Heavy Quark Masses
From Calibrated Uncertainty

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Work in collaboration with

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Outline

- Motivation and Introduction
- Using Sum Rules to extract m_Q
  - overview
  - our proposal for *charm* (and *bottom*)
- *Impact on the muon g-2*
- Conclusions and outlook

Motivation: why precise m_Q?

Higgs decay $\sim \overline{m_b}(M_H)^2$

"g-2" of the muon:

- Hadronic Vacuum Polarization needs quark masses

$$\Gamma(B \to X_u l\nu) \sim G_F^2 m_b^5 |V_{ub}|^2$$

$$B \to K(^*)\ell\ell$$

$$D \to K\ell\nu$$

(pQCD contributions on FFs depend on m_q)

Yukawa unification

$$\frac{\delta m_b}{m_b} \sim \frac{\delta m_t}{m_t} \quad \text{if } \delta m_t \sim 1\text{GeV} \Rightarrow \delta m_b \sim 25\text{MeV}$$

Motivation: why precise m_Q?

Υ-spectroscopy

$$m(\Upsilon(1S)) = 2M_b - \mathcal{C}\alpha^2 M_b + \cdots$$

Lattice QCD

$$M_{H^{(*)}} = m_h + \overline{\Lambda} + \frac{\mu_\pi^2}{2m_h} - d_{H^{(*)}}\frac{\mu_G^2(m_h)}{2m_h} + \mathrm{O}\left(m_h^{-2}\right)$$

QCD Sum Rules

$$\int \frac{\mathrm{d}s}{s^{n+1}} R_q(s) \sim \left(\frac{1}{m_q}\right)^{2n}$$

Motivation: why precise m_Q?

Snapshot from PDG: CHARM

VALUE (GeV)		*DOCUMENT ID*		*TECN*
1.27 ± 0.02		**OUR EVALUATION**		
1.266 ±0.006	1	NARISON	2020	THEO
$1.290\ ^{+0.077}_{-0.053}$	2	ABRAMOWICZ	2018	HERA
1.273 ±0.010	3	BAZAVOV	2018	LATT
1.2737 ±0.0077	4	LYTLE	2018	LATT
1.223 ±0.033	5	PESET	2018	THEO
1.279 ±0.008	6	CHETYRKIN	2017	THEO
1.272 ±0.008	7	ERLER	2017	THEO
1.246 ±0.023	8	KIYO	2016	THEO
1.288 ±0.020	9	DEHNADI	2015	THEO
1.348 ±0.046	10	CARRASCO	2014	LATT
$1.24 \pm 0.03\ ^{+0.03}_{-0.07}$	11	ALEKHIN	2013	THEO
1.159 ±0.075	12	SAMOYLOV	2013	NOMD
1.278 ±0.009	13	BODENSTEIN	2011	THEO
$1.28\ ^{+0.07}_{-0.06}$	14	LASCHKA	2011	THEO
1.196 ±0.059 ±0.050	15	AUBERT	2010A	BABR
1.25 ±0.04	16	SIGNER	2009	THEO

Motivation: why precise m_Q?

Snapshot from PDG: BOTTOM

VALUE (GeV)		*DOCUMENT ID*		*TECN*
$4.18^{+0.03}_{-0.02}$		**OUR EVALUATION** of $\overline{\mathrm{MS}}$ Mass.		
4.197 ±0.008	1	NARISON	2020	THEO
$4.049^{+0.138}_{-0.118}$	2	ABRAMOWICZ	2018	HERA
4.195 ±0.014	3	BAZAVOV	2018	LATT
4.186 ±0.037	4	PESET	2018	THEO
4.197 ±0.022	5	KIYO	2016	THEO
4.183 ±0.037	6	ALBERTI	2015	THEO
$4.203^{+0.016}_{-0.034}$	7	BENEKE	2015	THEO
4.196 ±0.023	8	COLQUHOUN	2015	LATT
4.176 ±0.023	9	DEHNADI	2015	THEO
4.21 ±0.11	10	BERNARDONI	2014	LATT
4.169 ±0.002 ±0.008	11	PENIN	2014	THEO
4.166 ±0.043	12	LEE	2013O	LATT
4.247 ±0.034	13	LUCHA	2013	THEO
4.171 ±0.009	14	BODENSTEIN	2012	THEO
4.29 ±0.14	15	DIMOPOULOS	2012	LATT
$4.18^{+0.05}_{-0.04}$	16	LASCHKA	2011	THEO
4.186 ±0.044 ±0.015	17	AUBERT	2010A	BABR
4.163 ±0.016	18	CHETYRKIN	2009	THEO
4.243 ±0.049	19	SCHWANDA	2008	BELL

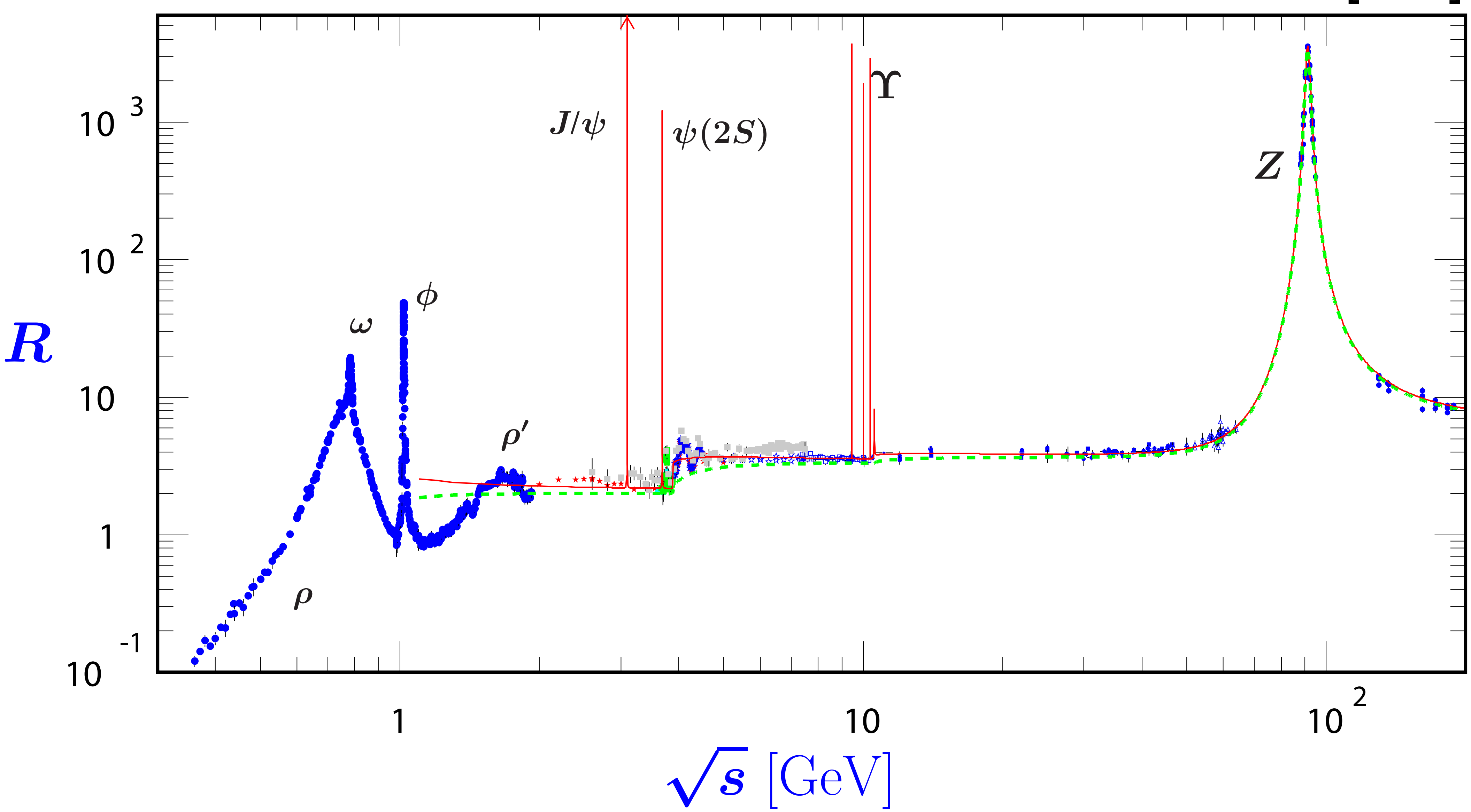
QCD Sum Rules

QCD Sum Rules

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

$$\sigma(e^+e^- \to \mu^+\mu^-) = 4\pi\alpha_{\rm em}(s)^2/3s$$

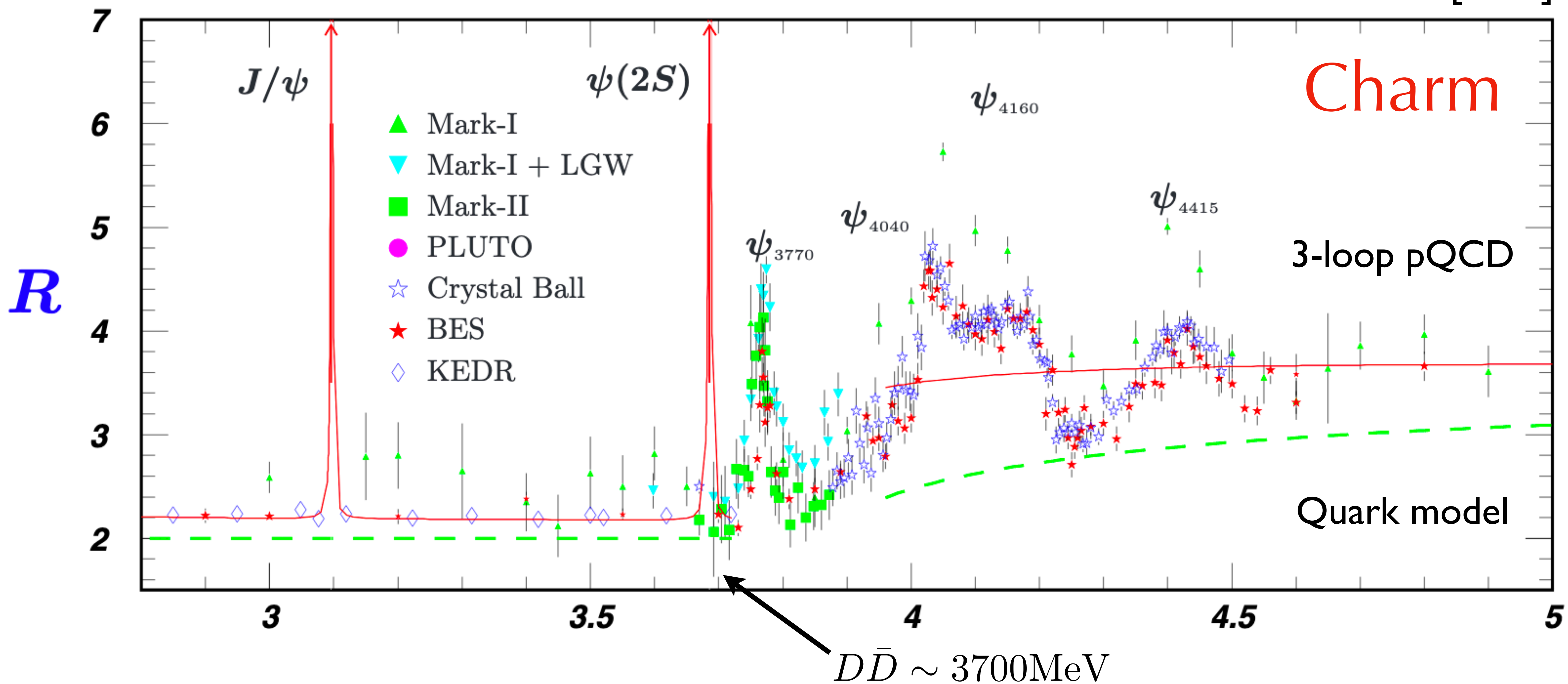
[PDG]



QCD Sum Rules

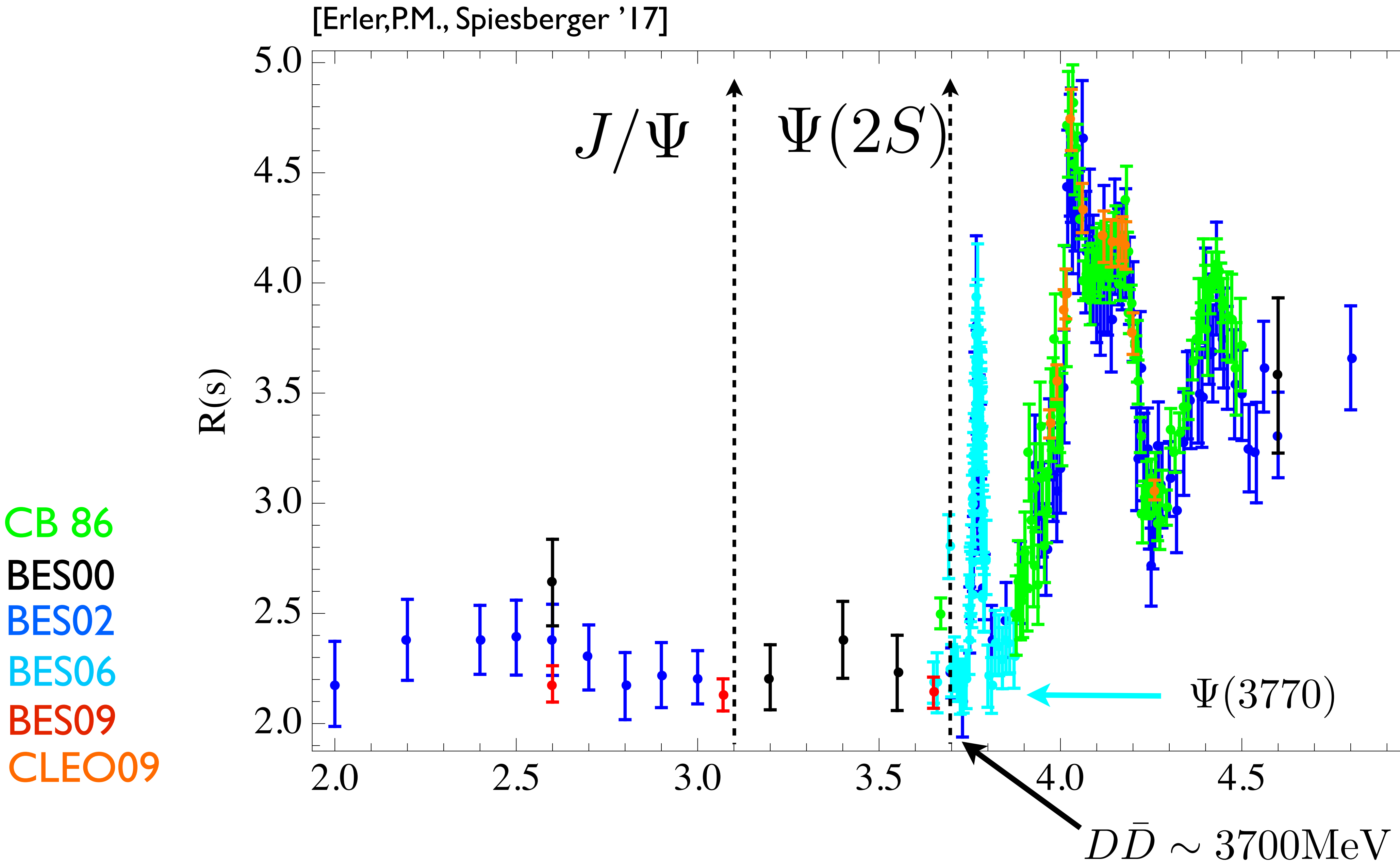
Zoom into the open-charm threshold

[PDG]



