Lecture notes China 2025 - Becattini

Bibliography

- Density operator → 1902.01089
- Entropy current → 1903.05422
- Spin and polarization → 2004.04050 and 2103.10917

Classical approach to relativistic hydrodynamics

Decomposition of currents onto a four-velocity

$$T^{\mu\nu} = \rho' u^\mu u^\nu + p' \Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \Pi^{\mu\nu}$$

$$\rho' \equiv T^{\mu\nu} u_\mu u_\nu \qquad \Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu \qquad u\cdot u = 1$$

$$q^\mu \equiv \Delta^\mu_\alpha T^{\alpha\beta} u_\beta \qquad \Delta^{\mu\alpha} \Delta^\nu_\alpha = \Delta^{\mu\nu}$$

$$\Pi^{\mu\nu} \equiv \left(\Delta^\mu_\alpha \Delta^\nu_\beta - \frac{1}{3} \Delta_{\alpha\beta} \Delta^{\mu\nu}\right) T^{\alpha\beta} \quad \Rightarrow \quad q\cdot u = 0$$

$$u_\mu \Pi^{\mu\nu} = \Pi^{\mu\nu} u_\nu = 0$$

$$p' = \frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu} \qquad \Delta_{\mu\nu} \Pi^{\mu\nu} = 0$$

$$j^\mu = n' u^\mu + J^\mu \qquad n' \equiv j\cdot u \qquad J^\mu \equiv \Delta^\mu_\alpha j^\alpha$$

All of them are functions of x

Equilibrium $\quad u \equiv u_{eq} = \text{constant}$

$$\lim_{u \to u_{eq}} T^{\mu\nu} = (\rho_{eq} + p_{eq})\, u^{\mu}_{eq} u^{\nu}_{eq} - p_{eq}\, g^{\mu\nu} \qquad \Rightarrow \lim_{u \to u_{eq}} q, \Pi = 0$$

$$\lim_{u \to u_{eq}} j^{\mu} = n_{eq} u^{\mu}_{eq}$$

ρ_{eq}, p_{eq}, n_{eq} are the thermodynamic equilibrium functions

$$\rho_{eq}(T, \mu) \qquad p_{eq}(T, \mu) \qquad n_{eq}(T, \mu) \qquad \rho_{eq} + p_{eq} = T s_{eq} + \mu n_{eq}$$

$$\Delta\rho \equiv \rho'(x) - \rho_{eq}(x) = \rho'(x) - \rho(x)$$

$$\pi \equiv p'(x) - p_{eq}(x) = p'(x) - p(x)$$

$\Delta n \equiv n'(x) - n_{eq}(x) = n'(x) - n(x)$ $\rightsquigarrow$ disregarded for the sake of simplicity

Derivation of constitutive equations

$$T ds = d\rho - \mu dn \quad \Rightarrow \quad \frac{\partial s}{\partial \rho} = \frac{1}{T} \qquad \frac{\partial s}{\partial n} = -\frac{\mu}{T} \quad \Rightarrow \quad \partial_\mu s = \frac{1}{T}\partial_\mu \rho - \frac{\mu}{T}\partial_\mu n$$

$$u^\mu \partial_\mu s \equiv \dot{s} \qquad \Rightarrow \quad T\dot{s} = \dot{\rho} - \mu \dot{n}$$

$$\partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \partial_\mu \left[(\rho + p) u^\mu u^\nu - p g^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \Pi^{\mu\nu} + \pi \Delta^{\mu\nu} + \Delta\rho\, u^\mu u^\nu \right] = 0$$

Contract with u_ν

$$u^\mu \partial_\mu (\rho + p) + (\rho + p)\partial \cdot u - u^\nu \partial_\nu p + \partial \cdot q + u_\nu u^\mu \partial_\mu q^\nu + u_\nu \partial_\mu \Pi^{\mu\nu} + \pi u_\nu \partial_\mu \Delta^{\mu\nu} + \partial_\mu (\Delta\rho\, u^\mu)$$

$$= \dot{\rho} + \dot{p} + (\rho + p)\partial \cdot u - \dot{p} + \partial \cdot q - q_\nu \underbrace{u^\mu \partial_\mu u^\nu}_{\substack{A_\nu \\ \text{acceleration field}}} - \Pi^{\mu\nu}\partial_\mu u_\nu - \pi \partial \cdot u + \partial \cdot (\Delta\rho\, u)$$

$$= \dot{\rho} + \cancel{\dot{p}} + (\rho + p)\,\partial\cdot u - \cancel{\dot{p}} + \partial\cdot q - q\cdot A - \Pi^{\mu\nu}\partial_\mu u_\nu - \pi\,\partial\cdot u + \partial\cdot(\Delta\rho\, u) = 0$$

$$\Rightarrow \quad \dot{\rho} = -(\rho + p)\,\partial\cdot u - \partial\cdot q + q\cdot A + \Pi : \partial u + \pi\,\partial\cdot u - \partial\cdot(\Delta\rho\, u)$$

Similarly

$$\partial_\mu j^\mu = 0 \quad \Rightarrow \quad \partial_\mu(n u^\mu + J^\mu) = 0 \qquad n\,\partial\cdot u + \dot{n} + \partial\cdot J = 0 \quad \text{and}$$

$$\dot{n} = -n\,\partial\cdot u - \partial\cdot J$$

Altogether

$$T\dot{s} = -(\rho + p)\,\partial\cdot u - \partial\cdot q + q\cdot A + \Pi : \partial u + \pi\,\partial\cdot u + \mu n\,\partial\cdot u + \mu\,\partial\cdot J - \partial\cdot(\Delta\rho\, u)$$

$$= (-\rho - p + \mu n)\,\partial\cdot u - \partial\cdot q + q\cdot A + \Pi : \partial u + \pi\,\partial\cdot u + \mu\,\partial\cdot J - \partial\cdot(\Delta\rho\, u)$$

