Dirac Equation

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From Schrödinger to Klein-Gordon to Dirac

Basic requirement of relativistic QM : Lorentz invariance of the associated wave-function

History: from Schrödinger to Klein-Gordon to Dirac

Non-relativistic formulation: $E = \frac{\mathbf{p}^2}{2m}.$ Schrödinger equation, obviously non invariant for Lorentz transformations (1st order in E, 2nd order in p)

Relativistic formulation (start of Klein-Gordon): $E^2 = \mathbf{p}^2 + m^2,$ 2nd order in both time and space

Using energy and momentum ***operators*** $\hat{\mathbf{p}} = -i\nabla$ and $\hat{E} = i\frac{\partial}{\partial t},$ transforms into:

$$\hat{E}^2\psi(\mathbf{x},t) = \hat{\mathbf{p}}^2\psi(\mathbf{x},t) + m^2\psi(\mathbf{x},t). \Rightarrow (\partial^\mu\partial_\mu + m^2)\psi = 0, \quad \partial^\mu\partial_\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2},$$

When applied to. $\psi(\mathbf{x},t) = Ne^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$, gives $E^2\psi = \mathrm{p}^2\psi + m^2\psi,$ $\Rightarrow$ Energy-momentum relationship OK

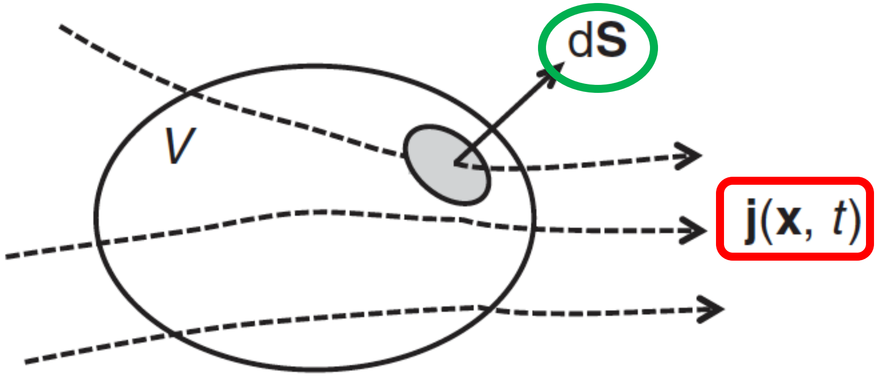
But negative energy solutions! $E = \pm\sqrt{\mathrm{p}^2 + m^2}.$

Klein-Gordon: Negative Energy Solutions

The physical interpretation of the wave-function is that

$$\psi^*(\boldsymbol{x},t)\psi(\boldsymbol{x},y)d^3\boldsymbol{x}$$

is the probability of finding the particle it represents in a volume $d^3\boldsymbol{x}$.



Introduce probability density:

$$\rho(x,t) = \psi^*(\boldsymbol{x},t)\psi(\boldsymbol{x},y)d^3\boldsymbol{x}\ /d^3\boldsymbol{x} = \psi^*(\boldsymbol{x},t)\psi(\boldsymbol{x},y)$$

If the particle doesn't decay or interacts → the probability stays constant.

Quantify the variation of probability = flux $j(\boldsymbol{x},t)$ of probability leaving the volume V through $\mathbf{dS}$ as

$$\frac{\partial}{\partial t}\int_V \rho(x,t)dV = \int_S j(\boldsymbol{x},t)d\boldsymbol{S}$$

It can be shown that → continuity equation $\nabla \cdot \mathbf{j} + \dfrac{\partial \rho}{\partial t} = 0.$ ɔ apply to

$$\boxed{\begin{aligned} i\frac{\partial\psi}{\partial t} &= -\frac{1}{2m}\nabla^2\psi, \\ -i\frac{\partial\psi^*}{\partial t} &= -\frac{1}{2m}\nabla^2\psi^* \end{aligned}}$$

Probability Density & Probability Current

Compute

$$\psi^*(\boldsymbol{x}, t) - \psi(\boldsymbol{x}, y)$$

And compare it to the continuity equation

$$-\frac{1}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = i\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right)$$

$$\Rightarrow \quad -\frac{1}{\boxed{2m}}\boldsymbol{\nabla}\cdot\boxed{\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right)} = i\frac{\partial}{\partial t}(\psi^*\psi) = \boxed{i\frac{\partial\rho}{\partial t}}.$$

$$\boldsymbol{\nabla}\cdot\boxed{\mathbf{j}} + \boxed{\frac{\partial\rho}{\partial t}} = 0.$$

that applied to a plane wave solution

$$\psi(\mathbf{x}, t) = Ne^{i(\mathbf{p}\cdot\mathbf{x} - Et)},$$

$$\rho = 2|N|^2 E$$

- Probability density goes like E → relativistic length contraction
- Negative energy solution ⇒ negative probability ⇒ impossible, unphysical

Dirac equation: both negative energy solutions and description of spin of particles

The Dirac Equation

Try writing an equation 1st order in both time and space → Dirac Equation
And fulfilling energy-momentum Einstein relation

Dirac $\hat{E}\psi = (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + \beta m)\psi,\qquad i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi.$

What are constants $\boldsymbol{\alpha}, \beta$??

