Dihadron angular correlations in the e^+e^- collision

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FFs & EEC workshop@IMP
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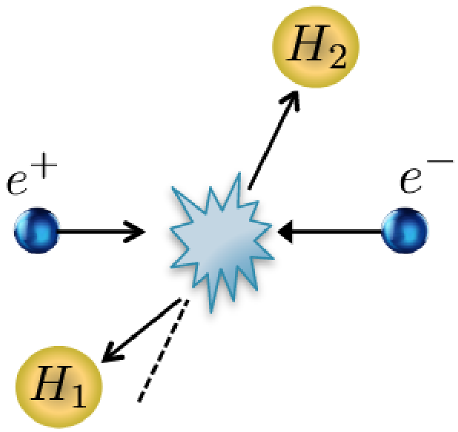
Based on Wan-Li Ju, Zhe-Yan Shu, Tong-Zhi Yang, Zhen-Hua Zhang, Hua Xing Zhu, arXiv: 2506.11463

Dihadron: Background

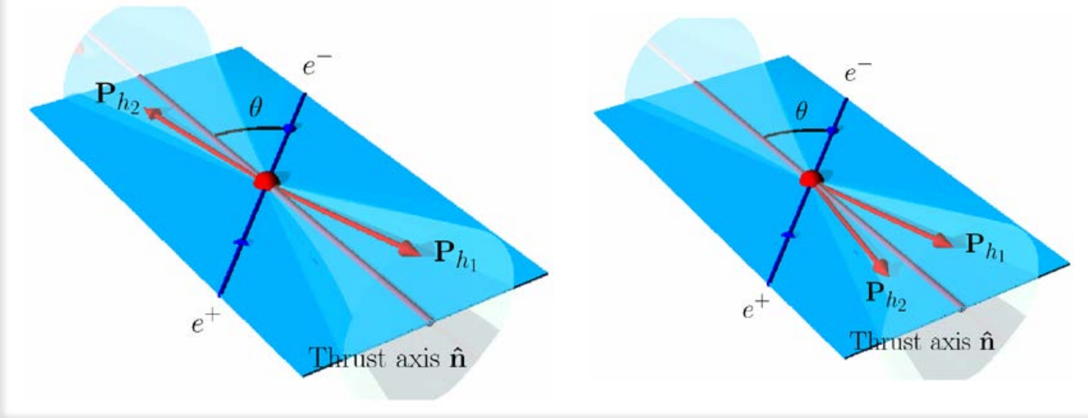


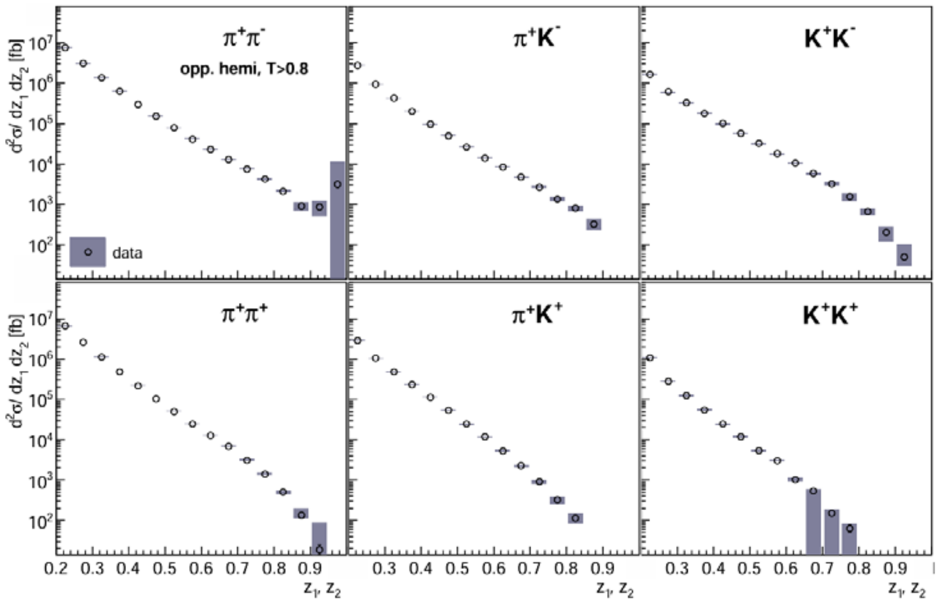
- $e^+e^- \to H_1 H_2 + X$ is a clean process to study the fragmentation functions
- Dihadron differential cross section

$$\frac{d\sigma^{H_1 H_2}}{dz_1 dz_2} = \sum_{ij} \hat{\sigma}^{ij} \otimes \underbrace{D_i^{H_1} \otimes D_j^{H_2}}_{\text{single-hadron FFs}} + \sum_i \hat{\sigma}^i \otimes \underbrace{DD_i^{H_1 H_2}}_{\text{di-hadron FF}}$$

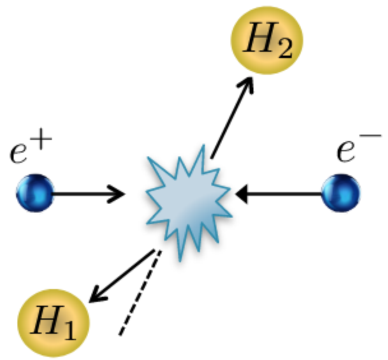
- Experimental study : Belle *PRD* 92 (2015) 9, 092007, *PRD* 96 (2017) 3, 032005, *PRD* 101 (2020) 9, 092004

Three topologies





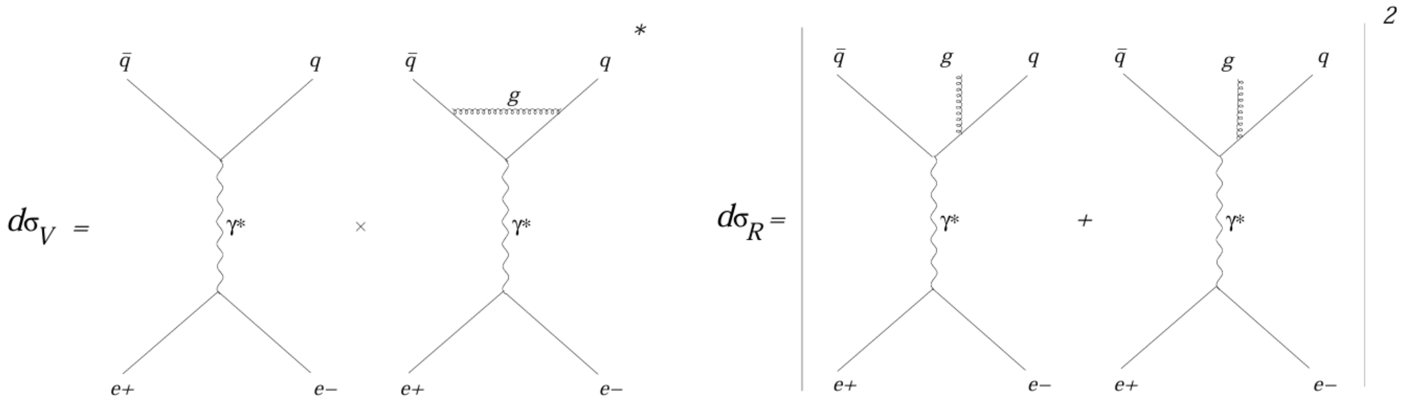
Dihadron: Background



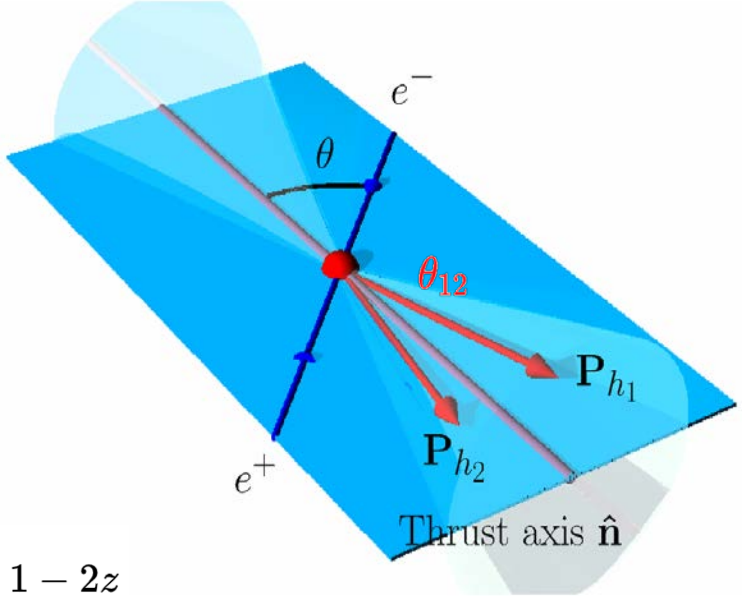
- Theoretical calculation :

$$\frac{d\sigma^{H_1H_2}}{dz_1dz_2} = \sum_{ij} \hat{\sigma}^{ij} \otimes \underbrace{D_i^{H_1} \otimes D_j^{H_2}}_{\text{single-hadron FFs}} + \sum_i \hat{\sigma}^i \otimes \underbrace{DD_i^{H_1H_2}}_{\text{di-hadron FF}}$$

- ☐ $\mathcal{O}(\alpha_S)$ D. de Florian, L.Vanni, *Phys.Lett.B* 578 (2004) 139-149
- ☐ $\mathcal{O}(\alpha_S)$ Guido Altarelli, R. Keith Ellis, G. Martinelli, So-Young Pi, *Nucl.Phys.B* 160 (1979) 301-329

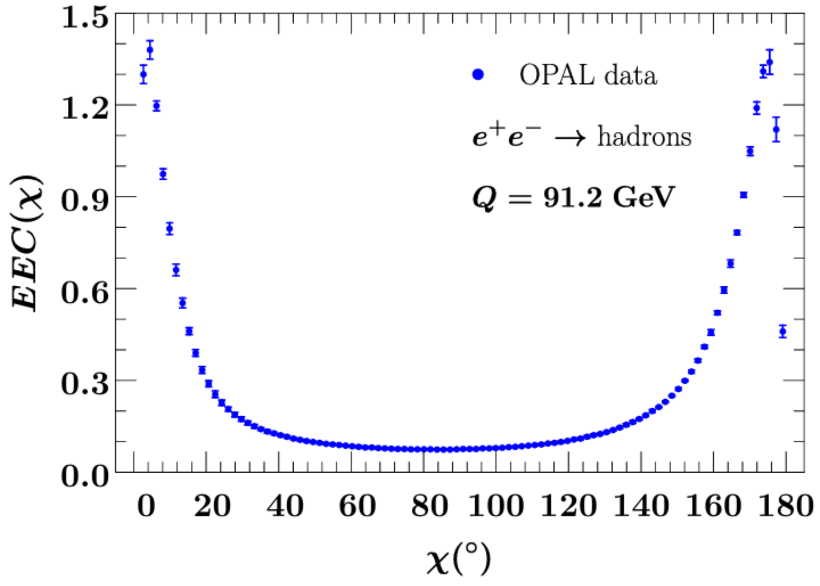


Dihadron: Background



- The topological classification of the dihadron production is rough
- Adding the angle ($z = \frac{1-\cos\theta_{12}}{2}$) between the two hadrons for precise topological classification
- Dihadron angular correlation $\frac{d^3\sigma}{d\tau_1 d\tau_2 dz}$ $\tau_1 = \frac{2E_1}{Q}$, $\tau_2 = \frac{2E_2}{Q}$, $\cos\theta_{12} = 1 - 2z$
- Success of energy-energy correlator (EEC) as an angular correlation function for precision QCD

  OPAL, Z. Phys. C 59, 1-19 (1993)



- Study of free hadron and transition region of EEC from DiFF

  K. Lee, I. Stewart, arXiv: 2507.11495
  C.-H. Chang, H. Chen, X. Liu, et al., arXiv:2507.15923
  Y. Guo, F. Yuan, W. Zhao, arXiv: 2507.15820
  Z.-B. Kang, A. Metz, D.Pitonyak, et al. , arXiv: 2507.17444

Dihadron angular correlation

$$\tau_1 = \frac{2E_1}{Q}, \quad \tau_2 = \frac{2E_2}{Q}, \quad \cos\theta_{12} = 1 - 2z$$