$$= -Ts\,\partial\cdot u - \partial\cdot q + q\cdot A + \Pi : \partial u + \pi\,\partial\cdot u + \mu\,\partial\cdot J - \partial\cdot(\Delta\rho\, u)$$

$$\dot{s} + s\,\partial\cdot u = -\frac{\partial\cdot q}{T} + \frac{q\cdot A}{T} + \frac{\Pi : \partial u}{T} + \frac{\pi}{T}\partial\cdot u + \frac{\mu}{T}\partial\cdot J - \frac{\partial\cdot(\Delta\rho\, u)}{T}$$

$$= -\partial\cdot\left(\frac{q}{T}\right) + q\cdot\partial\left(\frac{1}{T}\right) + \frac{q\cdot A}{T} + \Pi : \frac{\partial u}{T} + \frac{\pi}{T}\partial\cdot u + \partial\cdot\left(\frac{\mu}{T}J\right) - J\cdot\partial\left(\frac{\mu}{T}\right) - \frac{\partial\cdot(\Delta\rho\, u)}{T}$$

$$\partial_\mu(s u^\mu) = -\partial\cdot\left(\frac{q}{T}\right) + q\cdot\partial\left(\frac{1}{T}\right) + q\cdot A + \Pi : \frac{\partial u}{T} + \frac{\pi}{T}\partial\cdot u + \partial\cdot\left(\frac{\mu}{T}J\right) - J\cdot\partial\left(\frac{\mu}{T}\right)$$
$$- \partial\cdot\left(\frac{\Delta\rho\, u}{T}\right) + \Delta\rho\, u\cdot\partial\left(\frac{1}{T}\right)$$

$$\underbrace{\partial\cdot\left(s u + \frac{q}{T} - \frac{\mu}{T}J + \frac{\Delta\rho\, u}{T}\right)}_{\text{entropy current } s^\mu} = q\cdot\left(A + \partial\left(\frac{1}{T}\right)\right) + \Pi : \frac{\partial u}{T} + \frac{\pi}{T}\partial\cdot u - J\cdot\partial\left(\frac{\mu}{T}\right)$$
$$+ \Delta\rho\, u\cdot\partial\left(\frac{1}{T}\right)$$

Constitutive equations requiring $\partial_\mu s^\mu \geq 0$

E.g. $q = -\kappa \left(A + \partial\left(\frac{1}{T}\right) \right)$ $\kappa > 0$ thermal conductivity

Now:

$$\begin{matrix} T^{\mu\nu}[T, u, \mu] \\ j^\mu[T, u, \mu] \end{matrix} \longrightarrow \begin{matrix} \partial_\mu T^{\mu\nu} = 0 \\ \partial_\mu j^\mu = 0 \end{matrix}$$

5 equations for 5 unknowns

T, u, μ $(u \cdot u = 1)$

Two fundamental questions

1) Is it true that $\rho + p = Ts + \mu n$ out of equilibrium (or in different global equilibria)?

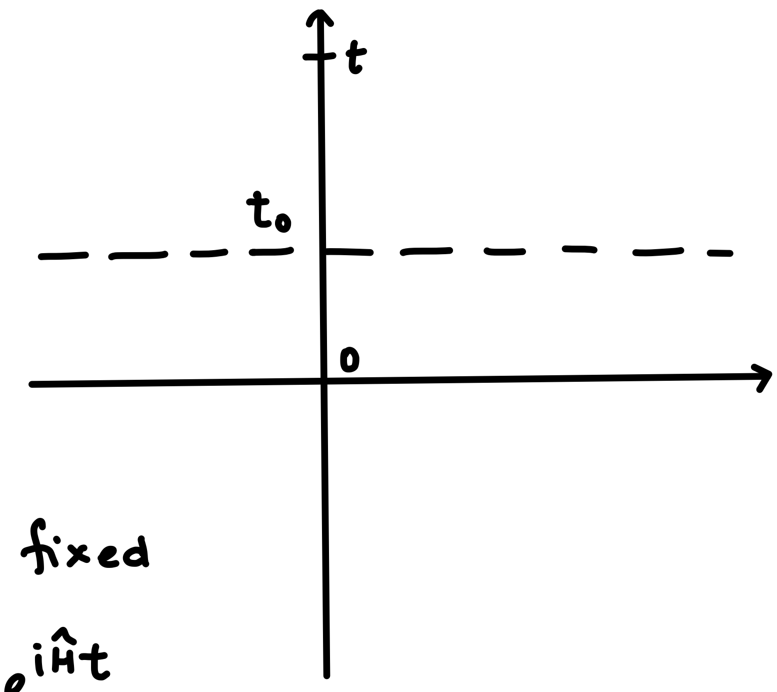
Does $T\,ds = d\rho - \mu\, dn$ hold?

2) How to deal with quantum observables? Spin

Density operator

$\hat{\rho} = ?$

$T^{\mu\nu}(\underline{x},t) = \mathrm{Tr}\left(\hat{\rho}(0)\,\hat{T}^{\mu\nu}(\underline{x},t)\right)$

