theta)(t) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' \frac{\mathrm{Im}(A,J,\Theta)(t')}{t'-t}$$

- Constraints:
  - Normalizations: mass m_N, spin $1/2$

$\Rightarrow$ sum rules saturated by $\pi\pi, K\bar{K}$ continuum and some higher mass states

Belushkin et al., PRC (2007); Hoferichter et al., EPJA (2016) …

$$\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}(A,J,\Theta)(t')}{t'} = \left(1, \frac{1}{2}, m_N\right)$$

Introduce S-wave $0^{++}$and D-wave (2^{++}) poles to the spectral functions:

$\pi c_{S,D} m_{S,D}^2 \delta\left(t - m_{S,D}^2\right)$ to satisfy the sum rules

✓ 0^{++} : $m_S \in (1.5, 1.8)$ GeV to cover $\mathrm{f}_0(1500)$ and $\mathrm{f}_0(1710)$

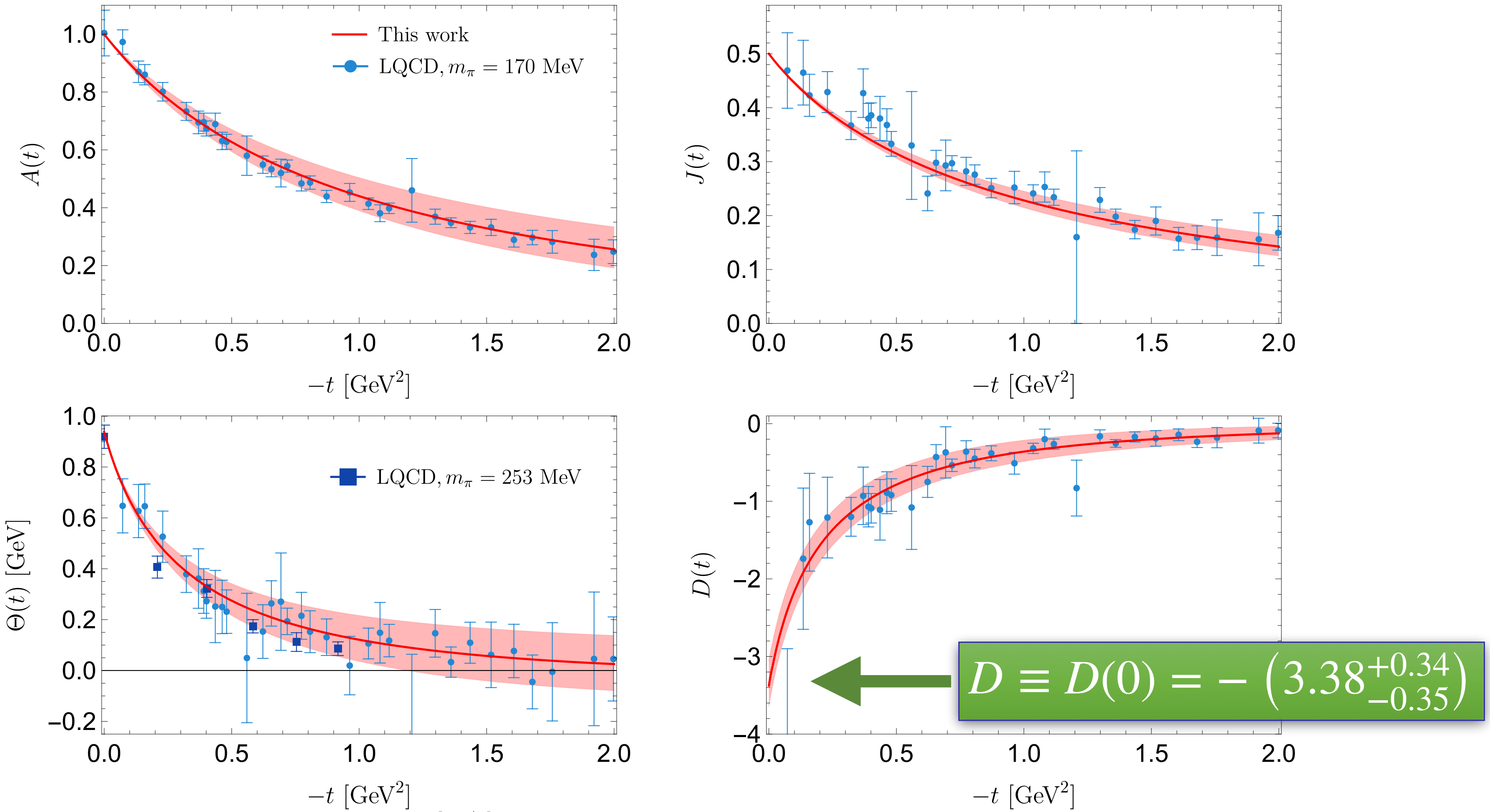
✓ 2^{++} : $m_D \in (1.5, 2.2)$ GeV to cover $\mathrm{f}_2(1500)$, $\mathrm{f}_2(1950)$ and $\mathrm{f}_2(2010)$