- Factorization:

$$\frac{\mathrm{d}^3\sigma}{\mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}z} = \frac{2\pi\alpha}{3Q^4} \sum_{i,j} \int_{\tau_1}^{1} \frac{\mathrm{d}x_1}{x_1} D_{H_1/i}\left(\frac{\tau_1}{x_1}, \mu\right) \int_{\tau_2}^{1} \frac{\mathrm{d}x_2}{x_2} D_{H_2/j}\left(\frac{\tau_2}{x_2}, \mu\right) C_{ij}(x_1, x_2, z, \mu)$$

- Wilson coefficients:

$$C_{ij}(x_1, x_2, z, \mu) = \sum_m \int \mathrm{d}\Phi_{[m]} \sum_{a \neq b \in X_m} \Theta_{ij}^{ab} \sum_{\mathrm{pol,col}} \left|\mathcal{M}(\gamma^* \to i + j + \cdots)\right|^2$$

- Measurement function:

$$\Theta_{ij}^{ab} = \delta_i^a \, \delta\left(x_1 - \frac{2q_\gamma \cdot k_a}{Q^2}\right) \delta_j^b \, \delta\left(x_2 - \frac{2q_\gamma \cdot k_b}{Q^2}\right) \delta\left(z - \frac{1}{x_1 x_2} \frac{2k_a \cdot k_b}{Q^2}\right)$$

- $\mathcal{O}(\alpha_s^0)$, $\mathcal{O}(\alpha_s)$ contributions:

$$\mathcal{C}_{ij}[\mathcal{O}(\alpha_s^0)], \mathcal{C}_{ij}^{\mathrm{VR}}[\mathcal{O}(\alpha_s)] \propto \delta(1 - x_1)\delta(1 - x_2)\delta(1 - z)$$

$$\mathcal{C}_{ij}^{\mathrm{RR}}[\mathcal{O}(\alpha_s)] \propto \delta(1 - x_1 - x_2 + x_1 x_2 z) \equiv \delta(m_X^2)$$

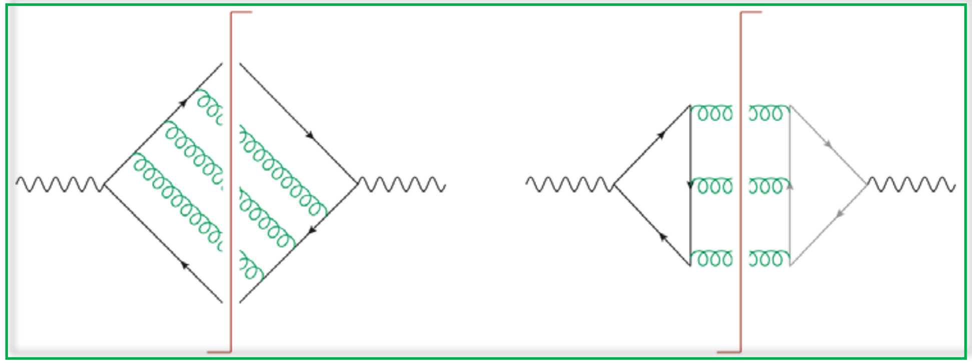
Non-contact contributions start at $\mathcal{O}(\alpha_s^2)$! Focus on the region $x_1, x_2, z \in (0,1)$

$\mathcal{O}(\alpha_s)$ results

- Wilson Coefficients: $C_{ij}(x_1, x_2, z, \mu) = 2\alpha N_C Q^2 \left\{ \sum_q e_q^2 \, \mathcal{C}_{ij;q} + \sum_{q,q'} e_q e_{q'} \, \mathcal{C}_{ij;qq'} \right\}$

$$\mathcal{C}_{ij;q}(x_1, x_2, z, \mu) = \sum_i \left[\frac{\alpha_s(\mu)}{4\pi} \right]^i \mathcal{C}^{(i)}_{ij;q}(x_1, x_2, z, \mu)$$

$C_{ij;qq'}$ contribute from $\mathcal{O}(\alpha_s^3)$

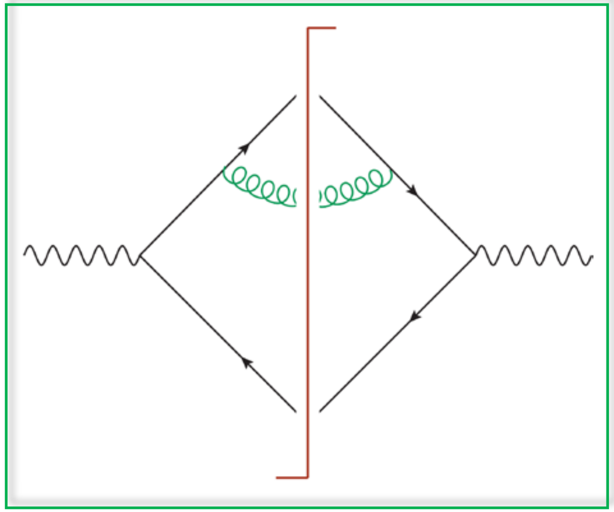


- Real emission at $\mathcal{O}(\alpha_s)$: $\gamma^*(q_\gamma) \to q(p_1) + \bar{q}(p_2) + g(p_3)$

$$\mathcal{C}^{(1)}_{q\bar{q};q}(x_1, x_2; \varepsilon = 0) = \delta(m_X^2) \frac{N_A}{N_C} \frac{x_1 x_2 (x_1^2 + x_2^2)}{(1 - x_1)(1 - x_2)}$$

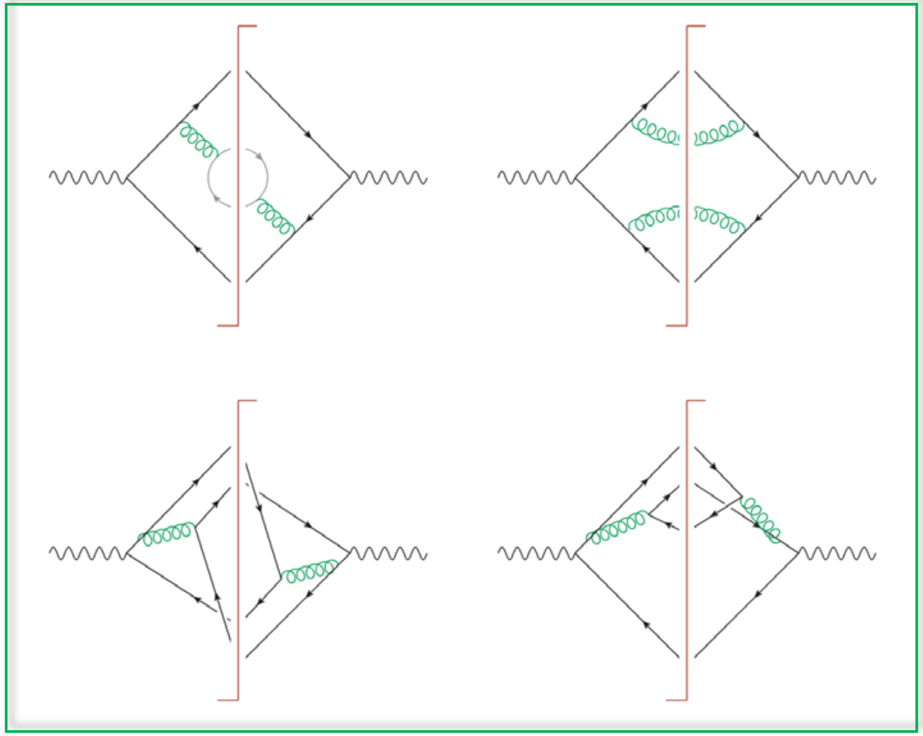
$$\mathcal{C}^{(1)}_{qg;q}(x_1, x_2; \varepsilon = 0) = \delta(m_X^2) \frac{N_A}{N_C} \frac{x_1 x_2 [x_1^2 + (2 - x_1 - x_2)^2]}{(1 - x_1)(x_1 + x_2 - 1)}$$

$$\mathcal{C}^{(1)}_{g\bar{q};q}(x_1, x_2; \varepsilon = 0) = \mathcal{C}^{(1)}_{qg;q}(x_2, x_1; \varepsilon = 0)$$



$m_X^2 = 1 - x_1 - x_2 + x_1 x_2 z$ $\qquad d = 4 - 2\varepsilon$

$\mathcal{O}(\alpha_s^2)$ Calculation: Double Real



- Three channels:

$$\gamma^*(q_\gamma) \to q(p_1) + \bar{q}(p_2) + q'(p_3) + \bar{q}'(p_4)$$
$$\gamma^*(q_\gamma) \to q(p_1) + \bar{q}(p_2) + g(p_3) + g(p_4)$$
$$\gamma^*(q_\gamma) \to q(p_1) + \bar{q}(p_2) + q(p_3) + \bar{q}(p_4)$$

- Phase space integral

$$f_{RR}(x_1, x_2, z, q_\gamma^2) = \int \frac{d^d p_3}{(2\pi)^{d-1}} \delta(p_3^2)\delta\big((q_\gamma - p_1 - p_2 - p_3)^2\big) \left(\sum_{\text{perm}(i,j,k,l)} |\mathcal{M}(p_i, p_j, p_k, p_l)|^2 \right)$$

- Strategy: Integration By Parts (IBP) + Differential Equation (DE)

Topological identification of the integrals -> IBP reduction to the MIs--> Evaluate the MIs using DE

$\mathcal{O}(\alpha_s^2)$ Calculation: Double Real

- MIs:

$$\mathbf{MI1} = \int \frac{d^d p_3}{(2\pi)^{d-1}} \delta_+(p_3^2)\delta_+((q-p_1-p_2-p_3)^2) \qquad \mathbf{MI2} = \int \frac{d^d p_3}{(2\pi)^{d-1}} \frac{\delta_+(p_3^2)\delta_+((q-p_1-p_2-p_3)^2)}{(p_1+p_2+p_3)^2}$$

$$\mathbf{MI3} = \int \frac{d^d p_3}{(2\pi)^{d-1}} \frac{\delta_+(p_3^2)\delta_+((q-p_1-p_2-p_3)^2)}{(p_1+p_4)^2(p_2+p_3)^2} \qquad \mathbf{MI4} = \int \frac{d^d p_3}{(2\pi)^{d-1}} \frac{\delta_+(p_3^2)\delta_+((q-p_1-p_2-p_3)^2)}{(p_1+p_4)^2(p_2+p_4)^2}$$

$$\mathbf{MI5} = \int \frac{d^d p_3}{(2\pi)^{d-1}} \frac{\delta_+(p_3^2)\delta_+((q-p_1-p_2-p_3)^2)}{(p_1+p_3)^2(p_1+p_2+p_3)^2} \qquad \mathbf{MI6} = \int \frac{d^d p_3}{(2\pi)^{d-1}} \frac{\delta_+(p_3^2)\delta_+((q-p_1-p_2-p_3)^2)}{(p_1+p_3)^2(p_1+p_2+p_4)^2}$$

- MI1 and MI2 can be evaluated directly

$$\mathbf{MI1} = \frac{2^{4\epsilon-5}\pi^{\epsilon-\frac{3}{2}} m_X^{-2\epsilon}}{\Gamma\left(\frac{3}{2}-\epsilon\right)} \qquad \mathbf{MI2} = \frac{2^{2\epsilon-3}\pi^{\epsilon-2} m_X^{-2\epsilon}\Gamma(1-\epsilon)}{\beta+x_1+x_2}\, {}_2\tilde{F}_1\left(1, 1-\epsilon; 2-2\epsilon; \frac{2\beta}{x_1+x_2+\beta}\right)$$