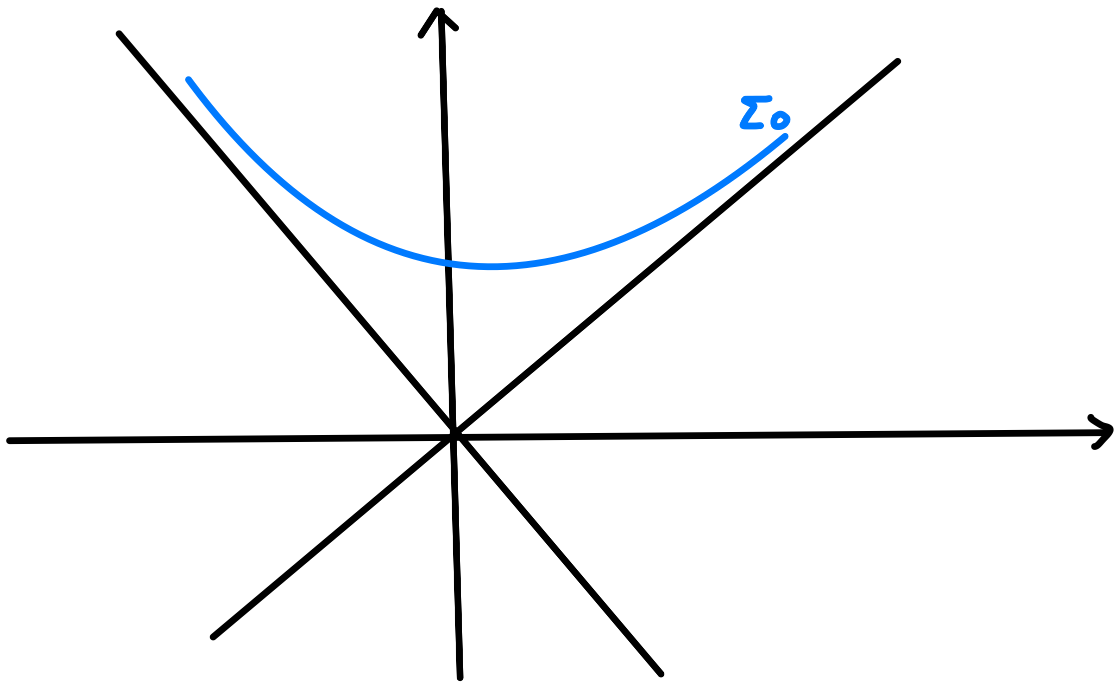


Heisenberg $\quad \hat{T}^{\mu\nu}(\underline{x},t) = e^{i\hat{H}t}\,\hat{T}^{\mu\nu}(\underline{x},0)\,e^{-i\hat{H}t} \quad \hat{\rho}$ fixed

Schrödinger $\quad \hat{T}^{\mu\nu}(\underline{x},0)$ fixed $\quad \hat{\rho}(t) = e^{-i\hat{H}t}\hat{\rho}(0)\,e^{i\hat{H}t}$

Suppose that a good approximation can be found of $\hat{\rho}(t_0) = e^{-i\hat{H}t_0}\hat{\rho}(0)\,e^{i\hat{H}t_0}$ at a suitable time t_0

$\rightarrow$ evolve $\hat{T}^{\mu\nu}$ or any other fields from t_0



$\Sigma_0 \longrightarrow \hat{\rho}(\Sigma_0)$

" local equilibrium is achieved "

$$S = -\mathrm{Tr}\,(\hat{\rho} \log \hat{\rho})$$

10 constants $\quad P_0^{\mu} \quad J_0^{\mu\nu}$ mean values + some charges Q_{0i}

Each constant $\longleftrightarrow$ Lagrange multiplier

$$F[\hat{\rho}] = -\mathrm{Tr}\,(\hat{\rho}\log\hat{\rho}) - b_\mu\left(\mathrm{Tr}(\hat{\rho}\hat{P}^\mu) - P_0^\mu\right) + \frac{\varpi_{\mu\nu}}{2}\left(\mathrm{Tr}(\hat{\rho}\hat{J}^{\mu\nu}) - J_0^{\mu\nu}\right)$$
$$+ \sum_i \zeta_i\left(\mathrm{Tr}(\hat{\rho}\,\hat{Q}_i) - Q_{0i}\right) + \lambda\left(\mathrm{Tr}(\hat{\rho}) - 1\right)$$

$$\frac{\delta F}{\delta \hat{\rho}} = 0 \qquad \text{Solution} \qquad \hat{\rho} = \frac{e^{-b\cdot\hat{P} + \frac{1}{2}\varpi : \hat{J} + \sum_i \zeta_i \hat{Q}_i}}{Z}$$

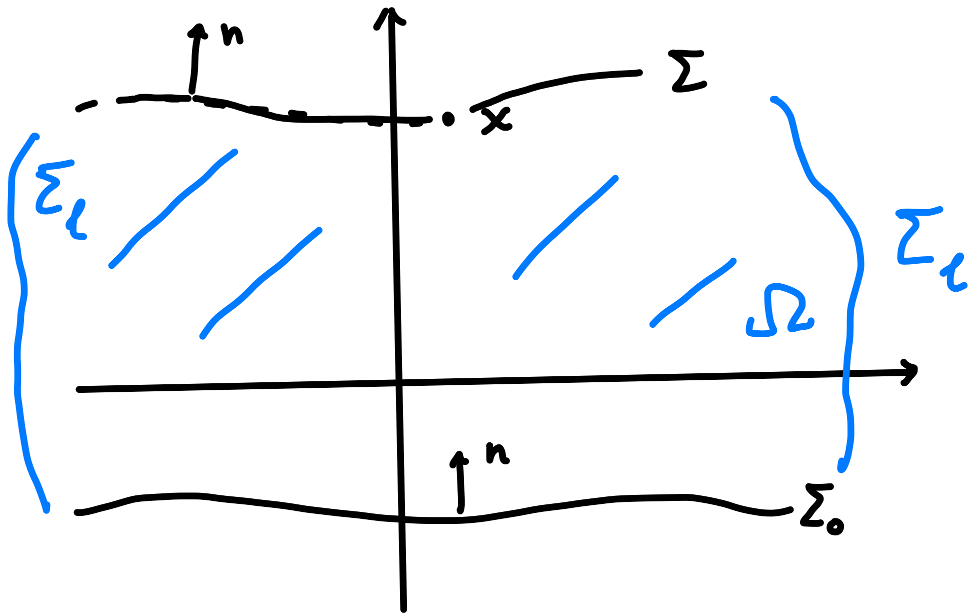
Problem: prove it

$$Z = \mathrm{Tr}\left(e^{-b\cdot\hat{P} + \frac{1}{2}\varpi : \hat{J} + \sum_i \zeta_i \hat{Q}_i}\right)$$

$\hat{\rho}$ approximated by $\hat{\rho}_{LE}(\Sigma_0)$

$$\hat{\rho}_{LE}(\Sigma_0) = \exp\left[-\int d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu - \zeta\,\hat{j}^\mu\right] \Big/ Z$$