Dirac equation **must** also satisfy Klein-Gordon equation!

$$-\frac{\partial^2\psi}{\partial t^2} = \left(i\alpha_x\frac{\partial}{\partial x} + i\alpha_y\frac{\partial}{\partial y} + i\alpha_z\frac{\partial}{\partial z} - \beta m\right)\left(i\alpha_x\frac{\partial}{\partial x} + i\alpha_y\frac{\partial}{\partial y} + i\alpha_z\frac{\partial}{\partial z} - \beta m\right)\psi,$$

$$\frac{\partial^2\psi}{\partial t^2} = \alpha_x^2\frac{\partial^2\psi}{\partial x^2} + \alpha_y^2\frac{\partial^2\psi}{\partial y^2} + \alpha_z^2\frac{\partial^2\psi}{\partial z^2} - \beta^2 m^2\psi$$

Looks like Klein-Gordon

Klein-Gordon

$$+ (\alpha_x\alpha_y + \alpha_y\alpha_x)\frac{\partial^2\psi}{\partial x\partial y} + (\alpha_y\alpha_z + \alpha_z\alpha_y)\frac{\partial^2\psi}{\partial y\partial z} + (\alpha_z\alpha_x + \alpha_x\alpha_z)\frac{\partial^2\psi}{\partial z\partial x}$$

$$+ i(\alpha_x\beta + \beta\alpha_x)m\frac{\partial\psi}{\partial x} + i(\alpha_y\beta + \beta\alpha_y)m\frac{\partial\psi}{\partial y} + i(\alpha_z\beta + \beta\alpha_z)m\frac{\partial\psi}{\partial z}.$$

α and β cannot be numbers → matrices

The Dirac Equation: α and β

For Dirac to satisfy Klein-Gordon then

$\boldsymbol{\alpha}, \beta$ *cannot be numbers*

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I,$$
$$\alpha_j \beta + \beta \alpha_j = 0,$$
$$\alpha_j \alpha_k + \alpha_k \alpha_j = 0 \quad (j \neq k),$$

α and β are 4 mutually anticommuting matrices

Lowest dimension: 4x4

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The Dirac equation is a 4x4 matrix of operators that act on a *four-component wave-function*

4 degrees of freedom

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

If all particles were massless then the β term would not be needed $\hat{E}\psi = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m)\psi,$

Particles would be described by a two-component object (Weyl spinor)

Dirac: Negative Probability Density?

As we did for Klein-Gordon compute difference between

A Hermitian matrix is a square matrix that is equal to the transpose of its conjugate matrix.

$$\begin{bmatrix} 3 & 3+i \\ 3-i & 2 \end{bmatrix} \xrightarrow{\text{conjugate}} \begin{bmatrix} 3 & 3-i \\ 3+i & 2 \end{bmatrix} \xrightarrow{\text{Transpose}} \begin{bmatrix} 3 & 3+i \\ 3-i & 2 \end{bmatrix}$$

$$-i\alpha_x\frac{\partial\psi}{\partial x} - i\alpha_y\frac{\partial\psi}{\partial y} - i\alpha_z\frac{\partial\psi}{\partial z} + m\beta\psi = +i\frac{\partial\psi}{\partial t},$$

Wave function

$$+i\frac{\partial\psi^\dagger}{\partial x}\alpha_x^\dagger + i\frac{\partial\psi^\dagger}{\partial y}\alpha_y^\dagger + i\frac{\partial\psi^\dagger}{\partial z}\alpha_z^\dagger + m\psi^\dagger\beta^\dagger = -i\frac{\partial\psi^\dagger}{\partial t}.$$

Hermitian conjugate Wave function

And compare to the continuity equation (omitting calculation)

Probability density = $\rho = \psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$ $\Rightarrow$ positive by definition

- Dirac formulation gives positive defined probability density;
- Dirac particles are more complex than Klein Gordon ones: four components wavefunctions
  - Additional degrees of freedom (spin, intrinsic angular momentum);
  - Can de shown to describe particles & antiparticles

Pauli Matrices

Hermitian matrix is a complex square matrix that is equal to its own conjugate transpose

$\boldsymbol{A}$ is Hermitian $\Leftrightarrow a_{ij} = \overline{aji}$ $\boldsymbol{A}$ is Hermitian $\Leftrightarrow A = \overline{A^T}$

The Pauli matrices are a set of three 2 ×2 complex matrices that are traceless, Hermitian, unitary.