QCD Sum Rules

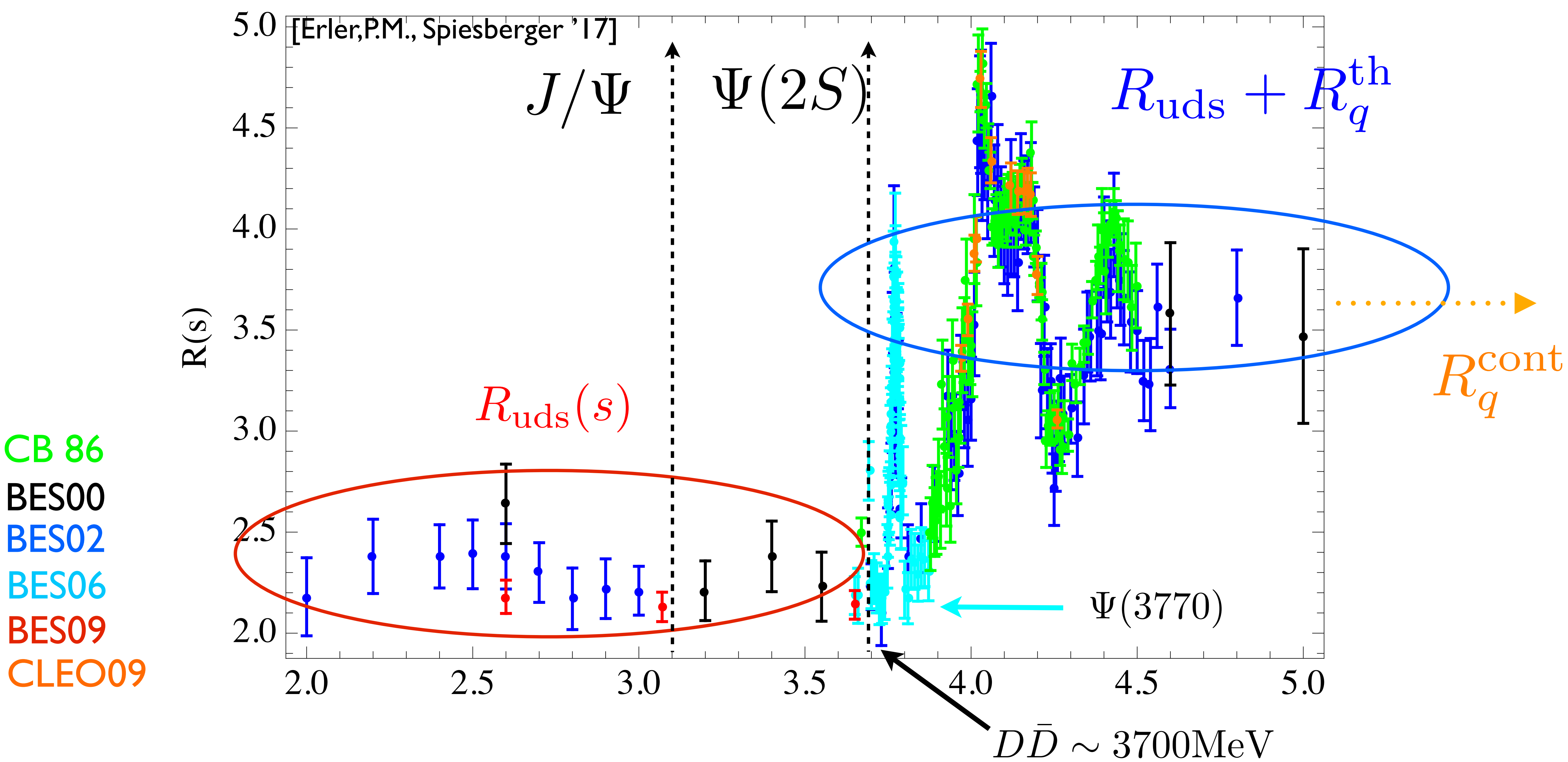
Zoom into the open-charm threshold



QCD Sum Rules

Zoom into the open-charm threshold

$$R(s) = R_{\rm uds}(s) + R_q(s) \qquad R_q(s) = R_q^{\rm Res}(s) + R_q^{\rm th}(s) + R_q^{\rm cont}(s)$$



QCD Sum Rules

Using the optical theorem: [SVZ,'79]

$$R(s) = 12\pi \mathrm{Im}[\Pi(s + i\epsilon)]$$

$\Pi_q(s)$ is the correlator of two heavy-quark vector currents which can be calculated in pQCD order by order in α_s and satisfies a Dispersion Relation:

$\hat{\Pi}_q(s)$ in $\overline{MS}$

$$R_q(s) = R_q^{\mathrm{Res}}(s) + R_q^{\mathrm{th}}(s) + R_q^{\mathrm{cont}}(s)$$

$$12\pi^2 \frac{\hat{\Pi}_q(0) - \hat{\Pi}_q(-t)}{t} = \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s} \frac{R_q(s)}{s+t}$$

For $t \to 0$

$$\mathcal{M}_n := \frac{12\pi^2}{n!} \frac{d^n}{dt^n} \hat{\Pi}_q(t)\Big|_{t=0} = \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s^{n+1}} R_q(s)$$

L.H.S.: theory R.H.S.: data

QCD Sum Rules

L.H.S.: $\hat{\Pi}_q(s)$ can be Taylor expanded:

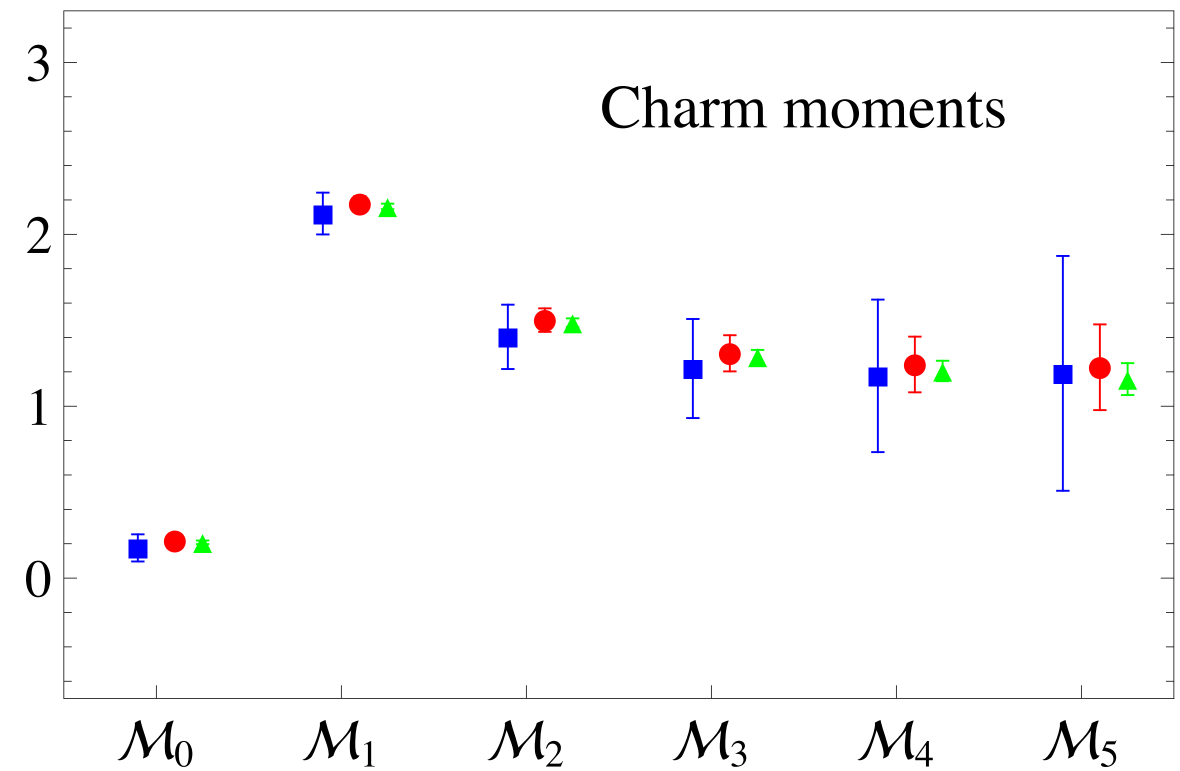
$$\Pi_q(t) = Q_q^2 \frac{3}{16\pi^2} \sum_{n\geq 0} \bar{C}_n \left(\frac{t}{4\hat{m}_q^2} \right)^n$$

QCD Sum Rules

L.H.S.: $\hat{\Pi}_q(s)$ can be Taylor expanded:

$$\Pi_q(t) = Q_q^2 \frac{3}{16\pi^2} \sum_{n\geq 0} \bar{C}_n \left(\frac{t}{4\hat{m}_q^2}\right)^n$$

$$\mathcal{M}_n^{\rm pQCD} = \frac{9}{4} Q_q^2 \left(\frac{1}{2\hat{m}_q(\hat{m}_q)}\right)^{2n} \bar{C}_n$$



$$\hat{\alpha} = \hat{\alpha}(\overline{m_q})$$

$$\bar{C}_n = \bar{C}_n^{(0)} + \left(\frac{\hat{\alpha}}{\pi}\right) \bar{C}_n^{(1)} + \left(\frac{\hat{\alpha}}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\hat{\alpha}}{\pi}\right)^3 \bar{C}_n^{(3)} + \mathcal{O}\left(\frac{\hat{\alpha}}{\pi}\right)^4$$

[Maier et al, '08]
[Chetyrkin, Steinhauser'06]
[Melnikov, Ritberger'03]

[Kiyo et al '09]
[Hoang et al '09]
[Greynat et al '09]

QCD Sum Rules

Sum Rules:

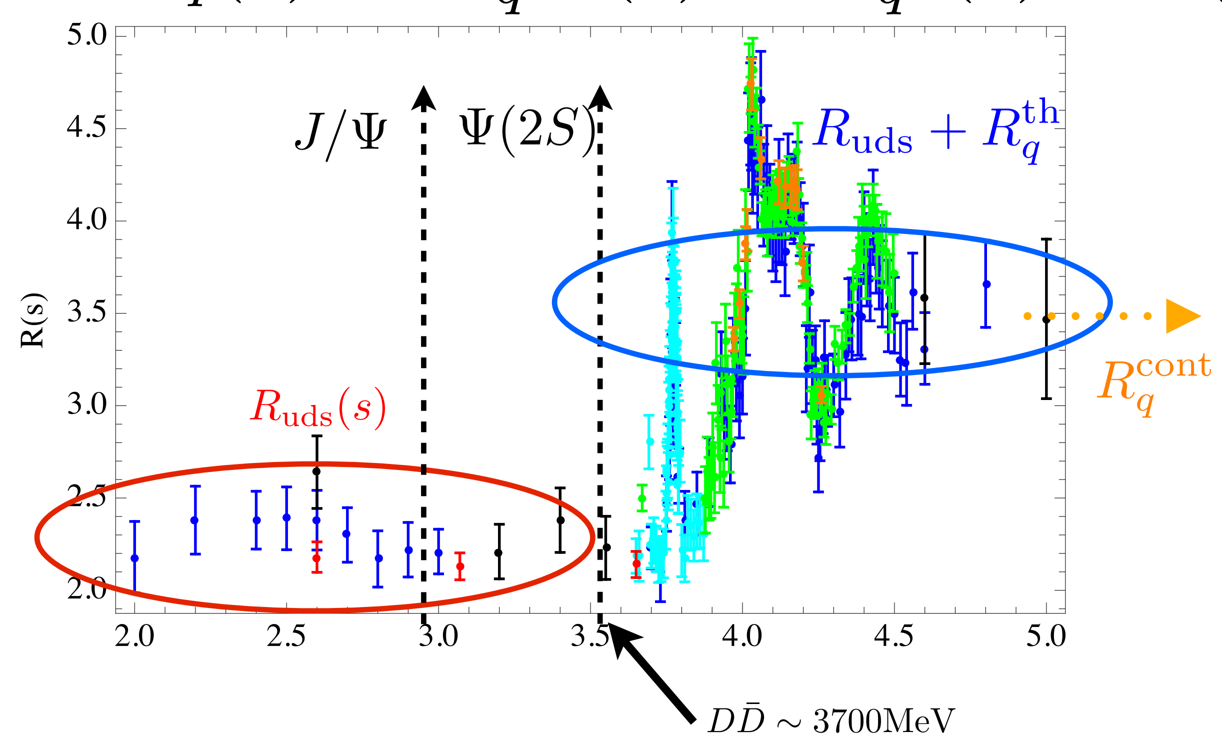
$$\mathcal{M}_n = \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s^{n+1}} R_q(s)$$

L.H.S. from theory

$$\mathcal{M}_n^{\mathrm{pQCD}} = \frac{9}{4} Q_q^2 \left(\frac{1}{2\hat{m}_q(\hat{m}_q)} \right)^{2n} \bar{C}_n$$

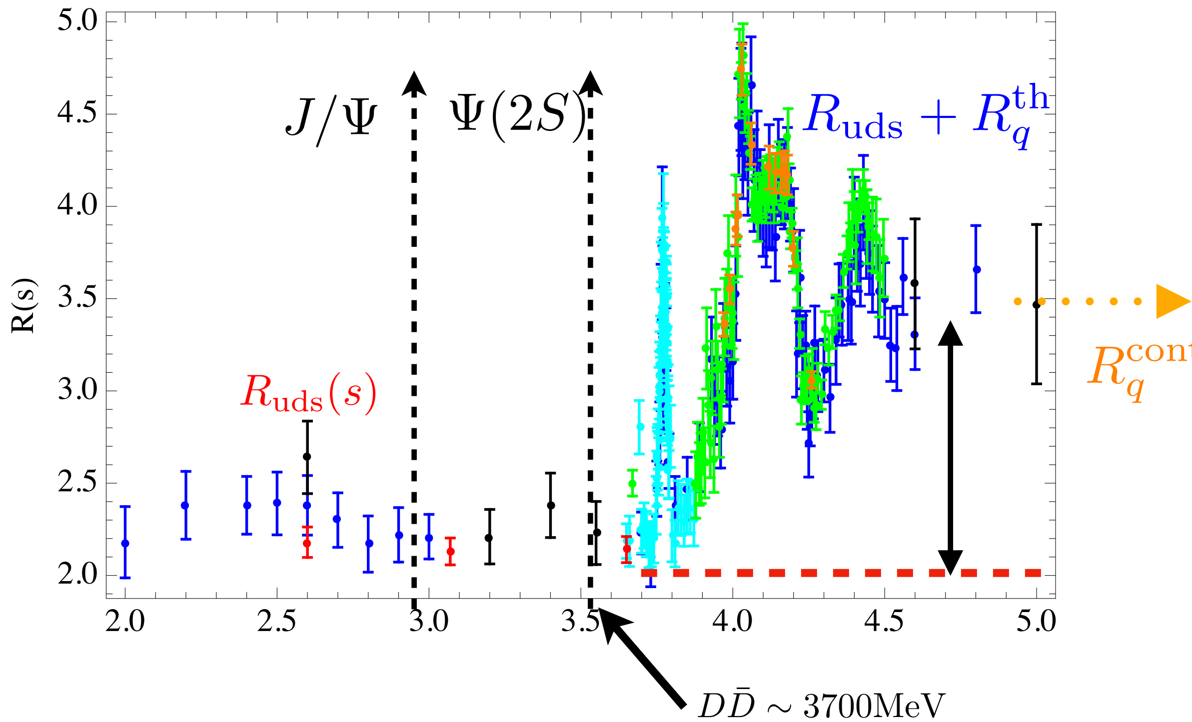
R.H.S. from experiment

$$R_q(s) = R_q^{\mathrm{Res}}(s) + R_q^{\mathrm{th}}(s) + R_q^{\mathrm{cont}}(s)$$



