De-projecting the EFT-hedron at loop level

Hangzhou Institute for Advanced Study, UCAS

Yongjun Xu In collaboration with Guangzhuo Peng, Longqi Shao and Anna Tokareva Based on arxiv:2501.09717

From EFT to EFT-hedron

- operators.
- Bottom-up EFT: write all possible terms in the Lagrangian with arbitrary Wilson coefficients allowed by symmetry.
- hedron. N.Arkani-Hamed et al JHEP 05 (2021) 259



• Top down EFT: integrate out the high energy d.o.fs. — a theory with higher dimension

• Not all of them are allowed to have a good UV completion if we assume "causality," locality, unitarity and Lorentz symmetry. N.Arkani-Hamed et al JHEP 10 (2006)

• The allowed region of Willson coefficients can form a non-trivial geometry called EFT-

UV and IR connected by dispersion relation

Causality — Analyticity in the upper half-plane • Locality $\longrightarrow |A(s,t)| \leq C s \log^2 s$ as $s \to \infty, t \to 0$ • $\frac{1}{2\pi i} \oint_{arc} \frac{A(\mu, t)}{(\mu - s)^{n+1}} = \int_{e^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \oint_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{d\mu}{\pi} \frac{(\mu - u)^{n+1}}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \int_{e^2 - t}^{\infty} \frac{d\mu}{\pi} \frac{d\mu}{\pi}$ UV

IR

• The left-hand side can be computed using low-energy (IR) amplitudes. The arc at infinity vanishes for $n \geq 2$



Partial wave expansion and unitarity • Unitarity implies optical theorem $\operatorname{Im} A(s, t = 0) = \sum_{X} \int d\Pi_X A^*(s \to X) A(s \to X)$

- This implies, from the n = 2 dispersion relation, that $g_2 > 0$. It is the first nontrivial result from positivity condition. N.Arkani-Hamed et al JHEP 10 (2006)
- Because of rotation symmetry, the amplitude can be expanded in partial waves: $A(\mu, t) = 16\pi \sum_{i}^{\infty} (2j+1) f_{j}(\mu) P_{j}\left(1 + \frac{2t}{\mu}\right)$
- Restricting to $2 \rightarrow 2$ scattering and using the partial wave expansion:

$$Im f_j > 2f_j^2 \text{ or } 0 < Im f_j < 2$$

The setup

- For a shift symmetric scalar: $\frac{1}{2}(\partial\phi)^2 + g_2(\partial\phi)^4 + g_3(\partial\partial\phi)^2(\partial\phi)^2 + g_4(\partial\partial\phi)^4 + \dots$
- Substituting $u \to -s t$, we get $A(s,t) = \sum g_{k,q} s^{k-q} t^{q} + (\text{loop part})...$ k - a > 2
- We expand both sides with respect to t and match their coefficients. (Be careful to account properly for loop effects.)

• For convenience, changing variable $\mu \rightarrow \frac{1}{7}$, we get:

• It can also be written as $g_{k,q} = \sum v_{k,q,j} a_{j,k}$, if we *j* EFT coefficients Double moment

• the amplitude can be approximated by $A(s, t, u) = \sum g_2(s^2 + t^2 + u^2) + g_3stu + g_4(s^2 + t^2 + u^2)^2 + ...$

$$\begin{aligned} &: g_{k,q} = \sum_{j} v_{k,q,j} \int_{0}^{1} \rho_{j}(z) z^{k-1} dz & 0 < \rho_{j}(z) < 2. \\ &\text{e define } a_{j,q} = \int_{0}^{1} \rho_{j}(z) z^{k-1} dz \end{aligned}$$

L-moment problem

- $a_k = \int_0^1 \rho(z) z^{k-1} dz \qquad 0 \le \rho(z) \le L$
- It is a problem solved long time ago L-moment problem. If we think $\rho(z)$ as of inertia...



Consider a symmetric toy model (e.g., a bird-shaped mass distribution)..., once we know the bird is made of ordinary matter, then:

 $0 < \rho_{bird} < \rho_{neuron star}$. (By unitarity)

What is my moment?

• We first neglect the sum of j and $v_{k,q,j}$ for illustration, then the problem becomes:

mass density and z as length, then a_1 is just the total mass, and a_2 is the moment

UV assumption

IR region



Minkowski sum

- The definition of the Minkowski sum of two areas A and B is:

 $A + B := \{a + b \mid a \in A, b \in B\}$

Notice $\partial(A + B)$ is not necessary $\partial A + \partial B$

- To get the right boundary of the sum, you need ' follow the boundaries placed in the right order.
- In this 2d example, the right order means ordering them by their slopes. For example:

• We will solve it by the method of Minkowski sums. Y-t H, L R, et al JHEP 05 (2024) 102





Solve the L-moment problem with Minkowski sums

sum:

$$a_{k} = \int_{0}^{1} \rho(z) z^{k-1} dz = \sum_{n=1}^{N} \rho\left(\frac{n}{N}\right) \left(\frac{n}{N}\right)^{k-1} \frac{1}{N} \qquad 0 < \rho(z) < L$$

• In the case of 2d: consider $k_i > k_j$, for some n, the vector is

$$\left(\rho\left(\frac{n}{N}\right)\left(\frac{n}{N}\right)^{k_i-1}\frac{1}{N}, \, \rho\left(\frac{n}{N}\right)\left(\frac{n}{N}\right)^{k_j-1}\frac{1}{N}\right) \quad 0 < \rho\left(\frac{n}{N}\right) < L.$$

• The slope is $\left(\frac{n}{N}\right)^{\kappa_j-\kappa_i}$, it becomes larger as we increase n.