$m_X = \sqrt{1-x_1-x_2+x_1x_2z}$: invariant mass of the unidentified particles

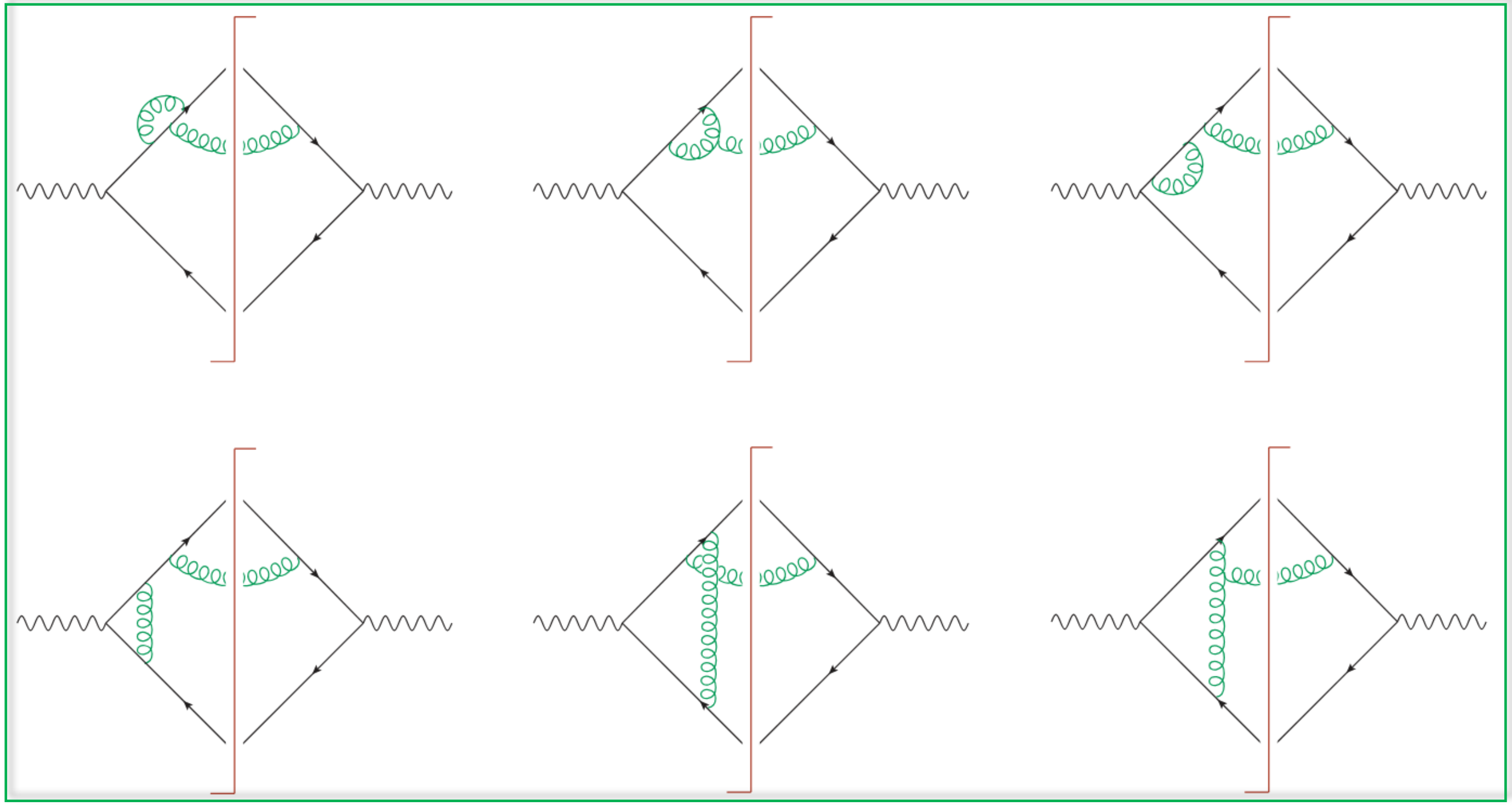
$\beta = \sqrt{(x_1+x_2)^2-4x_1x_2z}$: relative momentum between the identified and unidentified particles

- MI3-MI6 are calculated using DE to transcendental weight-4, with the boundary condition settled by AMFlow + PSLQ Fit

$\mathcal{O}(\alpha_s^2)$ Calculation: Virtual Real

- The virtual real contributions are only nonzero for $m_X \to 0$

$$\mathcal{C}_{ij}^{\mathrm{VR}} \propto \delta(m_X^2)\left|\mathcal{M}[\gamma^* \to q\bar{q}g]\right|^2$$



$\mathcal{O}(\alpha_s^2)$ Calculation: Renormalization

- α_s renormalization for UV divergence:

$$\alpha_s^b \to \alpha_s(\mu^2)\left[1 - \frac{\alpha_s(\mu^2)}{4\pi\epsilon}\left(\frac{11}{3}C_A - \frac{2}{3}N_F\right) + \ldots\right]$$

- Collinear IR divergence=>absorbed into fragmentation functions

Collinear counter term:

$$\mathcal{C}_{ij}^{\mathrm{CT}}(x_1, x_2, z; \varepsilon) = \frac{1}{\varepsilon}\sum_{i'} \underbrace{P_{ii'}^{\mathrm{T},(0)}(x_1)}_{\mathcal{O}(\alpha_s)} \otimes \underbrace{\mathcal{C}_{i'j}^{(1)}(x_1, x_2, z; \varepsilon)}_{\mathcal{O}(\alpha_s)} + \frac{1}{\varepsilon}\sum_{j'} P_{jj'}^{\mathrm{T},(0)}(x_2) \otimes \mathcal{C}_{ij'}^{(1)}(x_1, x_2, z; \varepsilon)$$

- ϵ pole canceled by the collinear counter term:

$$\mathcal{C}_{ij}(x_1, x_2, z) = \mathcal{C}_{ij}^{\mathrm{RR}}(x_1, x_2, z; \varepsilon) + \mathcal{C}_{ij}^{\mathrm{VR}}(x_1, x_2, z; \varepsilon) + \mathcal{C}_{ij}^{\mathrm{CT}}(x_1, x_2, z; \varepsilon)$$

$\mathcal{O}(\alpha_s^2)$ Calculation: Collinear counter term

- Collinear counter term:

$$\mathcal{C}_{ij}^{\mathrm{CT}}(x_1, x_2, z; \varepsilon) = \frac{1}{\varepsilon} \sum_{i'} \underbrace{P_{ii'}^{\mathrm{T},(0)}(x_1)}_{\mathcal{O}(\alpha_s)} \otimes \underbrace{\mathcal{C}_{i'j}^{(1)}(x_1, x_2, z; \varepsilon)}_{\mathcal{O}(\alpha_s)} + \frac{1}{\varepsilon} \sum_{j'} P_{jj'}^{\mathrm{T},(0)}(x_2) \otimes \mathcal{C}_{ij'}^{(1)}(x_1, x_2, z; \varepsilon)$$

- Time-like splitting function:

$$P_{ij}^{\mathrm{T},(0)}(x) = P_{ij,\mathrm{reg}}^{\mathrm{T},(0)}(x) + P_{ij,\delta}^{\mathrm{T},(0)}\,\delta(1-x) + P_{ij,+}^{\mathrm{T},(0)}\left[\frac{1}{1-x}\right]_+$$

- Collinear counter term in m_X distributions

$$\mathcal{C}_{ij}^{\mathrm{CT}} = \mathcal{C}_{ij,\mathrm{reg}}^{\mathrm{CT}} + \mathcal{C}_{ij,\delta}^{\mathrm{CT}}\,\delta(m_X^2) + \mathcal{C}_{ij,*}^{\mathrm{CT}}\left[\frac{1}{m_X^2}\right]_*$$

- star distribution

$$\int_0^\tau dm_X^2 f(m_X^2)\left[\frac{\ln^k(m_X^2)}{m_X^2}\right]_* \equiv \int_0^\tau \frac{dm_X^2}{m_X^2}\Big[f(m_X^2) - f(0)\Big]\ln^k(m_X^2) + \frac{\ln^{k+1}(\tau)}{k+1} f(0)$$

$\mathcal{O}(\alpha_s^2)$ Calculation: $m_X \to 0$ limit

- RR contribution at $m_X \to 0$: Expansion to $\frac{1}{m_X^2}$. MIs resummation over m_X^2 using DE

- IR divergence isolated with the star distribution

$$(m_X^2)^{-1+n\epsilon} = \frac{1}{n\epsilon}\delta(m_X^2) + \sum_{k=0}^{\infty} \frac{(n\epsilon)^k}{k!} \left[\frac{\ln^k(m_X^2)}{m_X^2}\right]_*$$

- Results in terms of distributions

$$\mathcal{C}_{ij}\big|_{m_X^2\to 0} = \left(\mathcal{C}_{ij}^{\mathrm{RR}} + \mathcal{C}_{ij}^{\mathrm{VR}} + \mathcal{C}_{ij}^{\mathrm{CT}}\right)\Big|_{m_X^2\to 0} = \mathcal{C}_{ij,\mathrm{reg}} + \mathcal{C}_{ij,\delta}\,\delta(m_X^2) + \mathcal{C}_{ij,*}\left[\frac{1}{m_X^2}\right]_*$$

- Final results with the distributions

$$\mathcal{C}_{ij} = \mathcal{C}_{ij}\big|_{m_X^2>0} + \mathcal{C}_{ij,\delta}\,\delta(m_X^2) + \mathcal{C}_{ij,*}\underbrace{\left(\left[\frac{1}{m_X^2}\right]_* - \frac{1}{m_X^2}\right)}_{\text{avoid double counting}}$$

$\mathcal{O}(\alpha_s^2)$ Calculation: Results

- 6 independent sectors at parton level: $\{q\bar{q}, qq', q'\bar{q}', qg, gg, qq\}$

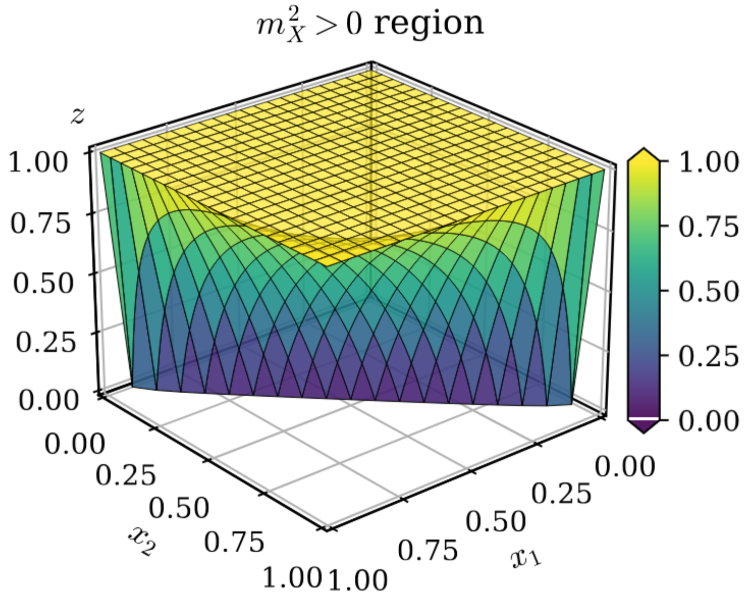
- The $\mathcal{O}(\alpha_s^2)$ Wilson coefficients:

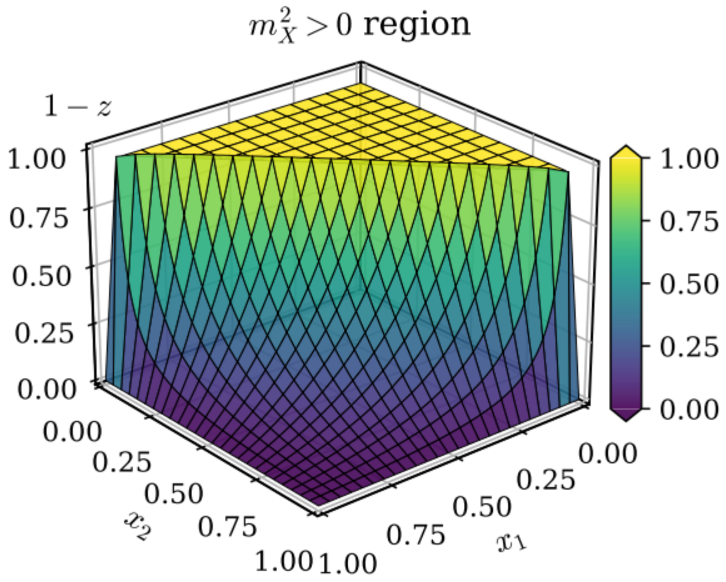
$$\tilde{x}_2 = \frac{1 - x_1}{1 - x_1 z}$$

$$\mathcal{C}^{(2)}_{ij;q}(x_1, x_2, z; \mu) = \theta(m_X^2)\, R_{ij;q}(x_1, x_2, z; \mu) + V_{ij;q}(x_1, \tilde{x}_2; \mu)\delta(m_X^2) + \theta(m_X^2) \sum_{k=0}^{1} U^{[k]}_{ij;q}(x_1, \tilde{x}_2; \mu) \left(\left[\frac{\ln^k(m_X^2)}{m_X^2} \right]_* - \frac{\ln^k(m_X^2)}{m_X^2} \right)$$