Background*

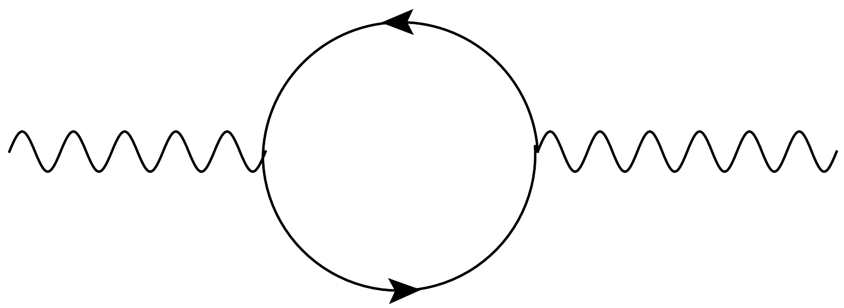
$$R_q(s) = R_q^{\text{Res}}(s) + R_q^{\text{th}}(s) + R_q^{\text{cont}}(s)$$



$$R_q^{\text{Res}}(s) = \frac{9\pi M_R \Gamma_R^e}{\alpha_{\text{em}}^2(M_R)}\delta(s - M_R^2)$$

$$R_q^{\text{th}}(s) = R_q(s) - R_{\text{background}}$$ $(2M_D \leq \sqrt{s} \leq 4.8\text{GeV})$

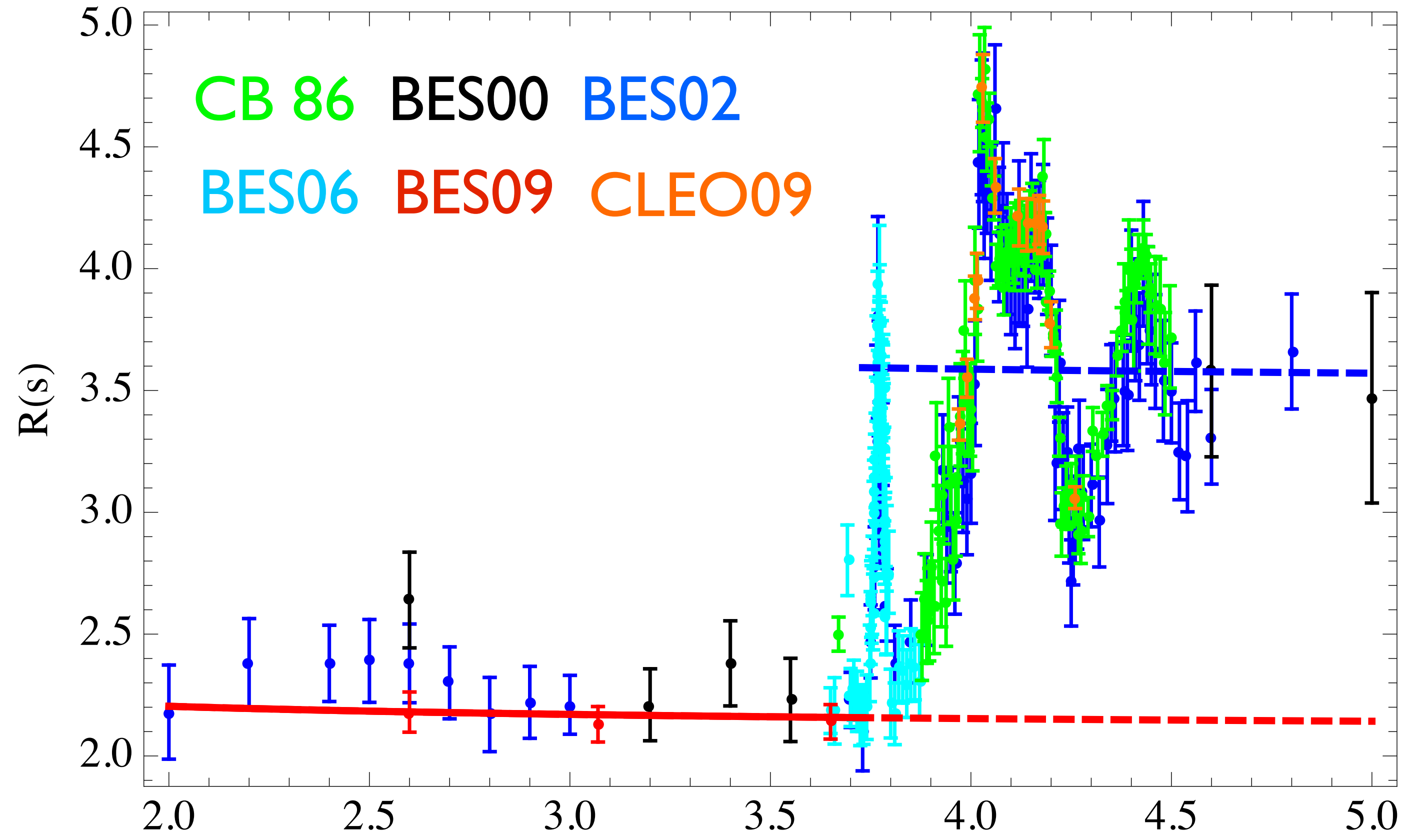
$R_q^{\text{cont}}(s)$ **calculated using pQCD** $(\sqrt{s} \geq 4.8\text{GeV})$



Background*

$$R_{\rm background} = R_{\rm uds} + R_{\rm uds(cb)} + R_{\rm sing} + R_{\rm QED}$$

Light flavor contribution in charm region
+
secondary production
+
singlet contribution

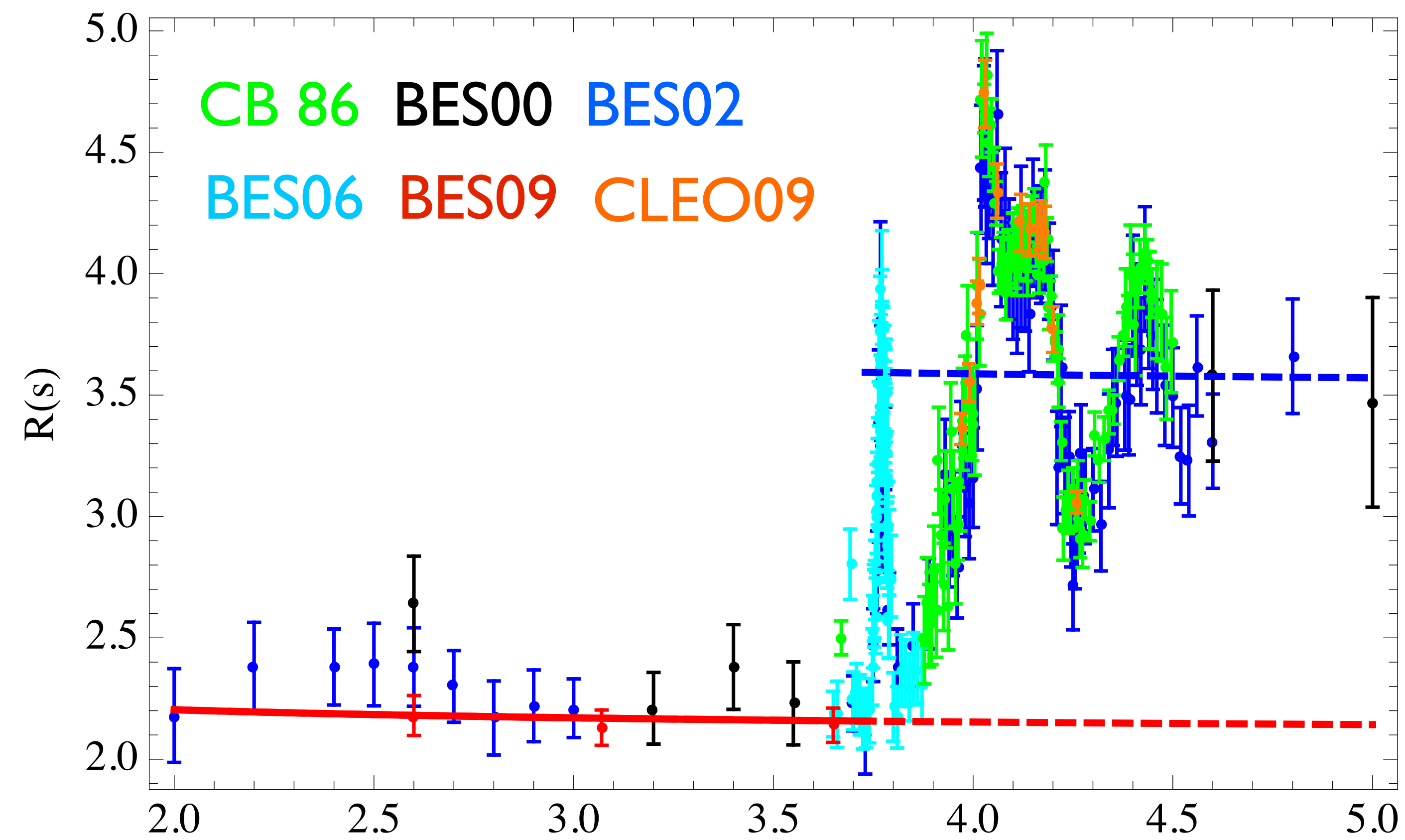


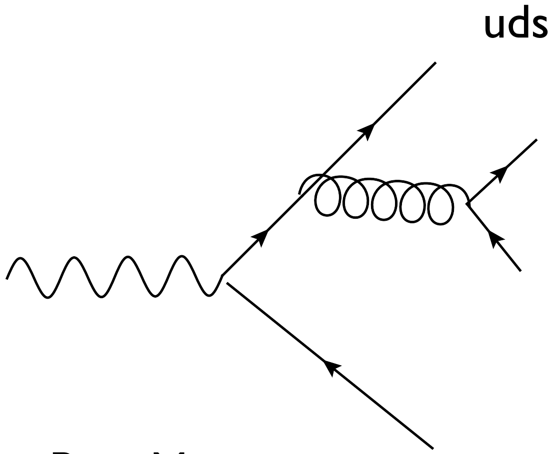
Using pQCD below threshold, calculate R, and extrapolate

Background*

$$R_{\text{background}} = R_{\text{uds}} + R_{\text{uds(cb)}} + R_{\text{sing}} + R_{\text{QED}}$$

Light flavor contribution in charm region
+
secondary production
+
singlet contribution



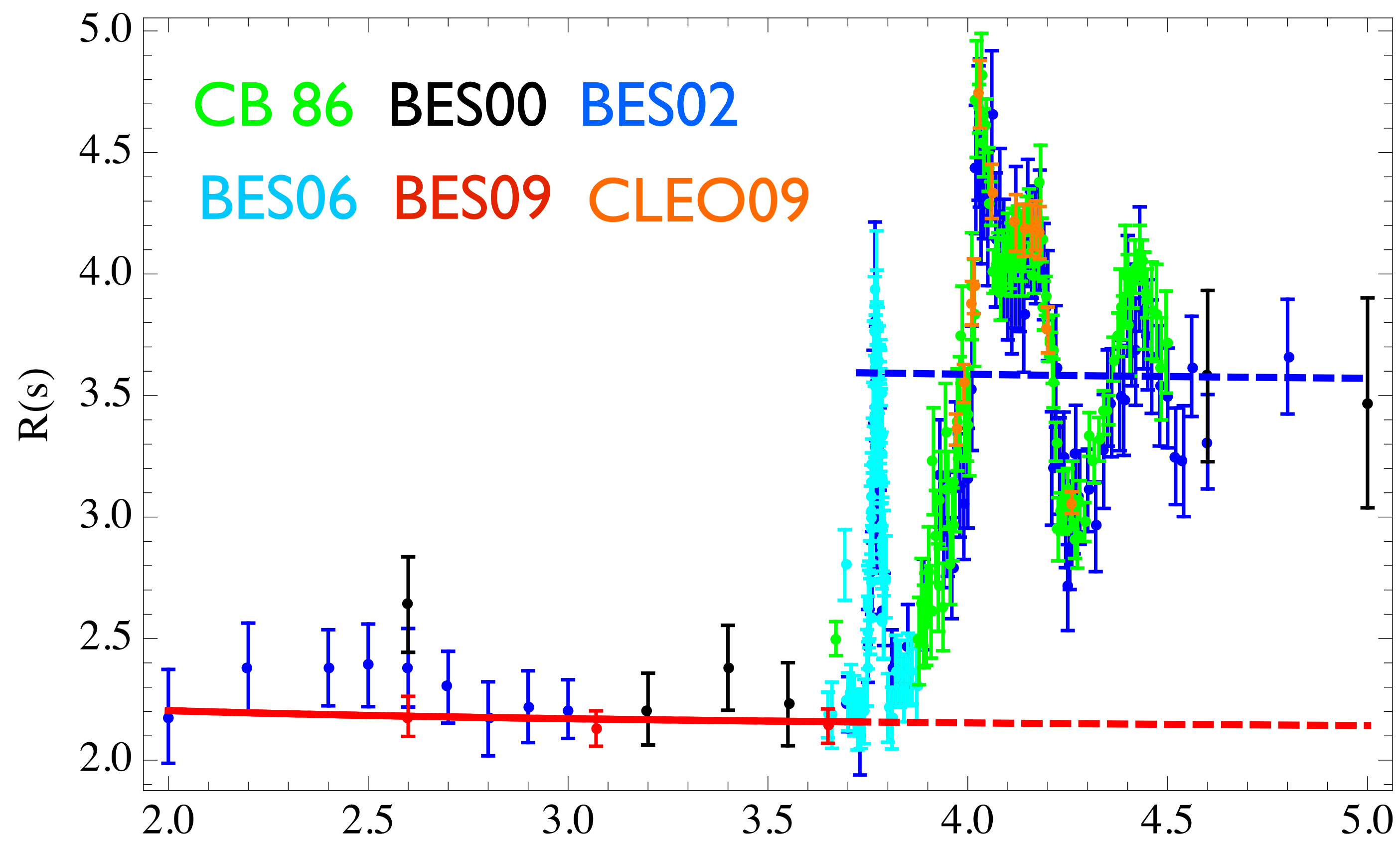


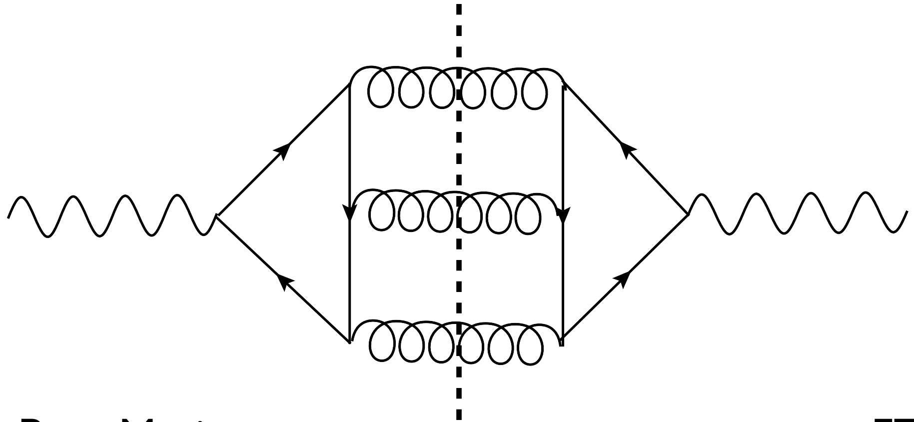
$\sim \alpha^2$

Background*

$$R_{\mathrm{background}} = R_{\mathrm{uds}} + R_{\mathrm{uds(cb)}} + R_{\mathrm{sing}} + R_{\mathrm{QED}}$$

Light flavor contribution in charm region
+
secondary production
+
singlet contribution



