Perturbations in Topological Star geometries: a self force approach

Giorgio Di Russo

Hangzhou Institute of Advanced Study, UCAS, Hangzhou

Talk based on: 2305.15105, 2405.06566, 2406.19330, 2411.19612, 2502.21040

April 28, 2025

Black Holes binary merger

- BHs in GR: infinte curvature singularities surrounded by infinite redshift surface i.e. the event horizon
- Much info about structure is revealed by GW signal following a binary merger.







Credits: Ligo collaboration (GW150914)

<ロト < 同ト < 回ト < 回ト = 三日

- Inspiral \implies Classical Gravity (PN-PM expansions) or quantum amplitudes
- Merger ⇒ Numerical Relativity
- Ring-down \implies GWs emission signal dominated by Quasi Normal Modes (QNMs)

The fuzzball proposal

- Black holes (BHs) violate the information paradox
- Fuzzball proposal: classical BHs arise as superpositions of smooth and horizonless SUGRA solutions (O.Lunin, S.Mathur 0109154)
- Fuzzball proposal: successfull for extremal geometries but very few microstate candidates are known in the non-extremal case
- Philosophy: fuzzballs will replace BHs mimiking their aspect from large distances
- Focus of Topological Stars (TS) solution:
 - TSs are not microstates but admitting a regular (fuzzball-like) regime
 - They are charged solution but reproduce Schwarcschild (Sch) in some limit
 - When possible TS will be compared with Sch



・ロト ・ 母 ト ・ ヨ ト ・ ヨ ・ うへつ

Topological star scalar perturbations

• Field equations of d = 5 Einstein-Maxwell theory $S = \int dx^5 \sqrt{-g} \left[\frac{1}{2\kappa_e^2} R + \frac{1}{2} F_{(m)}^2 + \frac{1}{2} F_{(e)}^2 \right]$

$$d \star F_{(m)} = d \star F_{(e)} = 0 \quad , \quad R_{\mu\nu} = \kappa_{\mathbf{5}}^2 \left(T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T_{\alpha}^{\ \alpha} \right)$$
$$T_{\mu\nu} = F_{(m)}_{\mu\alpha} F_{(m)}_{\nu}^{\ \alpha} - \frac{1}{4} g_{\mu\nu} F_{(m)}_{\alpha\beta} F_{(m)}^{\alpha\beta} + \frac{1}{2} \left[F_{(e)}_{\mu\alpha\beta} F_{(e)}_{\nu}^{\ \alpha\beta} - \frac{1}{6} g_{\mu\nu} F_{(e)}_{\alpha\beta\gamma} F_{(e)}^{\ \alpha\beta\gamma} \right]$$

Double Wick rotation (I.Bah, P.Heidmann 2012.13407)



Radial equation for scalars mapped to CHE (M.Bianchi,GDR et al. 2305.15105)

$$\Box \Psi = 0 \quad , \quad \Psi = e^{-i\omega t + ipy} R(r) Y_{\ell,m}(\theta,\phi)$$
$$(r - r_s)(r - r_b) R''(r) + (2r - r_s - r_b) R'(r) + \left(\frac{\omega^2 r^3}{r - r_s} - \frac{p^2 r^3}{r - r_b} - \ell(\ell + 1)\right) R(r) = 0$$

• $r_s < r_b < 2r_s$: smooth horizonless, without any classical linear instability

(M, O)

Top star Post Newtonian solutions

- Study massless scalar waves emission by a low mass probe moving in TS equatorial geodesic
- The inhomogeneous wave equation (ρ scalar source and q the scalar charge)

$$\Box \phi = -4\pi\rho, \quad \rho = q \int \frac{1}{\sqrt{-g}} \delta^{(\mathbf{5})}(x^{\alpha} - x^{\alpha}_{p}(\tau)) d\tau$$

