EFT-Hedron and positivity bounds at loop level

In collaboration with L.-Q. Shao, A. Tokareva, Y. Xu Based on 2501.09717 and ongoing work

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Effective field theory (EFT)

Effective Field Theories (EFTs) describe the physics at a given energy scale by including only the degrees of freedom that are relevant at that scale. In practice, we construct an EFT by writing down all possible operators consistent with the symmetries of the system, each multiplied by an arbitrary coefficient.





EFTs

IR



UV-IR connections

Partial wave expansion: $A(s,\theta) = 32\pi \sum_{l=1}^{\infty} (l + \frac{1}{2})f_l(s)\mathcal{P}_l(cos\theta)$ QFT axioms in a language of scattering amplitudes:

- Lorentz invariance: A = A(s, t, u)
- Unitarity: when t < 0, $Lim_{s \to \infty} \frac{A(s, t)}{s^2} = 0$
- Causality: A(s, t) is an analytic function outside the real axes
- Locality: $A(s, t) < s^n t^n$ for finite *n*

Separate the UV and IR part:

$$\frac{1}{2\pi i} (\oint_{+} + \oint_{-}) \frac{A(\mu, t)d\mu}{(\mu - s)^{n+1}} = 0, \quad n \ge 2$$

$$\frac{1}{2\pi i} \oint_{arc} \frac{A(\mu, t)}{(\mu - s)^{n+1}} = \int_{\epsilon^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{\epsilon^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - u)^{n+1}} \quad I_{UV} = \int_{\epsilon^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{\epsilon^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - u)^{n+1}} \quad I_{UV} = \int_{\epsilon^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{\epsilon^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - u)^{n+1}} \quad I_{UV} = \int_{\epsilon^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{\epsilon^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - u)^{n+1}} \quad I_{UV} = \int_{\epsilon^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{\epsilon^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - u)^{n+1}} \quad I_{UV} = \int_{\epsilon^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{\epsilon^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - u)^{n+1}} \quad I_{UV} = \int_{\epsilon^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{\epsilon^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - u)^{n+1}} \quad I_{UV} = \int_{\epsilon^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{\epsilon^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - u)^{n+1}} \quad I_{UV} = \int_{\epsilon^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{\epsilon^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - u)^{n+1}} \quad I_{UV} = \int_{\epsilon^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{\epsilon^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - u)^{n+1}} \quad I_{UV} = \int_{\epsilon^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{\epsilon^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{d\mu$$



$$I_{IR} = \frac{1}{2\pi i} \oint_{arc} \frac{A(\mu, t)}{(\mu - s)^{n+1}}$$





Not all EFTs are allowed!

$$L = -\frac{1}{2}(\partial_{\mu}\phi)^{2} + \frac{g_{2}}{2}(\partial_{\mu}\phi)^{4} + \frac{g_{3}}{3}(\partial_{\mu}\phi)^{2}(\partial_{\rho}\partial_{\sigma}\phi)^{2}$$
$$A(s,t) = g_{2}(s^{2} + t^{2} + u^{2}) + g_{3}stu + g_{4}(s^{2} + t^{2})$$

Positivity bound: $g_2 > 0$ (Arkani-Hamed et al, 2008)

Galileon symmetry: $\phi \rightarrow \phi + a_{\mu} x^{\mu} \longrightarrow g_2 = 0$

(Crossing symmetry of the IR theory implies the compact bounds on EFT coefficients, A.J Tolley, Z-Y Wang, S-Y Zhou 2011.02400)

Large hierarchy between EFT Wilson coefficients is typically forbidden!

 $(p)^2 + 4g_4(\partial_\rho\partial_\sigma\phi\partial_\rho\partial_\sigma\phi)^2 + \dots$

 $(+ u^2)^2 + ...$



Simon Caron-Huot, Vincent Van Duong 2021

$$\tilde{g}_3 := g_3 \frac{\Lambda^2}{g_2} \quad \tilde{g}_4 := g_4 \frac{\Lambda^4}{g_2} \quad \Lambda$$



Finding the constraints

From the relation

$$I_{IR} = I_{UV}$$

The arc integral can be computed in the IR EFT

$$M_n(t) \equiv \frac{1}{2\pi i} \oint_{arc} \frac{A(\mu, t)}{(\mu - s)^{n+1}} d\mu + (-1)^n \int_{\epsilon^2}^{\epsilon^2 - t} \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - \mu)^n} d\mu$$

