Spin hydrodynamics

Dirk H. Rischke

thanks to

Xin-li Sheng, Masoud Shokri, Enrico Speranza,
David Wagner, Nora Weickgenannt, Qun Wang

based on

PRD 106 (2022) 096014, PRD 106 (2022) L091901, PRD 106 (2022) 116021, Phys. Rev. Res. 6 (2024) 4, 4

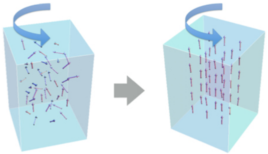
Seminar at USTC Hefei, China, October 16, 2025

Rotation leads to polarization

Condensed matter: **Barnett effect**

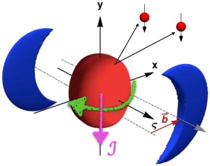
$\Longrightarrow$ spins align under rotation
(due to spin-orbit coupling)

$\Longrightarrow$ non-vanishing magnetization



Non-central heavy-ion collisions:
strong orbital angular momentum

Barnett effect in heavy-ion collisions?



$\Longrightarrow$ hadrons are polarized by vorticity of matter

Z.-T. Liang, X.-N. Wang, PRL 94 (2005) 102301

PRL **94,** 102301 (2005) PHYSICAL REVIEW LETTERS week ending 18 MARCH 2005

Globally Polarized Quark-Gluon Plasma in Noncentral $A + A$ Collisions

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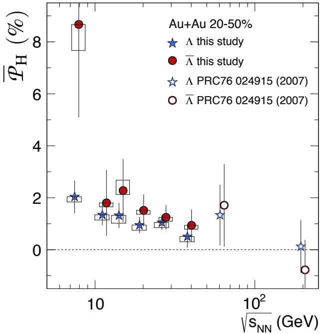
(Received 25 October 2004; published 14 March 2005)

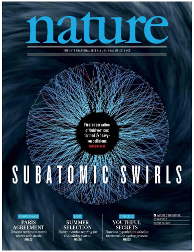
Λ polarization along angular-momentum direction ("global polarization")



L. Adamczyk et al. (STAR), Nature 548 (2017) 62

$\Longrightarrow$ QGP is "most vortical fluid ever observed"

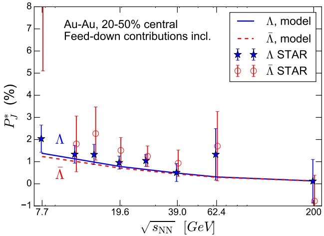
$\omega \simeq (9 + 1) \times 10^{21}\,\mathrm{s}^{-1}$



For comparison:

- Great Red Spot of Jupiter $\omega \simeq 10^{-4}\,\mathrm{s}^{-1}$
- turbulent flow in superfluid He-II $\omega \sim 150\,\mathrm{s}^{-1}$
- superfluid nanodroplets $\omega \sim 10^{7}\,\mathrm{s}^{-1}$

Assuming local equilibrium on freeze-out hypersurface, hydrodynamics describes global polarization quite well . . .



$$\Pi^\mu \sim \epsilon^{\mu\nu\rho\sigma} k_\nu \int_{\Sigma_{\text{f.o.}}} d\Sigma \cdot k \, \varpi_{\rho\sigma} f_{0\mathbf{k}}$$

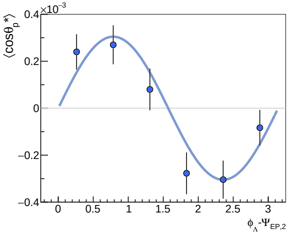
where $\varpi_{\rho\sigma} \equiv -\frac{1}{2}\left(\partial_\rho \beta_\sigma - \partial_\sigma \beta_\rho\right)$ thermal vorticity,

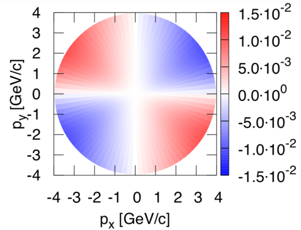
$\beta^\mu \equiv u^\mu / T$, $f_{0\mathbf{k}}$ local-equilibrium distribution function

I. Karpenko, F. Becattini, NPA 967 (2017) 764

. . . but fails to describe azimuthal-angle dependence of polarization along the beam direction (“local longitudinal polarization”)

F. Becattini, M.A. Lisa, Ann. Rev. Nucl. Part. Sci. 70 (2020) 395





Polarization and hydrodynamics

Further (non-dissipative?) contributions to the polarization were found

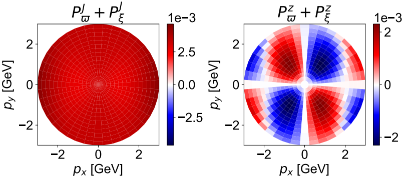
F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, A. Palermo, PRL 127 (2021) 272302

$$\Pi^\mu \sim \epsilon^{\mu\nu\rho\sigma} k_\nu \int_{\Sigma_{\text{f.o.}}} d\Sigma \cdot k \left(\varpi_{\rho\sigma} + 2\hat{t}_\rho \xi_{\sigma\lambda} \frac{k^\lambda}{k_0} \right) f_{0\mathbf{k}}$$