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Dirac Particles and Angular Momentum

Reminder: the time dependence of an observable $\hat{O}$ is given by

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{O}\rangle = i\langle\psi|[\hat{H},\hat{O}]|\psi\rangle.$$

if $\frac{d\hat{O}}{dt} = 0$ the observable is conserved ↔ $[\hat{H},\hat{O}] = 0$
→ the two operators commute

Angular momentum

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = (yp_z - zp_y, zp_x - xp_z, xp_y - yp_x).$$

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \quad \text{and} \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x.$$

Schrödinger equation (*non relativistic*)

$$\hat{H}_{SE} = \frac{\hat{\mathbf{p}}^2}{2m}, \quad [\widehat{H_{SE}},\hat{L}] = 0$$ →. Angular Momentum is conserved

Dirac equation (*relativistic*)

$$\hat{H}_D = \boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + \beta m, \quad [\hat{H}_D,\hat{\mathbf{L}}] = [\boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + \beta m, \hat{\mathbf{r}}\times\hat{\mathbf{p}}] = [\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}, \hat{\mathbf{r}}\times\hat{\mathbf{p}}].$$

$$[\hat{H}_D,\hat{\mathbf{L}}] = -i\boldsymbol{\alpha}\times\hat{\mathbf{p}}.$$

Angular Momentum is NOT conserved

Introduce a new operator:

$$\hat{\mathbf{S}} \equiv \tfrac{1}{2}\hat{\boldsymbol{\Sigma}} \equiv \frac{1}{2}\begin{pmatrix}\boxed{\boldsymbol{\sigma}} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}$$

$$\hat{\Sigma}_x = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, \quad \hat{\Sigma}_y = \begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix} \quad \text{and} \quad \hat{\Sigma}_z = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$$

Spin & Angular Momentum of Dirac Particles

It can be shown that also $\hat{S}$ doesn't commute with Hamiltonian $[\hat{H}_D, \hat{\mathbf{S}}] = i\alpha \times \hat{\mathbf{p}}.$

We have just seen that $[\hat{H}_D, \hat{\mathbf{L}}] = -i\alpha \times \hat{\mathbf{p}}.$

- It is natural to associate the operator $\hat{S}$ with the intrinsic angular momentum of the particle;

Which translates into $\left[\hat{H}_D, \hat{\mathbf{J}}\right] \equiv \left[\hat{H}_D, \hat{\mathbf{L}} + \hat{\mathbf{S}}\right] = 0.$

- The total angular momentum $\hat{L} + \hat{S}$ is a conserved quantity;
- Dirac particles have all intrinsic angular momentum $s = \frac{1}{2}$;
- The intrinsic magnetic moment of a Dirac particle is $\hat{\boldsymbol{\mu}} = \frac{q}{m}\hat{\mathbf{S}},$ where q *and* m are the charge and the mass of the Dirac particle

Dirac equation includes naturally the description of spin ½ particles.
This is NOT a mathematical consequence.
This is the consequence of requiring the wavefunction to satisfy a particular structure of the Dirac equation

Covariant Form of the Dirac Equation

The Dirac equation can be expressed in a covariant form (a few steps …)

1. Start from the standard equation $i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi.$
2. Multiply it by β $\quad i\beta\alpha_x\frac{\partial\psi}{\partial x} + i\beta\alpha_y\frac{\partial\psi}{\partial y} + i\beta\alpha_z\frac{\partial\psi}{\partial z} + i\beta\frac{\partial\psi}{\partial t} - \beta^2 m\psi = 0.$
3. Define $\gamma^0 \equiv \beta, \quad \gamma^1 \equiv \beta\alpha_x, \quad \gamma^2 \equiv \beta\alpha_y$ and $\gamma^3 \equiv \beta\alpha_z,$
4. And $\partial_\mu \equiv (\partial_0, \partial_1, \partial_2, \partial_3) \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$
5. You can rewrite the Dirac equation as $(i\gamma^\mu\partial_\mu - m)\psi = 0,$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Solutions of the Dirac Equation

Which is the physical meaning of the solutions of the Dirac equation? $(i\gamma^\mu\partial_\mu - m)\psi = 0.$

Free particle wavefunctions of spin ½ particles

- $u(E,\boldsymbol{p})$ is 4-component spinor
- No position and time dependence

$$\psi(\mathbf{x},t) = u(E,\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)},$$

Derivatives $\partial_\mu\psi(\boldsymbol{x},t)$ act only on the exponent

$$\partial_\mu \equiv (\partial_0,\partial_1,\partial_2,\partial_3) \equiv \left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

$$\partial_0\psi \equiv \frac{\partial\psi}{\partial t} = -iE\psi, \quad \partial_1\psi \equiv \frac{\partial\psi}{\partial x} = ip_x\psi, \quad \partial_2\psi = ip_y\psi \quad \text{and} \quad \partial_3\psi = ip_z\psi.$$

$$\boxed{(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u(E,\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)} = 0,}$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

$$(\gamma^\mu p_\mu - m)\,u = 0,$$

→ the expression doesn't contain derivatives
It is the free-particle Dirac equation for the spinor $u(E,\boldsymbol{p})$