Gauss theorem



$$-\int_{\Sigma_0} d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu + \int_{\Sigma} d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu =$$

$$= \int d\Omega \, \nabla_\mu \left(\hat{T}^{\mu\nu}\beta_\nu\right)$$

$$\text{if} \quad \int_{\Sigma_\ell} d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu = 0$$

$$\boxed{-\int_{\Sigma_0} d\Sigma \, \hat{T}^{\mu\nu}\beta_\nu = -\int d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu + \int d\Omega \, \hat{T}^{\mu\nu}\nabla_\mu\beta_\nu}$$

$$\Rightarrow \quad \hat{\rho}_{LE}(\Sigma_0) = \frac{1}{Z}\exp\left[-\int d\Sigma_\mu\left(\hat{T}^{\mu\nu}\beta_\nu - \zeta\hat{j}^\mu\right) + \int d\Omega\left(\hat{T}^{\mu\nu}\nabla_\mu\beta_\nu - \nabla_\mu\zeta\cdot\hat{j}^\mu\right]\right.$$

Linear response

$$\hat{\rho} = \frac{1}{Z} e^{\hat{A}+\hat{B}} \qquad Z = \mathrm{Tr}\left(e^{\hat{A}+\hat{B}}\right)$$

Kubo identity

$$e^{\hat{A}+\hat{B}} = \left[I + \int_0^1 dz\, e^{z(\hat{A}+\hat{B})} \hat{B} e^{-z\hat{A}} \right] e^{\hat{A}}$$ can be iterated

$$\Rightarrow e^{\hat{A}+\hat{B}} \cong \left[I + \int_0^1 dz\, e^{z\hat{A}} \hat{B} e^{-z\hat{A}} \right] e^{\hat{A}}$$ linear response

$$\hat{\rho} \cong \hat{\rho}_A \left(1 - \langle \hat{B} \rangle_A \right) + \int_0^1 dz\, e^{z\hat{A}} \hat{B} e^{-z\hat{A}} \hat{\rho}_A \qquad \hat{\rho}_A = \frac{e^{\hat{A}}}{\mathrm{Tr}(e^{\hat{A}})}$$

Example:

$$\hat{A} = -\int_\Sigma d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu$$

$$\hat{B} = \int d\Omega\, \hat{T}^{\mu\nu}\partial_\mu\beta_\nu - \partial_\mu\zeta\, \hat{j}^\mu$$

$$\hat{\rho}_{LE}(0) \cong \hat{\rho}_{LE}\left(1 - \langle\hat{B}\rangle_{LE}\right) + \int_0^1 dz\, e^{z\hat{A}}\hat{B}e^{-z\hat{A}}\hat{\rho}_{LE}$$

$$\langle \hat{T}^{\mu\nu}(x)\rangle = \mathrm{Tr}\left(\hat{\rho}\,\hat{T}^{\mu\nu}(x)\right)$$

$$\langle \hat{T}^{\mu\nu}(x)\rangle \cong \langle \hat{T}^{\mu\nu}(x)\rangle_{LE} + \int_0^1 dz \langle \hat{T}^{\mu\nu}(x)\, e^{z\hat{A}}\hat{B}e^{-z\hat{A}}\rangle_{LE} - \langle \hat{T}^{\mu\nu}(x)\rangle_{LE}\langle\hat{B}\rangle_{LE}$$

$$= \langle \hat{T}^{\mu\nu}(x)\rangle_{LE} + \int_0^1 dz \langle \hat{T}^{\mu\nu}(x)\, e^{z\hat{A}}\hat{B}e^{-z\hat{A}}\rangle_{LE,c}$$

$\rangle_c \rightarrow$ subtraction of mean values

$$= \langle \hat{T}^{\mu\nu}(x)\rangle_{LE} + \int_0^1 dz \int d\Omega(y) \langle \hat{T}^{\mu\nu}(x)e^{z\hat{A}}\hat{T}^{\rho\sigma}(y)e^{-z\hat{A}}\rangle_{LE,c}\,\partial_\rho\beta_\sigma(y) + \ldots\ldots$$

Hydro approx. $\partial_\rho\beta_\sigma(y) \simeq \partial_\rho\beta_\sigma(x)$

$$\Rightarrow \quad \langle \hat{T}^{\mu\nu}(x)\rangle_{LE} + \partial_\rho\beta_\sigma(x)\int_0^1 dz \int d\Omega(y) \langle \hat{T}^{\mu\nu}(x)e^{z\hat{A}}\hat{T}^{\rho\sigma}(y)e^{-z\hat{A}}\rangle_{LE,c} + \ldots\ldots$$

What about hydro equations?

① $n_\mu T^{\mu\nu} = n_\mu T^{\mu\nu}_{LE} \rightarrow$ find $\beta_n(x)$, all we need. Depends on the foliation

Similar to the definition of other frames

$T^{\mu\nu} u_\mu = \lambda u^\nu$ eigenvector

How to determine $T^{\mu\nu}$? $\longrightarrow$ $n_\mu \delta T^{\mu\nu}[\beta, g, \Sigma] = 0$

$\rightarrow$ campo n scelto

Il problema è che $n_\mu \delta T^{\mu\nu}[\beta, g, n] = 0$ è molto difficile da risolvere. Più facile

$\partial_\mu T^{\mu\nu}[\beta, g, n] = 0$

② Apparently $T^{\mu\nu}(x) = \mathrm{Tr}\left[\frac{e^{\hat{A}+\hat{B}}}{Z}\hat{T}^{\mu\nu}(x)\right] \longrightarrow T^{\mu\nu}(x) = T^{\mu\nu}[\beta_\Sigma, g, \Sigma]$

$\frac{\delta}{\delta\Sigma} T^{\mu\nu}[\beta_\Sigma, g, \Sigma] = 0$ eventually

Expand in gradients $\longrightarrow T^{\mu\nu}(\beta_\Sigma, \partial\beta_\Sigma, \partial^2\beta_\Sigma, \ldots\, g, \ldots, \Sigma)$