We can relate $M_n(t)$ to the positive-definite functions

$$B_{n}(t) \equiv \left\langle \frac{\mathscr{P}_{j}(1+\frac{2t}{\mu})}{\mu^{n+1}} \right\rangle + (-1)^{n} \left\langle \frac{\mathscr{P}_{j}(1+\frac{2t}{\mu})}{(\mu+t)^{n+1}} \right\rangle$$
$$Disc_{s}A(\mu,t) = 16\pi \sum_{j=even}^{\infty} (2j+1) \operatorname{Im} f_{j}(\mu) \mathscr{P}_{j}(1+\frac{2t}{\mu})$$
$$\left\langle X(\mu,j) \right\rangle \equiv \sum_{j=even} 16(2j+1) \int_{e^{2}}^{\infty} d\mu \operatorname{Im} f_{j}(\mu) X(\mu,j)$$
$$When s = 0, M_{n}(t) = B_{n}(t). \text{ i.e. } M_{2}(t) = \left\langle \left(\frac{1}{\mu^{3}}\right)^{n} \right\rangle$$

 $\frac{(\mu, t)}{n+1}$



$$0 < \operatorname{Im} f_l(\mu) < 2$$

$$\frac{1}{(\mu+t)^3} \mathcal{P}_j(1 + \frac{2t}{\mu}) \rangle..$$

Finding the constraints

$$A_{tree}(s,t) = g_2(s^2 + t^2 + u^2) + g_3stu + g_4(s^2 + t^2) + g_3stu + g_4(s^2 + t^2) + g_3stu + g_4(s^2 + t^2) + g_4(s^$$

$$g_2 = \left\langle \frac{1}{\mu^3} \right\rangle$$
, $g_3 = \left\langle \frac{1}{\mu^4} \right\rangle - \left\langle \frac{2\mathcal{J}^2}{\mu^4} \right\rangle$, $g_4 = \left\langle \frac{1}{\mu^4} \right\rangle$

By using crossing symmetry, the relation about g_4

Null constraints are crucial for obtaining the compact region of couplings!





Finding the constraints

The inequalities on a_{kq}

That g_n is relating on positive-definite functions e.g. $g_{20} = a_{20}$, $g_{31} = a_{30} - 2a_{31}$...



- "Relaxing inequalities" : $a_{20} \ge a_{30}$, $a_{30} \ge a_{40}$, $a_{31} \ge a_{41}$...

- Grams inequalities : k = 4 constraints

 $\begin{vmatrix} a_{20} & a_{30} & a_{31} \\ a_{30} & a_{40} & a_{41} \\ a_{31} & a_{41} & a_{42} \end{vmatrix} \ge 0 , \begin{vmatrix} a_{20} & a_{30} \\ a_{30} & a_{40} \end{vmatrix} \ge 0 , \begin{vmatrix} a_{20} & a_{31} \\ a_{30} & a_{40} \end{vmatrix} \ge 0 , \begin{vmatrix} a_{20} & a_{31} \\ a_{31} & a_{42} \end{vmatrix} \ge 0$

Moments of Positivedefinite Functions



$$a_{kq} = \left\langle \frac{\mathcal{J}^{2q}}{\mu^{k+1}} \right\rangle$$

Their values are lying in a compact space (Called a-geometry in Nima Arkani-Hamed et al 2012.15849, L-Y Chiang, Y-T Huang, L.Rodina et al 2015.02862, 2204.07140 and developed by S-L Wan, S-Y Zhou 2411.11964)

$$\geq 0 \ , \ \begin{vmatrix} a_{40} & a_{41} \\ a_{41} & a_{42} \end{vmatrix} \geq 0$$

The optimal bounds should

include k=5 ($a_{5,q}$) constraints

- Polytope bounds: $\left\langle \frac{(\mathcal{J}^2 - 6)(\mathcal{J}^2 - 20)}{\mu^5} \right\rangle \ge 0, \quad a_{42} - 26a_{41} + 120a_{40} \ge 0$