Dirac Equation: Solution for a Particle at Rest

$\psi(\mathbf{x},t) = u(E,\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$, a particle at rest $\psi = u(E,0)e^{-iEt}$,

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u(E,\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)} = 0,$$

$E\gamma^0 u = mu$

$$E\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{\gamma^0}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}.$$

γ^0 is diagonal $\Leftrightarrow$ 4 orthogonal solutions

The 4 states are also eigenstates of the $\hat{S}_z$ operator

$\boldsymbol{N}$ wavefunction normalisation

Spin up, Spin down

$$u_1(E,0) = N\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } u_2(E,0) = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

Positive energy solution $E = +m$

Spin up, Spin down

$$u_3(E,0) = N\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } u_4(E,0) = N\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

Negative energy solution $E = -m$

Particle at Rest: Dirac solution

$$\psi_1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt},\ \psi_2 = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt},\ \psi_3 = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt}\ \text{and}\ \psi_4 = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}.$$

- Dirac equation for a particle at rest has positive probability density;
- Represents well spinors with spin up and spin down;

- Has still not solved the problem with negative energy solutions!

Dirac Equation: Solution for a Free-Particle

Pauli matrices

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$E\gamma^0 u = mu$

Written in full → $(E\gamma^0 - p_x\gamma^1 - p_u\gamma^2 - p_z\gamma^3 - m)u = 0.$

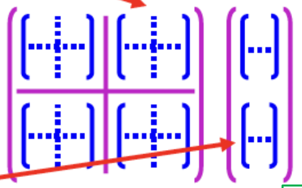
$$\left[\underbrace{\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}}_{\gamma^0} E - \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{pmatrix} - m\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] u = 0,$$

γ^0 $\quad$ $\gamma^{1,2,3}$

$$= \begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -(E+m)I \end{pmatrix}$$

4 solutions !

Note
$\vec{\sigma}.\vec{p} = p_x\sigma_x + p_y\sigma_y + p_z\sigma_z$



u_A^1
u_A^2
u_B^1
u_B^2

Note in the above equation the 4x4 matrix is written in terms of four 2x2 sub-matrices

- **Writing the four component spinor as** $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

Describe u as sum of u_a and u_b

$$\boxed{\boldsymbol{\sigma}\cdot\mathbf{p}} \equiv \sigma_x p_x + \sigma_x p_y + \sigma_x p_z = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}.$$

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

Explicit Positive & Negative Energy Solutions

$$\begin{pmatrix} (E-m)I & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -(E+m)I \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

Gives u_a as a function of u_b

$$u_A = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E-m}u_B,$$

$$u_B = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}u_A.$$

One more step: explicit u_a: u_A^1 and u_A^2.
(2 solutions for positive energy)

$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Orthogonal choice

The corresponding $u_B^{1,2}$ terms can be derived as

$$u_B = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} = \frac{1}{E+m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}u_A,$$

Explicit Positive & Negative Energy Solutions

The first two solutions of the Dirac equation for a free particle.

Positive or negative energy?

$$u_1(E,\mathbf{p}) = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad u_2(E,\mathbf{p}) = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix},$$

Compare with solutions for a particle at rest:
→ the spin operator $\hat{S}$ doesn't return 0 or 1

$$u_1(E,0) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2(E,0) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Other two solution are obtained with

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

And derive u_a from u_b

$$u_A = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E-m} u_B,$$

The choices are arbitrary; just like choosing one reference frame.
It is the simplest choice!

Explicit Positive & Negative Energy Solutions

- Explicitly write down 4 solutions;

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, \quad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}.$$

- Which energy do they correspond to? All these solutions satisfy Dirac equation:

$$\psi_i = u_i(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$$

If you put back any of these solutions into Dirac equation → get $E^2 = p^2 + m^2$

If you put $p = 0$ then you get

$$\begin{pmatrix} 1 \\ 0 \\ \cancel{\frac{p_z}{E+m}} \\ \cancel{\frac{p_x+ip_y}{E+m}} \end{pmatrix}$$

Spinors $u_{1,2}$ reduce to the positive energy solution of a Dirac particle at rest

$$E = +\left|\sqrt{\mathrm{p}^2 + m^2}\right|$$

And the same for $u_{3,4}$

$$E = -\left|\sqrt{\mathrm{p}^2 + m^2}\right|$$

- There are 4 independent solutions;
- We cannot avoid negative energy solutions

Antiparticles & Negative Energy Solutions

Dirac equation:

- Incredibly good framework for spin ½ particles;
- Spin and magnetic moments emerge naturally;
- Negative energy solutions cannot be excluded as '*unphysical*';
- Must provide a '*physical*' interpretation for these solutions.