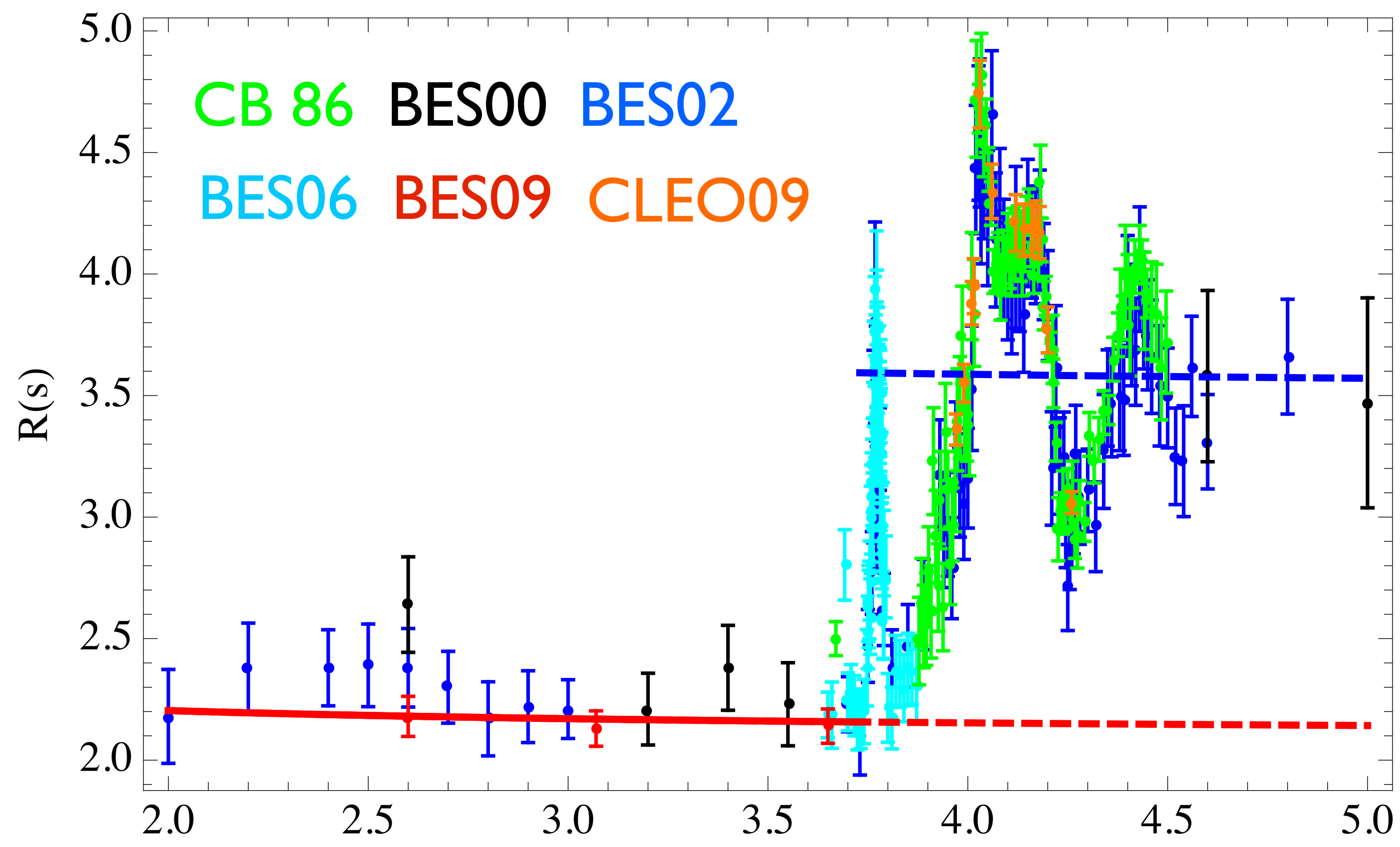


$\sim \alpha^3$

Background*

$$R_{\text{background}} = R_{\text{uds}} + R_{\text{uds(cb)}} + R_{\text{sing}} + R_{\text{QED}}$$

Light flavor
contribution in
charm region
+
secondary
production
+
singlet contribution
+
2loop QED



Non-perturbative effects

Non-perturbative effects due to gluon condensates to the moments are:

[Chetyrkin et al '12]

$$\mathcal{M}_n^{\rm nonp}(\mu^2) = \frac{12\pi^2 Q_q^2}{(4\hat{m}_q^2)^{n+2}} \mathrm{Cond}\, a_n \left(1 + \frac{\alpha_s(\hat{m}_q^2)}{\pi} b_n\right)$$

$a_n\ , b_n$ are numbers, and $\mathrm{Cond} = \langle \frac{\alpha_s}{\pi} G^2 \rangle = (5 \pm 5) \cdot 10^{-3} \mathrm{GeV}^4$ [Dominguez et al '14]

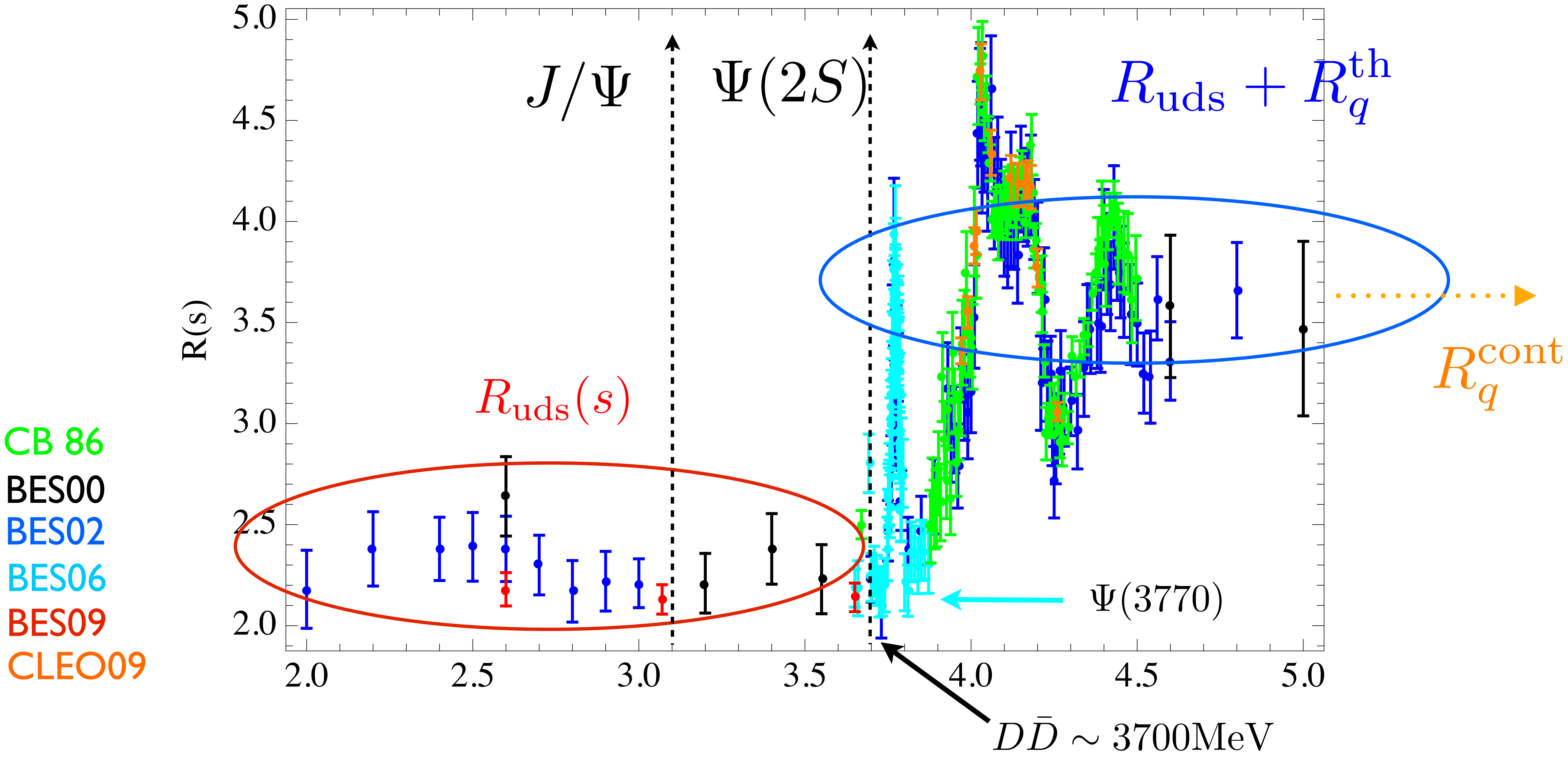
→ from fits to tau data

$$\frac{\mathcal{M}_n^{\rm nonp}(\hat{m}_c)}{\mathcal{M}_n^{\rm th}} \sim 0.5\% - 2\% \longrightarrow \Delta\hat{m}_c(\hat{m}_c) \sim 2\mathrm{MeV} - 8\mathrm{MeV}$$

QCD Sum Rules

$$R(s) = R_{\rm uds}(s) + R_q(s) \qquad R_q(s) = R_q^{\rm Res}(s) + R_q^{\rm th}(s) + R_q^{\rm cont}(s)$$

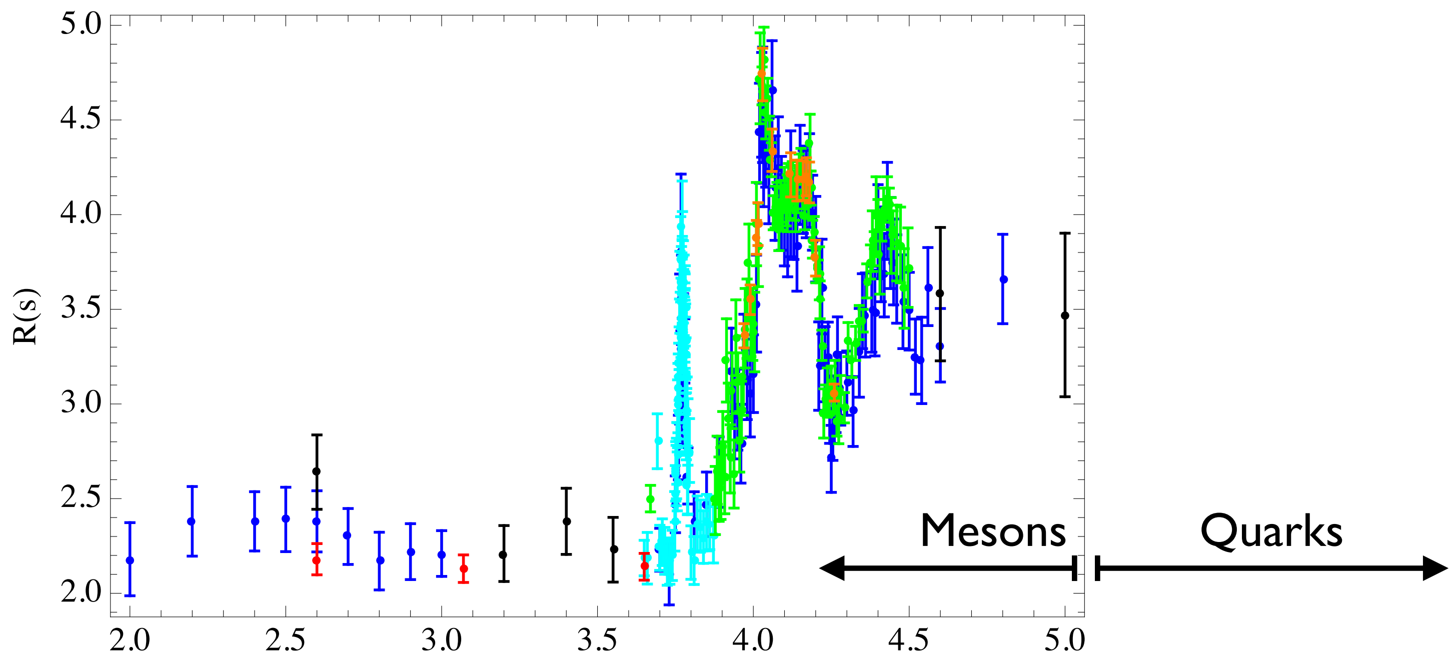
[Erler, P.M., Spiesberger '17]



QCD Sum Rules

Our approach is *different*

- We try to avoid *local* duality: consider *global* duality



Standard procedure: $$\mathscr{M}_n^{\mathrm{pQCD}} = R_q^{\mathrm{Res}} + \int_{D\bar{D}}^{s_0} \frac{\mathrm{d}s}{s^{n+1}} R_q(s) + pQCD(\mu)$$

We really want: $$\mathscr{M}_n^{\mathrm{pQCD}} = R_q^{\mathrm{Res}} + \int_{D\bar{D}}^{\infty} \frac{\mathrm{d}s}{s^{n+1}} R_q(s)$$

QCD Sum Rules

Our approach is *different*

- We try to avoid *local* duality: consider *global* duality
- Then, we do *not use experimental data* on threshold region, only resonances below threshold
  - Experimental data in threshold used for error estimation
- How you do it then? Use *two different moment* equations to determine the continuum requiring self-consistency:
  - extract the quark mass

$$\mathcal{M}_n^{\rm pQCD} = R_q^{\rm Res} + \int_{D\bar{D}}^{\infty} \frac{{\rm d}s}{s^{n+1}} R_q^{\rm cont}(s)$$

Charm

QCD Sum Rules

Our approach

For a global duality:

$\hat{\Pi}_q(s)$ in $\overline{MS}$

$$12\pi^2 \frac{\hat{\Pi}_q(0) - \hat{\Pi}_q(-t)}{t} = \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s} \frac{R_q(s)}{s+t}$$

$t \to \infty \quad \text{define the } \mathcal{M}_0$

[Erler, Luo ’03]

QCD Sum Rules

Our approach

For a global duality:

$\hat{\Pi}_q(s)$ in $\overline{MS}$

$$12\pi^2 \frac{\hat{\Pi}_q(0) - \hat{\Pi}_q(-t)}{t} = \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s} \frac{R_q(s)}{s+t}$$

$t \to \infty$ $\quad \text{define the } \mathcal{M}_0$ (but has a divergent part) [Erler, Luo '03]

$$\lim_{t\to\infty} \hat{\Pi}_q(-t) \sim \log(t) \quad \longleftrightarrow \quad \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s} R_q(s) \sim \log(\infty)$$

Fortunately, divergence given by the zero-mass limit of R(s), can be easily subtracted

[Chetyrkin, Harlander, Kühn, '00]

QCD Sum Rules

Our approach

zero-mass limit of R(s)

[Chetyrkin, Harlander, Kühn, '00]

$$\lambda_1^q(s) = 1 + \frac{\alpha_s(s)}{\pi}$$

$$+ \left[\frac{\alpha_s(s)}{\pi}\right]^2 \left[\frac{365}{24} - 11\zeta(3) + n_q\left(\frac{2}{3}\zeta(3) - \frac{11}{12}\right)\right]$$

$$+ \left[\frac{\alpha_s(s)}{\pi}\right]^3 \left[\frac{87029}{288} - \frac{121}{8}\zeta(2) - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5)\right.$$

$$+n_q\left(-\frac{7847}{216} + \frac{11}{6}\zeta(2) + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta(5)\right)$$

$$\left.+n_q^2\left(\frac{151}{162} - \frac{1}{18}\zeta(2) - \frac{19}{27}\zeta(3)\right)\right]$$

n_q active flavors

QCD Sum Rules

Our approach

Zeroth Sum Rule:

$$\sum_{\text{resonances}} \frac{9\pi\Gamma_R^e}{3Q_q^2 M_R \hat{\alpha}_{em}^2(M_R)} + \int_{4M^2}^{\infty} \frac{\mathrm{d}s}{s} \frac{R_q^{\text{cont}}}{3Q_q^2} - \int_{\hat{m}_q^2}^{\infty} \frac{\mathrm{d}s}{s} \lambda_1^q(s)$$

$$= -\frac{5}{3} + \frac{\hat{\alpha}_s}{\pi}\left[4\zeta(3) - \frac{7}{2}\right] \qquad \hat{\alpha}_s = \alpha_s(\hat{m}_q^2)$$

$$+ \left(\frac{\hat{\alpha}_s}{\pi}\right)^2 \left[\frac{2429}{48}\zeta(3) - \frac{25}{3}\zeta(5) - \frac{2543}{48} + n_q\left(\frac{677}{216} - \frac{19}{9}\zeta(3)\right)\right]$$

$$+ \left(\frac{\hat{\alpha}_s}{\pi}\right)^3 \left[-9.86 + 0.40\, n_q - 0.01\, n_q^2\right]$$

$$= -1.667 + 1.308\,\frac{\hat{\alpha}_s}{\pi} + 1.595\left(\frac{\hat{\alpha}_s}{\pi}\right)^2 - 8.427\left(\frac{\hat{\alpha}_s}{\pi}\right)^3$$