• To fit our intuition of doing Minkowski sums, we first discretize the integral to the



From projective to non-projective geometry

In the continuous limit, the corresponding boundaries are the following parametric curves:

lower boundary:
$$\left(L\frac{m^{k_i}}{k_i}, L\frac{m^{k_j}}{k_j}\right) \quad m \in [0, 1]$$

upper boundary: $\left(L\frac{1-m^{k_i}}{k_i}, L\frac{1-m^{k_j}}{k_j}\right)$

We give the picture of the allowed regions. In the limit $L \to \infty$ all the non-projective information has lost, only left with $a_1 > a_2$.



Add the different angular momentum contributions

• Once we get the allowed region of $a_{i,q}$ =

the remaining question is — how are we

- The answer is quite simple just do the Minkowski sum again for j.
- For the case interesting to us, we show it in the picture on the right.
- The Minkowski sum of j=2,0,4 regions for rescaled moments. The boundary of the sum can be obtained by ordering the individual boundaries by their slopes.

$$= \int_{0}^{1} \rho_{j}(z) z^{k-1} dz,$$

adding them together? $g_{k,q} = \sum_{i} v_{k,q,j} a_{j,k}$





N=4 null constraint

- Because of the full crossing-symmetry, we can get the same coefficient in different ways. It gives strong constraints on the UV states.
- The first non-trivial one is n = 4: $g_4(s^2 + t^2 + u^2)^2$

• we define
$$M_n = \frac{1}{2\pi i} \oint_{arc} \frac{A(\mu, t)}{(\mu)^{n+1}}$$
, and we have different

$$0 = 6M_4 - \partial_t^2 M_2 |_{t \to 0} = \sum_j 16(2j+1)(j^2(j+1)^2 - 8j(j+1)) \int_0^1 \rho_j(z) z^4 dz$$

A J. T et al JHEP 05 (2021) 255 S C-H at al JHEP 05 (2021) 280

EFT-hedron



ways to get g_4 .

Bound on
$$g_2$$
 f
• Once loop effects taken account, null con

$$\sum_{j} 16(2j+1)(j^2(j+1)^2 - 8j(j+1)) \int_{0}^{1}$$
Details will be explained by Gu
Also $g_2 = \sum_{j} 16(2j+1) \int_{0}^{1} \rho_j(z)zdz$

j



 n_4 500 $n_4 = -\frac{2353}{1968}\pi^2 g_2^2$ 20 40 60 80 100 120 - 500 Loop-corrected - 1000 |-Positivity only with IR loops 4 - ~ ~

Results

The (g_2, n_4) -space with boundary sections is shown in the picture below.

Tree level

 g_2

- section 1: j=2
- section 2: j=0
- **—** section 3: *j*=4, 6, 8, ...
- section ∞ : $j = \infty$
- k=4 null constraint
- L= ∞ , the lower boundary

Previous result:

 $0 < g_2 < 125.4$

Our result: $0 < g_2 < 37.8$

Loops are important !!!

Discussion

- From a geometric point of view, the positivity conditions define the allowed region of Wilson coefficients—known as the EFT-hedron—in projective space.
- When full unitarity is taken into account, this EFT-hedron is deprojected.
- We find that, once loop effects are considered, the geometry is further deformed and even only positivity condition can give non-projective structures.
- We are interested in how this method can be used in the gravity case. Especially how the full-unitarity condition can be used in the weak gravity conjecture. (work in progress.)



Think dispersion relation as a distribution

•
$$2M_4 - \partial_t M_3 = 0 + (-2b_1 - b^2) \log(-t) = \sum_j 0 \int_0^1 \rho_j(z) z^4 dz$$
 $0 \le \rho_j(z) \le 2$

Infinity equal to 0 is not always a good thing, in this case, we should really think it as a equality only true in a distribution sense. A. Martin, INABILITY OF FIELD THEORY TO EXPLOIT THE FULL UNITARITY CONDITION

• Α

After doing such justification, we can subtract
$$2M_4 - \partial_t M_3$$
 from
 $(-2b_2 - 6b_1)\log(-t) - 18b_1 = \sum_j 16(2j+1)(j^2(j+1)^2 - 8j(j+1)) \int_0^1 \rho_j(z)z^4 dz \qquad 0 \le \rho_j(z) \le 2$

with right combination to eliminate log(-t) from above, then we can safely set t goes to 0 limit.

The null constraint has been modified: $\sum_{i} 16(2j+1)(j^2(j+1)^2 - 8j(j+1)) \int_0^1 \rho_j(z) z^4 dz$

$$z = \frac{28b_1^2 + 28b_1b_2 + 5b_2^2}{2b_1 + b_2}$$