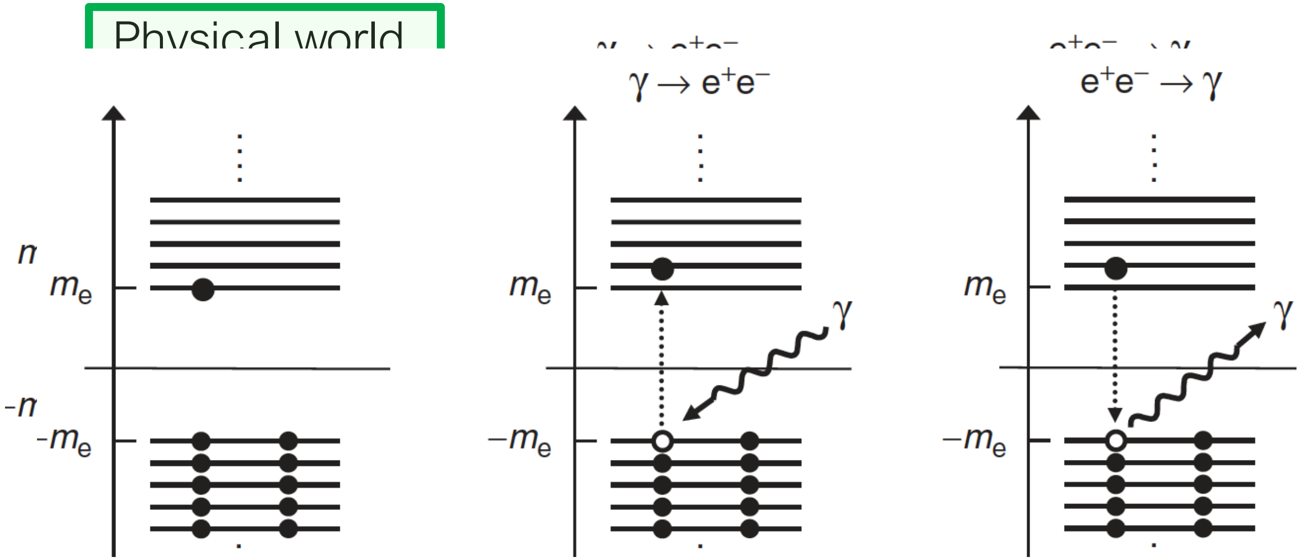
Difficulty:

- If really '***negative energy states***' existed, and were ***accessible***, then all positive energy electrons would fall into this lower energy states;

First attempt: the 'Dirac' sea

The vacuum is fully occupied by negative energy states

- → no hole is present for +energy electrons to go;
- → '***negative energy states***' are inaccessible
- Fermi exclusion principle prevents electrons from occupying the same position/energy



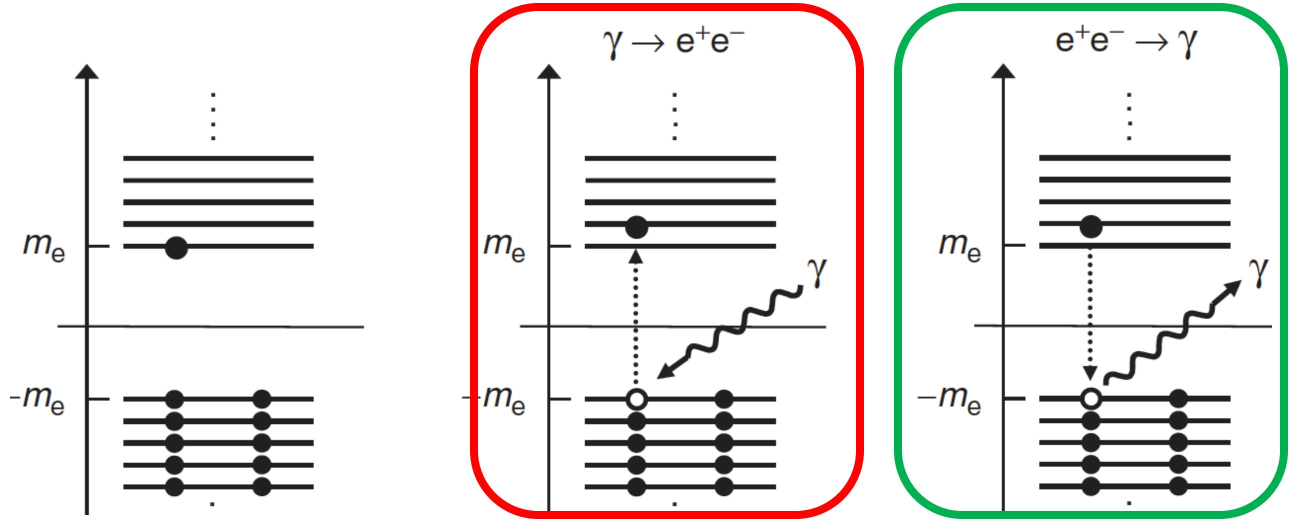
The Dirac see

This idea seemed not bad:

- If a photon, energy $> 2m_e$ excites one 'negative energy electron' would leave a 'hole' with
  - Less negative energy;
  - A loss of charge -1 → charge +1
- A positive energy *electron* with charge +1

- A positive energy electron falling into one available 'hole' would give
  - Disappearance of energy (negative energy);
  - Disappearance of a charge -1 (charge +1)
- Electron/positron annihilation.