- $m_X^2 > 0$ region

$$m_X^2 = 1 - x_1 - x_2 + x_1 x_2 z > 0$$





$\mathcal{O}(\alpha_s^2)$ Calculation: Results

➢The $m_X^2 > 0$ terms can be express as linear combinations of 13 logarithmic functions (including ln2) with rational coefficients

$$R_{ij}(x_1, x_2, z; \mu) = \sum_{k=0}^{14} r_{ij}^{(k)}(x_1, x_2, z; \mu)\, \ell^{(k)}$$

$$m_X = \sqrt{1 - x_1 - x_2 + x_1 x_2 z}$$
$$\beta = \sqrt{(x_1 + x_2)^2 - 4 x_1 x_2 z}$$

$$\ell^{(0)} = 1, \quad \ell^{(1)} = \ln(2), \quad \ell^{(3)} = \ln(1 - x_1), \quad \ell^{(4)} = \ln(1 - x_2),$$
$$\ell^{(5)} = \ln(x_1), \quad \ell^{(6)} = \ln(x_2), \quad \ell^{(7)} = \ln(1 - x_1 z), \quad \ell^{(8)} = \ln(1 - x_2 z),$$
$$\ell^{(9)} = \ln\left[x_1(1-z) + x_2(1 - x_1 z)\right], \quad \ell^{(10)} = \ln\left[x_2(1-z) + x_1(1 - x_2 z)\right],$$
$$\ell^{(11)} = \ln(x_1 + x_2 - \beta), \quad \ell^{(12)} = \ln(x_1 + x_2 + \beta), \quad \ell^{(13)} = \ln(m_X^2), \quad \ell^{(14)} = \ln(1 - z)$$

$$r_{ij}^{(11)}\Big|_{\beta \to -\beta} = r_{ij}^{(12)}$$

$$\text{for } \{ij\} \in \{q\bar{q},\, q'\bar{q}',\, gg,\, qq\},\ r_{ij}^{(3)} = r_{ij}^{(4)},\ r_{ij}^{(5)} = r_{ij}^{(6)},\ r_{ij}^{(7)} = r_{ij}^{(8)},\ r_{ij}^{(9)} = r_{ij}^{(10)}$$

$\mathcal{O}(\alpha_s^2)$ Calculation: Results

- The denominators of the coefficients $r_{ij}^{(k)}$ for the $m_X^2 > 0$ results have 11 irreducible factors in total

$$r_{ij}^{(k)}(x_1, x_2, z; \mu) = \mathcal{N}_{ij}^{(k)}(x_1, x_2, z; \mu) \prod_{(m,n)\in\mathcal{S}_{ij}^{(k)}} \left(\frac{1}{\mathcal{D}^{(m)}}\right)^n$$

$$\begin{aligned}
&\mathcal{D}^{(1)} = x_1, \quad \mathcal{D}^{(2)} = x_2, \quad \mathcal{D}^{(3)} = z, \quad \mathcal{D}^{(4)} = x_1 + x_2,\\
&\mathcal{D}^{(5)} = 1 - x_1, \quad \mathcal{D}^{(6)} = 1 - x_2, \quad \mathcal{D}^{(7)} = 1 - z,\\
&\mathcal{D}^{(8)} = 1 - x_1 z, \quad \mathcal{D}^{(9)} = 1 - x_2 z, \quad \mathcal{D}^{(10)} = m_X^2, \quad \mathcal{D}^{(11)} = \beta
\end{aligned}$$

$m \in [1, 11]$: index of different type of denominators

$n \in [0, 5]$: power of the denominators

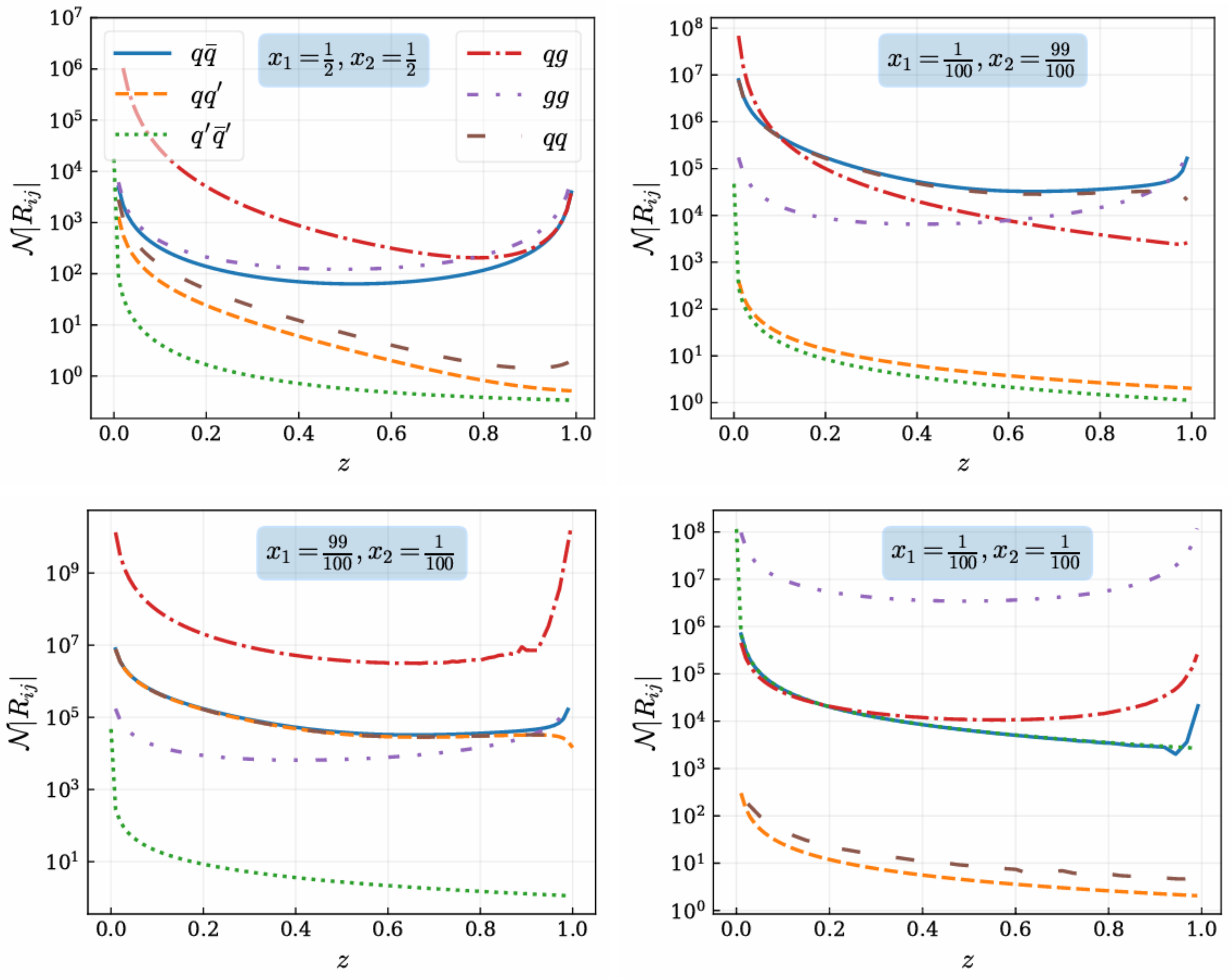
$\mathcal{O}(\alpha_s^2)$ results: $m_X^2 > 0$, fixed x_1, x_2

Physical region:

$$m_X^2 = 1 - x_1 - x_2 + x_1 x_2 z > 0$$

Divergence at

$$m_X^2 \to 0, z \to 0\&1$$



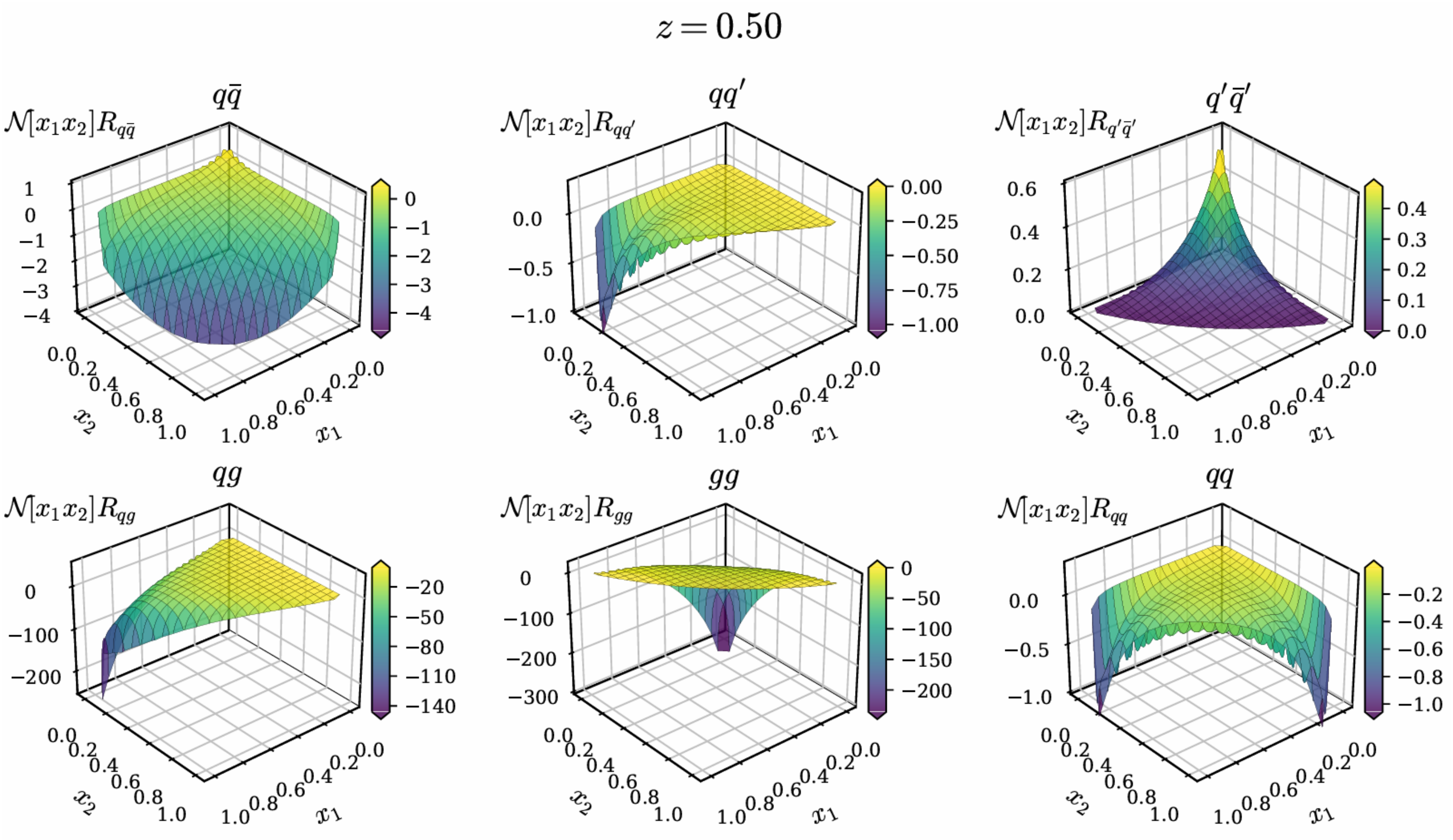
$\mathcal{O}(\alpha_s^2)$ results: $m_X^2 > 0, z = 1/2$

Physical region:

$m_X^2 = 1 - x_1 - x_2 + x_1 x_2 z > 0$

Divergence at

$m_X^2 \to 0, x_1 \to 0\&1, x_2 \to 0\&1$



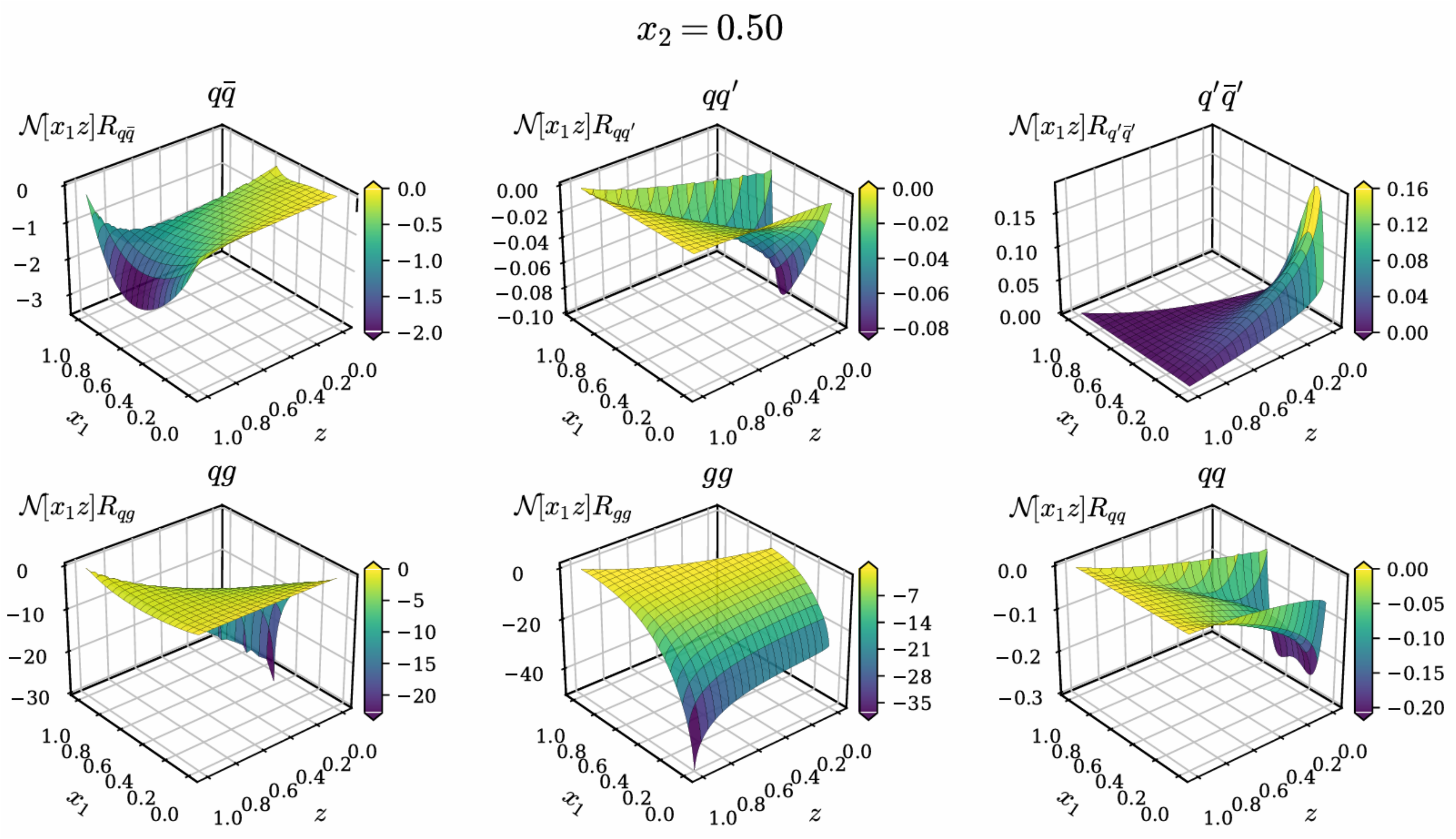
$\mathcal{O}(\alpha_s^2)$ results: $m_X^2 > 0, x_2 = 1/2$

Physical region:

$m_X^2 = 1 - x_1 - x_2 + x_1 x_2 z > 0$

Divergence at

$m_X^2 \to 0, x_1 \to 0\&1, z \to 0\&1$



$\mathcal{O}(\alpha_s^2)$ Calculation: Results

$$\mathcal{C}^{(2)}_{ij;q}(x_1, x_2, z; \mu) = \theta(m_X^2)\, R_{ij;q}(x_1, x_2, z; \mu) + V_{ij;q}(x_1, \tilde{x}_2; \mu)\, \delta(m_X^2) + \theta(m_X^2) \sum_{k=0}^{1} U^{[k]}_{ij;q}(x_1, \tilde{x}_2; \mu) \left(\left[\frac{\ln^k(m_X^2)}{m_X^2} \right]_* - \frac{\ln^k(m_X^2)}{m_X^2} \right)$$

- Two independent sectors with distributions: $\{q\bar{q}, qg\}$

- Coefficients of $[\ln(m_X^2)/m_X^2]_*$:

$$U^{[1]}_{q\bar{q}}(x_1, \tilde{x}_2) = \frac{4N_A(N_A - 1)x_1\tilde{x}_2\left(x_1^2 + \tilde{x}_2^2\right)}{N_C^2 w_1 \tilde{w}_2}$$

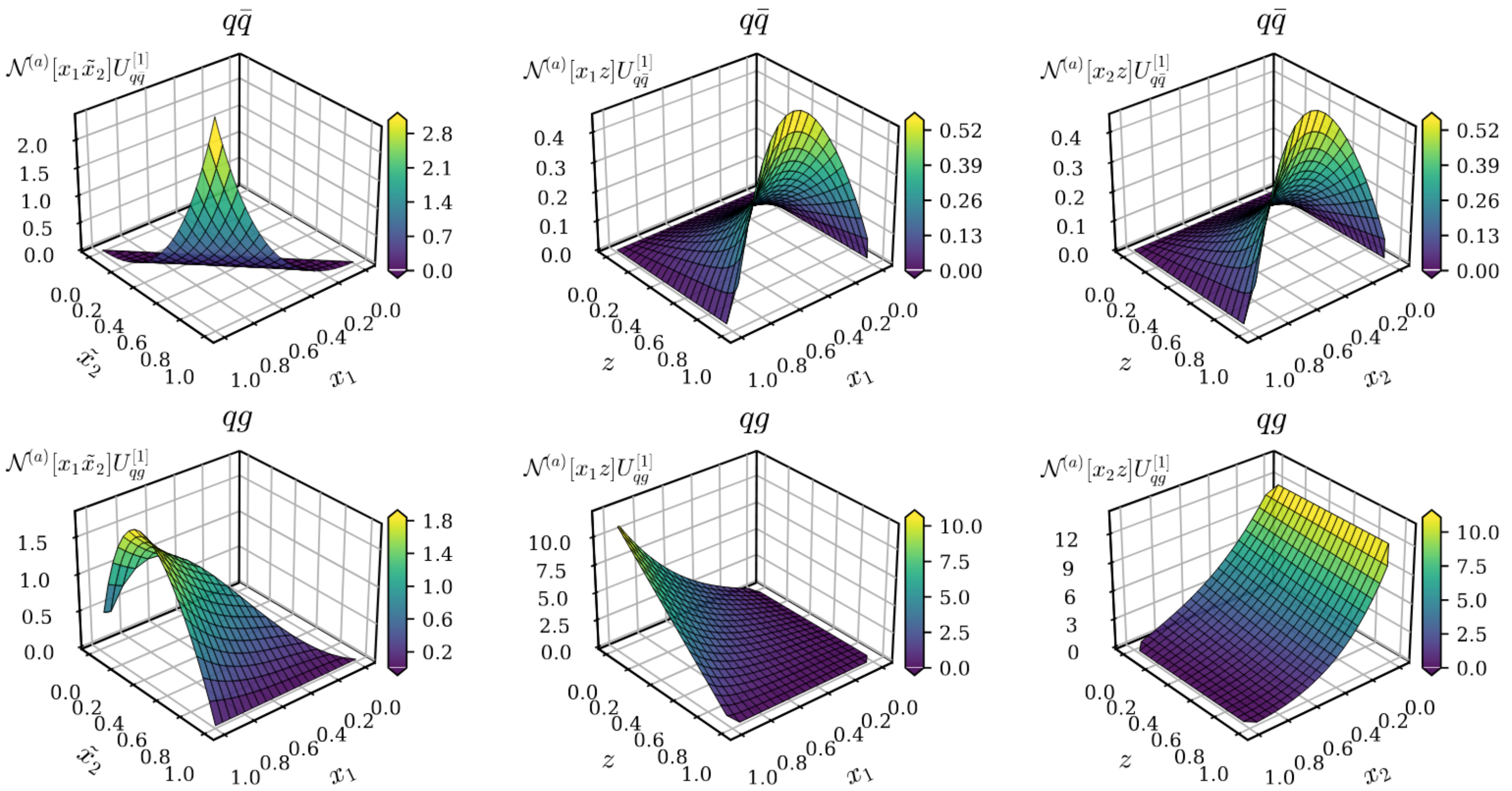
$$U^{[1]}_{qg}(x_1, \tilde{x}_2) = \frac{2N_A(5N_A + 4)x_1\tilde{x}_2\left(x_1^2 + x_X^2\right)}{N_C^2 w_1 w_X}$$

$$w_\varrho = 1 - x_\varrho \ (\varrho = \{1, X\}), \ \tilde{w}_2 = 1 - \tilde{x}_2$$

$$x_X = 2 - x_1 - \tilde{x}_2$$

$$\tilde{x}_2 = \frac{1 - x_1}{1 - x_1 z}$$

Coefficients of the $[\ln m_X^2/m_X^2]_*$



$\mathcal{O}(\alpha_s^2)$ Calculation: Results

$$\mathcal{C}^{(2)}_{ij;q}(x_1, x_2, z; \mu) = \theta(m_X^2)\, R_{ij;q}(x_1, x_2, z; \mu) + V_{ij;q}(x_1, \tilde{x}_2; \mu)\, \delta(m_X^2) + \theta(m_X^2) \sum_{k=0}^{1} U^{[k]}_{ij;q}(x_1, \tilde{x}_2; \mu) \left(\left[\frac{\ln^k(m_X^2)}{m_X^2} \right]_* - \frac{\ln^k(m_X^2)}{m_X^2} \right)$$

- ➢ Two independent sectors with distributions: $\{q\bar{q}, qg\}$

- ➢ Coefficients of $[1/m_X^2]_*$:

$$U^{[0]}_{q\bar{q}}(x_1, \tilde{x}_2) = \frac{N_A x_1 \tilde{x}_2 (x_1^2 + \tilde{x}_2^2) \left[N_C(2N_f - 11N_C) + 12\ell_*^{[0]} \right]}{3N_C^2 w_1 \tilde{w}_2} + L_h \frac{4N_A^2 x_1 \tilde{x}_2 (x_1^2 + \tilde{x}_2^2)}{N_C^2 w_1 \tilde{w}_2},$$

