On-shell Approach to Black Hole Mergers

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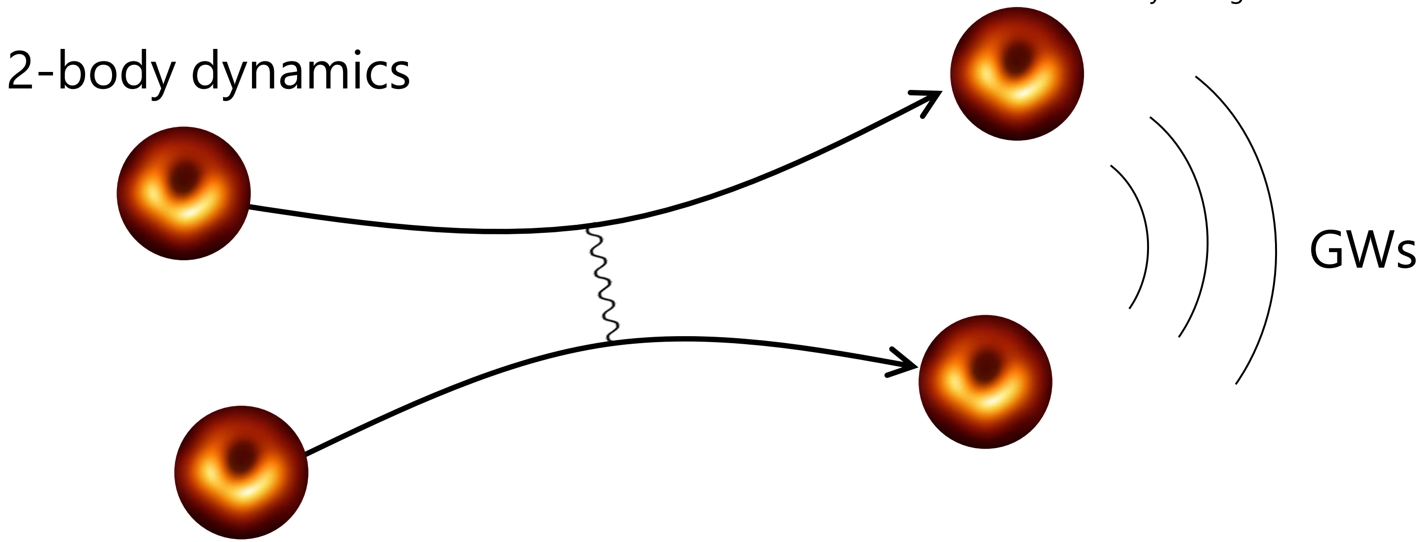
Introduction

❑ **Modern amplitude methods are quite powerful for classical dynamics.**

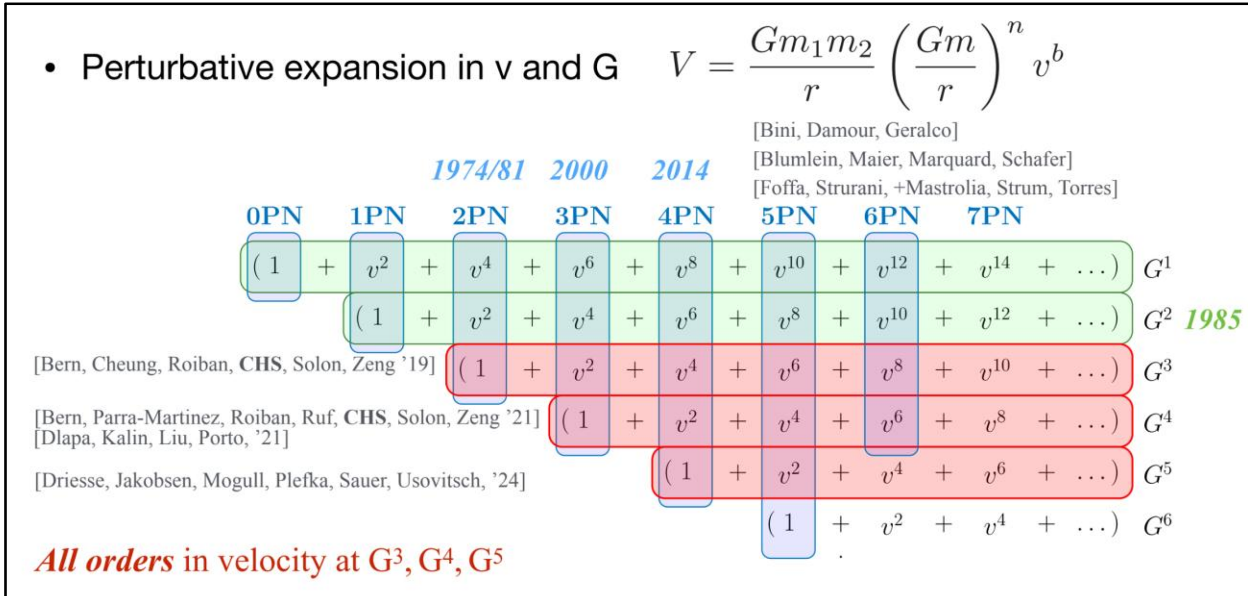
Cf. talks by Zhengwen Liu and Fei Teng yesterday