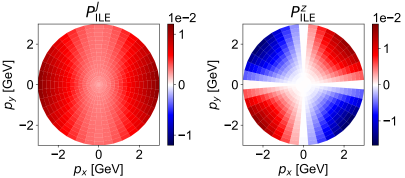
Here, $\xi_{\sigma\lambda} \equiv \frac{1}{2}\partial_{(\sigma}\beta_{\lambda)}$ thermal shear tensor, where $a_{(\sigma}b_{\lambda)} \equiv a_\sigma b_\lambda + a_\sigma b_\lambda$

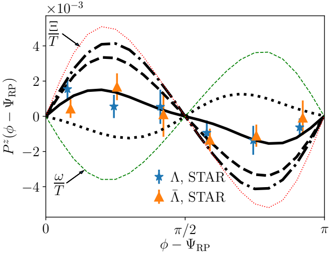
(see also S.Y.F. Liu, Y. Yin, JHEP 07 (2021) 188)

$\Longrightarrow$ this does not quite do the job ...



... but neglecting temperature gradients on $\Sigma_{\text{f.o.}}$ does!





Observations:

- derivation of formula for polarization ($\Pi^\mu \sim \varpi_{\rho\sigma}$) is strictly valid only for rotating global-equilibrium state
  F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Annals Phys. 338 (2013) 32

  $\Longrightarrow$ application of formula requires fast equilibration of spin degrees of freedom (relative to timescale of collision)
- dissipative effects influence (almost) all other observables in a heavy-ion collision, even seem to appear explicitly in new term $\sim \xi_{\sigma\lambda}$ in formula for polarization

$\Longrightarrow$ Questions:

(I) How fast do spin degrees of freedom equilibrate?

(II) How is polarization influenced by dissipative effects?

$\Longrightarrow$ requires a theory of second-order dissipative spin hydrodynamics!

N. Weickgenannt, D. Wagner, E. Speranza, DHR,
PRD 106 (2022) 096014, PRD 106 (2022) L091901

D. Wagner, PRD 111 (2025) 016008

Particle number conservation

$$\partial_\mu N^\mu = 0$$

where N^μ particle four-current

Energy-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

where $T^{\mu\nu}$ energy-momentum tensor

Angular-momentum conservation

$$\partial_\mu J^{\mu,\nu\lambda} = 0$$

where $J^{\mu,\nu\lambda}$ angular-momentum tensor

With $J^{\mu,\nu\lambda} \equiv x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu} + \hbar S^{\mu,\nu\lambda}$ and energy-momentum conservation:

Equation of motion for spin tensor

$$\hbar \partial_\mu S^{\mu,\nu\lambda} = T^{[\lambda\nu]}$$

where $a^{[\lambda} b^{\nu]} \equiv a^\lambda b^\nu - a^\nu b^\lambda$

Particle number conservation

$$\partial_\mu N^\mu = 0$$

1 equation, 4 unknowns

Energy-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

4 equations, (at least) 10 unknowns

Angular-momentum conservation

$$\partial_\mu J^{\mu,\nu\lambda} = 0$$

With $J^{\mu,\nu\lambda} \equiv x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu} + \hbar S^{\mu,\nu\lambda}$ and energy-momentum conservation:

Equation of motion for spin tensor

$$\hbar \partial_\mu S^{\mu,\nu\lambda} = T^{[\lambda\nu]}$$

6 equations, 24 unknowns

where $a^{[\lambda} b^{\nu]} \equiv a^\lambda b^\nu - a^\nu b^\lambda$

Assume local equilibrium

$\Longrightarrow$ **Single-particle distribution function (to order $\mathcal{O}(\hbar)$)**

$$f_{\text{eq},\mathbf{k}\mathfrak{s}} = f_{0\mathbf{k}} \left(1 + \frac{\hbar}{4} \Omega_{\mu\nu} \Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}} \right) \,, \quad f_{0\mathbf{k}} = \exp\left(\alpha - \beta \cdot k \right)$$

where

- $\alpha \equiv \dfrac{\mu}{T}$ Lagrange multiplier for particle-number conservation
- $\beta^\mu \equiv \dfrac{u^\mu}{T}$ Lagrange multiplier for energy-momentum conservation, u^μ fluid 4-velocity
- $\Omega_{\mu\nu}$ spin potential, Lagrange multiplier for angular-momentum conservation
- $\Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}} \equiv -\dfrac{1}{m} \epsilon^{\mu\nu\alpha\beta} k_\alpha \mathfrak{s}_\beta$ dipole-moment tensor of particle with mass m, (on-shell) 4-momentum k_α, and spin 4-vector $\mathfrak{s}_\beta$