• Computations are done at y = const. After expansion in spherical harmonics we have to solve

$$R''(r) + \frac{2r - r_{\rm s} - r_{\rm b}}{(r - r_{\rm b})(r - r_{\rm s})} R'(r) + \frac{\omega^2 r^3 - \ell(\ell+1)(r - r_{\rm s})}{(r - r_{\rm s})^2(r - r_{\rm b})} R(r) = -4\pi q Y_{\ell m}^* \left(\frac{\pi}{2}, 0\right) \int dt \frac{e^{i(\omega t - m\phi_p(t))}\delta(r - r_p(t))}{r^2 u^t(t)f_{\rm s}(r)\sqrt{f_{\rm b}(r)}} R(r) = -4\pi q Y_{\ell m}^* \left(\frac{\pi}{2}, 0\right) \int dt \frac{e^{i(\omega t - m\phi_p(t))}\delta(r - r_p(t))}{r^2 u^t(t)f_{\rm s}(r)\sqrt{f_{\rm b}(r)}} R(r) = -4\pi q Y_{\ell m}^* \left(\frac{\pi}{2}, 0\right) \int dt \frac{e^{i(\omega t - m\phi_p(t))}\delta(r - r_p(t))}{r^2 u^t(t)f_{\rm s}(r)\sqrt{f_{\rm b}(r)}} R(r) = -4\pi q Y_{\ell m}^* \left(\frac{\pi}{2}, 0\right) \int dt \frac{e^{i(\omega t - m\phi_p(t))}\delta(r - r_p(t))}{r^2 u^t(t)f_{\rm s}(r)\sqrt{f_{\rm b}(r)}} R(r) = -4\pi q Y_{\ell m}^* \left(\frac{\pi}{2}, 0\right) \int dt \frac{e^{i(\omega t - m\phi_p(t))}\delta(r - r_p(t))}{r^2 u^t(t)f_{\rm s}(r)\sqrt{f_{\rm b}(r)}} R(r) = -4\pi q Y_{\ell m}^* \left(\frac{\pi}{2}, 0\right) \int dt \frac{e^{i(\omega t - m\phi_p(t))}\delta(r - r_p(t))}{r^2 u^t(t)f_{\rm s}(r)\sqrt{f_{\rm b}(r)}} R(r) = -4\pi q Y_{\ell m}^* \left(\frac{\pi}{2}, 0\right) \int dt \frac{e^{i(\omega t - m\phi_p(t))}\delta(r - r_p(t))}{r^2 u^t(t)f_{\rm s}(r)\sqrt{f_{\rm b}(r)}} R(r)$$

Solutions using the Green function method

$$\phi(t,r,\theta,\phi) = -4\pi q \sum_{lm} Y_{lm}(\theta,\phi) Y_{lm}^*\left(\frac{\pi}{2},0\right) \int dt' \int \frac{d\omega}{2\pi} e^{-i\omega t} G_{l\omega}(r,r_p(t')) \frac{e^{i(\omega t'-m\phi_p(t'))}\sqrt{f_p(r_p(t'))}}{u^t(t')}$$

• Solutions of the homogeneous problem: Post Newtonian (PN) expansion $\eta=1/c$

$$\begin{split} v^{2} &\sim GM/r \quad , \quad GM/c^{2} \sim r_{s,b} \quad , \quad \omega \sim v/c \implies r_{s,b} \sim \eta^{2} \quad , \quad \omega \sim \eta \\ R_{\mathrm{PN,in}}(r;r_{b},r_{s},\ell,\omega) &= r^{\ell} - \left(\frac{(r_{b}+r_{s})\ell}{2} + \frac{r^{3}\omega^{2}}{2(2\ell+3)}\right) \eta_{v}^{2}r^{\ell-1} + \mathcal{O}(\eta_{v}^{4}) \\ R_{\mathrm{PN,up}}(r;r_{b},r_{s},\ell,\omega) &= R_{\mathrm{PN,in}}(r;r_{b},r_{s},-\ell-1,\omega) \end{split}$$