✓ Higher order potentials: no gauge redundancy, nice "hidden" properties,
✓ Spin: systematic treatment of higher spin,
✓ Tidal or horizon effects: clear separation between "IR" and "UV" physics,
✓ Radiation reactions: microscopic origin of calculations,
✓ …

Introduction

- ❑ **The amplitude approach has had great success so far.**



Taken from Chia-Hsien's talk@Gravity 2025, Kyoto

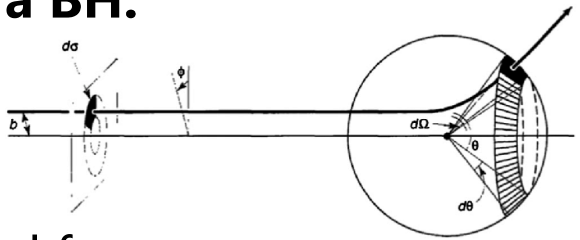
- ❑ **Let's apply this idea to the problem of black hole mergers.**

✓ BH mergers are the main targets of GW observations. We need more!
✓ How can we understand BH formations in S-matrix or quantum physics?

On-shell view of black holes

❑ **Let's consider how a distant observer will see a BH.**

✓ BH is just a localised object with mass & spin.
= one-particle state $|\mathrm{BH}\rangle = |M, J\rangle$



✓ Its structure is seen by how it responds to external force.
= (In)existence of interactions, e.g. BH has minimal coupling to gravity.

A. Guevara+ '18; M.-Z. Chung+ '18;
N. Arkani-Hamed+ '19; A. Aoude+ '19

→ **BH is just a particle with certain interactions.**

❑ **BH mergers are then seen as just a fusion of particles!**

KA, A. Cristofoli, Y.-t. Huang, 2410.13632.

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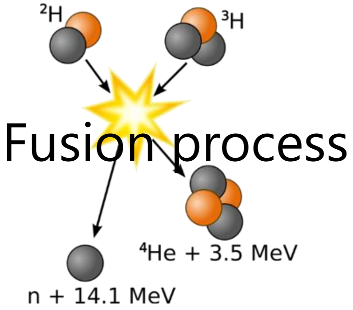
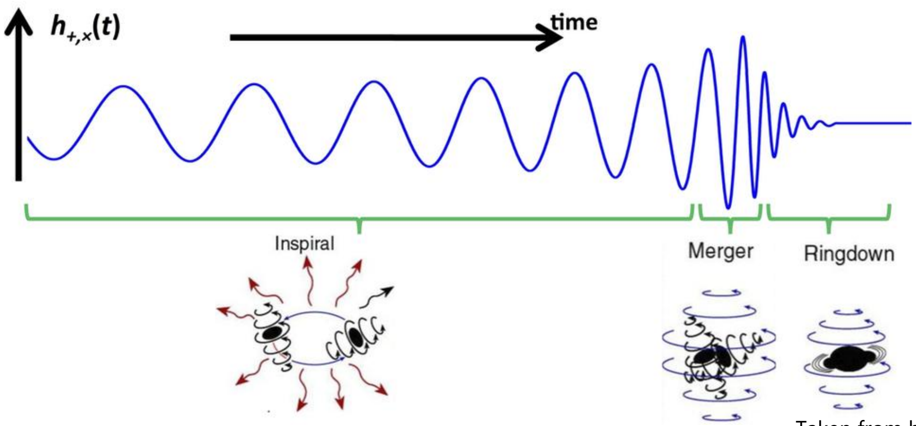


Information of BH merger is in on-shell 3pt & 4pt!

(at least without radiation reaction)



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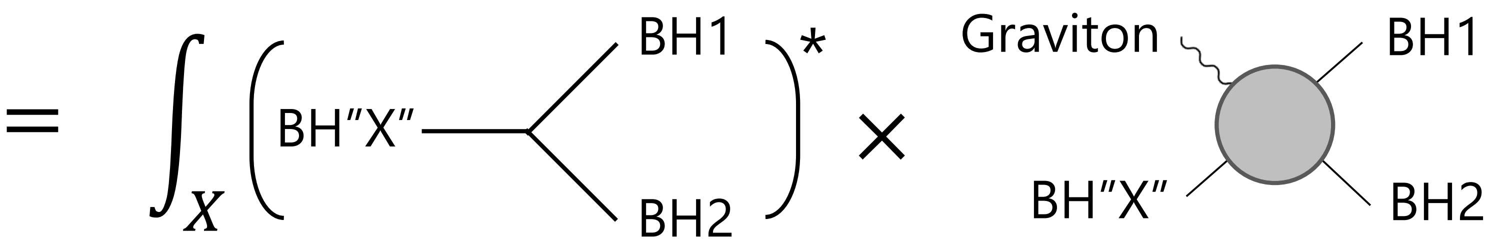

Fixed by kinematics

Encodes dynamics

KA, A. Cristofoli, Y.-t. Huang, 2410.13632.

Merger amplitude

- BH merger is non-perturbative. Maybe, no analytic control…
  No, it's just a local reaction and can be described by an effective operator.

  c.f. chiral EFT.