Assume local equilibrium

$\Longrightarrow$ **Single-particle distribution function (to order $\mathcal{O}(\hbar)$)**

$$f_{\text{eq},\mathbf{k}\mathfrak{s}} = f_{0\mathbf{k}} \left(1 + \frac{\hbar}{4} \Omega_{\mu\nu} \Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}} \right) \;, \quad f_{0\mathbf{k}} = \exp\left(\alpha - \beta \cdot k\right)$$

where

- $\alpha \equiv \dfrac{\mu}{T}$ Lagrange multiplier for particle-number conservation
  1 parameter
- $\beta^\mu \equiv \dfrac{u^\mu}{T}$ Lagrange multiplier for energy-momentum conservation, u^μ fluid 4-velocity
  4 parameters
- $\Omega_{\mu\nu}$ spin potential, Lagrange multiplier for angular-momentum conservation
  6 parameters
- $\Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}} \equiv -\dfrac{1}{m} \epsilon^{\mu\nu\alpha\beta} k_\alpha \mathfrak{s}_\beta$ dipole-moment tensor of particle with mass m, (on-shell) 4-momentum k_α, and spin 4-vector $\mathfrak{s}_\beta$

Extend phase space by spin degrees of freedom:

$$dK \equiv \frac{d^3\mathbf{k}}{(2\pi)^3 k^0} \longrightarrow d\Gamma \equiv dK\, dS(k)$$

with $\quad dS(k) \equiv \sqrt{\frac{k^2}{3\pi^2}}\, d^4\mathfrak{s}\, \delta(k\cdot\mathfrak{s})\, \delta(\mathfrak{s}\cdot\mathfrak{s}+3)$,

such that $\int dS(k) = 2\,,\ \int dS(k)\,\mathfrak{s}^\mu = 0\,,\ \int dS(k)\,\mathfrak{s}^\mu\mathfrak{s}^\nu = -2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$

$\Longrightarrow$

Fluid-dynamical currents

$$\begin{aligned} N^\mu &\equiv \langle k^\mu \rangle \\ T^{\mu\nu} &\equiv \langle k^\mu k^\nu \rangle + \mathcal{O}(\hbar^2) \\ S^{\mu,\nu\lambda} &\equiv \frac{1}{2}\left\langle k^\mu \Sigma^{\nu\lambda}_{\mathbf{k}\mathfrak{s}} \right\rangle + \mathcal{O}(\hbar^2) \end{aligned}$$

where $\langle \cdots \rangle \equiv \int d\Gamma \cdots f(x,k,\mathfrak{s})$

$\Longrightarrow$ for $\langle \cdots \rangle_{\text{eq}} \equiv \int d\Gamma \cdots f_{\text{eq},\mathbf{k}\mathfrak{s}}$ $\Longrightarrow$ equations of motion are closed!

see, e.g., W. Florkowski, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

Dissipative spin hydrodynamics

$\Longrightarrow$ provide additional equations of motion

$\Longrightarrow$ start from underlying microscopic theory, apply method of moments
see, e.g., G.S. Denicol, H. Niemi, E. Molnár, DHR, PRD 85 (2012) 114047

From quantum kinetic theory, in semi-classical approximation, i.e., to order $\mathcal{O}(\hbar)$, derive:
N. Weickgenannt, E. Speranza, X.-l. Sheng, Q. Wang, DHR,
PRL 127 (2021) 052301, PRD 104 (2021) 016022

Boltzmann equation for spin-1/2 particles with nonlocal collision term

$$k \cdot \partial f(x, k, \mathfrak{s}) = \mathfrak{C}[f]$$

$$\mathfrak{C}[f] = \int d\Gamma_1 d\Gamma_2 d\Gamma' d\bar{S}(k)\, \mathcal{W}_{\mathbf{k}\mathbf{k}' \to \mathbf{k}_1 \mathbf{k}_2}^{\bar{\mathfrak{s}}\mathfrak{s}' \to \mathfrak{s}_1 \mathfrak{s}_2} (2\pi\hbar)^4 \delta^{(4)}(k + k' - k_1 - k_2)$$

$$\times \left[f(x + \Delta_1 - \Delta, k_1, \mathfrak{s}_1) f(x + \Delta_2 - \Delta, k_2, \mathfrak{s}_2) - f(x, k, \bar{\mathfrak{s}}) f(x + \Delta' - \Delta, k', \mathfrak{s}') \right]$$

where nonlocal position shift $\quad \Delta^\mu \sim \dfrac{\hbar}{m}$ Compton wavelength!

$\Longrightarrow$ quantum length scale!