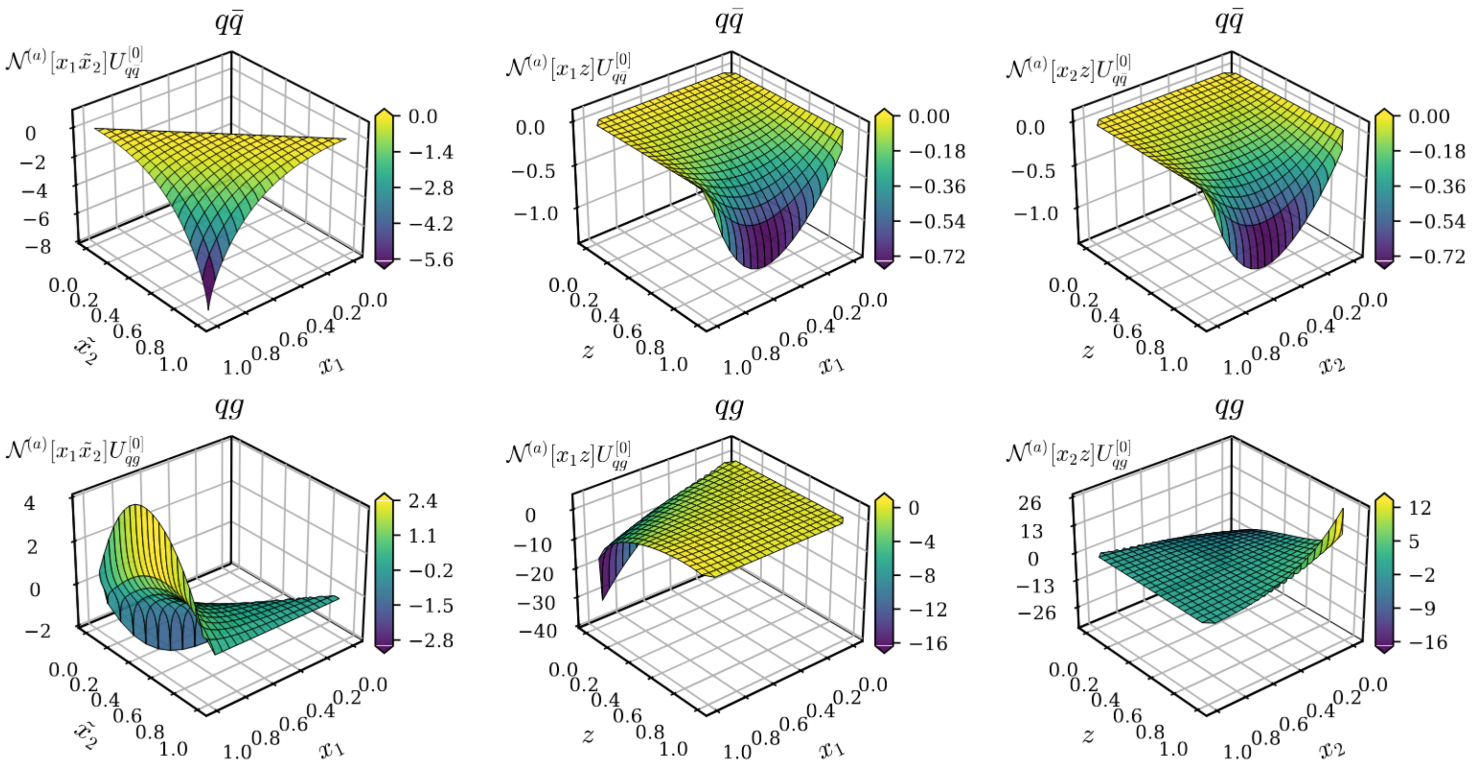
$$U^{[0]}_{qg}(x_1, \tilde{x}_2) = -\frac{N_A x_1 \tilde{x}_2 (x_1^2 + x_X^2) \left(3N_A + 8N_C^2 \ell_*^{[0]} \right)}{2N_C^2 w_1 w_X} + L_h \frac{2N_A(3N_A + 2) x_1 \tilde{x}_2 (x_1^2 + x_X^2)}{N_C^2 w_1 w_X}$$

$$\ell_*^{[0]} = \ln w_1 + \ln \tilde{w}_2 - \ln w_X$$

$$w_\varrho = 1 - x_\varrho \ (\varrho = \{1, X\}), \ \tilde{w}_2 = 1 - \tilde{x}_2$$

$$x_X = 2 - x_1 - \tilde{x}_2 \qquad \tilde{x}_2 = \frac{1 - x_1}{1 - x_1 z}$$

Coefficients of the $[1/m_X^2]_*$



$\mathcal{O}(\alpha_s^2)$ Calculation: Results

- Two independent sectors with distributions: $\{q\bar{q}, qg\}$

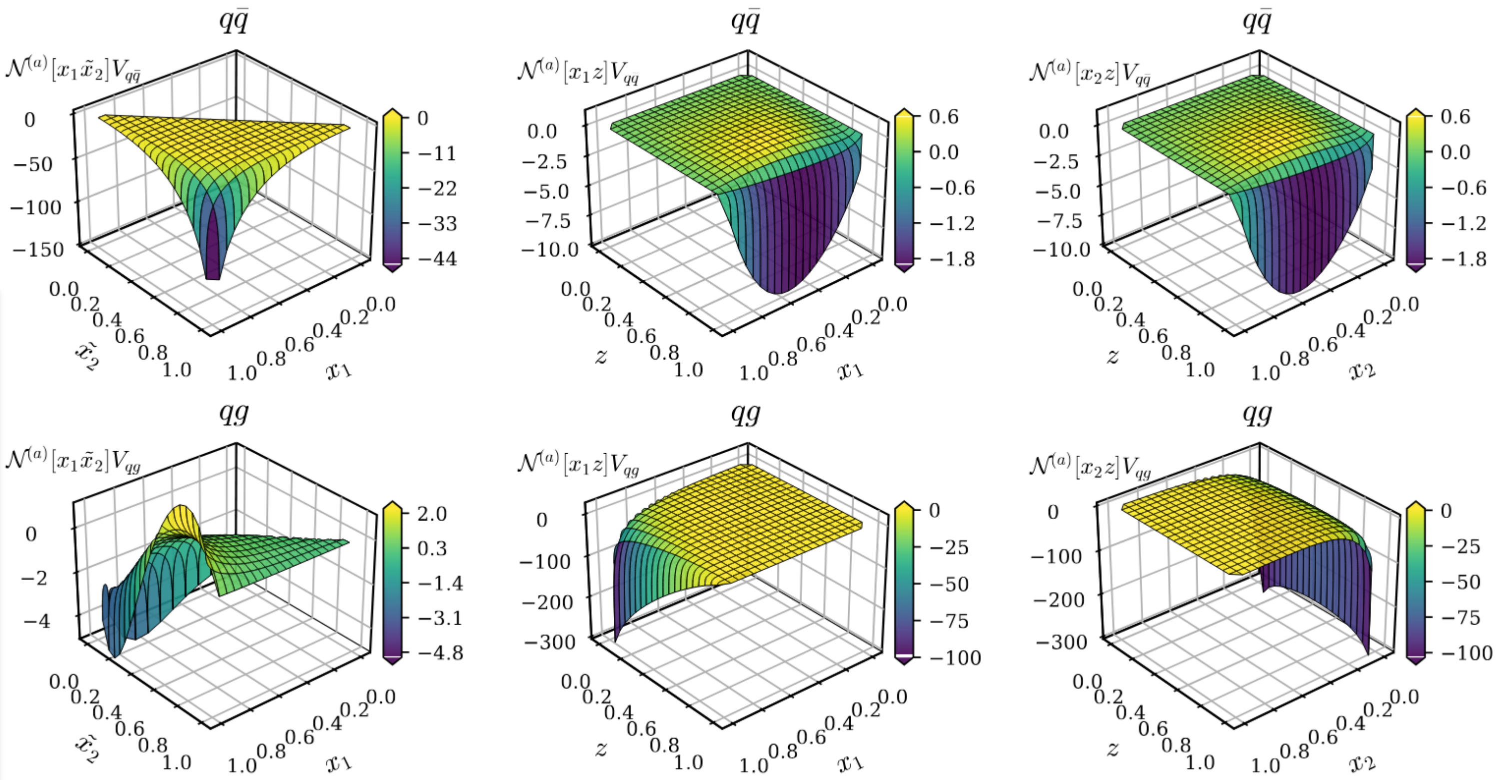
$$
\begin{aligned}
&\mathcal{C}^{(2)}_{ij;q}(x_1, x_2, z; \mu) \\
&= \theta(m_X^2)\, R_{ij;q}(x_1, x_2, z; \mu) + V_{ij;q}(x_1, \tilde{x}_2; \mu)\, \delta(m_X^2) \\
&+ \theta(m_X^2) \sum_{k=0}^{1} U^{[k]}_{ij;q}(x_1, \tilde{x}_2; \mu) \left(\left[\frac{\ln^k(m_X^2)}{m_X^2} \right]_* - \frac{\ln^k(m_X^2)}{m_X^2} \right)
\end{aligned}
$$

- Coefficients of $\delta(m_X^2)$:

$$
V_{ij}(x_1, \tilde{x}_2; \mu) = \sum_{k=0}^{3} (r_\delta)^{(k)}_{ij}(x_1, \tilde{x}_2; \mu)\, \ell^{(k)}_\delta + \sum_{k=1}^{8} (pr_\delta)^{(k)}_{ij}(x_1, \tilde{x}_2; \mu)\, (\ell\ell)^{(k)}_\delta
$$

$$
\begin{aligned}
&\ell^{(1)}_\delta = \ln w_1, \quad \ell^{(2)}_\delta = \ln \tilde{w}_2, \quad \ell^{(3)}_\delta = \ln w_X \\
&(\ell\ell)^{(1)}_\delta = \ln x_1 \ln w_1 + \mathrm{Li}_2(w_1), \\
&(\ell\ell)^{(2)}_\delta = \ln \tilde{x}_2 \ln \tilde{w}_2 + \mathrm{Li}_2(\tilde{w}_2), \\
&(\ell\ell)^{(3)}_\delta = \ln x_X \ln w_X + \mathrm{Li}_2(w_X), \\
&(\ell\ell)^{(4)}_\delta = \ln^2(w_1), \quad (\ell\ell)^{(5)}_\delta = \ln^2(\tilde{w}_2) \\
&(\ell\ell)^{(6)}_\delta = \ln w_1 \ln \tilde{w}_2, \quad (\ell\ell)^{(7)}_\delta = \ln w_1 \ln w_X, \\
&(\ell\ell)^{(8)}_\delta = \ln \tilde{w}_2 \ln w_X
\end{aligned}
$$

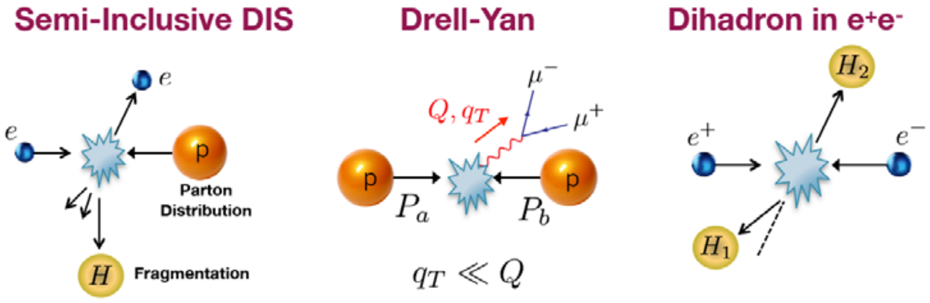
Coefficients of the $\delta(m_X^2)$



$$
w_\varrho = 1 - x_\varrho\ (\varrho = \{1, X\}),\ \tilde{w}_2 = 1 - \tilde{x}_2 \quad x_X = 2 - x_1 - \tilde{x}_2 \quad \tilde{x}_2 = \frac{1 - x_1}{1 - x_1 z}
$$

Summary and outlook

- We calculated the Wilson coefficients of the dihadron angular correlation in the e^+e^- collision at $\mathcal{O}(\alpha_s^2)$, and investigated its $m_X \to 0$ limit
- The results can be expressed in logarithmic functions and three Li2 functions with rational coefficients
- The results can be used in the global fit of the fragmentation functions at the e^+e^- collisions to provide more information of hadronization
- Future work:
  - Investigate the $x_1, x_2 \to 0{,}1$ and $z \to 1$, $z \to 0$ limit (related to the dihadron fragmentation functions)
  - EEC, QQC …
  - Physical results of the dihadron correlation function
  - Crossing to SIDIS, Drell-Yan process



Thank you for your attention!