QCD Sum Rules

Our approach: ansatz

Zeroth Sum Rule: invoke global quark-hadron duality

[Erler, Luo '03]

$$R_q^{\rm cont}(s) = 3Q_q^2\lambda_1^q(s)\sqrt{1-\frac{4\,\hat{m}_q^2(2M)}{s'}}\left[1+\lambda_3^q\frac{2\,\hat{m}_q^2(2M)}{s'}\right]$$

Simpler version of analytic reconstruction [Greynat, PM, Peris'12]

$$s' = s + 4(\hat{m}_q^2(2M) - M^2)$$

Two parameters to determine: m_q , λ_3^q

QCD Sum Rules

Our approach: ansatz

Zeroth Sum Rule: invoke global quark-hadron duality

[Erler, Luo '03]

$$R_q^{\rm cont}(s) = 3Q_q^2 \lambda_1^q(s) \sqrt{1 - \frac{4\,\hat{m}_q^2(2M)}{s'}} \left[1 + \lambda_3^q \frac{2\,\hat{m}_q^2(2M)}{s'}\right]$$

Simpler version of analytic reconstruction [Greynat, PM, Peris'12]

$$s' = s + 4(\hat{m}_q^2(2M) - M^2)$$

Two parameters to determine: $m_q \,,\, \lambda_3^q$

We need two equations: *zeroth* moment + *nth* moment

$$\frac{9}{4} Q_q^2 \left(\frac{1}{2\hat{m}_q(\hat{m}_q)}\right)^{2n} \bar{C}_n = \sum_{\rm resonances} \frac{9\pi \Gamma_R^e}{M_R^{2n+1}\hat{\alpha}_{em}^2(M_R)} + \int_{4M^2}^{\infty} \frac{{\rm d}s}{s^{n+1}} R_q(s)$$

$$n \geq 1$$

QCD Sum Rules

Our approach: ansatz

Zeroth Sum Rule: invoke global quark-hadron duality

[Erler, Luo '03]

$$R_q^{\rm cont}(s) = 3Q_q^2 \lambda_1^q(s) \sqrt{1 - \frac{4\,\hat{m}_q^2(2M)}{s'}} \left[1 + \lambda_3^q \frac{2\,\hat{m}_q^2(2M)}{s'}\right]$$

Simpler version of analytic reconstruction [Greynat, PM, Peris'12]

$$s' = s + 4(\hat{m}_q^2(2M) - M^2)$$

Two parameters to determine: m_q , λ_3^q

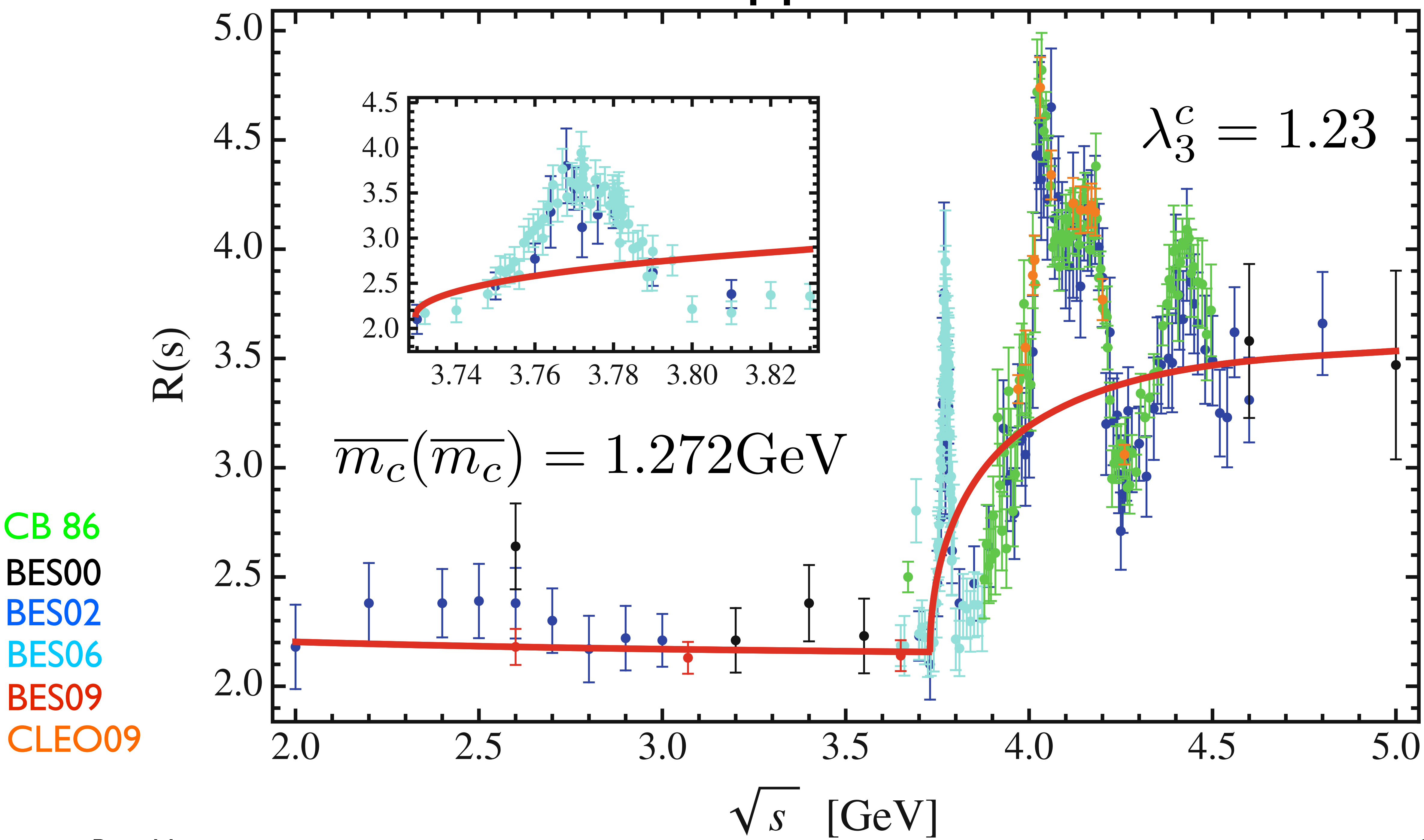
We use Zeroth + 2nd moments
(no experimental data on R(s) so far)

we require self-consistency among the 2 moments

n	Resonances	Continuum	Total	Theory
0	1.231 (24)	−3.229(+28)(43)(1)	−1.999(56)	**Input** (11)
1	1.184 (24)	0.966(+11)(17)(4)	2.150(33)	2.169(16)
2	1.161 (25)	0.336(+5)(8)(9)	1.497(28)	**Input** (25)
3	1.157 (26)	0.165(+3)(4)(16)	1.322(31)	1.301(39)
4	1.167 (27)	0.103(+2)(2)(26)	1.270(38)	1.220(60)
5	1.188 (28)	0.080(+1)(1)(38)	1.268(47)	1.175(95)

QCD Sum Rules

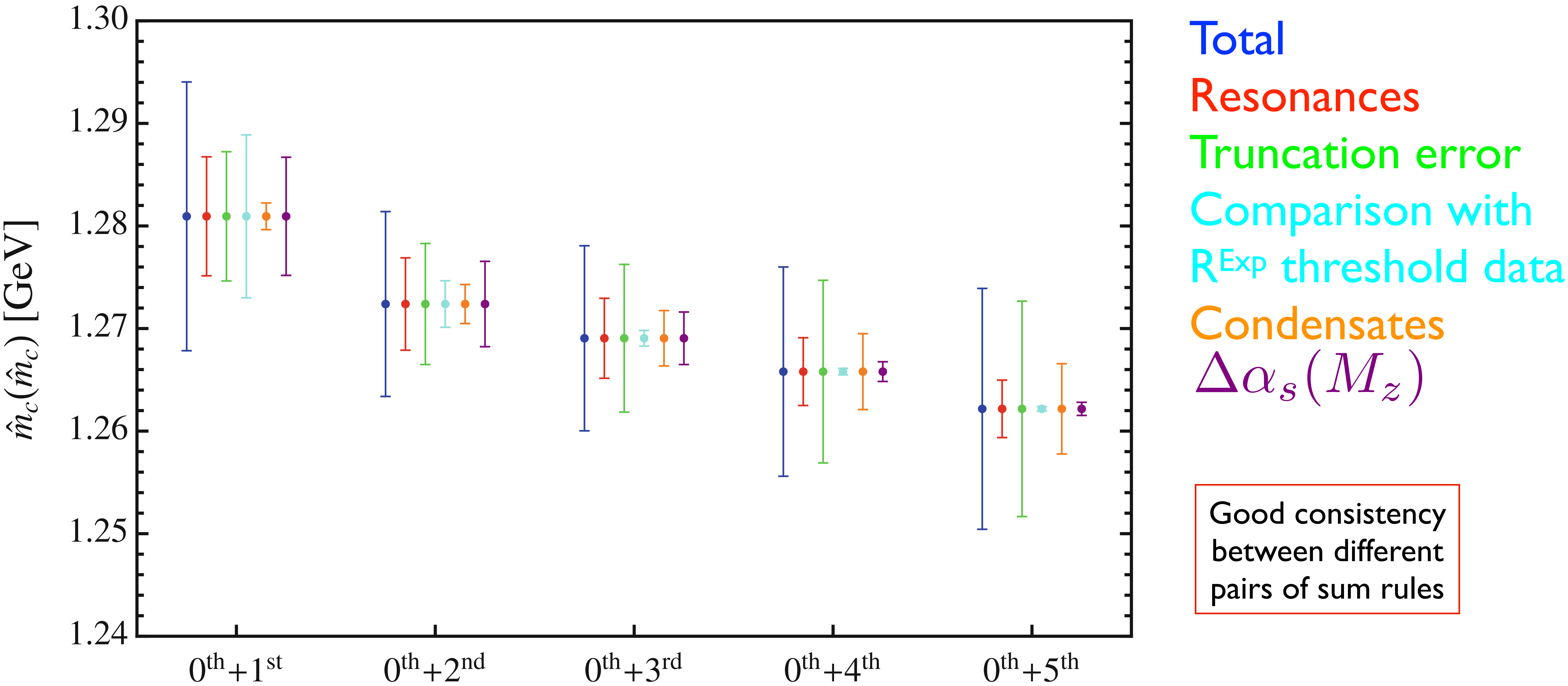
Our approach



QCD Sum Rules

Our approach

Repeat for each pair Zeroth+nth moment



Good consistency between different pairs of sum rules

QCD Sum Rules

Our approach: error budget

Resonances:

$$\frac{9}{4}Q_q^2\left(\frac{1}{2\hat{m}_q(\hat{m}_q)}\right)^{2n}\bar{C}_n = \sum_{\text{resonances}} \frac{9\pi\Gamma_R^e}{M_R^{2n+1}\hat{\alpha}_{em}^2(M_R)} + \int_{4M^2}^{\infty}\frac{\mathrm{d}s}{s^{n+1}}R_q(s)$$

from 6 MeV to 3 MeV

(0th+1st) (0th+5th)

(completely dominated by J/Ψ)

R	M_R [GeV]	Γ_R^e [keV]
J/Ψ	3.096916	5.55(14)
$\Psi(2S)$	3.686109	2.36(4)

QCD Sum Rules

Our approach: error budget

Truncation Error (theory error):

$$\mathcal{M}_n^{\mathrm{pQCD}} = \frac{9}{4} Q_q^2 \left(\frac{1}{2\hat{m}_q(\hat{m}_q)} \right)^{2n} \bar{C}_n$$

$$\bar{C}_n = \bar{C}_n^{(0)} + \left(\frac{\hat{\alpha}}{\pi}\right) \bar{C}_n^{(1)} + \left(\frac{\hat{\alpha}}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\hat{\alpha}}{\pi}\right)^3 \bar{C}_n^{(3)} + \mathcal{O}\left(\frac{\hat{\alpha}}{\pi}\right)^4$$

(use the largest group th. factor in the next uncalculated pert. order)

[Erler, Luo '03]

$$\Delta\mathcal{M}_n^{(4)} = \pm N_C C_F C_A^3 \, Q_q^2 \left[\frac{\hat{\alpha}_s(\hat{m}_q)}{\pi}\right]^4 \left(\frac{1}{2\hat{m}_q(\hat{m}_q)}\right)^{2n}$$

from 5 MeV to 10 MeV
(0th+1st) (0th+5th)

Example known orders

n	$\frac{\Delta\mathcal{M}_n^{(2)}}{\lvert\mathcal{M}_n^{(2)}\rvert}$	$\frac{\Delta\mathcal{M}_n^{(3)}}{\lvert\mathcal{M}_n^{(3)}\rvert}$
0	1.88	3.03
1	2.14	2.84
2	1.92	4.58
3	3.25	5.63
4	6.70	4.30
5	19.18	3.62

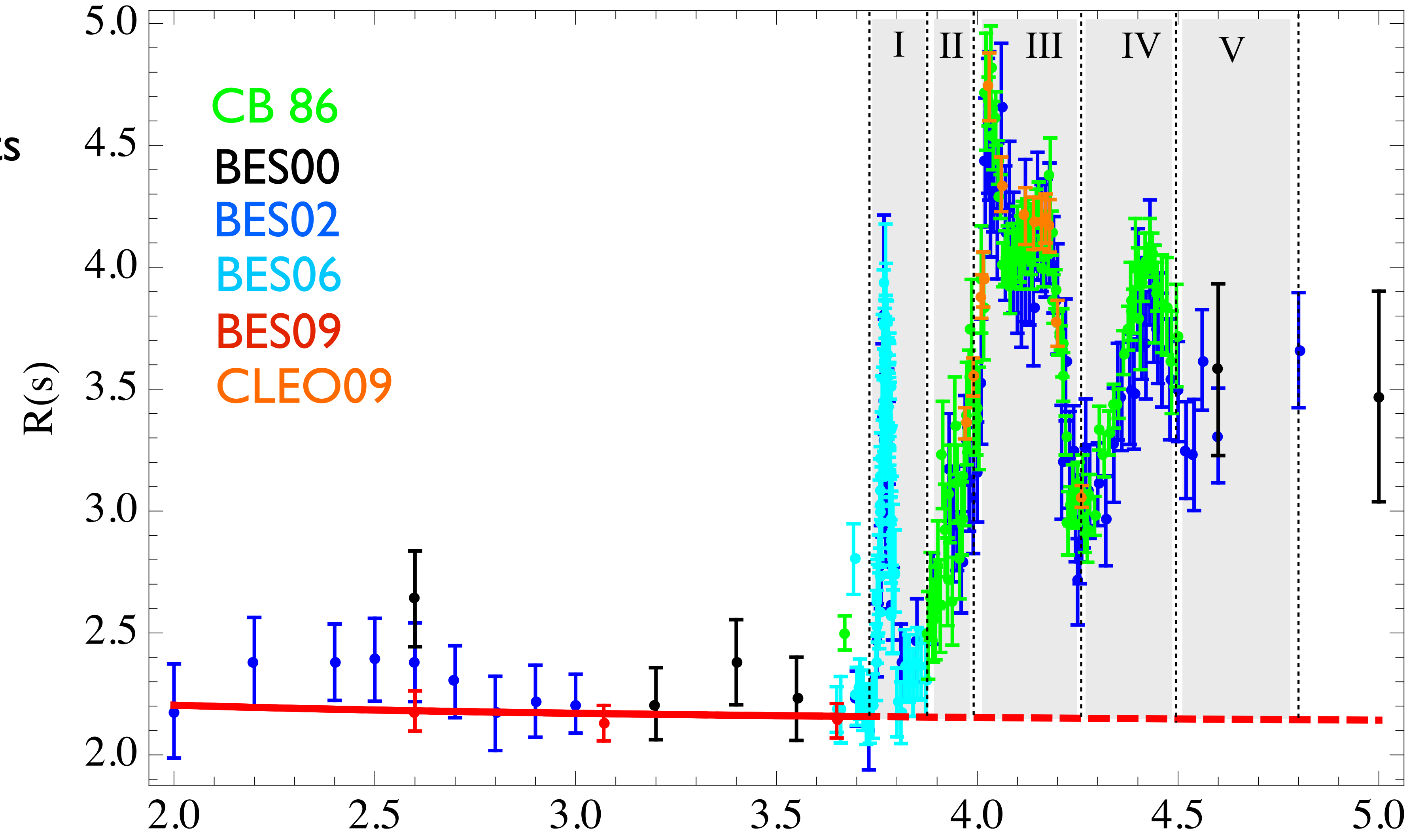
More conservative than varying the renorm. scale within a factor of 4