Nonlocal collisions: allow mutual conversion of orbital angular momentum and spin

D. Wagner, N. Weickgenannt, DHR, PRD 106 (2022) 116021

For NJL-type 4-fermion interaction $\sim (\bar{\psi}\Gamma^{(a)}\psi)^2$

covariant position shift

$$\Delta^\mu = \frac{\hbar}{m} \frac{\mathrm{Im}\left\{\mathrm{Tr}\left[h\gamma^\mu\Gamma^{(b)}h_2\Gamma^{(a)}\right]\mathrm{Tr}\left[\Gamma^{(b)}h_1\Gamma^{(a)}h'\right] - \mathrm{Tr}\left[h\gamma^\mu\Gamma^{(b)}h_1\Gamma^{(a)}h'\Gamma^{(b)}h_2\Gamma^{(a)}\right]\right\}}{4\,\mathrm{Re}\left\{\mathrm{Tr}\left[h\Gamma^{(d)}h_2\Gamma^{(c)}\bar{h}\right]\mathrm{Tr}\left[\Gamma^{(d)}h_1\Gamma^{(c)}h'\right] - \mathrm{Tr}\left[h\Gamma^{(d)}h_1\Gamma^{(c)}h'\Gamma^{(d)}h_2\Gamma^{(c)}\bar{h}\right]\right\}}$$

where $h \equiv h(k,\mathfrak{s}) \equiv \frac{1}{4m}(\mathbb{1} + \gamma_5 \not{\mathfrak{s}})(\not{k} + m)$

and similarly for $(k_1, \mathfrak{s}_1)$, $(k_2, \mathfrak{s}_2)$, $(k', \mathfrak{s}')$, $(k, \bar{\mathfrak{s}})$

Usually, local-equilibrium distribution function f_{eq} is defined by condition

Local equilibrium

$$\mathfrak{C}\left[f_{\text{eq}}\right] \equiv 0$$

However, as shown in N. Weickgenannt, E. Speranza, X.-l. Sheng, Q. Wang, DHR, PRL 127 (2021) 052301, PRD 104 (2021) 016022, nonlocal collision term vanishes only in

Global equilibrium

$$\begin{aligned} \alpha &= \text{const.} \\ \partial^{(\mu}\beta^{\nu)} &= 0 \\ \Omega_{\mu\nu} &= \varpi_{\mu\nu} = \text{const.} \end{aligned}$$

$\Longrightarrow$ appears too restrictive: nonlocality $\Delta \lesssim r_{\text{int}} \ll \lambda_{\text{mfp}} \ll L_{\text{hydro}}$

$\Longrightarrow$ (quantum) nonlocality scale Δ much smaller than hydrodynamic scale L_{hydro}

$\Longrightarrow$

Generalized local equilibrium

$$\mathfrak{C}\left[f_{\text{eq}}\right] \sim \mathcal{O}(\Delta / L_{\text{hydro}})$$

Define hydrodynamic scale L_{hydro} by $\frac{1}{m} k \cdot \partial f_{\text{eq},\mathbf{k}\mathfrak{s}} \sim \frac{1}{L_{\text{hydro}}} f_{\text{eq},\mathbf{k}\mathfrak{s}}$

$\Longrightarrow$

$$\begin{aligned} \partial_\mu \alpha &\sim \mathcal{O}(L_{\text{hydro}}^{-1}) \\ \partial^{(\mu}\beta^{\nu)} &\sim \mathcal{O}((kL_{\text{hydro}})^{-1}) \\ \partial_\lambda \Omega_{\mu\nu} &\sim \mathcal{O}(L_{\text{hydro}}^{-1}) \end{aligned}$$

$\Longrightarrow$ using conservation of total angular momentum $J^{\mu\nu} \equiv \Delta^{[\mu} k^{\nu]} + \frac{\hbar}{2}\Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}}$ in binary collisions, order $\mathcal{O}(\hbar)$ contribution to nonlocal collision term:

$$\sim \frac{\hbar}{4}\Omega_{\mu\nu}\Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}} + \Delta^\mu \partial_\mu(\alpha - \beta_\nu k^\nu) = \frac{1}{2}\Delta^{[\mu} k^{\nu]}(\varpi_{\mu\nu} - \Omega_{\mu\nu}) + \mathcal{O}(\Delta/L_{\text{hydro}})$$

$\Longrightarrow$ for generalized local equilibrium: $\Omega_{\mu\nu} - \varpi_{\mu\nu} \sim \mathcal{O}((kL_{\text{hydro}})^{-1})$

$\Longrightarrow$ consistent, as in global equilibrium $L_{\text{hydro}} \to \infty$ and thus $\Omega_{\mu\nu} \to \varpi_{\mu\nu}$

Usually, all gradients of fluid-dynamical quantities are $\mathcal{O}(L_{\text{hydro}}^{-1})$

However, global-equilibrium conditions do not restrict value of thermal vorticity $\varpi_{\mu\nu}$
see, e.g., F. Becattini, L. Tinti, Annals Phys. 325 (2010) 1566

$\Longrightarrow$ vorticity does not follow usual power counting, $\varpi_{\mu\nu} \not\sim \mathcal{O}((kL_{\text{hydro}})^{-1})$

$\Longrightarrow$ define scale ℓ_{vort} set by vorticity: $\boxed{\varpi_{\mu\nu} \sim \mathcal{O}((k\ell_{\text{vort}})^{-1})}$