Back up

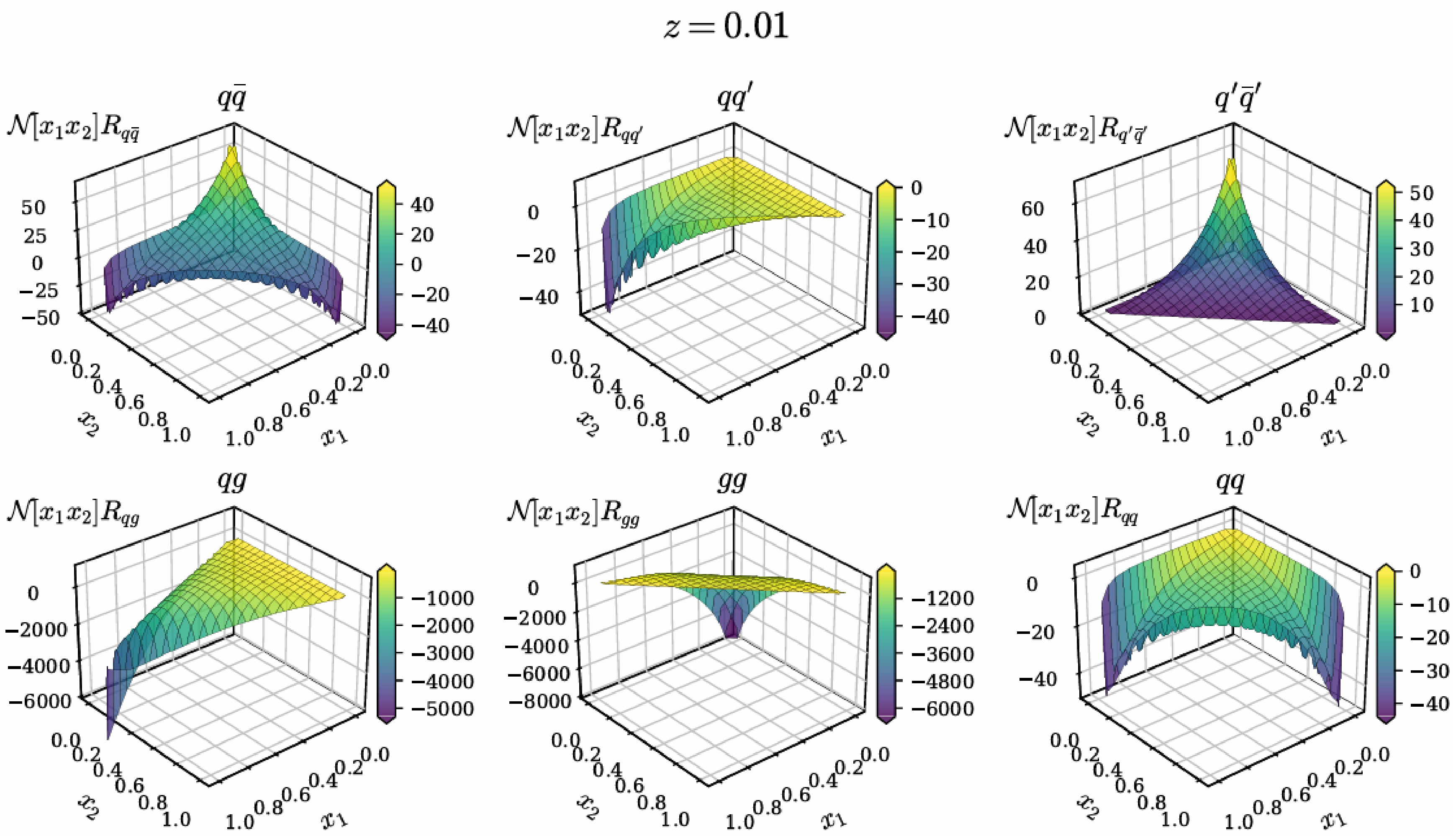
$\mathcal{O}(\alpha_s^2)$ results: $m_X^2 > 0, z = 1/100$

Physical region:

$m_X^2 = 1 - x_1 - x_2 + x_1 x_2 z > 0$

Divergence at

$m_X^2 \to 0, x_1 \to 0\&1, x_2 \to 0\&1$



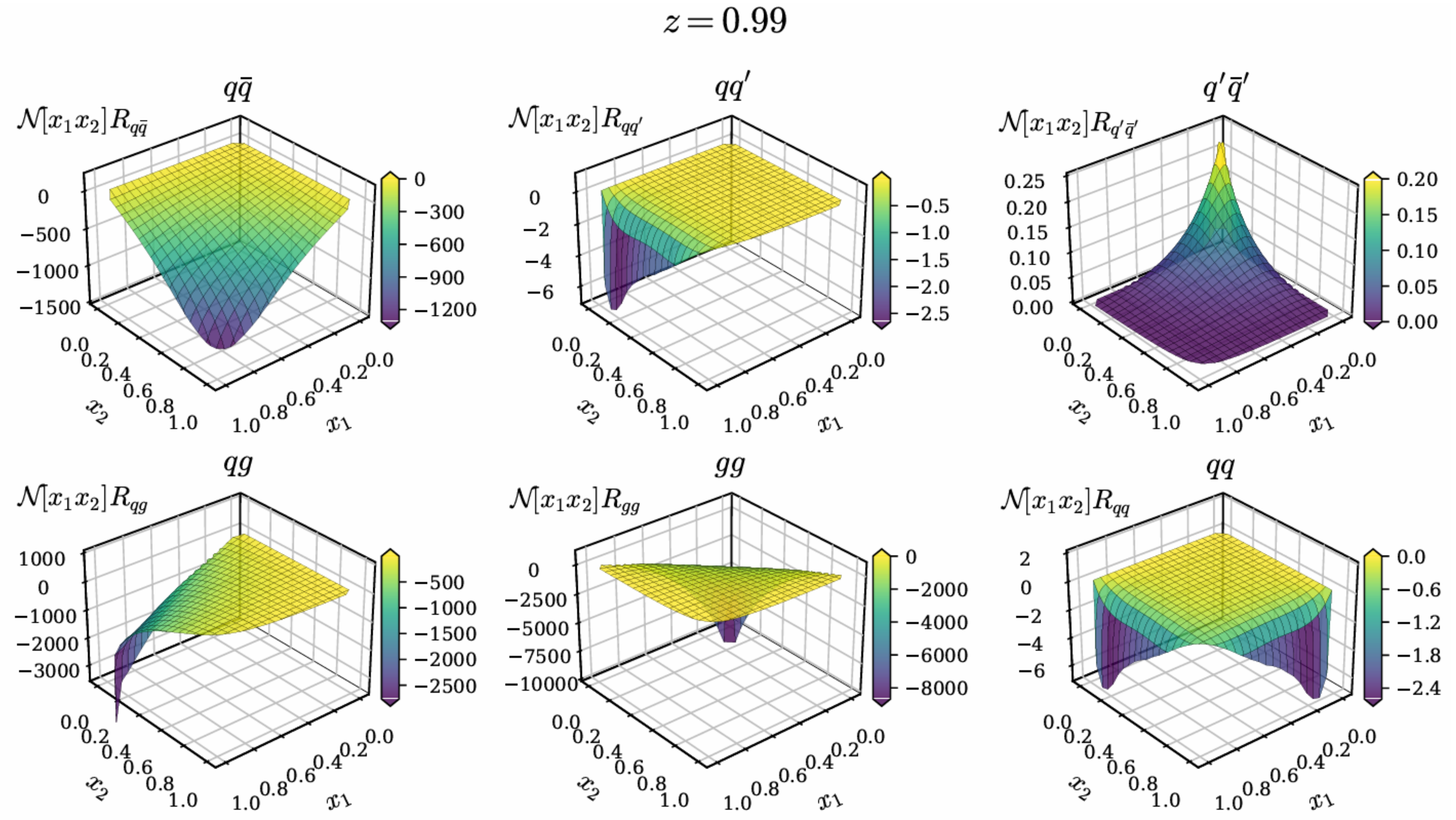
$\mathcal{O}(\alpha_s^2)$ results: $m_X^2 > 0, z \to 99/100$

Physical region:

$$m_X^2 = 1 - x_1 - x_2 + x_1 x_2 z > 0$$

Divergence at

$$m_X^2 \to 0, x_1 \to 0\&1, x_2 \to 0\&1$$



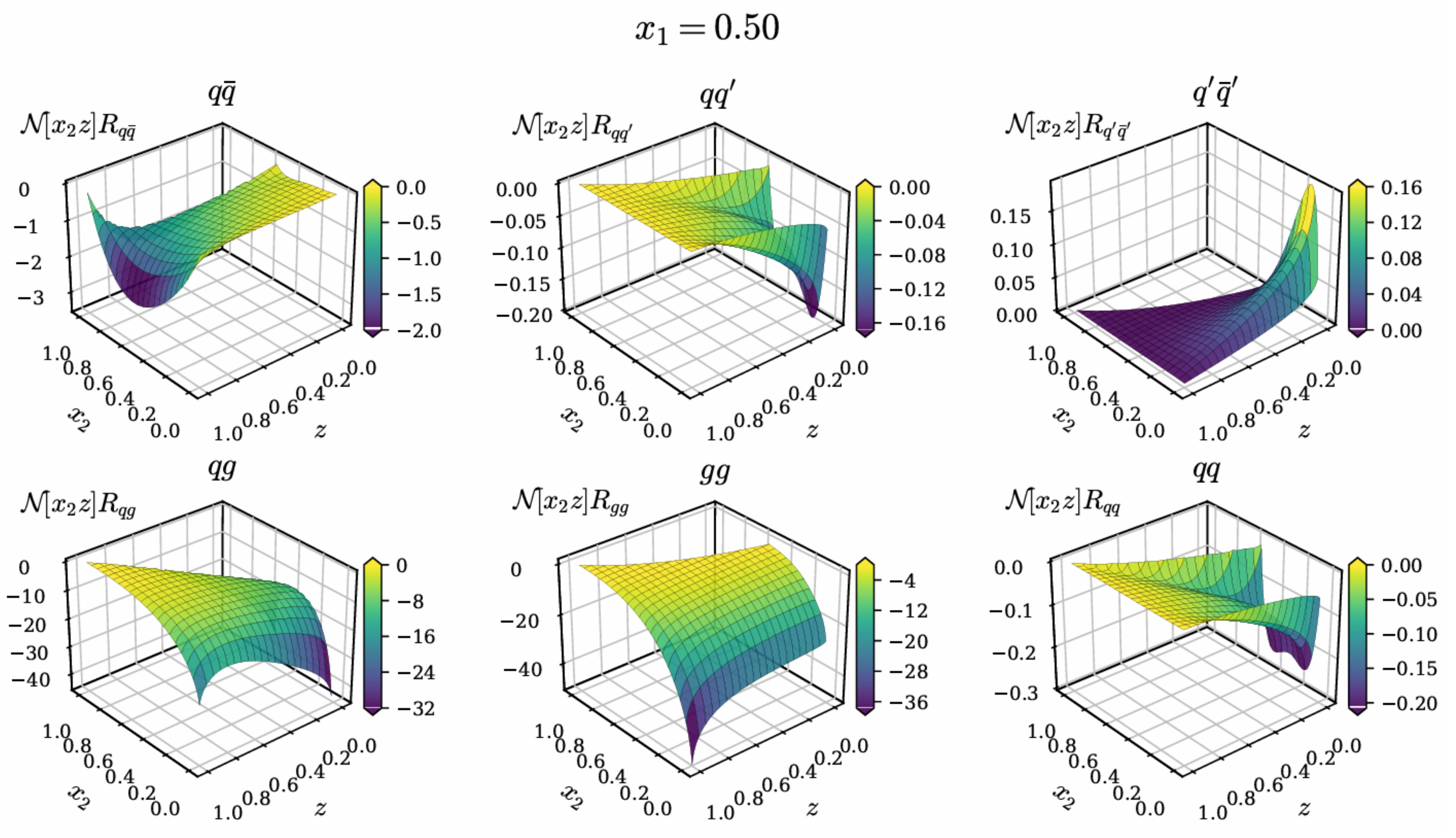
$\mathcal{O}(\alpha_s^2)$ results: $m_X^2 > 0, x_1 = 1/2$

Physical region:

$m_X^2 = 1 - x_1 - x_2 + x_1 x_2 z > 0$

Divergence at

$m_X^2 \to 0, z \to 0\&1, x_2 \to 0\&1$



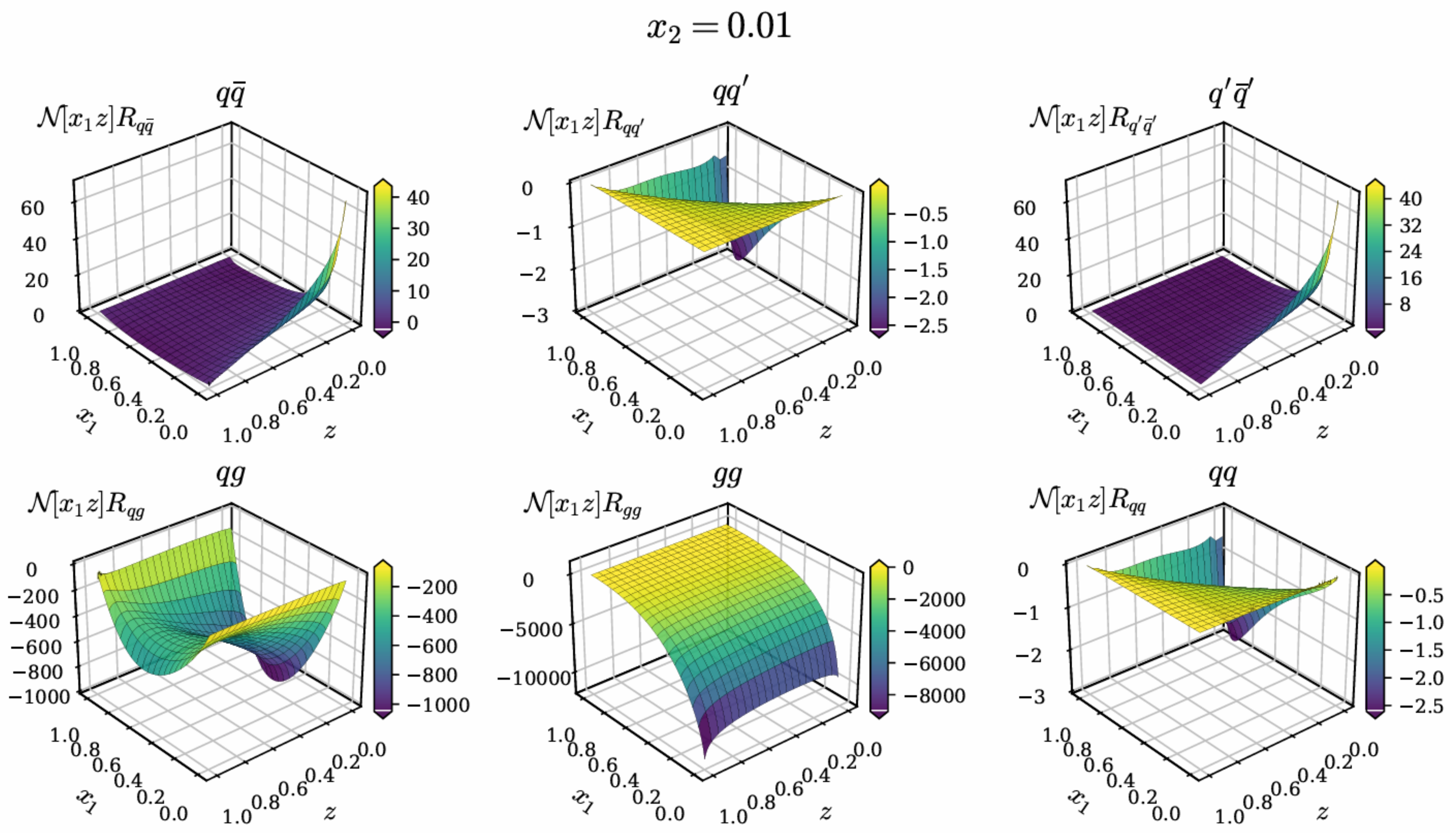
$\mathcal{O}(\alpha_s^2)$ results: $m_X^2 > 0, x_2 = 1/100$

Physical region:

$m_X^2 = 1 - x_1 - x_2 + x_1 x_2 z > 0$

Divergence at

$m_X^2 \to 0, x_1 \to 0\&1, z \to 0\&1$



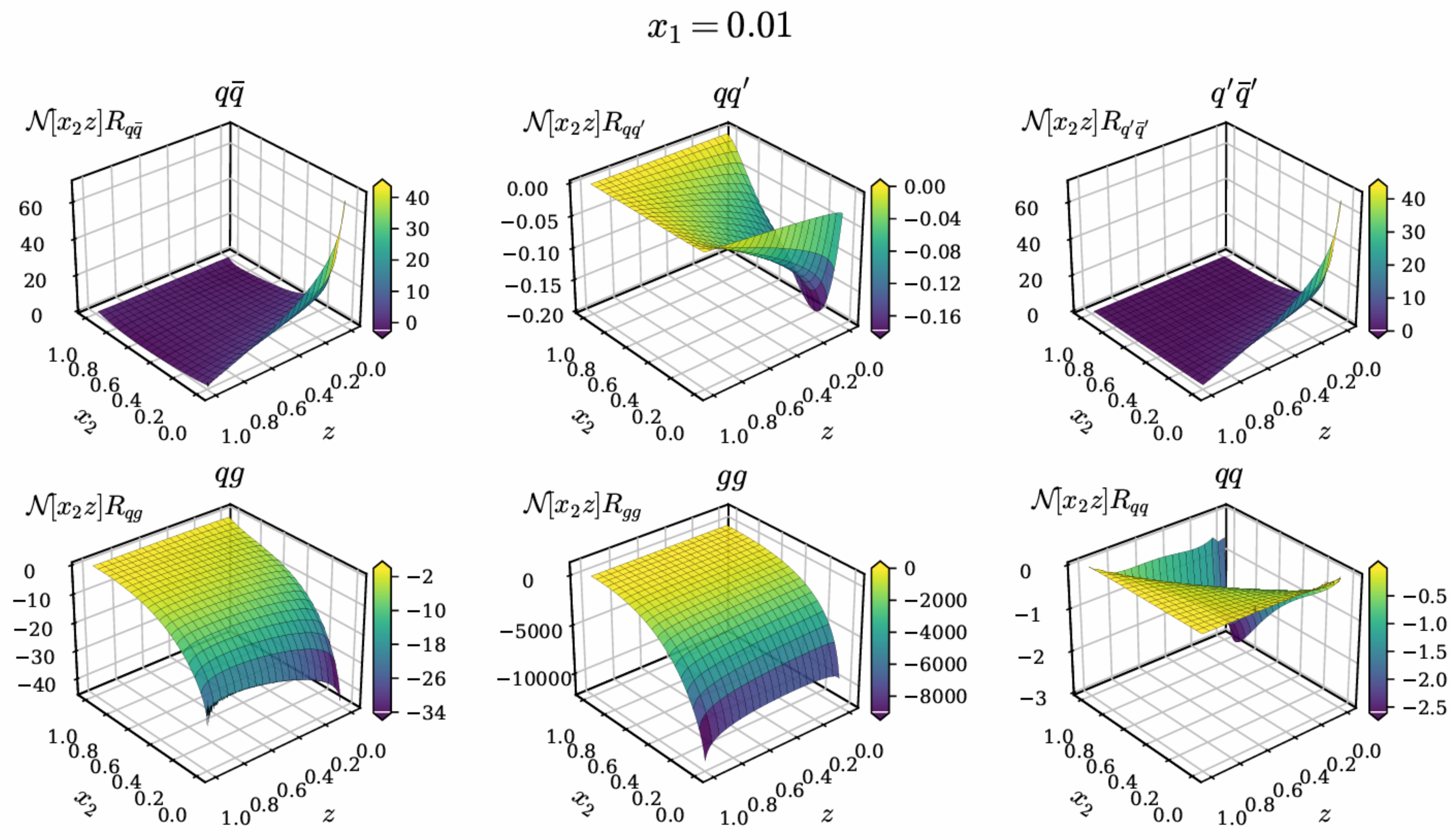
$\mathcal{O}(\alpha_s^2)$ results: $m_X^2 > 0, x_1 = 1/100$

Physical region:

$m_X^2 = 1 - x_1 - x_2 + x_1 x_2 z > 0$

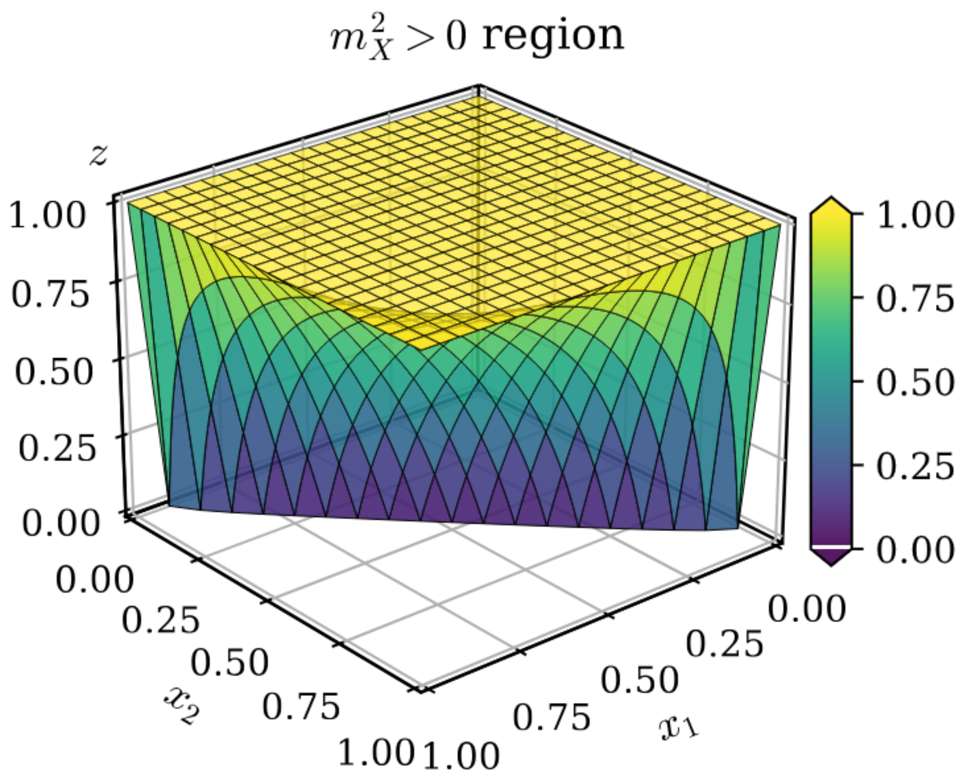
Divergence at

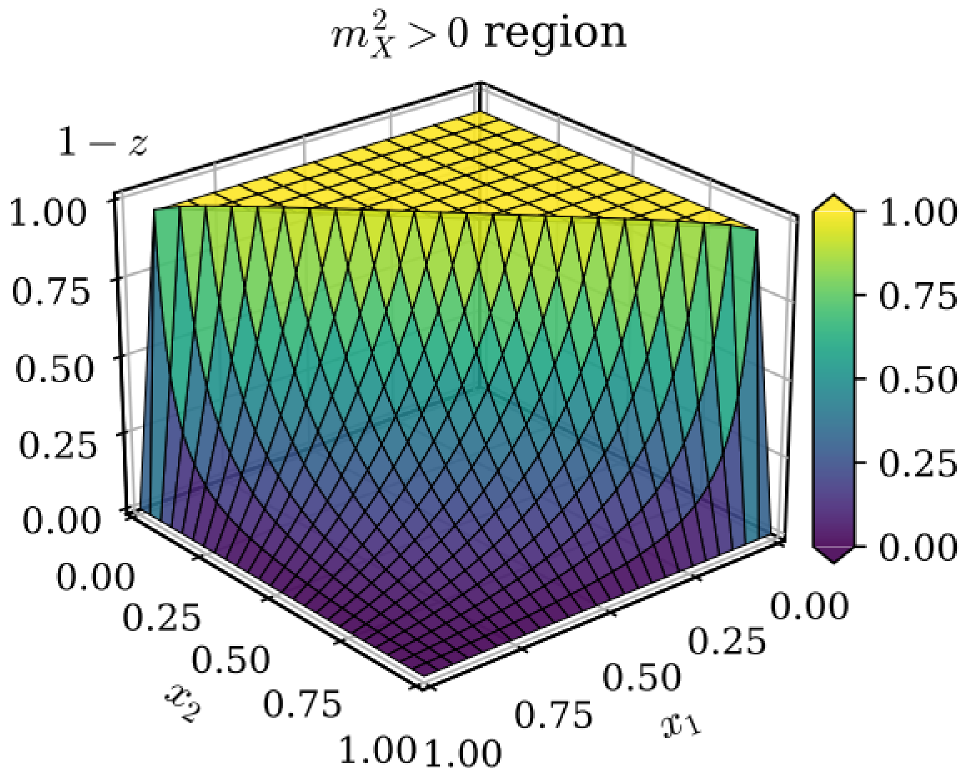
$m_X^2 \to 0, z \to 0\&1, x_2 \to 0\&1$



Physical region: $m_X^2 > 0$

Physical region: $m_X^2 = 1 - x_1 - x_2 + x_1 x_2 z > 0$





$\mathcal{O}(\alpha_s^2)$ results: $m_X^2 \to 0$

Physical region:

$$z = \frac{x_1 + x_2 - 1}{x_1 x_2} \in (0,1)$$

Divergence at

$$x_1 \to 0\&1, x_2 \to 0\&1$$