QCD Sum Rules

Our approach: error budget

Comparison with R^{Exp} threshold data: $(2M_D \leq \sqrt{s} \leq 4.8\text{GeV})$

Calculate Exp moments



QCD Sum Rules

Our approach: error budget

Comparison with R^{Exp} threshold data:

Collab.	n	$[2M_{D^0}, 3.872]$	[3.872, 3.97]	[3.97, 4.26]	[4.26, 4.496]	[4.496, 4.8]
CB86	0	–	0.0339(22)(24)	0.2456(25)(172)	0.1543(27)(108)	–
	1	–	0.0220(14)(15)	0.1459(16)(102)	0.0801(14)(56)	–
	2	–	0.0143(9)(10)	0.0868 (9)(61)	0.0416(7)(29)	–
BES02	0	0.0334(24)(17)	0.0362(29)(18)	0.2362(41)(118)	0.1399(38)(70)	0.1705(63)(85)
	1	0.0232(17)(12)	0.0235(19)(12)	0.1401(24)(70)	0.0726(20)(36)	0.0788(30)(39)
	2	0.0161(12)(8)	0.0152(13)(8)	0.0832(15)(42)	0.0378(10)(19)	0.0365(14)(18)
BES06	0	0.0311(16)(15)	–	–	–	–
	1	0.0217(11)(11)	–	–	–	–
	2	0.0151(8)(7)	–	–	–	–
CLEO09	0	–	–	0.2591(22)(52)	–	–
	1	–	–	0.1539(13)(31)	–	–
	2	–	–	0.0915(8)(18)	–	–
Total	0	0.0319(14)(11)	0.0350(18)(15)	0.2545(18)(46)	0.1448(27)(59)	0.1705(63)(85)
	1	0.0222(9)(8)	0.0227(12)(10)	0.1511(11)(27)	0.0752(14)(31)	0.0788(30)(39)
	2	0.0155(6)(6)	0.0147(8)(6)	0.0899(6)(16)	0.0391(7)(16)	0.0365(14)(18)

QCD Sum Rules

Our approach: error budget

Comparison with R^{Exp} threshold data:

$$\int_{(2M_{D^0})^2}^{(4.8\,\mathrm{GeV})^2} \frac{\mathrm{d}s}{s} R_c^{\mathrm{cont}}(s)\Big|_{\hat{m}_c=1.272\,\mathrm{GeV}} = \mathcal{M}_0^{\mathrm{Data}} = 0.6367(195) \longrightarrow \lambda_3^{\mathrm{c,exp}} = 1.34(17)$$

$$(2M_D \leq \sqrt{s} \leq 4.8\mathrm{GeV})$$

Error induced to Quark mass:

I) $\lambda_3^c = 1.23 \to \lambda_3^{\mathrm{c,exp}} = 1.34$

from $+6.4$ MeV to $+0.2$ MeV

II) $\Delta\lambda_3^{\mathrm{c,exp}} = 0.17$

from 4.7 MeV to 0.1 MeV

n	Data	$\lambda_3^c = 1.34(17)$	$\lambda_3^c = 1.23$
0	0.6367(195)	0.6367(195)	0.6239
1	0.3500(102)	0.3509(111)	0.3436
2	0.1957(54)	0.1970(65)	0.1928
3	0.1111(29)	0.1127(38)	0.1102
4	0.0641(16)	0.0657(23)	0.0642
5	0.0375(9)	0.0389(14)	0.0380

QCD Sum Rules

Our approach: error budget

Comparison with R^{Exp} threshold data:

$$\int_{(2M_{D^0})^2}^{(4.8\,\mathrm{GeV})^2} \frac{\mathrm{d}s}{s} R_c^{\mathrm{cont}}(s)\Big|_{\hat{m}_c=1.272\,\mathrm{GeV}} = \mathcal{M}_0^{\mathrm{Data}} = 0.6367(195) \longrightarrow \lambda_3^{\mathrm{c,exp}} = 1.34(17)$$

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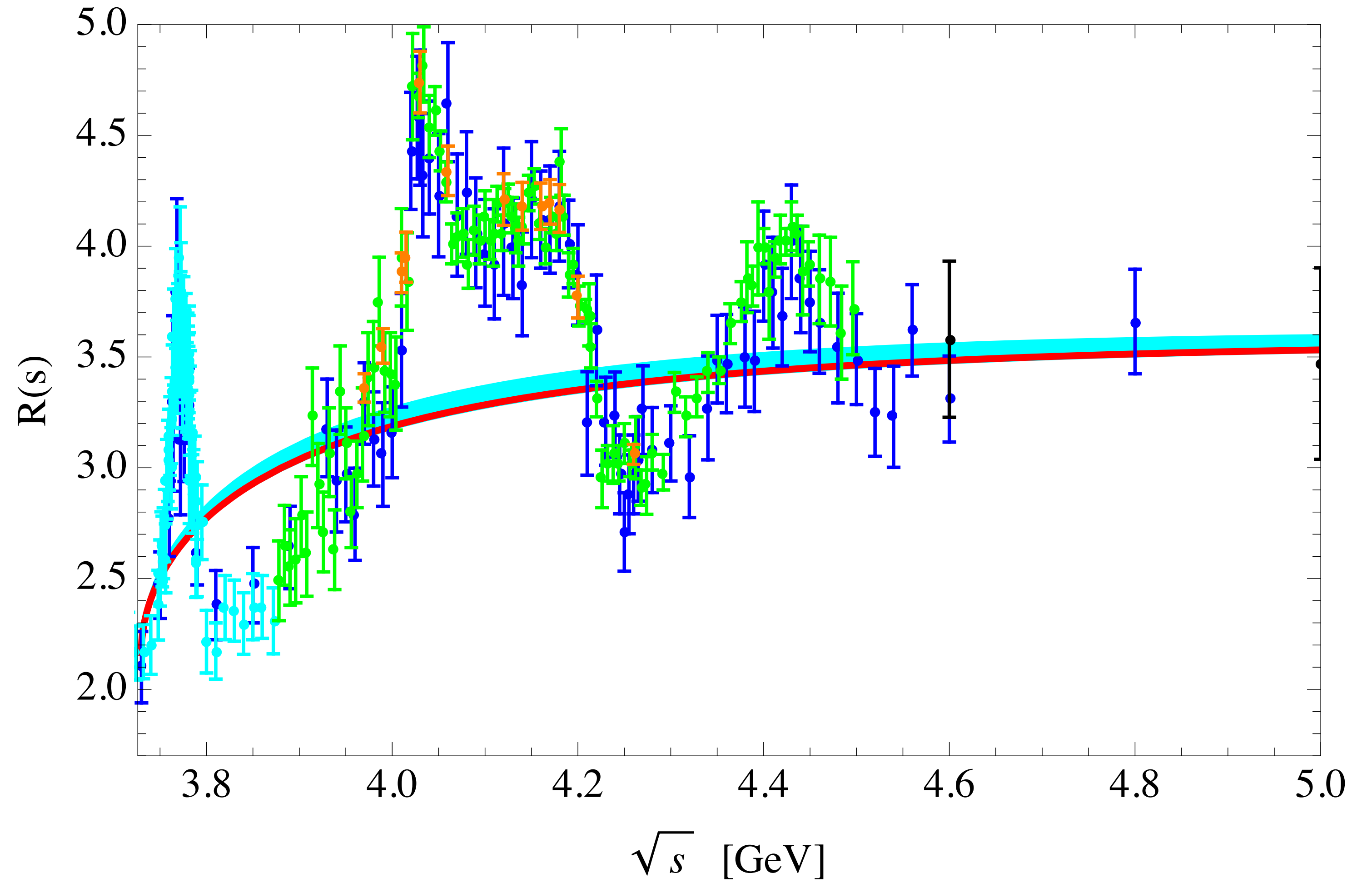


QCD Sum Rules

Our approach: error budget

Condensates:

Non-perturbative effects due to gluon condensates to the moments are: [Chetyrkin et al '12]

$$\mathcal{M}_n^{\rm nonp}(\mu^2) = \frac{12\pi^2 Q_q^2}{(4\hat{m}_q^2)^{n+2}} \mathrm{Cond}\, a_n \left(1 + \frac{\alpha_s(\hat{m}_q^2)}{\pi} b_n\right)$$

$a_n\,,b_n$ are numbers, and $\mathrm{Cond} = \langle \frac{\alpha_s}{\pi} G^2 \rangle = (5 \pm 5)\cdot 10^{-3}\mathrm{GeV}^4$ [Dominguez et al '14]

$\Delta\langle \frac{\alpha_s}{\pi} G^2 \rangle = 5 \cdot 10^{-3}\mathrm{GeV}^4 \longrightarrow$ from 1 MeV to 4 MeV

(0th+1st) (0th+5th)

Parametric error:

$$\Delta\overline{m_c}(\overline{m_c})[\mathrm{MeV}] = -0.5\cdot 10^3 \frac{\mathrm{MeV}}{\mathrm{GeV}^4} \Delta\langle \frac{\alpha_s}{\pi} G^2 \rangle$$

(but this is only the first condensate)

QCD Sum Rules

Our approach: error budget

$\Delta\alpha_s(M_z)$ $\alpha_s(M_z) = 0.1182(16)$ from PDG16

$$\Delta\alpha_s(M_z) = 0.0016 \longrightarrow \text{from 6 MeV to 1 MeV}$$

Parametric error:

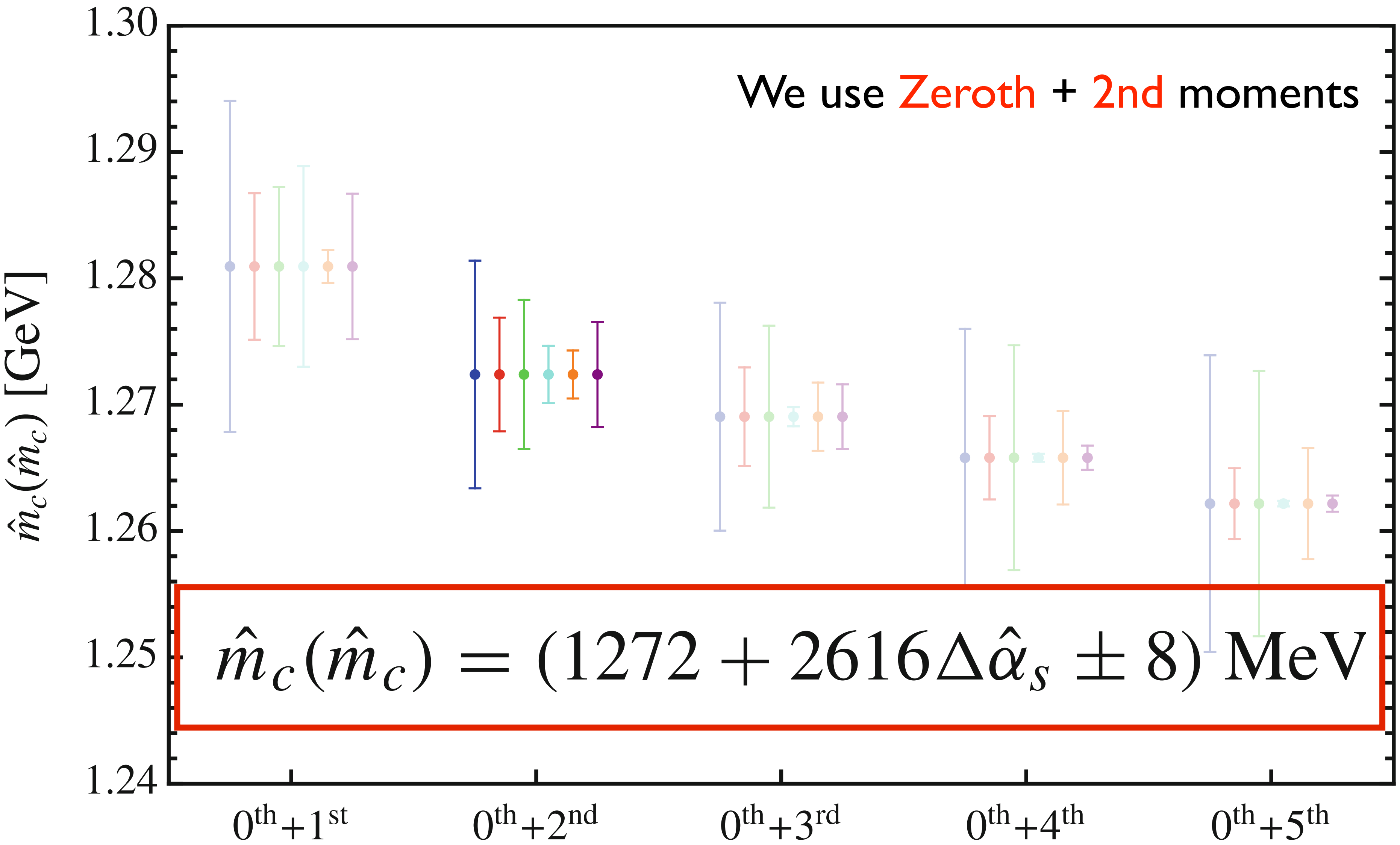
(0th+1st) $\Delta\overline{m_c}(\overline{m_c})[\text{MeV}] = 3.6 \cdot 10^3 \Delta\alpha_s(M_z)$

(0th+5th) $\Delta\overline{m_c}(\overline{m_c})[\text{MeV}] = -0.4 \cdot 10^3 \Delta\alpha_s(M_z)$

QCD Sum Rules

Our approach: final result

[J.Erler, P.M., H. Spiesberger'17]



Resonances
Truncation error
Comparison with R^{Exp} threshold data
Condensates
$\Delta\alpha_s(M_z)$