Green function can be computed in PN-sense

Top star MST solutions

- Solutions of the homogeneous problem: MST method ightarrow applied to Teukolsky eq. (Kerr) which is a CHE.
- MST-type IN solution: boundary conditions at $r = r_b$. In the variable $z = z_{in} = \frac{r_b r}{r_b r_s}$:

$$z(1-z)R''(z) + [\gamma - (\gamma + \delta)z]R'(z) + (\beta - \tilde{\beta})R(z) = -\eta z(1-z)R'(z) + (\alpha z - \tilde{\beta})R(z)$$

· Consider the source (RHS) as a perturbation . If we choose

$$\frac{\delta^2}{4} - \tilde{\beta} + \beta = \left(\nu + \frac{1}{2}\right)^2$$

• No square roots will appear in the the dictionary. u is the renormalized angular momentum $u = \ell + \mathcal{O}(\omega)$

$$\begin{split} \alpha &= 2(i+\tau-i\kappa)\epsilon, \quad \beta = \ell(\ell+1) + i\epsilon + \kappa - \kappa^2 - \frac{1}{3}(\epsilon+2\tau)^2 + 2\epsilon\tau, \quad \gamma = 1, \quad \delta = 1 - 2\kappa, \quad \eta = 2i\epsilon\\ \epsilon &= -(r_b - r_s)\omega, \quad \tau = \omega(r_s + r_b/2), \quad 27\kappa^2\epsilon + (\epsilon+2\tau)^3 = 0 \end{split}$$

• The regular solution $R(r) \underset{r \to r_b}{\sim} (r - r_b)^{\mathbf{0}}$ of the LHS is

$$F_{\nu}(z) = {}_{\mathbf{2}}F_{\mathbf{1}}(1 + \nu - \kappa, -\nu - \kappa, \mathbf{1}, z)$$

We search for solutions of the full ODE of the form

$$R(z) = \sum_{n=-\infty}^{\infty} C_n F_{n+\nu}(z)$$

Since for 2F1-hypergeometric these identities hold

$$z(1-z)\frac{dF_{\nu+n}}{dz} = A_{+}F_{\nu+n+1} + A_{0}F_{\nu+n} + A_{-}F_{\nu+n-1}$$
$$zF_{\nu+n} = B_{+}F_{\nu+n+1} + B_{0}F_{\nu+n} + B_{-}F_{\nu+n-1}$$
$$(\Box \triangleright \langle \neg \Box \rangle \langle \neg \Box \rangle \rangle \langle \neg \Box \rangle \rangle = 0$$

Top star MST solutions

• As a consequence the ODE becomes a 3-terms recursion relation

$$\alpha_{n} c_{n+1} + \beta_{n} c_{n} + \gamma_{n} c_{n-1} = 0$$

$$\alpha_{n} = -\frac{i\epsilon (\kappa + n + \nu + 1) (-\kappa + n + \nu + 1) (i\tau + n + \nu + 1)}{(n + \nu + 1)(2n + 2\nu + 3)}$$

$$\beta_{n} = \ell(\ell + 1) - (n + \nu)(n + \nu + 1) + \frac{\epsilon \kappa^{2} \tau}{(n + \nu)(n + \nu + 1)} - \frac{1}{3} (4\tau^{2} + \tau\epsilon + \epsilon^{2})$$

$$\gamma_{n} = \frac{i\epsilon (\kappa + n + \nu) (-\kappa + n + \nu) (-i\tau + n + \nu)}{(n + \nu)(2n + 2\nu - 1)}$$

 $\alpha \in \mathcal{L} \setminus \mathcal{B} \subseteq \mathcal{L} \cap \mathcal{L} = \mathbf{0}$