Schwarzschild (spin-0)

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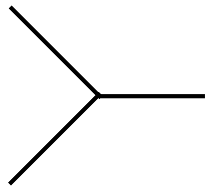


Kerr (spin-ℓ)

$$= g_\ell \langle \boldsymbol{X} | p_1 p_2 | \boldsymbol{X} \rangle^\ell \sim g_\ell Y_{\ell,m}$$

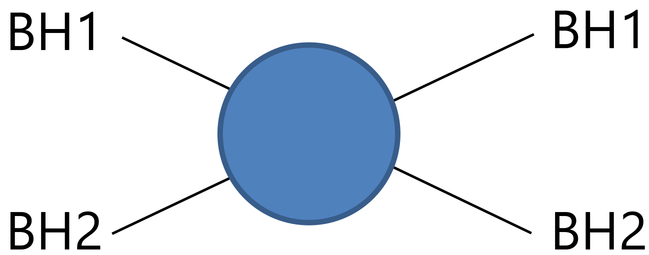
This 3pt is uniquely fixed by kinematics except for overall coupling.

N. Arkani-Hamed, T.-C. Huang, Y.-t. Huang, 2017.

- The coupling is determined to recover the fundamental feature of BH:
  nothing can escape the event horizon = complete absorption process

BH formation = Complete absorption

- Let's consider the elastic scattering of BHs. S.B. Giddings & M. Srednicki 2008; S.B. Giddings & R.A. Porto 2009.



$$\mathcal{A}_{12\to 12}(s,\cos\theta) = 16\pi \sum_{\ell=0}^{\infty} (2\ell+1) a_\ell(s) P_\ell(\cos\theta)$$

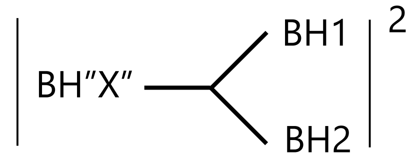
$$a_\ell(s) = \frac{E}{2P}\frac{\eta_\ell e^{2i\delta_\ell}-1}{2i}$$

elasticity $0 \leq \eta_\ell \leq 1$

- Classically, two BHs collide and form a new BH for $\ell < L_c = O(G m_1 m_2)$.
  The system never comes back to 2-body → $\eta_\ell = 0 \quad \text{for} \quad \ell < L_c$

- The absorption cross-section for BH formation is computed by the cut.

$$\sigma_\ell^{\text{abs}} = \sigma_\ell^{\text{tot}} - \sigma_\ell^{\text{el}} = \frac{\pi(2\ell+1)}{P^2}, \qquad (\ell \leq L_c)$$



$$\sigma_\ell^{\text{abs}} = \frac{\pi}{2EP}\sum_{m=-\ell}^{\ell}\int_{(m_1+m_2)^2}^{+\infty} \mathrm{d}m_X^2\, \delta(s-m_X^2)\rho_\ell(m_X^2)|\mathcal{A}(p_X,\ell,m|p_1;p_2)|^2 \sim |g_\ell|^2\rho_\ell$$

Cf. wave absorption: Aoude+ '23; Jones+ '23; Chen+ '23.

⇒ **The coupling (× spectral density) is completely fixed.**

"Quantum" BH → "Classical" BH

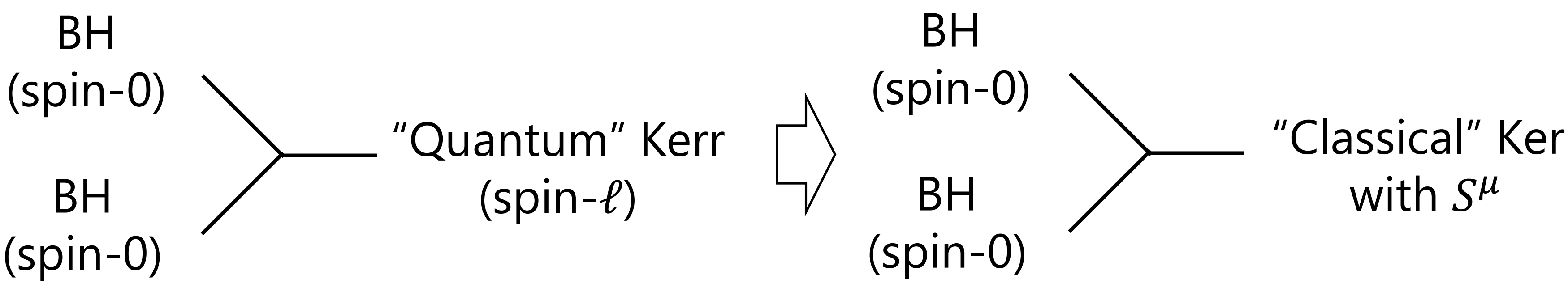
- ❑ The final state must be a classical Kerr with classical spin.
  We describe the final state by the coherent spin state R. Aoude & A. Ochirov 2021.

$$|\alpha\rangle := e^{-\frac{1}{2}\tilde{\alpha}_J \alpha^J} e^{\alpha^I \hat{a}_I^\dagger} |0\rangle$$

$$= e^{-\frac{1}{2}\|\alpha\|^2} \sum_{2\ell=0}^{\infty} \frac{1}{\sqrt{(2\ell)!}} \alpha^{I_1} \cdots \alpha^{I_{2\ell}} |\ell, \{I_1 \cdots I_{2\ell}\}\rangle$$