$T^{\mu\nu} = a\,\beta_\Sigma^\mu \beta_\Sigma^\nu + \ldots$

Entropy current

1) $S = -\mathrm{Tr}(\hat{\rho} \log \hat{\rho}) \longrightarrow -\mathrm{Tr}(\hat{\rho}_{LE} \log \hat{\rho}_{LE})$ $\qquad \dfrac{d\hat{\rho}_{LE}}{dt} \neq 0$

$\dfrac{d\hat{\rho}}{dt}$

maximized

$\downarrow$ constrained ignorance

2) Extensivity

Sia $\quad \hat{\rho}(\lambda) = \dfrac{1}{Z(\lambda)} \exp\left[-\lambda \int d\Sigma_\mu \hat{T}^{\mu\nu} \beta_\nu\right]$ $\quad (\zeta = 0$ per semplicità$)$ $\quad \hat{\rho} \equiv \hat{\rho}(1)$

Corrisponde a moltiplicare la temperatura comovente $1/\sqrt{\beta^2}$ per $1/\lambda$.

$$Z(\lambda) = \mathrm{Tr}\left(\exp\left[-\lambda \int d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu\right]\right) \quad \Rightarrow \quad \frac{\partial \log Z}{\partial \lambda}$$

$$= \frac{1}{Z(\lambda)}\frac{\partial}{\partial\lambda} \mathrm{Tr}\left(\exp\left[-\lambda\int d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu\right]\right) = \frac{\mathrm{Tr}}{Z(\lambda)}\left(\exp\left[-\lambda\int d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu\right](-1)\int d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu\right) =$$

$$= -\int d\Sigma_\mu \langle \hat{T}^{\mu\nu}\rangle(\lambda)\beta_\nu$$

$\langle \hat{T}^{\mu\nu}\rangle(\lambda)$ **mean value with** $\hat{\rho}(\lambda)$

Integrating between 1 and λ_0

$$\int_1^{\lambda_0} d\lambda \frac{\partial}{\partial\lambda}\log Z(\lambda) = \log Z(\lambda_0) - \log Z = -\int_1^{\lambda_0} d\lambda \int d\Sigma_\mu \langle \hat{T}^{\mu\nu}\rangle(\lambda)\beta_\nu = \int d\Sigma_\mu\left(\int_{\lambda_0}^1 d\lambda \langle \hat{T}^{\mu\nu}\rangle(\lambda)\beta_\nu\right)$$

If λ_0 **such that** $\log Z(\lambda_0) = 0$ avrò

$$\log Z = \int d\Sigma_\mu \phi^\mu \qquad \text{con} \quad \phi^\mu = \int_1^{\lambda_0} d\lambda \langle \hat{T}^{\mu\nu}\rangle(\lambda)\beta_\nu$$

Entropy current

$$S = \int d\Sigma_\mu \, \phi^\mu + T^{\mu\nu}_{LE} \beta_\nu - \zeta j^\mu_{LE}$$

but $n_\mu T^{\mu\nu} = n_\mu T^{\mu\nu}_{LE}$

$n_\mu j^\mu = n_\mu j^\mu_{LE}$

$$S = \int d\Sigma_\mu \, \phi^\mu + T^{\mu\nu} \beta_\nu - \zeta j^\mu$$

$$\Rightarrow \quad s^\mu = \phi^\mu + T^{\mu\nu} \beta_\nu - \zeta j^\mu$$

Entropy production

$$\boxed{\frac{\delta}{\delta \Sigma}\left(\int d\Sigma_\mu V^\mu\right)_\xi = \frac{1}{2}\int d\tilde{S}_{\mu\nu}\,\xi^\nu V^\mu + \int d\Sigma_\mu \xi^\mu \nabla \cdot V}$$

$$Z_{\phi_\varepsilon(\Sigma)} - Z_\Sigma = \int_{\phi_\varepsilon(\Sigma)} d\Sigma_\mu \phi^\mu - \int d\Sigma_\mu \phi^\mu \quad \text{ovvero} \quad \mathcal{L}_\xi \int d\Sigma_\mu \phi^\mu \ \text{domain derivative} = \int d\Sigma_\mu \xi^\mu \ \nabla\cdot\phi \quad \text{se} \int d\tilde{S} = 0$$

$$Z_{\phi_\varepsilon(\Sigma)} = \mathrm{Tr}\left(e^{-\int_{\phi_\varepsilon(\Sigma)} d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu}\right) \cong \mathrm{Tr}\left(e^{-\int d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu - \varepsilon\int d\Sigma_\mu \xi^\mu \ \nabla\cdot(\hat{T}^{\mu\nu}\beta_\nu)}\right) \quad \text{se} \int d\tilde{S} = 0$$

$$\cong \quad Z - \varepsilon\,\mathrm{Tr}\left(e^{-\int\dots}\int d\Sigma\cdot\xi \ \ \nabla_\mu(\hat{T}^{\mu\nu}\beta_\nu)\right)$$

Ora divido per Z e ho

$$\lim_{\varepsilon\to 0}\frac{Z_\varepsilon - Z}{\varepsilon Z} = \mathcal{L}_\xi(\log Z) = -\int d\Sigma\cdot\xi \ \ \langle\hat{T}^{\mu\nu}\rangle_{LE}\nabla_\mu\beta_\nu \qquad \text{e pertanto}$$

$$\mathcal{L}_\xi \int d\Sigma_\mu \phi^\mu = \int d\Sigma\cdot\xi\ \nabla\cdot\phi = -\int d\Sigma\cdot\xi \ \ \langle\hat{T}^{\mu\nu}\rangle_{LE}\nabla_\mu\beta_\nu$$

being ξ arbitrary $\Rightarrow \quad \nabla\cdot\phi = -\langle \hat{T}^{\mu\nu}\rangle_{LE}\, \nabla_\mu \beta_\nu$