In principle, ℓ_{vort} can take any value from $\ell_{\text{vort}} \ll L_{\text{hydro}}$ to $\ell_{\text{vort}} \sim L_{\text{hydro}}$

However, in order for $\hbar-$expansion to apply: $\hbar\Omega_{\mu\nu}\Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}} \sim \dfrac{\hbar}{m}\varpi_{\mu\nu}\epsilon^{\mu\nu\alpha\beta}k_\alpha\mathfrak{s}_\beta \sim \dfrac{\Delta}{\ell_{\text{vort}}} \ll 1$

$\Longrightarrow$ ℓ_{vort} cannot be arbitrarily small (as in global equilibrium)

$\Longrightarrow$ remember $\Delta \lesssim r_{\text{int}} \ll \lambda_{\text{mfp}} \ll L_{\text{hydro}}$ $\Longrightarrow$ ℓ_{vort} could be as small as λ_{mfp}!

$\Longrightarrow$ for the sake of simplicity assume $\boxed{\dfrac{\Delta}{\ell_{\text{vort}}} \sim \dfrac{\lambda_{\text{mfp}}}{L_{\text{hydro}}} \equiv \text{Kn} \ll 1}$

Extend method of moments developed in G.S. Denicol, H. Niemi, E. Molnár, DHR, PRD 85 (2012) 114047 by spin degrees of freedom

Single-particle distribution function

$$f(x,k,\mathfrak{s}) = f_{\text{eq},\mathbf{k}\mathfrak{s}} + \delta f_{\mathbf{k}\mathfrak{s}}$$

$$\delta f_{\mathbf{k}\mathfrak{s}} = f_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n \in \mathbb{S}_\ell} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \left[\rho_n^{\mu_1 \cdots \mu_\ell} - \tau_n^{\mu,\mu_1\cdots\mu_\ell} \left(\Delta_{\mu\nu} + \frac{k_{\langle\mu\rangle} k_{\langle\nu\rangle}}{E_\mathbf{k}^2} \right) \mathfrak{s}^\nu \right] k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle}$$

where

- $k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle}$ irreducible tensors
- $\rho_n^{\mu_1\cdots\mu_\ell} \equiv \left\langle E_\mathbf{k}^n k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \right\rangle_\delta$ irreducible moments, $\langle \cdots \rangle_\delta \equiv \langle \cdots \rangle - \langle \cdots \rangle_{\text{eq}}$
- $\tau_n^{\mu,\mu_1\cdots\mu_\ell} \equiv \left\langle \mathfrak{s}^\mu E_\mathbf{k}^n k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \right\rangle_\delta$ spin moments
- $E_\mathbf{k} \equiv k \cdot u$ energy of particle in fluid rest frame
- $A^{\langle \mu \rangle} \equiv \Delta^{\mu\nu} A_\nu$, with $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ projector onto 3-space orthogonal to u^μ
- $A^{\langle \mu_1 \cdots \mu_\ell \rangle} \equiv \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} A^{\nu_1\cdots\nu_\ell}$,
  $\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}$ symmetric, traceless rank-2ℓ projection operators built from $\Delta^{\mu\nu}$
- $\mathcal{H}_{\mathbf{k}n}^{(\ell)}$ polynomials in $E_\mathbf{k}$ of rank N_ℓ

Define local-equilibrium state via

Landau matching conditions

$$\begin{aligned} N^\mu u_\mu &= N^\mu_{\rm eq} u_\mu \\ T^{(\mu\nu)} u_\nu &= T^{(\mu\nu)}_{\rm eq} u_\nu \\ S^{\mu,\nu\lambda} u_\mu &= S^{\mu,\nu\lambda}_{\rm eq} u_\mu \end{aligned}$$

$\Longrightarrow$ relates α, β^μ, and $\Omega_{\mu\nu}$ in $f_{\mathrm{eq},\mathbf{k}\mathfrak{s}}$ to fluid-dynamical variables

$\Longrightarrow$ determine α, β^μ, and $\Omega_{\mu\nu}$ via conservation laws! (more details follow later)

Equations of motion for standard dissipative currents

$\Longrightarrow$ see G.S. Denicol, H. Niemi, E. Molnár, DHR, PRD 85 (2012) 114047

$\Longrightarrow$ relaxation-type equations, e.g., for shear-stress tensor:

$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \ldots$, with $\dot{\pi}^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} u^\lambda \partial_\lambda \pi^{\alpha\beta}$

where $\sigma^{\mu\nu} \equiv \partial^{\langle\mu} u^{\nu\rangle}$ fluid shear tensor

Take spin moments of Boltzmann equation:

Equations of motion for spin moments

$$\Longrightarrow \quad \dot{\tau}_n^{\langle\mu\rangle,\langle\mu_1\cdots\mu_\ell\rangle} - \mathfrak{C}_{n-1}^{\langle\mu\rangle,\mu_1\cdots\mu_\ell} = \ldots$$

$$\mathfrak{C}_{n-1}^{\mu,\mu_1\cdots\mu_\ell} = \int d\Gamma\, E_{\mathbf{k}}^{n-1} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\mathfrak{s}^\mu\, \mathfrak{C}[f]$$

Linearized collision integral

$$\begin{aligned}
\Longrightarrow \quad \mathfrak{C}_{n-1}^{\mu,\mu_1\cdots\mu_\ell} &= \mathfrak{C}_{n-1,\text{local}}^{\mu,\mu_1\cdots\mu_\ell} + \mathfrak{C}_{n-1,\text{nonlocal}}^{\mu,\mu_1\cdots\mu_\ell} \\
\mathfrak{C}_{n-1,\text{local}}^{\mu,\mu_1\cdots\mu_\ell} &= -\sum_{r\in\mathbb{S}_\ell} B_{nr}^{(\ell)} \tau_r^{\mu,\mu_1\cdots\mu_\ell} \\
\mathfrak{C}_{n-1,\text{nonlocal}}^{\mu,\mu_1\cdots\mu_\ell} &= \int d\Gamma_1 d\Gamma_2 d\Gamma d\Gamma'\, \mathcal{W}_{\mathbf{k}\mathbf{k}'\to\mathbf{k}_1\mathbf{k}_2}^{\mathfrak{s}\mathfrak{s}'\to\mathfrak{s}_1\mathfrak{s}_2} E_{\mathbf{k}}^{n-1} f_{0\mathbf{k}} f_{0\mathbf{k}'} \\
&\times\; k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\mathfrak{s}^\mu \left[\frac{\hbar}{4}(\varpi_{\alpha\beta} - \Omega_{\alpha\beta})\Sigma_{\mathbf{k}\mathfrak{s}}^{\alpha\beta} + \xi_{\alpha\beta}\Delta^\alpha k^\beta\right]
\end{aligned}$$

- $\mathfrak{C}_{n-1,\text{local}}^{\mu,\mu_1\cdots\mu_\ell} \Longrightarrow$ inverting $B_{nr}^{(\ell)}$ yields relaxation times
- $\mathfrak{C}_{n-1,\text{nonlocal}}^{\mu,\mu_1\cdots\mu_\ell} \Longrightarrow$ gives rise to Navier-Stokes terms

Infinite set of moment equations needs to be truncated

$\Longrightarrow$ lowest-order truncation:

14 standard fluid-dynamical moments + 24 moments for components of spin tensor

$\Longrightarrow$ (14+24)–moment approximation

Without dissipation, spin tensor determined by 6 components of spin potential $\Omega_{\mu\nu}$

Independent dissipative spin moments: $\mathfrak{p}^\mu \equiv \tau_0^{\langle\mu\rangle}$, $\mathfrak{z}^{\mu\nu} \equiv \tau_1^{\langle\mu\rangle,\nu} + \tau_1^{\langle\nu\rangle,\mu}$, $\mathfrak{q}^{\mu\nu\lambda} \equiv \tau_0^{\langle\mu\rangle,\nu\lambda}$

2 + 6 + 10 components

Equations of motion for independent dissipative spin moments

$\Longrightarrow$

$$\begin{aligned}
\tau_{\mathfrak{p}}\, \dot{\mathfrak{p}}^{\langle\mu\rangle} + \mathfrak{p}^\mu &= \epsilon^{\mu\nu\alpha\beta}\left(\varpi_{\alpha\beta} - \Omega_{\alpha\beta}\right) u_\nu + \ldots \\
\tau_{\mathfrak{z}}\, \dot{\mathfrak{z}}^{\langle\mu\rangle\langle\nu\rangle} + \mathfrak{z}^{\mu\nu} &= \ldots \\
\tau_{\mathfrak{q}}\, \dot{\mathfrak{q}}^{\langle\mu\rangle\langle\nu\lambda\rangle} + \mathfrak{q}^{\mu\nu\lambda} &= \frac{1}{T}\sigma_\alpha{}^{\langle\nu}\epsilon^{\lambda\rangle\mu\alpha\beta} u_\beta + \ldots
\end{aligned}$$

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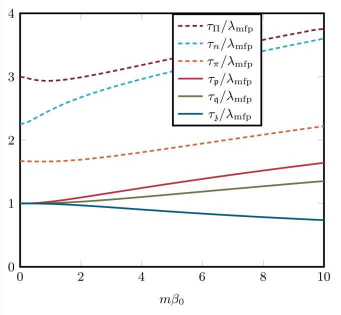
D. Wagner, PRD 111 (2025) 016008

Spin relaxation times



$\Longrightarrow$ spin relaxation times of the same order (but somewhat smaller) than relaxation times for Π, n^μ, $\pi^{\mu\nu}$