= a superposition of all spin$-\ell$ states with a wight α

$$S_X^\mu = \langle p_X, \tilde{\alpha}| \, \mathbb{S}_X^\mu \, |p_X, \alpha\rangle = \frac{\hbar}{2} \tilde{\alpha}_I [\sigma_X^\mu]^I{}_J \alpha^J$$



$$\mathcal{A}^{I_1 \cdots I_{2\ell}}(p_X, \ell | p_1; p_2) = m_X \frac{\sqrt{(2\ell)!}}{\ell!} \left(\frac{\langle \boldsymbol{X} | \, p_1 p_2 \, | \boldsymbol{X} \rangle}{m_X \lambda^{1/2}(m_1^2, m_2^2, m_X^2)} \right)^\ell$$

$$\mathcal{A}(p_X, \tilde{\alpha} | p_1, p_2) = m_X e^{-\frac{1}{2}\|\alpha\|^2 + z}$$

$$z(p_1, p_2) := \frac{\tilde{\alpha}_I \, \langle p_{12}^I | \, p_1 p_2 \, | p_{12}^J \rangle \, \tilde{\alpha}_J}{m_X \lambda^{1/2}(m_1^2, m_2^2, m_X^2)}$$

We now have the building block to describe BH mergers!

Classical physics from amplitudes

- ❑ **KMOC formalism** Kosower, Maybee & O'Connel 2018; A. Cristofoli et al, 2021.

✓ Observables in quantum physics are expectation values.

Expectation value at out Expectation value at in

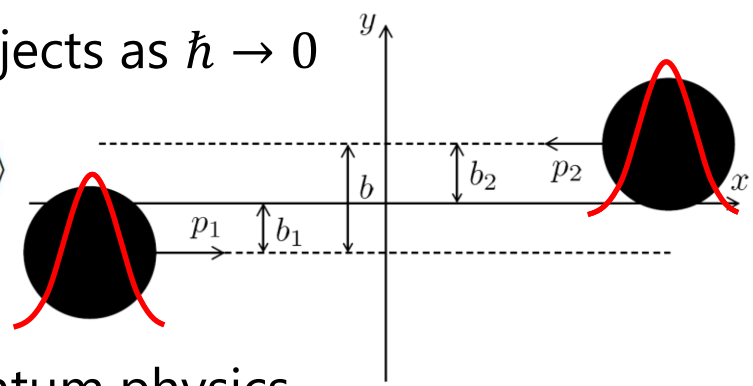
$$\Delta O = \langle\Psi|S^\dagger \mathcal{O} S|\Psi\rangle - \langle\Psi|\mathcal{O}|\Psi\rangle \qquad |\mathrm{out}\rangle = S|\Psi\rangle$$

✓ The initial state is given by localised wavepackets.

= classical localised objects as $\hbar \to 0$

$$|\Psi\rangle := \int d\Phi(p_1, p_2)\phi_1(p_1)\phi_2(p_2)e^{i(b_1\cdot p_1 + b_2\cdot p_2)}\,|p_1; p_2\rangle$$

2-body initial states



✓ Classical physics is just $\hbar \to 0$ limit of quantum physics.

Classical physics is recovered by on-shell S-matrix and states only!
No classical equations of motion nor classical fields are needed.

3pt = momentum conservations

- BH + BH → BH amplitude: $\langle p_X, \tilde{\alpha}|S|p_1, p_2\rangle = i\hat{\delta}^{(4)}(p_1+p_2-p_X)\mathcal{A}(p_X,\tilde{\alpha}|p_1,p_2)$

$$\mathcal{A}(p_X,\tilde{\alpha}|p_1,p_2) = m_X e^{-\frac{1}{2}\|\alpha\|^2+z} \qquad z(p_1,p_2) := \frac{\tilde{\alpha}_I \langle p_{12}^I|\, p_1 p_2\, |p_{12}^J\rangle \tilde{\alpha}_J}{m_X \lambda^{1/2}(m_1^2, m_2^2, m_X^2)}$$

with $\alpha \sim \hbar^{-1/2}$

Classical spin

The 3pt vanishes unless $-|\alpha|^2$ is cancelled by the kinematic function z.

→ 3pt is just delta function in classical limit.

$$\mathcal{I}_3(p_1,p_2,p_{12}) = \frac{\hbar^{5/2}}{4}\pi P \delta_\hbar(\mathrm{Im}\,\alpha^1)\delta_\hbar(\mathrm{Re}\,\alpha^2)\delta(S_X^z - bP)\delta(S_X^y)$$ at CoM.

Fourier transform of 3pt $\sim \delta^{(4)}(S_X^\mu - L_{\text{in}}^\mu)$

$$\mathcal{I}_3(p_1,p_2,p_X) := \int \prod_{i=1,2} \hat{\mathrm{d}}^4 q_i \hat{\delta}(2p_i\cdot q_i) e^{-ib_i\cdot q_i} \langle p_1+q_1; p_2+q_2|\, T^\dagger \,|p_X,\alpha\rangle$$

- Momentum & Spin of final states:

$$p_f^\mu = \langle\Psi|S^\dagger \mathbb{P}_X^\mu S|\Psi\rangle \overset{\hbar\to 0}{=} p_1+p_2, \;\; S_f^\mu = \langle\Psi|S^\dagger \mathbb{S}_X^\mu S|\Psi\rangle \overset{\hbar\to 0}{=} L_{\text{in}}^\mu$$

Classical conservations are reproduced from microscopic conservations.

