# New understandings on low energy $\pi\pi$ and $\pi N$ scatterings

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#### 2 The sub-threshold resonance

- PKU representation
- Roy-Steiner equation analysis
- (3)  $\pi\pi$  phase shift at un-physical pion masses

**O**(N)  $\sigma$  model revisited at different masses and temperatures

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## THEORETICAL DISCUSSIONS

#### • Problems to study

Low energy properties:

 $\pi N \sigma\text{-term},$  subthreshold expansions

[C. Ditsche et. al. 2012 JHEP][Y. H. Chen et. al. 2013 PRD][Hoferichter et. al. 2016 Phys. Rept.]

- Intermediate resonances:  $\Delta(1232)$ ,  $N^*(1440)$ ,  $N^*(1535)$  · · ·
- Methods
  - Perturbative calculation

 $O(p^3)$  [J.M. Alarcón et. al. 2012 RPD];  $O(p^4)$  [Y. H. Chen et. al. 2013 PRD]

- Couple channel Lippmann-Schwinger Equation
- Dispersion technique
- Roy-Steiner equation

[C. Ditsche et. al. 2012 JHEP][Hoferichter et. al. 2016 Phys.Rept.]

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# $S_{11}$ and $P_{11}$ channels

•  $S_{11}$  channel ( $L_{2I \ 2J}$  convention):  $N^*(1535)$ 

[N. Kaiser et. al. 1995 PLB][J. Nieves et. al. 2000 PRD]

- lies above the *P* wave first resonance  $N^*(1440)$
- large couple channel effects with  $\pi N$  and  $\eta N$
- $P_{11}$  channel:  $N^*(1440)$  (Ropper resonance), various puzzles
  - low mass, large decay width, coupling to  $\sigma N$  channel... [O. Krehl et. al. 2000 PRC]
  - two-pole structure? [R. A. Arndt et. al. 1985 PRD]
  - second sheet complex branch cut in  $P_{11}$  channel?

[S. Ceci et. al. 2011 PRC]

- A method is needed to examine the relevant channels carefully and to exhume more physics behind
  - low energy
  - model independent

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## PKU REPRESENTATION

• Production representation, or PKU representation: elastic two-body scattering amplitude

$$S = \prod_{i} S_i \times S_{cut}$$

•  $S_i$ : pole terms,  $S_{cut} = e^{2i\rho(s)f(s)}$ : left-hand cuts and right hand inelastic cut – background.

$$f(s) = \frac{s}{2\pi \mathrm{i}} \int_{\mathsf{L}} ds' \frac{\mathrm{disc} f(s')}{(s'-s)s'} + \frac{s}{2\pi \mathrm{i}} \int_{\mathsf{R}'} ds' \frac{\mathrm{disc} f(s')}{(s'-s)s'}$$

•  $f(0) \equiv 0$  [Z. Y. Zhou and H. Q. Zheng 2006 NPA]

QFT version of Ning Hu representation

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# PKU REPRESENTATION

- f(s) perturbatively calculated, poles as parameters (input or fit)
- Corresponding to the Ning Hu representation in QM

[N. Hu 1948 PR] [T. Regge 1958 Nuovo Cimento]

- Advantages
  - rigorous and universal
  - $\bullet~{\rm separated}~S \rightarrow {\rm additive~phase~shift}$
  - sensitive to (not too) distant poles
  - $\bullet\,$  definite sign of the phase shifts  $\rightarrow\,$  figuring out hidden contributions
- Applications
  - the  $\pi\pi$  elastic scattering  $\rightarrow$  existence of the  $\sigma$  particle ( $f_0(500)$ ) [Z. G. Xiao and H. Q. Zheng 2001 NPA]
  - the  $\pi K$  elastic scattering  $\rightarrow \kappa$  resonance  $(K^*(800))$

[H. Q. Zheng et. al. 2004 NPA]

• resonance sum rules (narrow width approximation)

[Guo Z.H. et al., JHEP 2007 NPA]

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# Branch cut structure of partial wave $\pi N$ elastic scattering amplitude

[S. W. MacDowell 1959 PR][J. Kennedy and T. D. Spearman 1961 PR]



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## PHASE SHIFT COMPONENTS

- PKU representation  $\rightarrow$  conventionally additive phase shift
- Phase shift contributions
  - $\bullet\,$  bound states  $\rightarrow\,$  negative phase shift
  - virtual states (usually hidden !)  $\rightarrow$  positive phase shift
  - $\bullet~{\rm resonances} \rightarrow {\rm positive~phase~shift}$
  - left hand cut  $\rightarrow$  (empirically) negative phase shift (proved in quantum mechanical potential scatterings)