and, with current, $\quad \nabla\cdot\phi = -\langle \hat{T}^{\mu\nu}\rangle_{LE}\, \nabla_\mu \beta_\nu + \langle \hat{j}^{\mu}\rangle_{LE}\, \nabla_\mu \zeta$

$$\Rightarrow \nabla\cdot s = \nabla\cdot\left(\phi + T^{\mu\nu}\beta_\nu - j^\mu\right) = \nabla\cdot\phi + \cancel{\nabla_\mu T^{\mu\nu}}\,\beta_\nu + T^{\mu\nu}\nabla_\mu\beta_\nu - \cancel{\nabla\cdot j}\,\zeta$$

$$-\nabla\zeta\cdot j = -T^{\mu\nu}_{LE}\,\nabla_\mu\beta_\nu + j^\mu_{LE}\,\nabla_\mu\zeta + T^{\mu\nu}\nabla_\mu\beta_\nu - \nabla_\mu\zeta\, j^\mu \Rightarrow$$

$$\nabla\cdot s = \left(T^{\mu\nu} - T^{\mu\nu}_{LE}\right)\nabla_\mu\beta_\nu - \left(j^\mu - j^\mu_{LE}\right)\nabla_\mu\zeta$$

Comments on entropy

1) $s^\mu = \phi^\mu + T^{\mu\nu}\beta_\nu - \zeta j^\mu$

is it independent of Σ (better: the foliation?)

No, because $\zeta(n)$ and $\beta(n)$. It can be made independent only if the foliation is fixed or, of course, at global equilibrium.

example : enforce $n = \hat{\beta}$

2) The relation $Ts + \mu n = \rho + p$ is not frame-independent.

At global equilibrium β is a Killing vector and entropy current is unique

Landau frame $\quad s \cdot u_L = \phi \cdot u_L + u_{L\nu} T^{\mu\nu}\beta_\nu - \zeta j \cdot u_L$

$$u_{L\nu} T^{\mu\nu}\beta_\nu = u_{L\nu} T^{\mu\nu}[(\beta \cdot u_L) u_{L\nu} + \beta_{T\nu}] = (\beta \cdot u_L)\,\rho_L$$

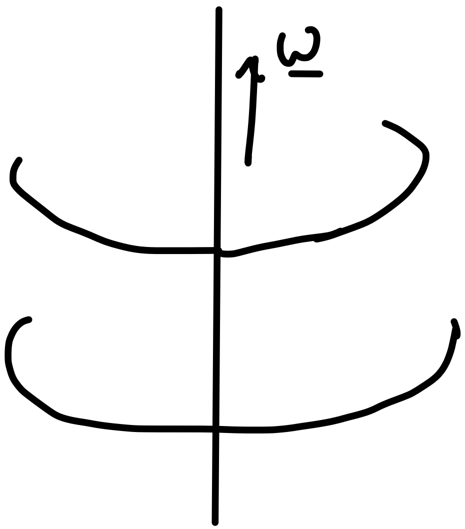
So we have

$$s \cdot u_L \equiv s_L = \phi \cdot u_L + \rho_L \frac{1}{T_L} - \zeta n_L$$

$$T_L = \frac{1}{\beta \cdot u_L} < \frac{1}{\sqrt{\beta^2}} \qquad \beta \cdot u_L = \sqrt{\beta^2}\, u \cdot u_L > \sqrt{\beta^2}$$

At global equilibrium with rotation we have $\beta = \frac{1}{T_0}\left(1, \underline{\omega} \times \underline{x}\right)$ a Killing vector.

$$T^2 = \frac{T_0^2}{\sqrt{1-\omega^2 r^2}} \qquad \text{Tolman's law} \qquad T_L^2 = \frac{1}{\beta^2}\frac{1}{(\hat{\beta}\cdot\hat{u}_L)^2} = T_0^2 \frac{1}{\sqrt{1-\omega^2 r^2}} \cdot \frac{1}{(\hat{\beta}\cdot\hat{u}_L)^2}$$



At global equilibrium with rotation we have $\beta = \frac{1}{T_0}(1, \underline{\omega} \times \underline{x})$ a Killing vector.

However

$$u^\mu = \frac{\beta^\mu}{\sqrt{\beta^2}}, \quad \alpha^\mu = \varpi^{\mu\nu} u_\nu, \quad w^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}u_\sigma, \quad l^\mu = \epsilon^{\mu\nu\rho\sigma}w_\nu\alpha_\rho u_\sigma.$$

$$\begin{aligned}\langle : T^{\mu\nu}(x) : \rangle =& \rho\, u^\mu u^\nu - p\,\Delta^{\mu\nu} + W\, w^\mu w^\nu + \mathcal{A}\,\alpha^\mu\alpha^\nu + G^l\, l^\mu l^\nu + G\,(l^\mu u^\nu + l^\nu u^\mu) + \mathbb{A}\,(\alpha^\mu u^\nu + \alpha^\nu u^\mu)\\ &+ G^\alpha\,(l^\mu\alpha^\nu + l^\nu\alpha^\mu) + \mathbb{W}\,(w^\mu u^\nu + w^\nu u^\mu) + A^w\,(\alpha^\mu w^\nu + \alpha^\nu w^\mu) + G^w\,(l^\mu w^\nu + l^\nu w^\mu),\end{aligned}$$

$$T^{\mu\nu}u_\nu = \rho u^\mu + \underbrace{G l^\mu + \mathbb{A}\alpha^\mu + \mathbb{W} w^\mu}_{\text{heat flux? NO}}$$

$$T^{\mu\nu}u_\nu = \frac{\pi^2}{30}T^4 u^\mu + \frac{1-70\xi}{240\pi^2}T^4 l^\mu$$