$\Longrightarrow$ dissipative spin d.o.f.'s equilibrate (i.e., approach their Navier-Stokes values) as fast (or even faster) than Π, n^μ, $\pi^{\mu\nu}$

$\Longrightarrow$ answers Question (I) for dissipative spin d.o.f's

Pauli-Lubanski vector (spin polarization vector!) in Navier-Stokes limit

$$\Pi^\mu_{\rm NS} \sim \int_{\Sigma_{\rm f.o.}} d\Sigma \cdot k\, f_{0\mathbf{k}} \Big\{ \epsilon^{\mu\nu\rho\sigma} k_\nu \Omega_{\rho\sigma} + \left(\delta^\mu_\nu - \frac{u^\mu k_{\langle\nu\rangle}}{E_\mathbf{k}} \right) \times \left[\kappa_{\mathfrak{p}} \epsilon^{\nu\rho\alpha\beta} \left(\Omega_{\alpha\beta} - \varpi_{\alpha\beta} \right) u_\rho + \frac{\kappa_{\mathfrak{q}}}{T} \sigma_\alpha{}^{\langle\rho} \epsilon^{\sigma\rangle\nu\alpha\beta} u_\beta k_{\langle\rho} k_{\sigma\rangle} \right] \Big\}$$

$\Longrightarrow$ novel dissipative(?) corrections $\sim \Omega_{\alpha\beta} - \varpi_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ $\Longrightarrow$ answers Question (II)

Conservation equations of spin hydrodynamics

$$\partial_\mu N^\mu = 0\,, \quad \partial_\mu T^{\mu\nu} = 0\,, \quad \hbar\partial_\mu S^{\mu,\nu\lambda} = T^{[\lambda\nu]}$$

Idea: Solve them perturbatively in powers of $\hbar$

$\Longrightarrow$ Decompose $T^{\mu\nu} = \frac{1}{2}T^{(\mu\nu)} + \frac{1}{2}T^{[\mu\nu]}$

Note: $\frac{1}{2}T^{(\mu\nu)} = \varepsilon\, u^\mu u^\nu + (P+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \sim \mathcal{O}(1)$
while: $\hbar S^{\mu,\nu\lambda} \sim \mathcal{O}(\hbar^2)$, $T^{[\mu\nu]} = -\frac{\hbar}{2}\int \mathrm{d}\Gamma\, \Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}}\mathfrak{C}[f] \sim \mathcal{O}(\hbar^2)$

Conservation equations to $\mathcal{O}(\hbar)$

$$\partial_\mu N^\mu = 0\,, \quad \partial_\mu T^{(\mu\nu)} = 0$$

$\Longrightarrow$ To order $\mathcal{O}(\hbar)$, conservation equations of spin hydrodynamics are identical to conservation equations of ordinary fluid dynamics!

Moreover, to order $\mathcal{O}(\hbar)$, spin degrees of freedom do not couple to relaxation equations for dissipative quantities!

$\Longrightarrow$ To order $\mathcal{O}(\hbar)$, EoM's of dissipative spin hydrodynamics are identical to EoM's of ordinary dissipative fluid dynamics!

$\Longrightarrow$ To order $\mathcal{O}(\hbar)$, spin dynamics decouples from bulk evolution of the fluid!

At order $\mathcal{O}(\hbar^2)$, spin dynamics is given by

Conservation equation for total angular momentum

$$\hbar \partial_\mu S^{\mu,\nu\lambda} = T^{[\lambda\nu]}$$

plus relaxation equations for dissipative spin d.o.f.'s

To first approximation, neglect dissipative spin d.o.f.'s, solve "ideal-spin" hydrodynamics

Ideal spin tensor

$$S^{\mu,\nu\lambda} = Au^\mu \Omega^{\nu\lambda} + Bu^\mu u_\alpha \Omega^{\alpha[\nu} u^{\lambda]} + Cu^\mu \Omega^{\alpha[\nu} \Delta^{\lambda]}_{\ \alpha} + E\Delta^\mu_{\ \alpha} \Omega^{\alpha[\nu} u^{\lambda]}$$

with thermodynamic coefficients $A, \ldots, E \sim \mathcal{O}(\hbar)$

$\Longrightarrow$ conservation equation for total angular momentum leads to coupled system of relaxation equations for components of $\Omega^{\mu\nu} \equiv u^{[\mu}\kappa^{\nu]} + \epsilon^{\mu\nu\alpha\beta} u_\alpha \omega_\beta$

Relaxation equations for spin potential (static background)

$$\begin{aligned} \tau_\kappa \dot{\boldsymbol{\kappa}} + \boldsymbol{\kappa} &= \mu_\kappa \boldsymbol{\nabla} \times \boldsymbol{\omega} \\ \tau_\omega \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} &= -\mu_\omega \boldsymbol{\nabla} \times \boldsymbol{\kappa} \end{aligned}$$