• The consistency of the 3-terms recursion can be written in the form of a continuous fraction fraction

$$\beta_{\mathbf{0}} - \frac{\alpha_{-1}\gamma_{\mathbf{0}}}{\beta_{-1} - \frac{\alpha_{-2}\gamma_{-1}}{\beta_{-2} - \frac{\alpha_{-3}\gamma_{-2}}{\beta_{-3} - \dots}}} - \frac{\alpha_{\mathbf{0}}\gamma_{\mathbf{1}}}{\beta_{\mathbf{1}} - \frac{\alpha_{\mathbf{0}}\gamma_{\mathbf{1}}}{\beta_{2} - \frac{\alpha_{2}\gamma_{3}}{\beta_{3} - \frac{\alpha_{3}\gamma_{4}}{\beta_{3} - \dots}}} = \mathbf{0} \,,$$

· which can be perturbatively solved in

$$\nu = \ell + \sum_{n=1}^{\infty} \nu_n \omega^n \, .$$

• UP solution: outgoing b.c.s at $R(r) \underset{r \to \infty}{\sim} e^{i\omega r}$. Introduce $z = \omega(r - r_s)$ and $R(z) = z^{-1}f(z)$

$$\begin{aligned} z^{2} \Big[f''(z) + f(z) \Big] &- \left(\frac{(2r_{s} + r_{b})\epsilon z}{r_{b} - r_{s}} + \nu(\nu + 1) \right) f(z) = -\epsilon z f''(z) + \epsilon f'(z) \\ &+ \Big[-\epsilon z + \ell(1 + \ell) - \nu(1 + \nu) - \frac{3r_{s}^{2}\epsilon^{2}}{(r_{b} - r_{s})^{2}} - \frac{\epsilon}{z} \left(1 - \frac{r_{s}^{3}\epsilon^{2}}{(r_{b} - r_{s})^{3}} \right) \Big] f(z) \,. \end{aligned}$$

Top star MST solutions

- As for the IN solution we can manage the RHS of the previous diff.eq. as the (small) source
- · Solution of the LHS can be written in terms of Whittacker functions

$$f_{\nu}(z) = (-2iz)^{\mathbf{1}+\nu} e^{iz} U \Big[\nu + 1 - i\tau, 2(\nu+1), -2iz \Big] = W_{\nu}(z)$$
$$U(a, b; x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \mathbf{1} F_{\mathbf{1}}(a, b, x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{\mathbf{1}-b} \mathbf{1} F_{\mathbf{1}}(a-b+1, 2-b, x)$$

We search for solution of the full differential equation of the form

$$f(z) = \sum_{n=-\infty}^{\infty} \tilde{C}_n W_{\nu+n}(z) ,$$

Analogously to the IN, identities for confluent hypergeometric hold, so again we have a 3-terms recursion

$$\tilde{\alpha}_n \tilde{C}_{n+1} + \tilde{\beta}_n \tilde{C}_n + \tilde{\gamma}_n \tilde{C}_{n-1} = 0$$

- IN and UP must bu matched. This happens if the coefficients of the recursion coincide
- This happens automatically with IN and UP constructed in such way
- In Teukolsky matching is achieved multiplying the UP with n-dependent constant (MST 9603020)
- We specialized to circular and hyperbolic orbits (M.Bianchi, D.Bini, GDR 2407.10868, 2502.21040)

$$x_{p}(\tau) = (r_{p}(\tau), r_{p}(\tau), \frac{\pi}{2}, \phi_{p}(\tau)), \quad u^{\mu}(\tau) = \frac{dx_{p}^{\mu}(\tau)}{d\tau} = (u^{t}(\tau), u^{r}(\tau), 0, u^{\phi}(\tau))$$

Circular orbits

$$u^{t} = \frac{E}{1 - \frac{0}{r_{s}}}, \quad u^{\phi} = \frac{J}{r_{0}^{2}}, \quad U \cdot U = -1, \quad \frac{1}{2} \frac{\partial \mathcal{L}}{\partial u^{t}} = -E, \quad \frac{1}{2} \frac{\partial \mathcal{L}}{\partial u^{\phi}} = J$$