$$\Downarrow$$

$$u_L \neq \hat{\beta} \quad !$$

free scalar field $m=0$

$$\begin{aligned}
\rho =& \frac{\pi^2}{30\beta^4} + \frac{4\xi-1}{12\beta^4}w^2 + \frac{6\xi-1}{12\beta^4}\alpha^2 + \frac{4\xi-1}{48\pi^2\beta^4}w^4 + \frac{60\xi-11}{480\pi^2\beta^4}\alpha^4 + \frac{270\xi-61}{720\pi^2\beta^4}\alpha^2 w^2,\\
p =& \frac{\pi^2}{90\beta^4} - \frac{\xi}{6\beta^4}w^2 + \frac{1-6\xi}{18\beta^4}\alpha^2 - \frac{\xi}{24\pi^2\beta^4}w^4 + \frac{19-120\xi}{1440\pi^2\beta^4}\alpha^4,\\
W =& \frac{2\xi-1}{12\beta^4} + \frac{2\xi-1}{48\pi^2\beta^4}w^2 + \frac{120\xi-29}{360\pi^2\beta^4}\alpha^2,\\
\mathcal{A} =& \frac{1-6\xi}{12\beta^4} + \frac{1}{360\pi^2\beta^4}w^2 + \frac{1-6\xi}{48\pi^2\beta^4}\alpha^2,\\
G^l =& \frac{1-70\xi}{240\pi^2\beta^4},\\
G =& \frac{6\xi+1}{36\beta^4} + \frac{10\xi-1}{240\pi^2\beta^4}w^2 + \frac{30\xi-7}{720\pi^2\beta^4}\alpha^2,\\
\mathbb{A} =& 0,\\
G^\alpha =& 0,\\
G^w =& A^w = \mathbb{W} = 0.
\end{aligned}$$

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PARTICLES AND SPIN

$$[a(p), a^\dagger(p')] = 2\varepsilon\, \delta^3(\underline{p}-\underline{p}')$$

Spectrum $\qquad \mathrm{Tr}\left(\hat{\rho}\, a^\dagger(p)\, a(p)\right) = 2\varepsilon \dfrac{dN}{d^3p}$

Spin $\qquad \dfrac{\mathrm{Tr}\left(\hat{\rho}\, a_r^\dagger(p)\, a_s(p)\right)}{\sum_r \mathrm{Tr}\left(\hat{\rho}\, a_r^\dagger(p)\, a_r(p)\right)} = \Theta_{rs}(p) \qquad$ spin density matrix of a particle

Spin polarization vector

$$S^\mu(p) = \sum_{\substack{rs\\i}} D^S(J_i)_{rs}\, \Theta_{sr}(p)\, n_i^\mu(p) = \sum_i \mathrm{tr}\left(D^S(J_i)\, \Theta(p)\right)[p]_i^\mu$$

$$\hat{\rho} = \hat{\rho}_{LE}(0) = \frac{\exp}{Z}\left[-\int_{F_0} d\Sigma_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu\right) + \int d\Omega\, \hat{T}^{\mu\nu}\nabla_\mu \beta_\nu\right]$$

local equilibrium $\qquad$ dissipative

$\mathrm{Tr}(\hat{\rho}_{LE}(FO)\, a^\dagger(p) a(p))$ how to expand β_ν in the exponent?

$$\int d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu \cong \beta_\nu(x)\hat{P}^\nu + \frac{1}{2}\varpi : \hat{J}_x + \frac{1}{2}\xi : \hat{Q}_x$$

$$\hat{J}^{\mu\nu}_x = \int d\Sigma_\lambda \hat{T}^{\lambda\nu}x^\mu - (\mu\leftrightarrow\nu) \qquad \hat{Q}^{\mu\nu}_x = \int d\Sigma_\lambda \hat{T}^{\lambda\nu}x^\mu + (\mu\leftrightarrow\nu)$$

It is necessary to have a local operator!

Covariant Wigner operator

Scalar
$$\widehat{W}(x,k) = \frac{2}{(2\pi)^4}\int d^4y \ :\widehat{\psi}^\dagger(x+y/2)\widehat{\psi}(x-y/2):\mathrm{e}^{-iy\cdot k}$$

Dirac
$$\widehat{W}(x,k)_{AB} = -\frac{1}{(2\pi)^4}\int d^4y\, \mathrm{e}^{-ik\cdot y} : \Psi_A(x-y/2)\overline{\Psi}_B(x+y/2) :$$
$$= \frac{1}{(2\pi)^4}\int d^4y\, \mathrm{e}^{-ik\cdot y} : \overline{\Psi}_B(x+y/2)\Psi_A(x-y/2) :$$

General idea: re-express a^+a as a function of $\hat{W}(x,k)$

$$W(x,k) = W(x,k)\theta(k^2)\theta(k^0) + W(x,k)\theta(k^2)\theta(-k^0) + W(x,k)\theta(-k^2) \equiv \underbrace{W_+(x,k)}_{\text{particle}} + \underbrace{W_-(x,k)}_{\text{antiparticle}} + \underbrace{W_S(x,k)}_{\text{mixing}}$$

1) use field expansions in plane waves to expand $\hat{W}$

2) Show that $k^\mu \partial_\mu \hat{W} = 0 \Rightarrow \int_\Sigma d\Sigma_\mu k^\mu \hat{W}(x,k)$ independent of Σ and k on-shell!

3) Show the relation

$$\frac{dN}{d^3p} = \int dp_0 \int d\Sigma_\mu p^\mu W_+(x,p) \qquad W_+(x,p) = \mathrm{Tr}\left(\hat{\rho}_{LE}(0)\, \hat{W}_+(x,p)\right)$$

$$\frac{dN}{d^3p} = \int dp_0 \int d\Sigma_\mu p^\mu \, \mathrm{tr}\, W_+(x,p) \frac{1}{m}$$