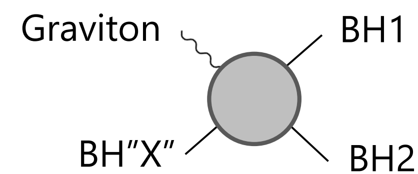
GWs from BH merger

- Waveform in KMOC: $h_{\mu\nu} = \langle \mathrm{out}|\mathbb{H}_{\mu\nu}|\mathrm{out}\rangle = \langle \Psi|S^\dagger \mathbb{H}_{\mu\nu} S|\Psi\rangle$

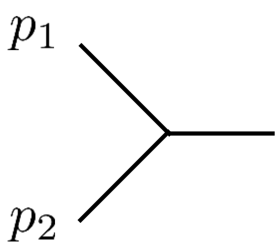
“Leading” order waveform (neglecting radiation reaction) is

$$iW^\sigma = \left\langle\!\!\left\langle \int_X \int \prod_{i=1,2} \hat{\mathrm{d}}^4 q_i \hat{\delta}(2p_i \cdot q_i) e^{-ib_i\cdot q_i} \underbrace{\langle p_1 + q_1; p_2 + q_2|\, T^\dagger \,|p_X, \alpha\rangle}_{\text{massive 3pt}} \underbrace{\langle p_X, \tilde{\alpha}; k^\sigma|\, T\, |p_1; p_2\rangle}_{\text{4pt with one graviton}} \right\rangle\!\!\right\rangle$$

“Waveform in frequency space = ∫ (Fourier transform of 3pt) × 4pt”

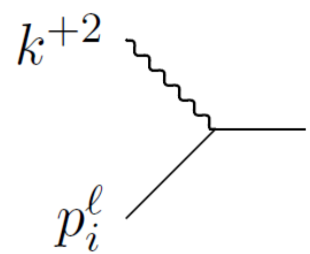


- We can compute the 4pt by gluing on-shell 3pts.



$p_X^\ell = g_\ell \langle \boldsymbol{X}|p_1 p_2|\boldsymbol{X}\rangle^\ell$

BH 3pt



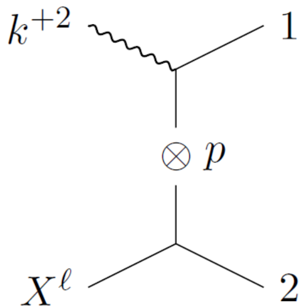
$p_i'^{\,\ell} = \dfrac{\kappa}{2} m_i^2 x_i^2 (\epsilon_{\alpha\beta})^{2\ell}$

Minimal coupling to graviton

A. Guevara+ ’18; M.-Z. Chung+ ’18;
N. Arkani-Hamed+ ’19; A. Aoude+ ’19

4pt amplitude at $O(G)$

- Tree-level 4pt is computed by on-shell gluing. N. Arkani-Hamed, T.-C. Huang, Y.-t. Huang, 2017.
  = Residue of 4pt must be factorised into (on-shell 3pt)^2.



$$
\begin{aligned}
\tilde{A}_4^{+} = & -\frac{[k|p_1p_2|k]^2\,\langle \boldsymbol{X}|\,p_1p_2\,|\boldsymbol{X}\rangle^{\ell}}{(t-m_1^2)(u-m_2^2)(s-m_X^2)} - \frac{\ell\,\langle \boldsymbol{X}|\,p_1p_2\,|\boldsymbol{X}\rangle^{\ell-1}\,\langle \boldsymbol{X}|\,p_1|k][k|p_2\,|\boldsymbol{X}\rangle\,[k|p_1p_2|k]}{(t-m_1^2)(u-m_2^2)} \\
& + \left[\sum_{n=2}^{\ell}(-1)^n\binom{\ell}{n}\langle \boldsymbol{X}|\,p_1p_2\,|\boldsymbol{X}\rangle^{\ell-n}\left(\langle \boldsymbol{X}k\rangle\,[k|p_2\,|\boldsymbol{X}\rangle\right)^{n-2}\left(\langle \boldsymbol{X}|\,p_1|k][k|p_2\,|\boldsymbol{X}\rangle\right)^2\right]\frac{1}{t-m_1^2} \\
& + \left[\quad (1\leftrightarrow 2)\quad\right]\frac{1}{u-m_2^2}\,. \qquad (3.22)
\end{aligned}
$$

- We further need to map it into the coherent spin state.

  coherent spin = infinite series sum of all spin-ℓ states

4pt 3pt × “soft factor”

$$
\mathcal{A}(p_X,\tilde{\alpha};k^{\pm}|p_1;p_2) = \mathcal{A}(p_{12},\tilde{\alpha}|p_1;p_2)S_{\alpha}^{\pm}
$$

$$
\begin{aligned}
S_{\alpha}^{\pm} = \frac{\kappa}{2}e^{\Delta z^{\pm}}\Bigg[& -\frac{(A^{\pm})^2}{(t-m_1^2)(u-m_2^2)(s-m_X^2)} - \frac{A^{\pm}v^{\pm}}{(t-m_1^2)(u-m_2^2)} \\
& - \frac{(v^{\pm})^2(1-w_2^{\pm}-e^{-w_2^{\pm}})}{(w_2^{\pm})^2(t-m_1^2)} + \frac{(v^{\pm})^2(1-w_1^{\pm}-e^{-w_1^{\pm}})}{(w_1^{\pm})^2(u-m_2^2)}\Bigg]
\end{aligned}
$$

*No soft limit is required for this factorisation.