→ antiparticles ??!!



Difficulties:

- The Dirac see would be populated by an infinite number of antielectrons → infinite energy! How to handle this?
- Today we know that also anti-bosons exist and the Fermi exclusion principle would not exclude occupying the same 'hole'

The Feynman–Stückelberg interpretation

We know today that:
Each spin ½ fermion has a spin ½ partner with exactly same characteristics BUT opposite charge.

Solution:

Negative energy fermions that propagate backward in time $\equiv$ Positive energy antifermions that propagate forward in time

Wavefunction doesn't change when
$(E) \rightarrow (-E) + (-t) \rightarrow (-t)$

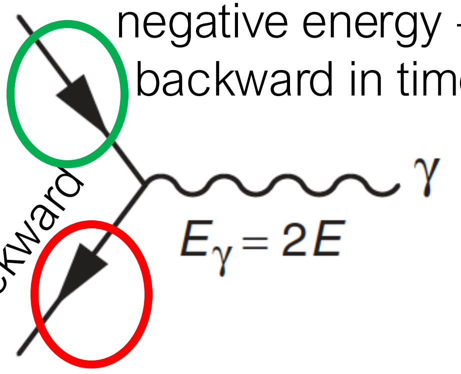
$$e^{[-iEt]} \equiv e^{[-i(-E)(-t)]}$$

Graphic convention:

Antiparticles are drawn as travelling back in time

electron emits γ + electron with negative energy → backward in time

e⁻($E > 0$)
electron



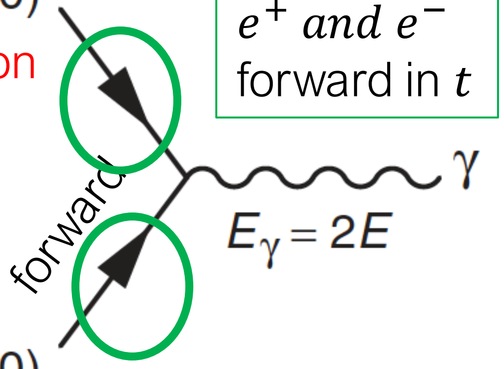
e⁻($E < 0$)
electron

Time →

electron (+E) annihilates with positron (+E) and emits γ

e⁻($E > 0$)
electron

e^+ and e^- forward in t



e⁺($E > 0$)
positron

Time →

Physical Antiparticle Spinors

Use 'physical spinors': physical energy and momentum: go from

$u_3 \to v_1$ and $u_4 \to v_2$

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}.$$

$$v_1(E,\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)} = u_4(-E,-\mathbf{p})e^{i[-\mathbf{p}\cdot\mathbf{x}-(-E)t]}$$

$$v_2(E,\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)} = u_3(-E,-\mathbf{p})e^{i[-\mathbf{p}\cdot\mathbf{x}-(-E)t]}.$$

$N_i = \sqrt{E+m}$ wavefunction normalisation (Lorentz contraction) to give $2E$ particles per unit volume

Same procedure as for u_1 and u_2

$$u_1(p) = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad u_2(p) = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix},$$

$$\boxed{\text{Dirac particle solution} \quad \psi_i = u_i e^{+i(\mathbf{p}\cdot\mathbf{x}-Et)}}$$

$$v_1(p) = \sqrt{E+m} \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2(p) = \sqrt{E+m} \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}.$$

$$\boxed{\text{Dirac antiparticle solution} \quad \psi_i = v_i e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}}$$

$$E = +\left|\sqrt{\mathbf{p}^2 + m^2}\right|$$

Operators Acting on Antiparticles

Operators that return physical energy and momentum of antiparticles have to modified:

$$\hat{H}^{(v)} = -i\frac{\partial}{\partial t} \quad \text{and} \quad \hat{\mathbf{p}}^{(v)} = +i\boldsymbol{\nabla},$$

Feynman–Stückelberg interpretation: $(E, \mathbf{p}) \to (-E, -\mathbf{p})$.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \to -\mathbf{L}.$$

To maintain $[\widehat{H_D}, \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}] = 0$ for antiparticles $\quad \hat{\mathbf{S}}^{(v)} = -\hat{\mathbf{S}},$

Dirac sea picture: a spin-up hole in the negative energy Dirac sea, leaves the vacuum in a net spin-down state.