D. Wagner, M. Shokri, DHR, Phys. Rev. Res. 6 (2024) 4, 4

τ_κ , τ_ω relaxation times, μ_κ , μ_ω length scales given by $T^{[\mu\nu]} = -\frac{\hbar}{2} \int \mathrm{d}\Gamma \, \Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}} \mathfrak{C}[f]$

Solving spin hydrodynamics III

Taking divergence, one sees that $\nabla \cdot \boldsymbol{\kappa}$, $\nabla \cdot \boldsymbol{\omega}$ relax to zero on timescales given by τ_κ , τ_ω

Assuming $\nabla \cdot \boldsymbol{\kappa} = \nabla \cdot \boldsymbol{\omega} = 0$ and taking time derivative, eqs. decouple and one obtains

Damped wave equations for spin potential

$$\ddot{\boldsymbol{X}} + a\dot{\boldsymbol{X}} + b\boldsymbol{X} - c_{\mathfrak{s}}^2 \Delta \boldsymbol{X} = 0$$

$\boldsymbol{X} = \boldsymbol{\kappa}, \boldsymbol{\omega}$,

$a = \frac{1}{\tau_\kappa} + \frac{1}{\tau_\omega}$, $\quad b = \frac{1}{\tau_\kappa \tau_\omega}$,

$c_{\mathfrak{s}}^2 = \frac{\mu_\kappa \mu_\omega}{\tau_\kappa \tau_\omega}$ spin-wave velocity

Dispersion relation

$$\mathfrak{w}_\pm(k) = \frac{i(\tau_\kappa + \tau_\omega) \pm \sqrt{4c_{\mathfrak{s}}^2 \tau_\kappa^2 \tau_\omega^2 k^2 - (\tau_\kappa - \tau_\omega)^2}}{2\tau_\kappa \tau_\omega}$$

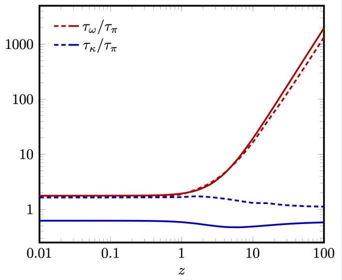
For $k \geq \frac{|\tau_\kappa - \tau_\omega|}{2c_{\mathfrak{s}} \tau_\kappa \tau_\omega}$: spin waves propagate and are damped on timescale $\tau_\star \sim \min(\tau_\kappa, \tau_\omega)$

For $k < \frac{|\tau_\kappa - \tau_\omega|}{2c_{\mathfrak{s}} \tau_\kappa \tau_\omega}$: spin waves do not propagate and are damped on timescale $\tau_\star \sim \max(\tau_\kappa, \tau_\omega)$

$\Omega_{\mu\nu} \to \varpi_{\mu\nu}$ on timescale $\tau_\star$

$\Longrightarrow$ answers Question (I) for non-dissipative spin d.o.f's

Spin-wave damping times



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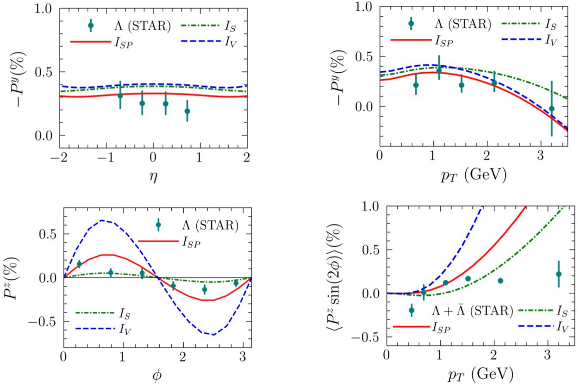
Timescale ordering

$\tau_{\text{diss. spin}} \lesssim \tau_{\text{diss. fluid}} \lesssim \tau_{\text{non-diss. spin}}$

$\Longrightarrow$ dynamical solution of spin hydrodynamics on non-trivial background fluid evolution

Sapna, S. K. Singh, D. Wagner, arXiv:2503.22552 [hep-ph]



$\Longrightarrow$ spin hydrodynamics can solve sign puzzle of local longitudinal polarization!

Conclusions and Outlook

- Starting from Boltzmann equation with nonlocal collision term, and using method of moments, derived equations of motion of relativistic second-order dissipative spin hydrodynamics in (14+24)–moment approximation
- $\hbar$ power-counting: dynamics of spin degrees of freedom decouples from bulk evolution of the fluid!
- Ordering of timescales:

$$\tau_{\text{diss. spin}} \lesssim \tau_{\text{diss. fluid}} \lesssim \tau_{\text{non-diss. spin}}$$

$\Longrightarrow$ Dissipative spin degrees of freedom relax as fast (or even faster) as usual dissipative quantities

$\Longrightarrow$ Non-dissipative spin degrees of freedom follow damped wave equations; damping timescale can be larger than that for usual dissipative quantities

$\Longrightarrow$ Need to dynamically solve spin hydrodynamics (on given background fluid evolution)!

- Spin hydrodynamics can solve sign problem of local longitudinal polarization!