All-order spin memory from amplitudes

- The waveform at this order is the classical limit of the soft factor.

$$iW^\sigma = \left\langle\!\!\left\langle \int_X \int \prod_{i=1,2} \hat{\mathrm{d}}^4 q_i \hat{\delta}(2p_i\cdot q_i) e^{-ib_i\cdot q_i} \underbrace{\langle p_1+q_1; p_2+q_2|\, T^\dagger \,|p_X, \alpha\rangle}_{\text{massive 3pt}} \underbrace{\langle p_X, \tilde{\alpha}; k^\sigma|\, T\, |p_1;p_2\rangle}_{\text{4pt with one graviton}} \right\rangle\!\!\right\rangle$$

$$= S_\alpha^\pm|_{S_X=L} \ \text{= spin memory waveform}$$

Cachazo and Strominger '14; Pasterski and Strominger '14

- Small spin expansion: $J_1 = -L_1, J_2 = -L_3, J_3 = S_f$

$$iW^\pm = \frac{\kappa}{2}\sum_{i=1}^{3}\left[\frac{(\varepsilon^\pm\cdot p_i')^2}{k\cdot p_i'} - i\frac{(\varepsilon^\pm\cdot p_i')(\varepsilon^\pm\cdot J_i\cdot k)}{k\cdot p_i'} - \frac{1}{2}\frac{(\varepsilon^\pm\cdot J_i\cdot k)^2}{k\cdot p_i'}\right] + \mathcal{O}(\omega^2 S^3)$$

Gravitational memory

Braginsky and Throne, 1987.

Classically proved

Laddha & Sen '19

Classically conjectured

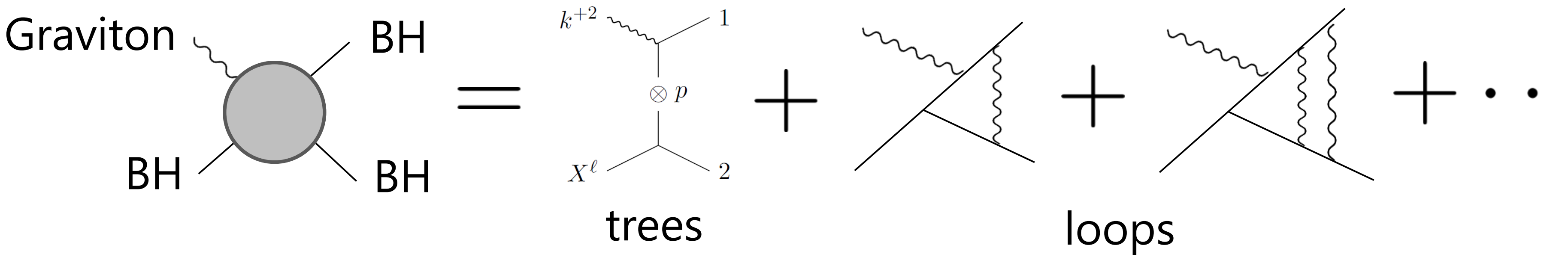
Our result

= all spin orders

On-shell proof of "memory = soft theorem" in all spin orders!

Beyond $O(G)$

- ❑ Can we go beyond memory?
- ❑ The tree-level approximation should be valid only for small ω.



We are now developing a classical resummation of loop diagrams.

- ❑ Before that, let's consider a simpler setup.
  If $m_2 \ll m_1$, the problem can be reduced to a potential scattering problem.

cf. QFT scattering → QM scattering, or "0 self-force"

It also gives a clear comparison btw our approach and the standard BHP.