Charge Conjugation

Symmetries are very important in Particle Physics. We will discuss more

Charge conjugation is a discrete transformation of particles into antiparticles

- Classical dynamics → how the Charge conjugation operator is defined.
- Motion of a charged particle in an electromagnetic field $A^{\mu} = (\phi, \boldsymbol{A})$

$$E \to E - q\phi \quad \text{and} \quad \mathbf{p} \to \mathbf{p} - q\mathbf{A},$$

4 vector notation $p_{\mu} \to p_{\mu} - qA_{\mu}.$

$\phi, \boldsymbol{A}$ scalar and vector potentials, q is the charge of the particle

Classical Physics → Quantum Mechanics $\hat{E} = i\partial/\partial t, \quad \hat{\mathbf{p}} = -i\nabla$ ⇒ $i\partial_{\mu} \to i\partial_{\mu} - qA_{\mu}.$

Dirac equation motion of a charged *particle* $q = -e$ in an EM field becomes

$$\boxed{\gamma^{\mu}(\partial_{\mu} - ieA_{\mu})\psi + im\psi = 0.}$$

particle

Dirac equation motion of a charged *antiparticle* $q = +e$ in an EM field becomes

$$\boxed{\gamma^{\mu}(\partial_{\mu} + ieA_{\mu})\psi' + im\psi' = 0.}$$

antiparticle

$\psi' = i\gamma^2\psi^*,$ ⇒ $\psi' = \hat{C}\psi = i\gamma^2\psi^*.$

Z-component of the spin operator
$$\hat{S}_z = \tfrac{1}{2}\Sigma_z = \tfrac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$u_1(E,0) = N\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2(E,0) = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Particle at rest spinors *(E>0)*
Eigenstates of $\widehat{S}_z$

Clearly not true for $p \neq 0$.

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix},$$

$$v_1(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}.$$

In the special case $p_z = \pm p, p_{x,y} = 0$.

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{\pm p}{E+m} \\ 0 \end{pmatrix}, \; u_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp p}{E+m} \end{pmatrix}, \; v_1 = N\begin{pmatrix} 0 \\ \frac{\mp p}{E+m} \\ 0 \\ 1 \end{pmatrix} \text{ and } v_2 = N\begin{pmatrix} \frac{\pm p}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Spin Effect

$$\hat{\mathbf{S}}^{(v)} = -\hat{\mathbf{S}},$$

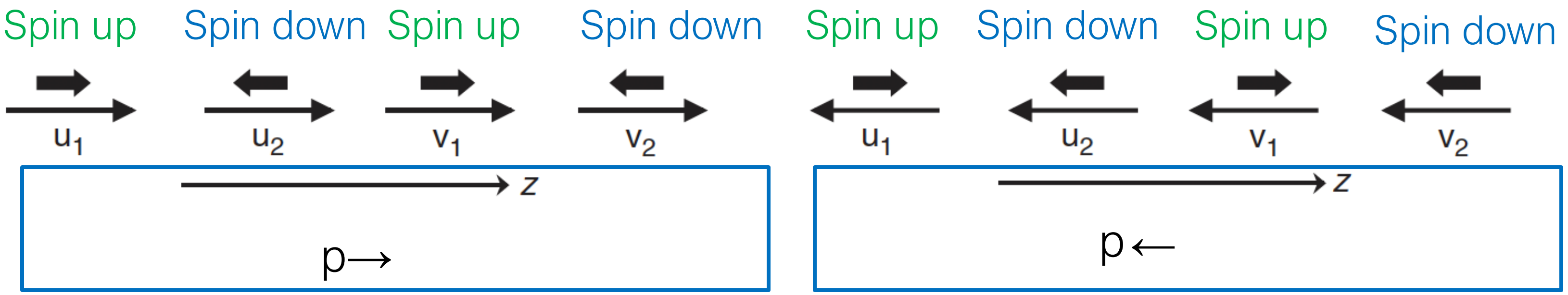
The action of $\widehat{S_Z}$ $(particles)$

$$\hat{S}_z\, u_1(E,0,0,\pm p) = +\tfrac{1}{2} u_1(E,0,0,\pm p). \quad \text{Spin up}$$
$$\hat{S}_z\, u_2(E,0,0,\pm p) = -\tfrac{1}{2} u_2(E,0,0,\pm p). \quad \text{Spin down}$$

and of $\widehat{S_Z^v}$ $(antiparticles)$

$$\hat{S}_z^{(v)} v_1(E,0,0,\pm p) \equiv -\hat{S}_z v_1(E,0,0,\pm p) = +\tfrac{1}{2} v_1(E,0,0,\pm p). \quad \text{Spin up}$$
$$\hat{S}_z^{(v)} v_2(E,0,0,\pm p) \equiv -\hat{S}_z v_2(E,0,0,\pm p) = -\tfrac{1}{2} v_2(E,0,0,\pm p). \quad \text{Spin down}$$

Possible configurations:



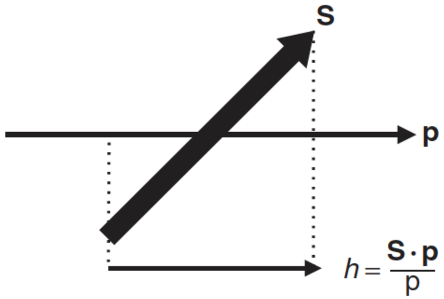
Helicity

- Cross sections calculation depends on spin states;
- The z component of the Spin operator is of limited use;
- The z component of the Spin operator does not commute with the Dirac Hamiltonian;