▲□▶ ▲□▶ ▲三▶ ▲三▶ - 三三 - のへの

Geodesic motion

• Hyperbolic orbits require $b = J/\sqrt{E^2 - 1} >> 1$ or equivalently G << 1 expansion a.k.a. Post Minkowskian (PM)

• It is convenient expand in the dimensionless quantity $\epsilon = \frac{r_s}{2bv^2}$ and introduce a dimensionless time $T = \frac{vt}{b}$

и

-

$$\begin{split} r(T) = & b\sqrt{1+T^2} + \epsilon r_1 + \epsilon^2 r_2 + O(\epsilon^3) \,, \\ \phi(T) = & \arctan(T) + \epsilon \phi_1 + \epsilon^2 \phi_2 + O(\epsilon^3) \,, \\ t = & \frac{dt}{d\tau} = & \gamma + \epsilon u_1^t + \epsilon^2 u_2^t + O(\epsilon^3) \,, \end{split}$$

$$\begin{array}{rcl} \frac{T_{b}}{b} & -1 + \frac{T \operatorname{arcsinh}(T)}{\sqrt{1+T^{2}}} - v^{2} \eta^{2} \frac{(\alpha+3)T \operatorname{arcsinh}(T)}{\sqrt{1+T^{2}}} \\ \frac{T_{b}}{\sqrt{1+T^{2}}} & \frac{1}{2\sqrt{7^{2}+1}} + \frac{T \operatorname{arcsinh}(T)}{1+T^{2}} + \frac{\operatorname{arcsinh}(T)}{2(1+T^{2})^{3/2}} + v^{2} \eta^{2} \Big[-\frac{2}{\sqrt{1+T^{2}}} - \frac{2(\alpha+3)T \operatorname{arcsinh}(T)}{1+T^{2}} - \frac{(\alpha+3)\operatorname{arcsinh}(T)^{2}}{(1+T^{2})^{3/2}} \Big] \\ & + v^{4} \eta^{4} \Big[-\frac{3(\alpha^{2}+2\alpha+5)T \operatorname{arcsinh}(T)}{2\sqrt{1+T^{2}}} + \frac{(\alpha+3)^{2}T \operatorname{arcsinh}(T)}{1+T^{2}} + \frac{(\alpha+3)^{2} \operatorname{arcsinh}(T)}{1+T^{2}} \Big] \\ \phi_{1} & \frac{T}{\sqrt{1+T^{2}}} + \frac{\operatorname{arcsinh}(T)}{1+T^{2}} + v^{2} \eta^{2} \Big[\frac{(\alpha+1)T}{\sqrt{1+T^{2}}} - \frac{(\alpha+3)\operatorname{arcsinh}(T)}{1+T^{2}} \Big] \\ \phi_{2} & \frac{2\operatorname{arcsinh}(T)}{(1+T^{2})^{3/2}} - \frac{T\operatorname{arcsinh}(T)^{2}}{(1+T^{2})^{2}} + v^{2} \eta^{2} \Big[\frac{(\alpha+3)T}{1+T^{2}} + \frac{2(\alpha+3)\operatorname{arcsinh}(T)}{(1+T^{2})^{2}} - \frac{2(\alpha+4)\operatorname{arcsinh}(T)}{(1+T^{2})^{3/2}} + (\alpha+3)\operatorname{arctan}(T) \Big] \\ v^{4} \eta^{4} \Big[\frac{(3\alpha^{2}+2\alpha+3)T}{4(1+T^{2})} - \frac{3(\alpha^{2}+2\alpha+5)\operatorname{arctan}(T)}{2(1+T^{2})} - \frac{(\alpha+3)^{2}T \operatorname{arcsinh}(T)^{2}}{(1+T^{2})^{2}} + \frac{2(\alpha+3)\operatorname{arcsinh}(T)}{(1+T^{2})^{3/2}} \\ + \frac{1}{4} \Big(3\alpha^{2}+2\alpha+3 \Big) \operatorname{arctan}(T) \Big] \\ u_{1}^{t} & -\frac{(\alpha-2)\eta^{2}v^{2}}{\sqrt{1+T^{2}}} - \frac{(\alpha-2)\eta^{4}v^{4}}{2\sqrt{1+T^{2}}} \\ u_{2}^{t} & \eta^{2}v^{2} \left(-\frac{(\alpha-2)}{1+T^{2}} + \frac{(\alpha-2)T \operatorname{arcsinh}(T)}{(1+T^{2})^{3/2}} \right) + v^{4} \eta^{4} \Big[- \frac{\alpha^{2}+5\alpha-10}{2(1+T^{2})} - \frac{(2\alpha^{2}+\alpha-10)T \operatorname{arcsinh}(T)}{2(1+T^{2})^{3/2}} \Big] \end{aligned}$$