From BH spacetime to on-shell amplitudes

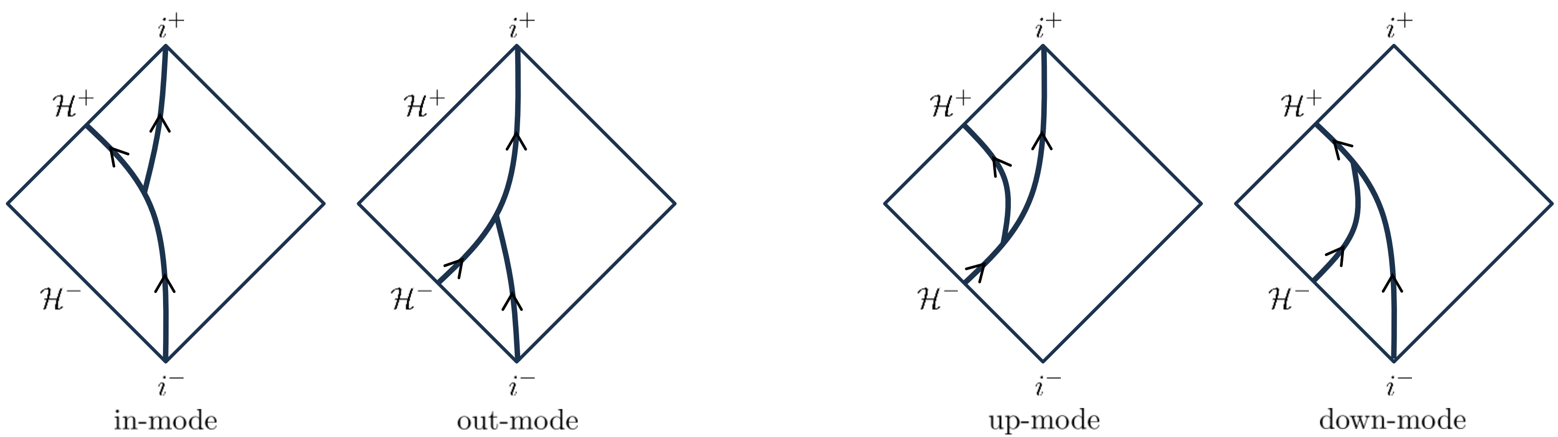
- ❑ Let's consider a scattering problem in GR and interpret it by on-shell.

$$\mathrm{d}s^2 = f(r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{f(r)} - r^2\mathrm{d}\Omega^2\,, \quad f(r) := 1 - \frac{2GM}{r} \qquad (\nabla_\mu\nabla^\mu + \mu^2)\varphi = 0$$

Black hole 1 (or EOB metric) "Black hole" 2 (or relative motion)

Heavier BH is always treated as classical while lighter can be quantum.

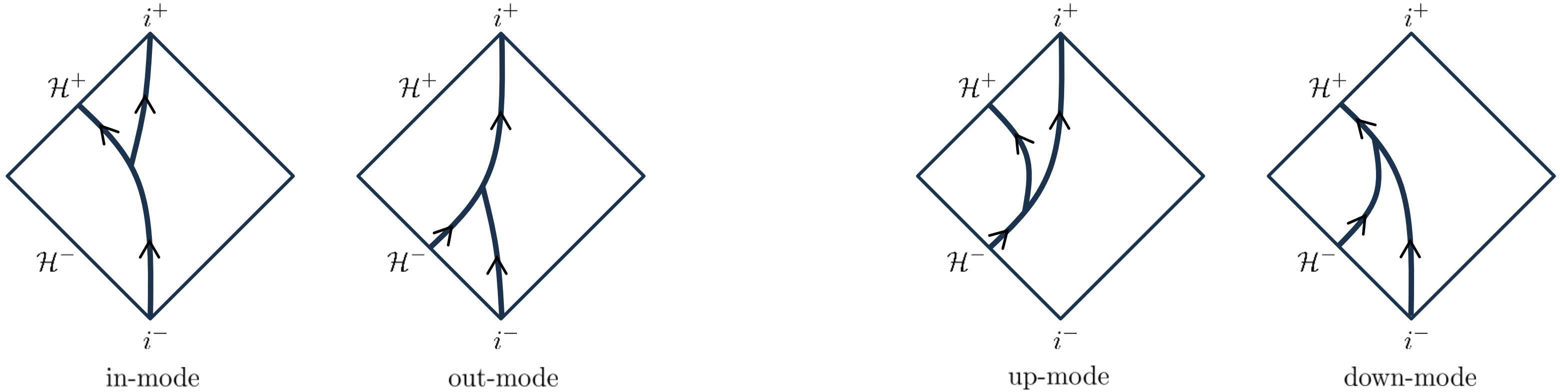
- ❑ As usual, we introduce four mode functions of φ (only external region).



2-particle (BH1&BH2) in/out states How do distant observers see them?

Absorption by BH = Formation of BH"X"

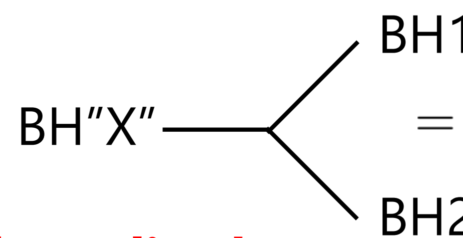
- No wave is observed at $i^{\pm}$ for down/up modes.
  A distant observer will see this state as a (composite) one-particle state!



2-particle (BH1&BH2) in/out states　　1-particle (BH"X"=1+2) in/out states

- Following quantum mechanics, the S-matrix is defined by inner products.

$$\langle\alpha|S|\beta\rangle := \begin{pmatrix} (\varphi^-_{P'},\varphi^+_{P}) & (\varphi^-_{P'},X^+_J) \\ (X^-_{J'},\varphi^+_P) & (X^-_{J'},X^+_J) \end{pmatrix} \Longleftrightarrow$$



$$= g_\ell \langle \boldsymbol{X}|p_1 p_2|\boldsymbol{X}\rangle^\ell$$

Exactly matches with on-shell amplitude

Radiation emission 4pt

- The waveform based on KMOC through the int. $\mathcal{L}_{\text{int}} = \frac{\lambda}{2} h \varphi^2$