Nucleon GFFs

XHC, F.-K. Guo, Q.-Z. Li and D.-L. Yao, Nature Commun. (2025)

- Predictions：

LQCD ($m_\pi \simeq 170$ MeV): Hackett et al., PRL (2024)



LQCD ($m_\pi \simeq 253$ MeV): Wang et al. [χQCD], PRD (2024)

☑ GFFs be model-independently predicted using data-driven dispersion relations + ChPT

☑ Not a fit!

Nucleon GFFs: D-term & radii

- Various radii in the Breit frame
  - Radius of the scalar trace density

$$\langle r_\Theta^2 \rangle = \frac{6\dot{\Theta}(0)}{m_N} = 6\dot{A}(0) - \frac{9D}{2m_N^2}$$

  - Radius of the mass (energy) density

$$\langle r_{\text{Mass}}^2 \rangle = 6\dot{A}(0) - \frac{3D}{2m_N^2}$$

  - Mechanical radius

Polyakov, PLB (2003)
Polyakov, & Schweitzer, IJMPA (2018)

$$\langle r_{\text{Mech}}^2 \rangle = \frac{6D}{\int_{-\infty}^{0} \mathrm{d}t D(t)}$$

  - Radius of the angular momentum (spin) density $J(t) + \frac{2}{3}t\frac{\mathrm{d}J(t)}{\mathrm{d}t}$

$$\langle r_J^2 \rangle = 20J'(0)$$

Polyakov, PLB (2003)
Lorcé et al., PLB (2018)

Quantity	Result	Error budget
D-term	$-3.38^{+0.34}_{-0.35}$	$+(0.18)_{\text{ChPT}}(0.12)_{\text{PWA}}(0.26)_{\text{eff}}$
		$-(0.16)_{\text{ChPT}}(0.12)_{\text{PWA}}(0.29)_{\text{eff}}$
$\sqrt{\langle r_\Theta^2 \rangle}$	$0.97^{+0.03}_{-0.03}$	$+(0.01)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.03)_{\text{eff}}$
		$-(0.02)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.26)_{\text{eff}}$
$\sqrt{\langle r_{\text{Mass}}^2 \rangle}$	$0.70^{+0.03}_{-0.04}$	$+(0.02)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.02)_{\text{eff}}$
		$+(0.02)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.03)_{\text{eff}}$
$\sqrt{\langle r_{\text{Mech}}^2 \rangle}$	$0.72^{+0.09}_{-0.08}$	$+(0.02)_{\text{ChPT}}(0.00)_{\text{PWA}}(0.09)_{\text{eff}}$
		$-(0.03)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.07)_{\text{eff}}$
$\sqrt{\langle r_J^2 \rangle}$	$0.70^{+0.02}_{-0.02}$	$+(0.01)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.01)_{\text{eff}}$
		$-(0.01)_{\text{ChPT}}(0.00)_{\text{PWA}}(0.02)_{\text{eff}}$

ChPT: NLO ChPT inputs
pwa: $\pi\pi/K\bar{K} \to N\bar{N}$
eff: effective poles m_S, m_D

Physics of the scalar (trace) radius

- The scalar (trace) radius is larger than the mass radius as long as $D < 0$

$T_\mu^{\ \mu}$ $\qquad$ T^{00}

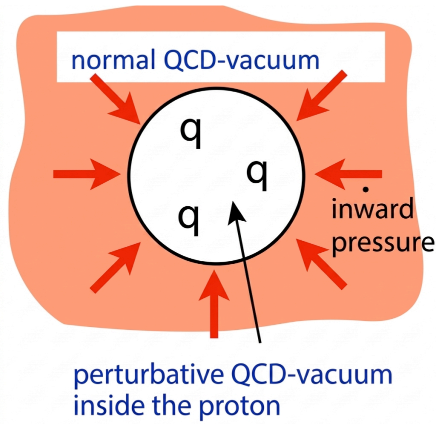
$$\sqrt{\langle r_\Theta^2 \rangle} = 0.97^{+0.03}_{-0.03}\mathrm{fm} \quad > \quad \sqrt{\langle r_{C,p}^2 \rangle} = 0.84\mathrm{fm} \quad > \quad \sqrt{\langle r_{\mathrm{Mass}}^2 \rangle} = 0.70^{+0.03}_{-0.04}\mathrm{fm}$$