→ introduce

Helicity: projection of the spin along the direction of motion

$$h \equiv \frac{\mathbf{S}\cdot\mathbf{p}}{\mathrm{p}}.$$



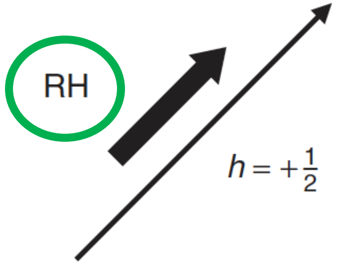
The corresponding helicity operator is $\hat{h} = \frac{\hat{\mathbf{\Sigma}}\cdot\hat{\mathbf{p}}}{2\mathrm{p}} = \frac{1}{2\mathrm{p}}\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix}$

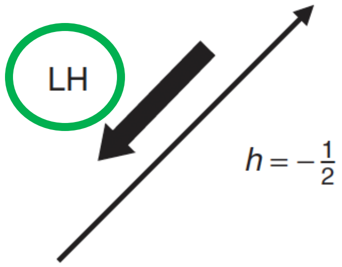
It can be shown that the Dirac Hamiltonian commutes with $\hat{h}$ $[H_D, \hat{\mathbf{\Sigma}}\cdot\hat{\mathbf{p}}] = 0$

simultaneous eigenstates of the free particle Dirac Hamiltonian and the helicity operator.

For a fermion the eigenvalues of the helicity operator ±1. These states called

- right-handed and
- left-handed helicity states





Helicity Eigenstates

Need explicit solutions of Dirac Equations that are also eigenstates of Helicity

$$(\boldsymbol{\sigma}\cdot\mathbf{p})u_A = 2\mathrm{p}\,\lambda u_A,$$
$$(\boldsymbol{\sigma}\cdot\mathbf{p})u_B = 2\mathrm{p}\,\lambda u_B.$$

$$\boxed{\frac{1}{2\mathrm{p}}\begin{pmatrix}\boldsymbol{\sigma}\cdot\mathbf{p} & 0\\ 0 & \boldsymbol{\sigma}\cdot\mathbf{p}\end{pmatrix}}\begin{pmatrix}u_A\\ u_B\end{pmatrix} = \lambda\begin{pmatrix}u_A\\ u_B\end{pmatrix}$$

$$\mathbf{p} = (\mathrm{p}\sin\theta\cos\phi,\ \mathrm{p}\sin\theta\sin\phi,\ \mathrm{p}\cos\theta),$$

$$\frac{1}{2\mathrm{p}}(\boldsymbol{\sigma}\cdot\mathbf{p}) = \frac{1}{2\mathrm{p}}\begin{pmatrix}p_z & p_x - ip_y\\ p_x + ip_y & -p_z\end{pmatrix}$$
$$\boxed{= \frac{1}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}}.$$

$s = \sin\left(\frac{\theta}{2}\right)$ and $c = \cos\left(\frac{\theta}{2}\right)$

Particles, right-handed spinor $u_\uparrow$ left-handed spinor $u_\downarrow$

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix}c\\ se^{i\phi}\\ \frac{\mathrm{p}}{E+m}c\\ \frac{\mathrm{p}}{E+m}se^{i\phi}\end{pmatrix} \quad u_\downarrow = \sqrt{E+m}\begin{pmatrix}-s\\ ce^{i\phi}\\ \frac{\mathrm{p}}{E+m}s\\ -\frac{\mathrm{p}}{E+m}ce^{i\phi}\end{pmatrix},$$

Antiparticles, right/left-handed spinor $v_\uparrow$ $v_\downarrow$

$$v_\uparrow = \sqrt{E+m}\begin{pmatrix}\frac{\mathrm{p}}{E+m}s\\ -\frac{\mathrm{p}}{E+m}ce^{i\phi}\\ -s\\ ce^{i\phi}\end{pmatrix} \quad v_\downarrow = \sqrt{E+m}\begin{pmatrix}\frac{\mathrm{p}}{E+m}c\\ \frac{\mathrm{p}}{E+m}se^{i\phi}\\ c\\ se^{i\phi}\end{pmatrix}.$$

In the ultra-relativistic region, $E \gg m$, the 4 spinors can be approximated as

$$u_\uparrow \approx \sqrt{E}\begin{pmatrix}c\\ se^{i\phi}\\ c\\ se^{i\phi}\end{pmatrix},\ u_\downarrow \approx \sqrt{E}\begin{pmatrix}-s\\ ce^{i\phi}\\ s\\ -ce^{i\phi}\end{pmatrix},\ v_\uparrow \approx \sqrt{E}\begin{pmatrix}s\\ -ce^{i\phi}\\ -s\\ ce^{i\phi}\end{pmatrix}\ \text{and}\ v_\downarrow \approx \sqrt{E}\begin{pmatrix}c\\ se^{i\phi}\\ c\\ se^{i\phi}\end{pmatrix}.$$

Helicity Eigenstates

$$u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_\downarrow = N\begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$