▲□▶ ▲□▶ ▲ 三▶ ▲ 三▶ - 三 - のへで

Regularization of the reconstructed field and self force

- Reconstructed field can be computed in PN (also PM if needed) expansion
- Unfortunately using only PN solution, the sum over ℓ is divergent and needs mode sum regularization

$$\phi = \sum_{\ell=0}^{\infty} \phi_{\ell} \Longrightarrow \phi_{\ell} = \underbrace{\ell \phi_{1} + \phi_{0} + \frac{\phi_{-1}}{\ell}}_{B_{\ell}} + \frac{\phi_{-2}}{\ell^{2}} + \frac{\phi_{-3}}{\ell^{3}} + \dots$$

The regularized field is still divergent for some choices of l. These l-contributions can be added using MST

$$\phi_{reg} = (\phi_{\ell=\mathbf{0}}^{MST} - B_{\ell}) + (\phi_{\ell=\mathbf{1}}^{MST} - B_{\ell}) + \sum_{\ell=\mathbf{2}}^{\infty} (\phi_{\ell}^{PN} - B_{\ell}) + \mathcal{O}(\eta^{7})$$

- The scalar self force along the source world-line $F_{\alpha} = q(\delta^{\beta}_{\alpha} + u_{\alpha}u^{\beta})\partial_{\beta}\phi_{reg}(x^{\alpha}_{p}(\tau))$
- From the self force definition the energy and angular momentum losses by the particle along the orbit

$$\Delta E = \int_{-\infty}^{\infty} F_t(\tau) d\tau \quad , \quad \Delta J = \int_{-\infty}^{\infty} F_{\phi}(\tau) d\tau$$

From the previous it is possible to distinguish the conservative and the dissipative sectors

$$F_{t,\phi}^{(cons)} = \frac{F_{t,\phi}(\tau) - F_{t,\phi}(-\tau)}{2} \quad , \quad F_{t,\phi}^{(diss)} = \frac{F_{t,\phi}(\tau) + F_{t,\phi}(-\tau)}{2} \quad , \quad F^{r} = \frac{u_{t}}{u_{r}}F^{t} - \frac{u_{\phi}}{u_{r}}F^{\phi}$$

• Deviation from the geodesic motion $x^{\mu}(\tau) = x^{\mu}_{geo}(\tau) + \delta x^{\mu}(\tau), \quad u^{\mu}(\tau) = u^{\mu}_{geo}(\tau) + \frac{d\delta x^{\mu}(\tau)}{d\tau}, \quad u^{2} = -1$