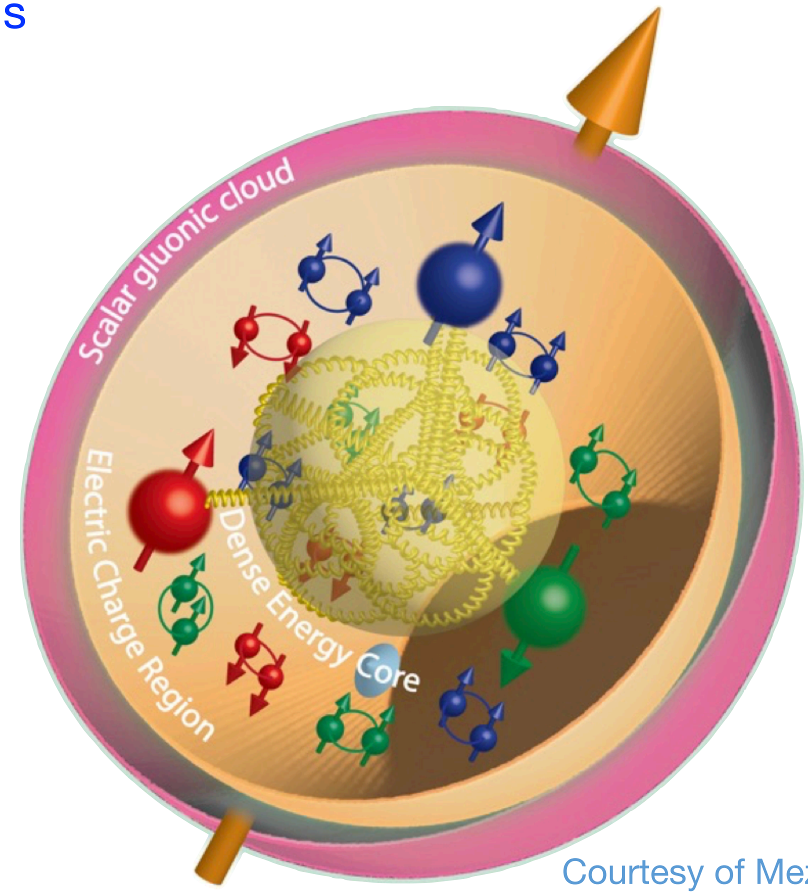
proton charge radius

"the largest radius in strong interaction"

- In the MIT bag model, scalar radius may be considered as the bag (confinement) radius ~1 fm (physical boundary of confinement)

Ji, Front.Phys.(Beijing) (2021)





Courtesy of Meziani

Summary and outlook

- ☑ The unity of dispersive techniques and lattice QCD data is powerful to investigate low energy hadron physics

- ☑ Widely-used unitarization methods such as K-matrix, etc., are not good in light meson & baryon studies

- ☑ The pion, kaon and nucleon GFFs are precisely determined using model-independent dispersive method

- ☑ Nucleon static properties:

$$D = -3.38^{+0.34}_{-0.35}, \quad \sqrt{\langle r^2_{\Theta} \rangle} = 0.97^{+0.03}_{-0.03}\,\text{fm} \quad > \quad \sqrt{\langle r^2_{C,p} \rangle} = 0.84\,\text{fm} \quad > \quad \sqrt{\langle r^2_{\text{Mass}} \rangle} = 0.70^{+0.03}_{-0.04}\,\text{fm}$$

proton charge radius

Thank you for your attention!