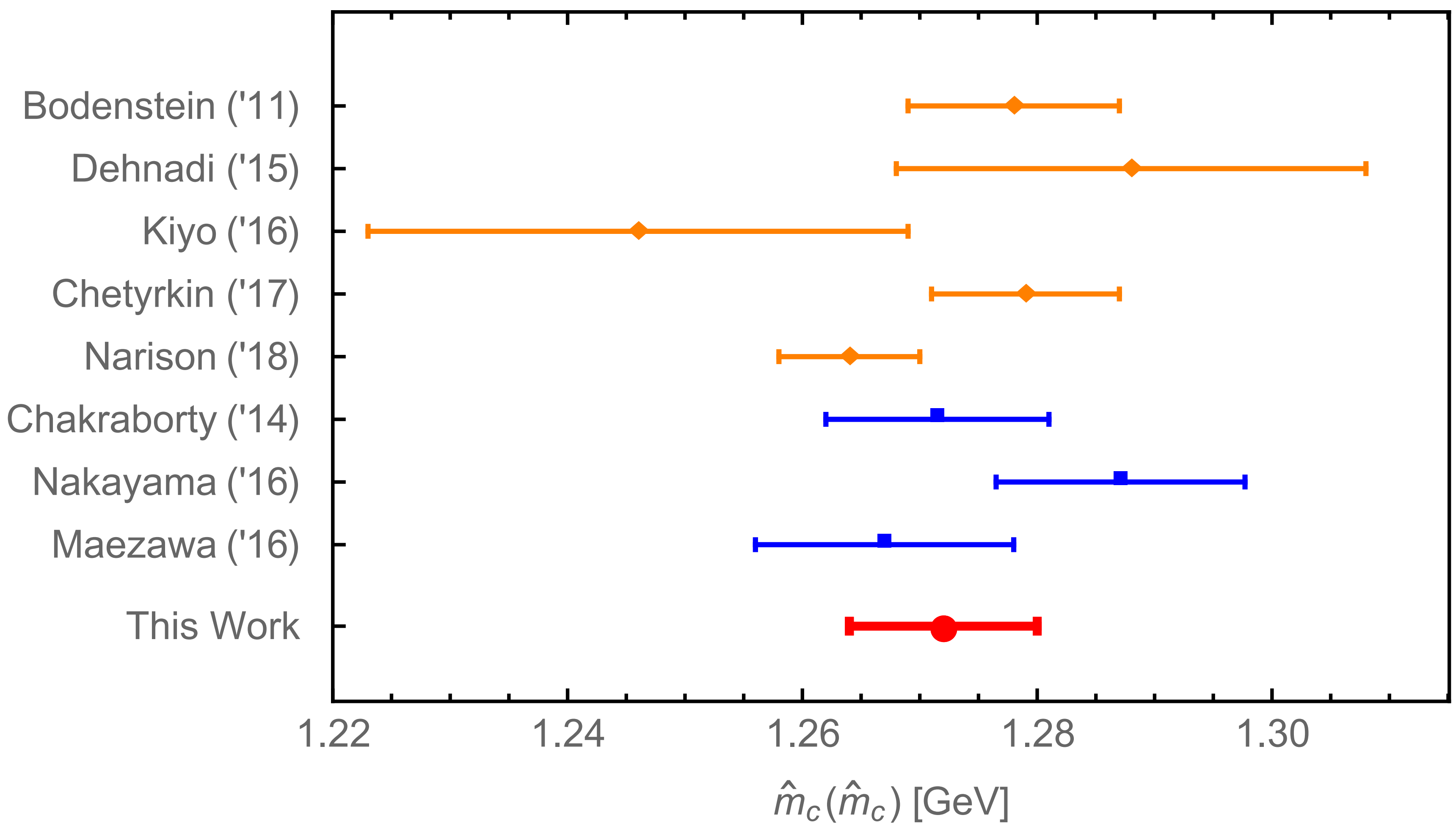
$$\hat{m}_c(\hat{m}_c) = (1272 + 2616\Delta\hat{\alpha}_s \pm 8)\ \text{MeV}$$

$$\Delta\hat{\alpha}_s = \hat{\alpha}_s(M_z) - 0.1182$$

QCD Sum Rules

results for the charm quark mass

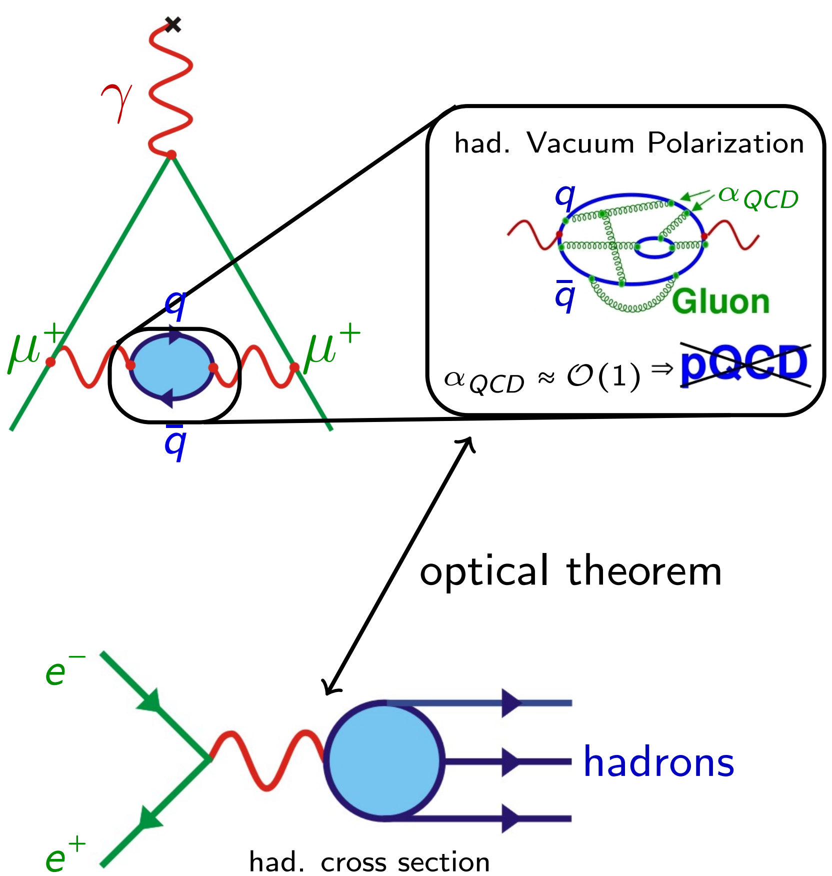
[Eur. Phys. J. C (2017) 77:99]

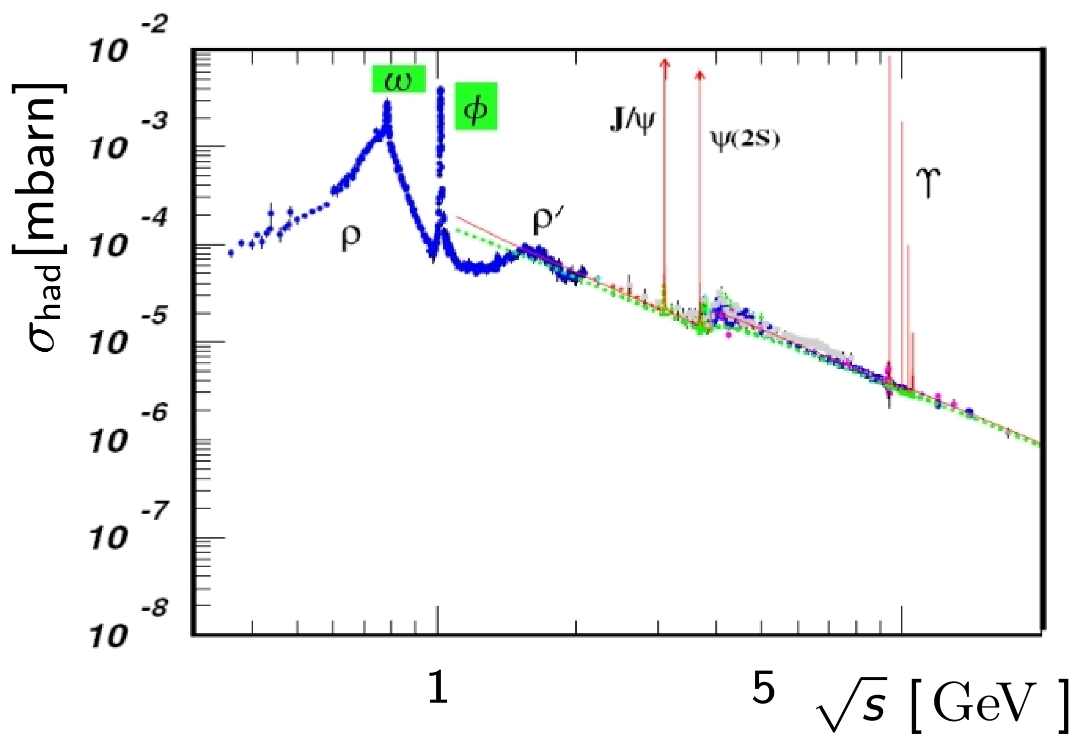


Heavy-quark contribution to (g-2)$_\mu$

Hadronic Vacuum Polarization: largest source of uncertainty in (g-2)$_\mu$

Flavor decomposition may help, specially to compare with lattice QCD estimates

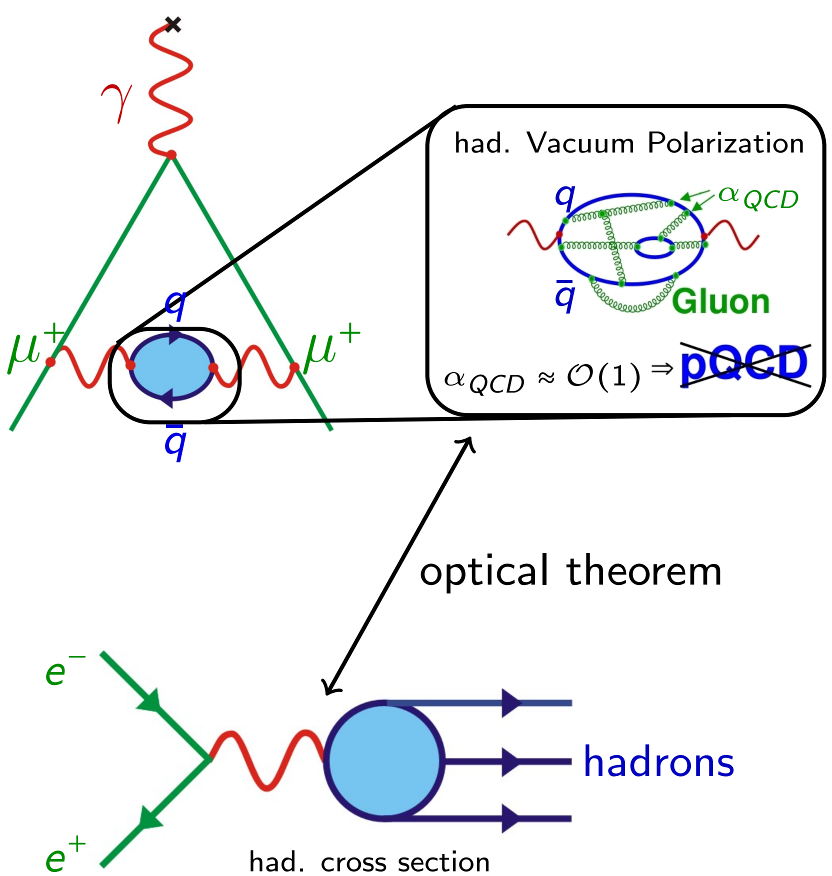




$$a_{\mu,LO}^{had} = \frac{1}{4\pi^3}\int_{m^2_{\pi^0}}^{\infty} ds\, K(s)\, \sigma_{had}(s)$$

Heavy-quark contribution to $(g\text{-}2)_\mu$

Hadronic Vacuum Polarization: largest source of uncertainty in $(g\text{-}2)_\mu$
Flavor decomposition may help, specially to compare with lattice QCD estimates



$$a^{\text{had}}_{\mu,LO} = \frac{1}{4\pi^3}\int_{m^2_{\pi^0}}^{\infty} ds\, K(s)\, \sigma_{\text{had}}(s)$$

$$a_\mu^{\text{charm}} = 14.36(23)\times 10^{-10} \qquad a_\mu^{\text{bottom}} = 0.30(2)\times 10^{-10}$$

$$a_\mu^{\text{charm-lattice}} = 14.6(1)\times 10^{-10} \qquad a_\mu^{\text{bottom-lattice}} = 0.27(4)\times 10^{-10}$$

from Borsanyi et al, *Nature* **593,** 51–55 (2021)

	central value	total error	resonances	$\Delta\lambda_3$	$\Delta\alpha_s$	Condensates	Truncation
a_μ^{charm}	1.436	0.023	0.012	0.018	0.005	0.001	0.004
a_μ^{bottom}	2.978	0.171	0.012	0.170	0.005	—	0.004

Conclusions and Outlook

- Using SR technique + *zeroth moment* (very sensitive to the continuum) + data on charm resonances below threshold + continuum exploiting self-consistency among different moments:

$$\hat{m}_c(\hat{m}_c) = 1.272(9)\text{GeV} \qquad \hat{m}_b(\hat{m}_b) = 4.180(8)\text{GeV}$$

- Error sources are understood: seems a clear roadmap for improvements
- Impact on $(g\text{-}2)_\mu$ from heavy quarks: $a_\mu^{\text{charm+bottom}} = 14.66(23) \times 10^{-10}$

- Opportunities for FTCF:
  - α_s at low energies, $R_{uds}(s)$, condensates at low energies

Thanks!

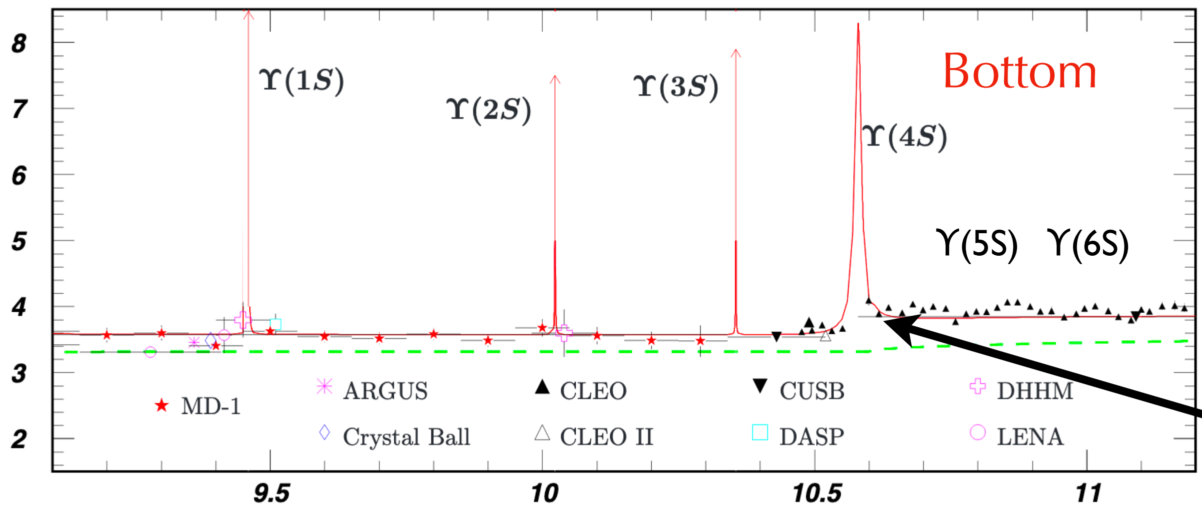
QCD Sum Rules

Bottom case

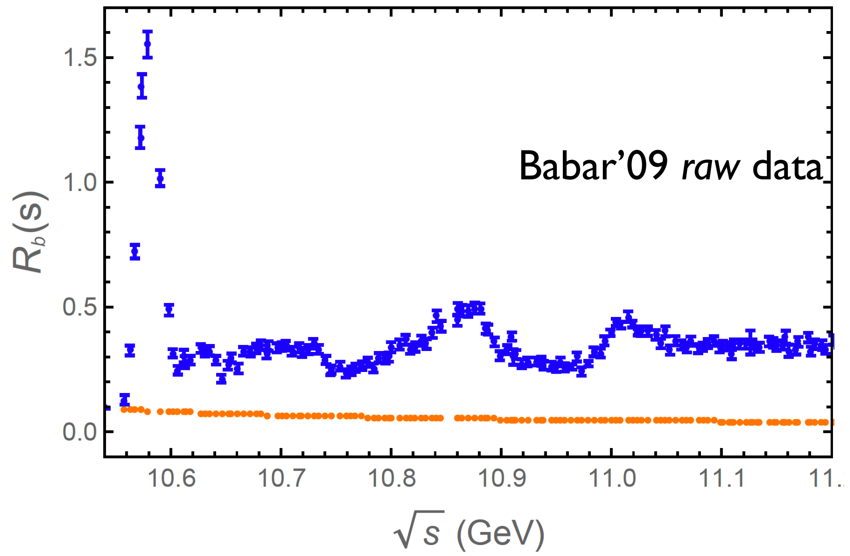
Procedure: the same as in the charm case