Top star self force approach: Results and comparison

• Results for the circular orbit (M.Bianchi, D.Bini, GDR 2411.19612)

$$\frac{dE}{dt} = -\left(\frac{\tilde{q}^2}{\pi r_s}\right)^2 \frac{u^4}{3} \left[1 - 2u - \frac{3117}{175}u^2 + \frac{121984}{2205}u^3 + \alpha \left(-2u - 5u^2 + \frac{8768}{105}u^3\right) + \alpha^2 \left(u^2 + \frac{256}{15}u^3\right)\right] + \mathcal{O}(u^8)$$
$$dJ = \frac{m}{\omega} dE \quad , \quad u = \frac{r_s}{2r_0} \quad , \quad \alpha = \frac{r_b}{r_s}$$

- Results for circular orbit have been numerically checked by (M.Melis, F.Corelli, R. Croft, P.Pani 2412.14259)
- Results for the hyperbolic orbit (M.Bianchi, D.Bini, GDR 2502.21040)

$$\Delta E = \frac{\pi \eta^3 q^2 r_s^2}{24b^3 v} + \eta^5 r_s^2 \left(\frac{\pi \alpha q^2 v}{16b^3} + \frac{31\pi q^2 v}{720b^3} \right) + \mathcal{O}(\eta^7, b^{-4})$$

$$\Delta J = \eta^2 \left(-\frac{\pi q^2 r_s^2}{12b^2 v^2} - \frac{q^2 r_s}{3b} \right) + \eta^4 \left(r_s^2 \left(-\frac{\pi \alpha q^2}{24b^2} - \frac{\pi q^2}{10b^2} \right) + r_s \left(-\frac{\alpha q^2 v^2}{3b} - \frac{17q^2 v^2}{30b} \right) \right) + \mathcal{O}(\eta^6, b^{-2})$$

• Self force deviations from the geodesic scattering angle $\epsilon = \frac{r_s}{2bv^2}$ (M.Bianchi, D. Bini, GDR next to appear)

$$\delta\chi^{\text{cons}} = \dot{q}^2 \frac{r_s}{b} \left[\epsilon \pi (2+\alpha) \left(-\frac{1}{8} \eta^2 v^2 - \frac{17}{24} \eta^4 v^4 + \frac{25}{192} \eta^6 v^6 \right) \right] + \mathcal{O}(\epsilon^2, \eta^8)$$

iss = $\epsilon^2 \dot{q}^2 \frac{r_s}{b} \left(\frac{2\eta v}{3} + \frac{\eta^3 v^3 (17+20\alpha)}{15} + \frac{\eta^5 v^5}{120} (-80+56\alpha+80\alpha^2+\pi^2(3+20\alpha)) \right) + \mathcal{O}(\eta^3, \eta^7)$

(日) (日) (日) (日) (日) (日) (日) (日)

• Schwarzschild, TS deformation, different b.c.s

 $\delta \chi^{\rm d}$

 Comparison with Schwarzschild (D.Bini, A.Geralico, C.Kavanagh, A.Pound 2406.15878), (D.Bini, T.Damour, A.Geralico 2407.02076)

Final comments

SO...WHY?

- It should be evident that computations are convoluted
- Comparison with equivalent results from amplitudes
- PM expansion is equivalent to loop expansion
- Notice that emission due to particles moving con circular orbit are exact in PM.



▲ロ ▶ ▲ □ ▶ ▲ □ ▶ ▲ □ ▶ ▲ □ ▶ ● ○ ○ ○

Future Projects

- After circular and hyperbolic orbits we could study elliptic orbits
 - Double expansion: PN (small v) and small eccentricity
 - Compute the periastron advance
 - Analytical continuation from periastron advance (bound) and scattering angle (unbound) (G.Kalin, R.Porto 1910.03008)
- From a mathematical point of view it would be interesting try to generalize the MST method in different directions
 - To any kind of CHE: relevant for Kerr-Newman (KN) family BHs and top stars (and not only)
 - HE relevant for BHs of AdS-KN family
 - DCHE relevant for extremal-KN family. This case probably share some similarities with the CHE case since in some cases there exist a non trivial coordinate transformation between DCHE and CHE.

Thank you!