Muskhelishvili-Omnes representation

- **Coupled-channel**: solution known as the Muskhelishvili-Omnes (MO) representation
  - Take isoscalar-scalar $\pi\pi - K\bar{K}$ as example (matching point~1.3 GeV)

$$\mathbf{\Omega}_0^0(t) = \frac{1}{\pi}\int_{t_\pi}^{\infty} \frac{\mathrm{d}t'}{t'-t}\left[\mathbf{T}_0^0(t)\right]^* \mathbf{\Sigma}_0^0(t)\mathbf{\Omega}_0^0(t')$$

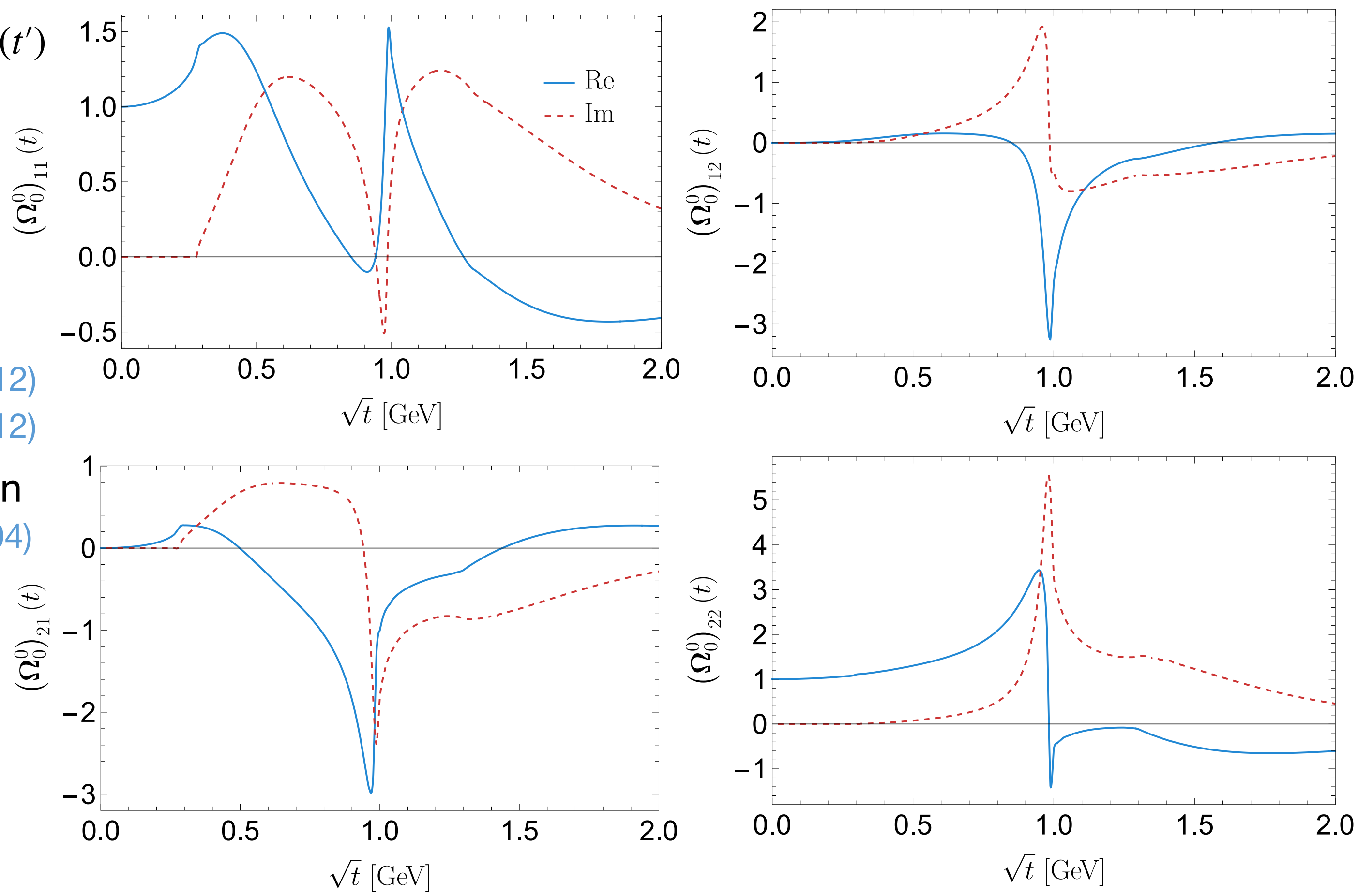
$\pi\pi$ phase shift: Roy equation

Ananthanarayan et al., Phys. Rept. (2012)

Caprini et al., EPJC (2012)