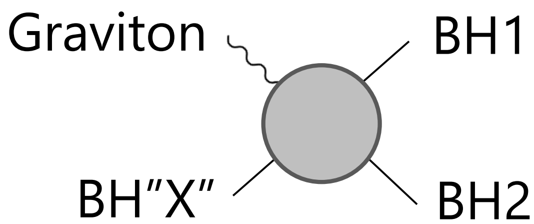
$$iW := \langle \Psi | S^\dagger a_k S | \Psi \rangle = \int_{P,P'} \phi^*(P')\phi(P) e^{i\vec{b}\cdot(\vec{P}'-\vec{P})} \langle P' | S^\dagger a_k S | P \rangle$$

$$\langle P' | S^\dagger a_k S | P \rangle = \int_{P''} \langle P' | S^\dagger | P'' \rangle \langle P''; k | S | P \rangle + \int_J \langle P' | S^\dagger | J \rangle \langle J; k | S | P \rangle + \mathcal{O}(\lambda^2)$$

4pt × 5pt from BH scattering

3pt × **4pt** **from BH merger**

- 4pt is now given by



down⊗out modes

in mode

$\Leftrightarrow$

down⊗out ← in

$$\langle X^-; h^- | S | \varphi^+ \rangle = \lambda \int_{r > r_S} \mathrm{d}^4 x \sqrt{-g} (h_k^-)^* (X_J^-)^* (\varphi_P^+)$$

*For simplicity, we consider massless scalar emission.
*We postulate amplitudes can be computed by on-shell action.
≃ distorted-wave Born approximation?

Classical vs. Quantum calculations

- ❑ We can solve the same problem in the classical way:

No radiation reaction

$$(\Box + m_2^2)\varphi \approx 0\,, \qquad \Box h = \frac{\lambda}{2}\varphi^2$$

Motion of lighter BH — Radiation sourced by BH

Point particle ≃ classical wavepacket

- ❑ The wave emission is computed by using the retarded Green's function.

- ❑ **The classical waveform exactly agrees with the KMOC waveform!**

$$\lambda \int_{r>r_S} \mathrm{d}^4x \sqrt{-g}(h_k^-)^*(\varphi_{P'}^+)^*\varphi_P^+ = \int_{P''} \langle P'|\, S^\dagger\, |P''\rangle \langle P''; k|\, S\, |P\rangle + \int_J \langle P'|\, S^\dagger\, |J\rangle \langle J; k|\, S\, |P\rangle$$

EoM-based computation
(integral of the source term)

Amplitude-based computation
(integral of the on-shell action)

→ **BH mergers can be computed by scattering amplitudes!**

Summary

- We initiated a program describing BH mergers by on-shell amplitudes.
- The central idea: **black holes are particles!**

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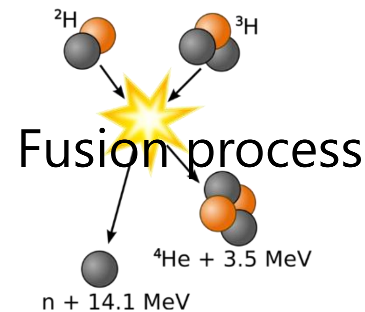


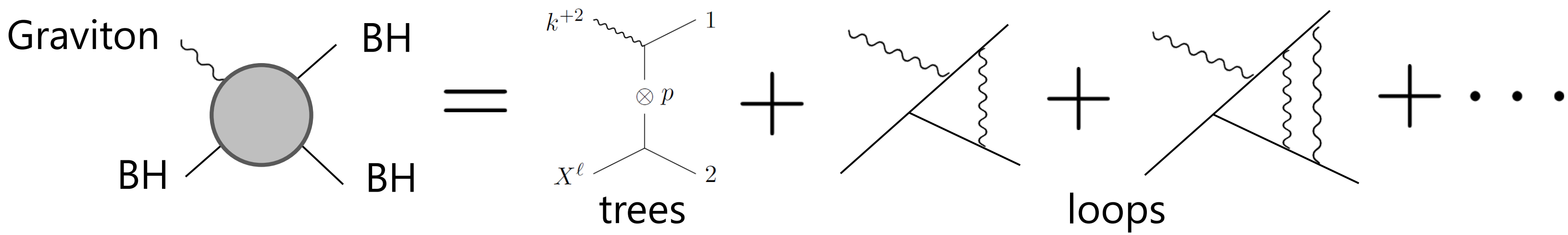
- Non-perturbative physics of merger can be packaged into massive 3pt.
- Waveforms are computed in two complementary cases.

1. Linear in G but no assumption about mass ratio (final spin)
   → all-order spin memory waveform (new prediction!).
2. Non-perturbative in G but leading in mass ratio
   → exact agreement with classical physics (proof of concept)

Discussions

- This serves as proof of concept that the entire process of BH merger can be explained by on-shell amplitudes. But, how practically?

- We need a resummation of loops for pre-merger dynamics.



- What about the ringdown after a merger? Mass-changing amplitudes! Amplitudes may naturally unify the PM and BHP computations?

Cf. Aoude+ '23; Jones+ '23; Chen+ '23; Aoude+ '24; Bautista+ '24.

- More directions: merger of Kerrs, matter collapse, Hawking radiations… **Can we understand black holes by on-shell amplitudes?**