Region III: $\tilde{g}_{31} = \frac{30}{7}\tilde{g}_{40} + \frac{37}{42}\sqrt{\tilde{g}_{40}(21 - 20\tilde{g}_{40})},$

2015.02862





- What happens if the loop-level amplitude is included? How can we deal with the non-analytic terms when $t \rightarrow 0$ (e.g. *logt* terms)?
- The loop-level amplitude contained the non-linear term such as g_{20}^2 , so we can no longer project out g_{20} . Is it still possible to determine the allowed region for g_{20} ?

$$\begin{array}{l} \textbf{Including loops} \\ A_{low}(s,t,u) &= g_2 \left(s^2 + t^2 + u^2\right) + g_3 \, stu + g_4 \left(s^2 + t^2 + u^2\right)^2 + g_5 \, stu \left(s^2 + t^2 + u^2\right) \\ &+ b_1 \left(s^4 \log(-s) + t^4 \log(-t) + u^4 \log(-u)\right) \\ &+ b_2 \left(s^2 tu \log(-s) + st^2 u \log(-t) + stu^2 \log(-u)\right) \\ &+ c_1 \left(s^5 \log(-s) + t^5 \log(-t) + u^5 \log(-u)\right) \\ &+ c_2 \left(s^3 tu \log(-s) + st^3 u \log(-t) + stu^3 \log(-u)\right) + \dots \end{array} \qquad \begin{array}{l} \textbf{b}_1 = -\frac{21g_2^2}{240\pi^2} \quad b_2 = \frac{g_2}{24} \\ b_1 = -\frac{g_2g_3}{60\pi^2} \quad c_2 = -\frac{g_2}{24} \end{array}$$

$$M_{2}(t)\Big|_{t\to0} = \frac{1}{3} \left(6g_{2} + 3b_{1}\epsilon^{4} + 2c_{1}\epsilon^{6} \right) + \frac{1}{2}t \left(-2g_{3} - 10b_{1}\epsilon^{2} - 4b_{2}\epsilon^{2} - 7c_{1}\epsilon^{4} - 2c_{2}\epsilon^{4} \right) + t^{2} \left(7b_{1} + 2b_{2} + 12g_{4} + 14c_{1}\epsilon^{2} + 3c_{2}\epsilon^{2} + 6b_{1}\log(-t) + b_{2}\log(-t) - b_{2}\log(\epsilon^{2}) \right)$$

$$M_3(t)\Big|_{t\to 0} = t\bigg(2b_1 + 8g_4 + 6c_1\epsilon^2 + 4b_1\log(-t) - b_2\big(\log(\epsilon^2) - \log(-t)\big)\bigg)$$

$$M_4(t)\Big|_{t\to 0} = 4g_4 + 2c_1\epsilon^2 + b_1\log(-t) + b_1\log(\epsilon^2)$$

There are plenty of nonanalytic terms!







Constructing the IR-finite terms

1)
$$\frac{1}{12b_1 + 2b_2} \frac{d^2 M_2(t)}{dt^2} - \frac{1}{4b_1 + b_2} \frac{dM_3(t)}{dt}$$

2)
$$\frac{1}{4b_1 + b_2} \frac{dM_3(t)}{dt} - \frac{M_4(t)}{b_1} = \left\langle -\frac{2(2b_1)}{b_1(4b_1)} \right\rangle$$

Solving the g_4 and Null-constraint

 $= \left\langle \frac{4b_1 \mathcal{J}^2 (-8 + \mathcal{J}^2) + b_2 (4 + \mathcal{J}^2 (-8 + \mathcal{J}^2))}{2(4b_1 + b_2)(6b_1 + b_2)\mu^5} \right\rangle$









g_2 dependence



Negative g_4 is allowed



■ k=4,*ε*²=1 □ k=5, ϵ^2 =1

12

Conclusion

- Not everything is allowed in low-energy EFT
- The constraints are imposed by the assumptions that the ultimate theory of Nature is unitary, causal, Lorentz-invariant, and local
- After fixing the cutoff scale of EFT we typically obtain bounds which are compact
- At tree level there is a state of art technique which allows to obtain the most optimal bounds analytically - EFThedron, theory of moments
- We provide a modification of these methods for theories with massless loops, and show that outside the weakly coupling limit the shapes of constraints are significantly modified