$\pi\pi \to K\bar{K}$: Roy-Steiner equation

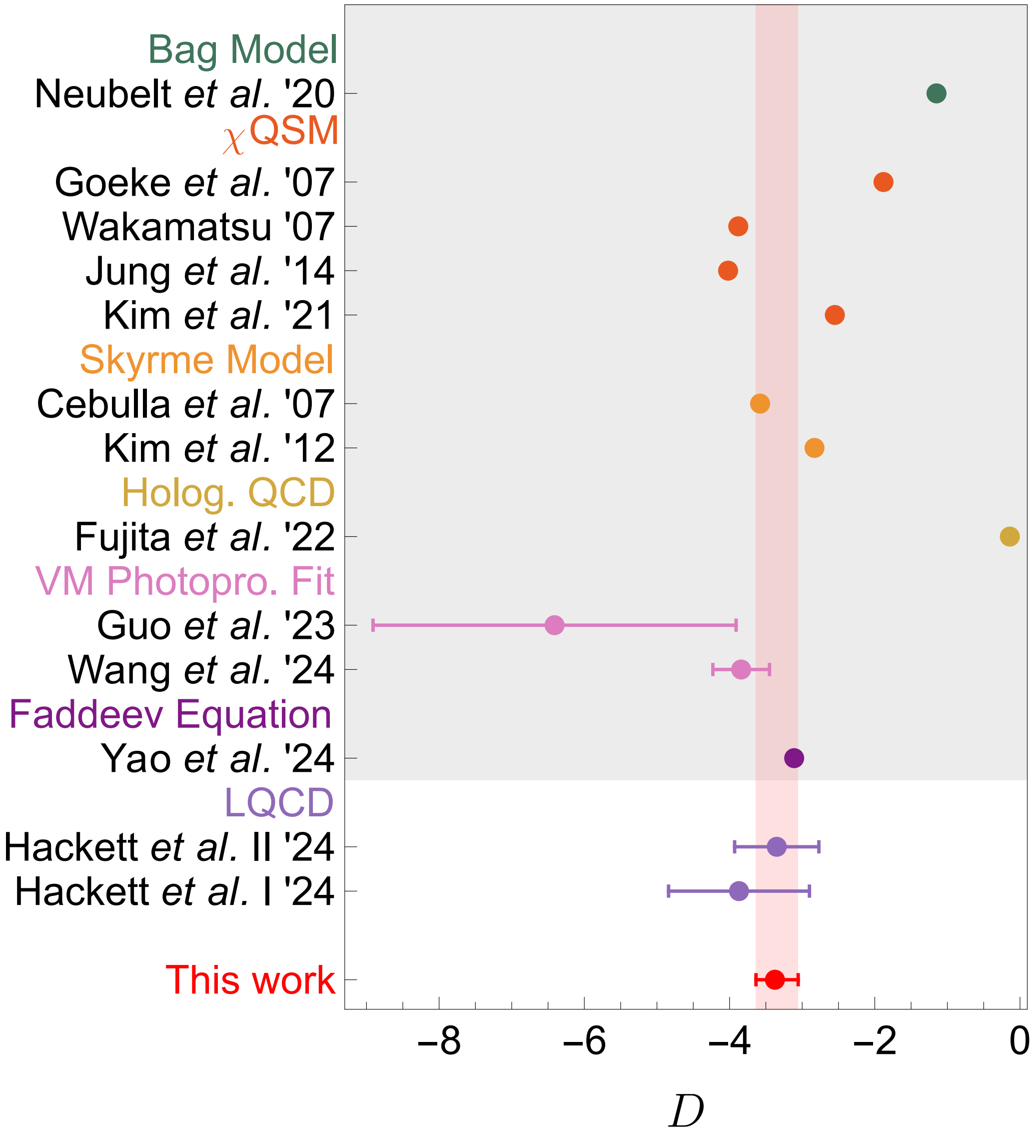
Büttiker et al., EPJC (2004)



See also Hoferichter et al., JHEP (2012)

Nucleon D-term

- Nucleon D-term $D \equiv D(0)$: $D = -\left(3.38^{+0.34}_{-0.35}\right)$



- Positivity bound:

$$D < -0.2$$

Gegelia & Polyakov, PLB (2021)

"The stability condition"

normal force $\frac{2}{3}s(r) + p(r) \geq 0$

$$\widetilde{D}(r) = \int \frac{d^3\mathbf{\Delta}}{(2\pi)^3} e^{-i\mathbf{\Delta r}} D\left(-\mathbf{\Delta}^2\right) < 0$$

Comparison with other works