Main differences:

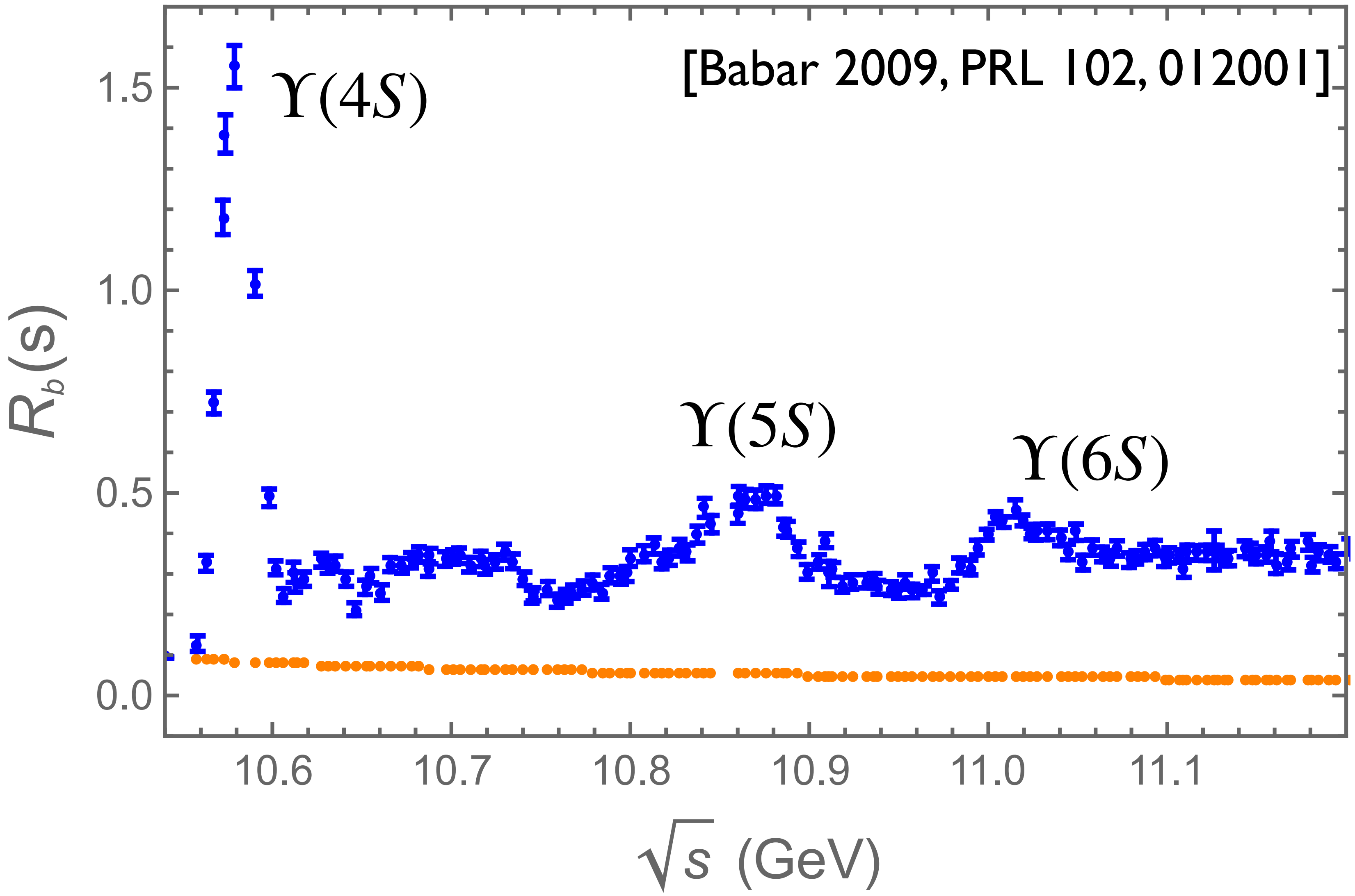
- Data from Babar '09 and Belle '15 for $R_b(s) = \sigma_b(s)/\sigma^0_{\mu\mu}$
- Condensates negligible
- Add systematically the $\Upsilon(4S), \Upsilon(5S), \Upsilon(6S)$



$B\bar{B} \sim 10.6\,\mathrm{GeV}$



QCD Sum Rules



Vacuum polarization

$$(\alpha(0)/\alpha(M_R))^2 \equiv 0.93$$

Radiative tails

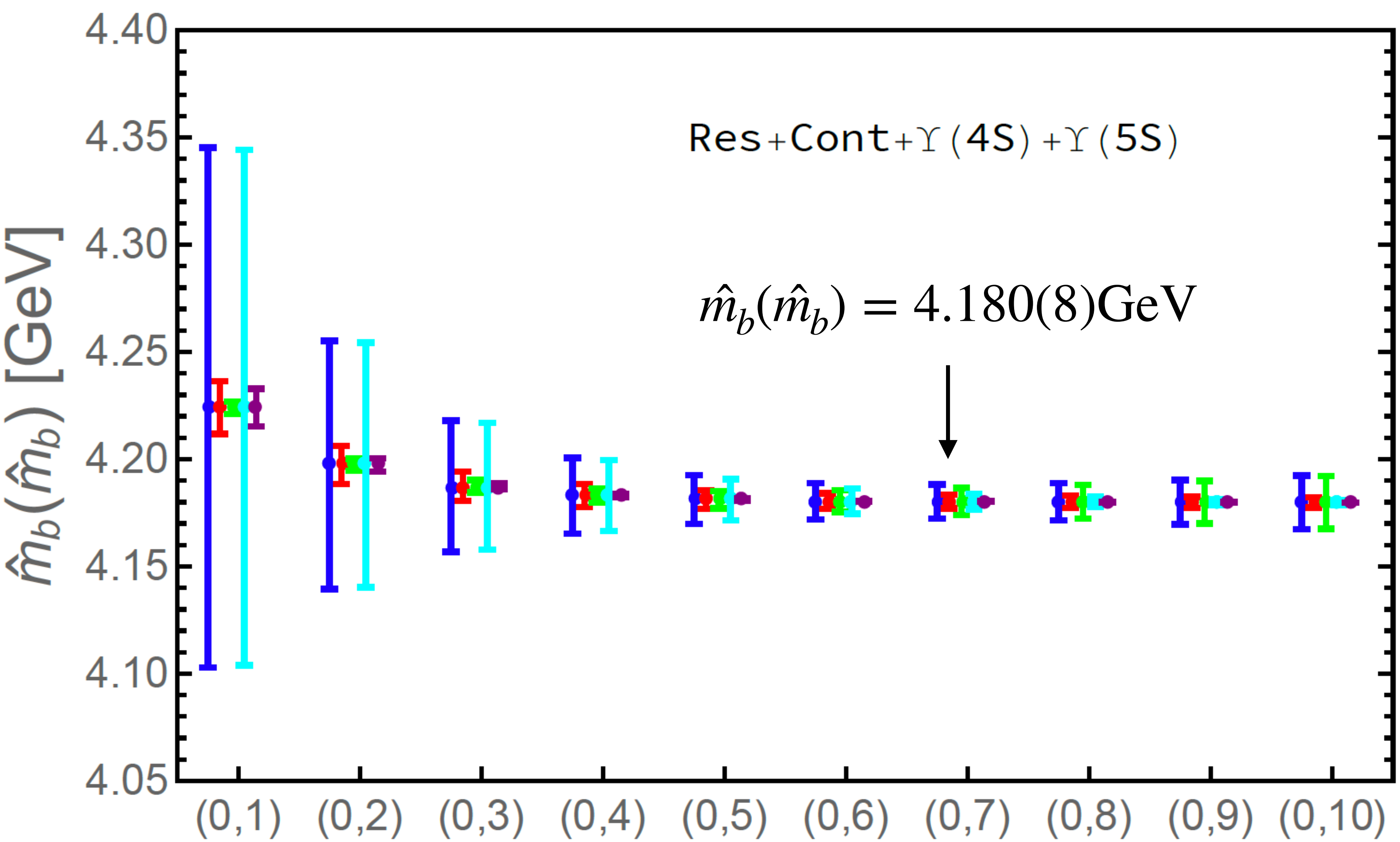
ISR corrections

$$\hat{R}(s) = \int_{z_0}^{1} \frac{\mathrm{d}z}{z} G(z,s) R(zs)$$

$$z_0 = 10.6^2/s$$

QCD Sum Rules

Our approach



Total
Resonances
Truncation error
Comparison with R^{Exp} threshold data
$\Delta\alpha_s(M_z)$

Good consistency between different pairs of sum rules

QCD Sum Rules

Our approach

Explore systematically $R_b(s) = R_b^{\text{res}}(s) + R_b^{\text{cont}}(s) + R_b^{\text{res,Gamma}}(s)$

	$\hat{m}_b(\hat{m}_b)$ [MeV]	Pair of moments
Only resonances below threshold	$4186.7 - 39.5\,\Delta\hat{\alpha}_s \pm 12.7$	$(\mathcal{M}_0, \mathcal{M}_9)$
$+\ \Upsilon(4S)$	$4183.8 - 68.0\,\Delta\hat{\alpha}_s \pm 9.7$	$(\mathcal{M}_0, \mathcal{M}_8)$
$+\ \Upsilon(4S) + \Upsilon(5S)$	$4180.2 - 108.5\,\Delta\hat{\alpha}_s \pm 7.9$	$(\mathcal{M}_0, \mathcal{M}_7)$
$+\ \Upsilon(4S) + \Upsilon(5S) + \Upsilon(6S)$	$4178.9 - 64.0\,\Delta\hat{\alpha}_s \pm 9.7$	$(\mathcal{M}_0, \mathcal{M}_8)$

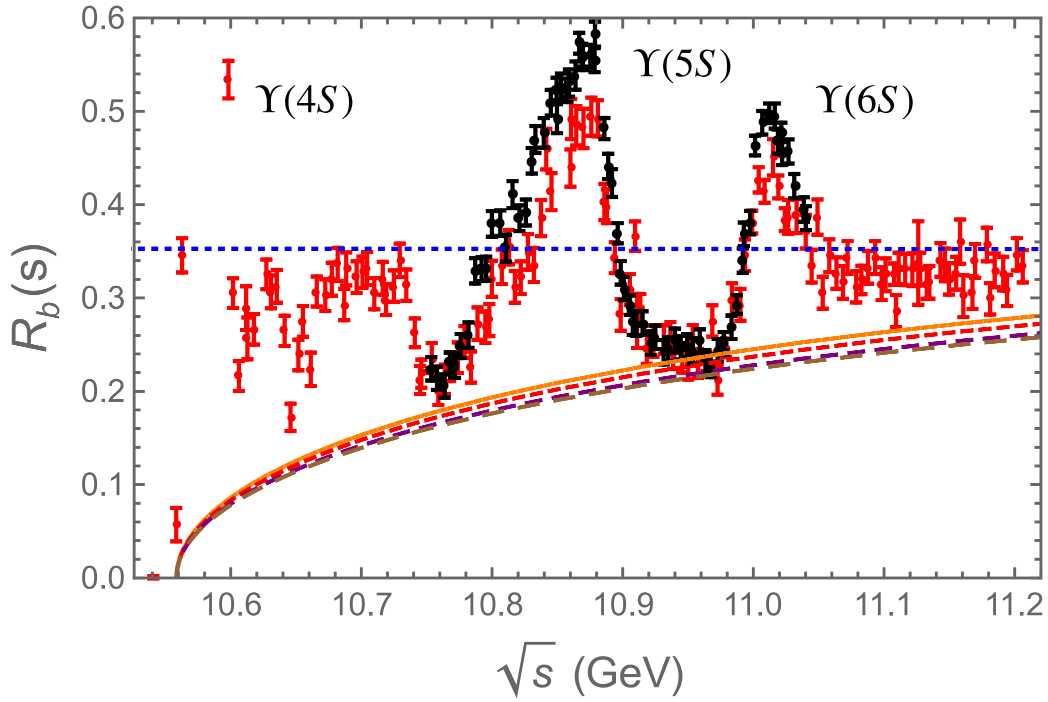
$$R_b^{\text{res,Gamma}}(s) = \sum_{R=\Upsilon(4S),\Upsilon(5S)} \frac{9\pi}{\alpha_{\text{em}}^2(M_R)} \frac{\Gamma_R^e}{M_R} \text{Gamma}(s - 4M_B^2|\alpha, \beta)$$

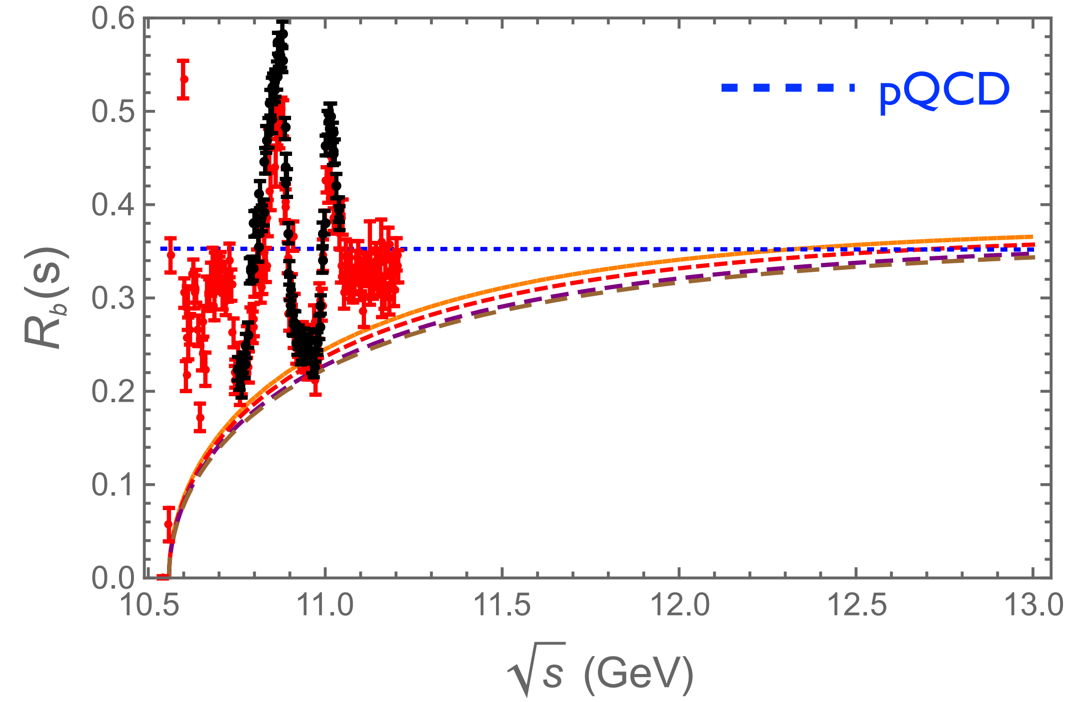
$$\alpha = 1 + \frac{2}{\sqrt[3]{\pi}} \frac{(M_R^2 - 4M_B^2)^2}{\Gamma_R^2 M_R^2} \qquad \beta = \frac{\alpha - 1}{M_R^2 - 4M_B^2}$$

QCD Sum Rules

Our approach

Data beyond 11.2 GeV will help reducing error: pQCD reaching at 13 GeV





$$R_b(s) = R_b^{\mathrm{res}}(s) + R_b^{\mathrm{cont}}(s) + R_b^{\mathrm{res,Gamma}}(s)$$