Example

$$W(x,k) = \frac{2}{(2\pi)^3}\int \frac{d^3p}{2\varepsilon}\frac{d^3p'}{2\varepsilon'} e^{i(p-p')\cdot x}\left[\delta^4(k-(p+p')/2)\langle \widehat{a}^\dagger(p)\widehat{a}(p')\rangle + \delta^4(k+(p+p')/2)\langle \widehat{b}^\dagger(p)\widehat{b}(p')\rangle\right]$$
$$+\,\delta^4(k-(p-p')/2)\left[e^{i(p+p')\cdot x}\langle \widehat{a}^\dagger(p)\widehat{b}^\dagger(p')\rangle + e^{-i(p+p')\cdot x}\langle \widehat{b}(p')\widehat{a}(p)\rangle\right]$$

$$k^\mu \partial_\mu W_{\pm/s} = 0 \qquad \Rightarrow \int d\Sigma_\mu k^\mu W_+(x,k) = \int d^3x\, k^0 W_+(x,k)$$

$$\int d^3x \longrightarrow (2\pi)^3 \delta^3(\underline{p}-\underline{p}') \Rightarrow$$

$$\int d\Sigma\cdot k W_+(x,k) = 2\int \frac{d^3p}{2\varepsilon}\frac{k^0}{2\varepsilon}\delta^4(k-p)\langle a^\dagger(p)a(p)\rangle = \int \frac{d^3p}{2\varepsilon}\delta^3(\underline{p}-\underline{k})\delta(k^0-p^0)\langle a^\dagger_p a_p\rangle =$$

$$= \frac{1}{2\varepsilon}\delta\left(k^0-\sqrt{\underline{k}^2+m^2}\right)\langle a^\dagger_{\underline{k}} a_{\underline{k}}\rangle = \delta\left(k^0-\sqrt{\underline{k}^2+m^2}\right)\frac{dN}{d^3k}$$

$$\Rightarrow \frac{dN}{d^3k} = \int dk^0 \int d\Sigma\cdot k\, W(x,k)$$

Spin : more elaborate

Derivation :
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$$S^\mu(p) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma \cdot p \; \mathrm{tr}_4(\gamma^\mu \gamma^5 W_+(x,p))}{\int \mathrm{d}\Sigma \cdot p \; \mathrm{tr}_4 W_+(x,p)}$$

$$S^\mu(p) = -\frac{1}{4}\epsilon^{\mu\beta\gamma\delta} p_\delta \frac{\int \mathrm{d}\Sigma_\lambda \mathrm{tr}_4(\{\gamma^\lambda, \Sigma_{\beta\gamma}\} W_+(x,p))}{\int \mathrm{d}\Sigma_\lambda p^\lambda \mathrm{tr}_4 W_+(x,p)}$$

$$S^\mu(p) = -\frac{1}{2m}\epsilon^{\mu\beta\gamma\delta} p_\delta \frac{\int \mathrm{d}\Sigma_\lambda p^\lambda \mathrm{tr}_4(\Sigma_{\beta\gamma} W_+(x,p))}{\int \mathrm{d}\Sigma_\lambda p^\lambda \mathrm{tr}_4 W_+(x,p)}$$

Calculation at LE

$$\hat{\rho}_{LE}(0) \approx \hat{\rho}_{LE}(\Sigma_{FO}) \longrightarrow \text{dissipation neglected}$$

$$W(x,k) = \mathrm{Tr}\left(\frac{e^{-\int_{FO} d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu}}{Z} \hat{W}_+(x,k) \right) \longrightarrow \text{Taylor expand } \beta \text{ from } x$$

$$\simeq \frac{\mathrm{Tr}}{Z}\left(e^{-\beta(x)\cdot\hat{P} + \frac{1}{2}\varpi : \hat{J}_x + \frac{1}{2}\xi : \hat{Q}_x} \hat{W}_+(x,k) \right)$$

Retaining the main term

$$\mathrm{Tr}\left(\frac{e^{-\beta(x)\cdot\hat{P}}}{Z} \hat{W}_+(x,k) \right) = \frac{2}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon}\frac{d^3p'}{2\varepsilon'}\, e^{i(p-p')\cdot x}\, \delta^4\!\left(k - \frac{p+p'}{2}\right) \mathrm{Tr}\left(\frac{e^{-\beta(x)\cdot\hat{P}}}{Z} a_p^\dagger a_{p'} \right)$$

$$= \frac{2}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon}\frac{d^3p'}{2\varepsilon'}\, e^{i(p-p')\cdot x}\, \delta^4\!\left(k - \frac{p+p'}{2}\right) \frac{2\varepsilon\, \delta^3(\underline{p} - \underline{p}')}{e^{\beta(x)\cdot p} - 1} = \frac{1}{\varepsilon_k} \frac{\delta(k^0 - \varepsilon_{\underline{k}})}{(2\pi)^3} n_B(\beta(x))$$

$$\Rightarrow \quad \varepsilon \frac{dN}{d^3p} = \int d\Sigma \cdot p \, \frac{1}{e^{\beta(x)\cdot p} - 1}$$

Spin

$$S^\mu(p) = \frac{1}{2} \frac{\int \mathrm{d}\Sigma \cdot p \, \mathrm{tr}_4(\gamma^\mu \gamma^5 W_+(x,p))}{\int \mathrm{d}\Sigma \cdot p \, \mathrm{tr}_4 W_+(x,p)}$$

We cannot stop at the leading order (0^{th} in the gradients) but include 1st order terms

$$\frac{\mathrm{Tr}}{Z}\left(e^{-\beta(x)\cdot \widehat{P} + \frac{1}{2}\varpi : \widehat{J}_x + \frac{1}{2}\xi : \widehat{Q}_x} \, \widehat{W}_+(x,k) \right)$$

For the detailed and most up-to-date derivation

see F.B., M.Buzzegoli, A. Palermo PLB 820 (2021) 136519

Two terms arise:

$$S^\mu_\varpi(p) = -\frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\int_\Sigma \mathrm{d}\Sigma_\lambda p^\lambda \, n_F(1-n_F)\varpi_{\nu\rho}}{\int_\Sigma \mathrm{d}\Sigma_\lambda p^\lambda \, n_F},$$

$$\boxed{S^\mu_\xi(p) = -\frac{1}{4m}\epsilon^{\mu\nu\sigma\tau}\frac{p_\tau p^\rho}{\varepsilon}\frac{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F(1-n_F)\hat{t}_\nu \xi_{\sigma\rho}}{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F},}$$

Also,