[T. Regge 1958 Nuovo Cimento]

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#### BACKGROUND PHASE SHIFTS

Estimated both at  $O(p^2)$  and  $O(p^3)$  level. (Tree level plotted).  $L_{2I\ 2J}$  convention,  $W = \sqrt{s}$ , data: green triangles [SAID: WI 08]



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#### BACKGROUND PHASE SHIFTS

 $L_{2I \ 2J}$  convention,  $W = \sqrt{s}$ , data: green triangles [SAID: WI 08]



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# Discrepancies in $S_{11}$ and $P_{11}$ channels

Large missing positive contributions

# $P_{11}$ :

• Analytical continuation:  $S^{II} = 1/S^{I}$ .

Second sheet poles  $\rightarrow$  first sheet zeros.

- Expansion:  $S^{\mathsf{I}} \sim a/(s M_N^2) + b + \cdots$
- ${\scriptstyle \bullet}$  Arbitrary non-zero  $b \rightarrow$  the virtual state
- $\bullet$  Perturbation calculation  $\rightarrow$  virtual state at 976 MeV; fit  $\rightarrow$  980 MeV



#### Finding $S_{11}$ hidden pole

 $O(p^2)$ [YF Wang et al., 2018 EPJC];

 $O(p^3)$ [YF Wang et al., 2019 CPC]

| $s_c \; (\text{GeV}^2)$ | Pole position (MeV) | $\chi^2/{ m d.o.f}$ |
|-------------------------|---------------------|---------------------|
| -0.08                   | 814(3) - i141(8)    | 1.46                |
| -1.00                   | 882(2) - i190(4)    | 1.31                |
| -9.00                   | 960(2) - i192(2)    | 1.14                |
| -25.0                   | 976(2) - i187(1)    | 1.14                |

• Hidden pole  $\rightarrow$  a "crazy resonance" below threshold 0.895(81)(2) - 0.164(23)(4)i GeV.

Table 2: The  $S_{11}$  hidden pole fit with different choices of  $s_c$ .



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# Positivity constraints for $\pi N$ scatterings

[Sanz-Cillero, Guo ZH, ZHQ, Eur.Phys.J.C 74 (2014) 2763]



**Fig. 1** Mandelstam plane (v, t). The Mandelstam triangle is the region contoured by the  $s = (m_N + M_\pi)^2$ ,  $u = (m_N + M_\pi)^2$  and  $t = 4M_\pi^2$  lines. Our region of study  $\mathcal{R}$  is the trapezium formed by the three previous lines and t = 0, which is marked in *red* 

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#### Accumulation of Singularities

[Li QZ and ZHQ, Commun. Theor. Phys. 74 (2022) 11, 115203]

# Singularities and Accumulation of Singularities of $\pi N$ Scattering amplitudes

July 18, 2022

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#### Abstract

It is demonstrated that for the isospin  $I = 1/2 \pi N$  scattering amplitude,  $T^{I=1/2}(s,t)$ ,  $s = (m_N^2 - m_\pi^2)^2/m_N^2$  and  $s = m_N^2 + 2m_\pi^2$  are two accumulation points of poles on the second sheet of complex *s* plane, and are hence accumulation of singularities of  $T^{I=1/2}(s,t)$ . For  $T^{I=3/2}(s,t)$ ,  $s = (m_N^2 - m_\pi^2)^2/m_N^2$  is the accumulation point of poles on the second sheet of complex *s* plane. The proof is valid up to all orders of chiral expansions.

[A. Martin, F. Cheung, 1970's]

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#### Positivity constraints for $\pi\pi$ scatterings

[Guo ZH, Zhang O, ZHQ, archive: 0911.4447]

#### Positivity constraints on LECs of $\chi PT$ lagrangian at $\mathcal{O}(p^6)$ level

Zhi-Hui Guo<sup>1</sup>, Ou Zhang<sup>2</sup> and H. Q. Zheng<sup>2</sup>

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2: Department of Physics, Peking University, Beijing 100871, P. R. China.

(Dated: June 23, 2018)

Positivity constraints on the LECs of  $O(p^6) \chi PT$  lagrangian are discussed. We demonstrate that the constraints are automatically satisfied inside the Mandelstam triangle for  $\pi\pi$  scatterings, when  $N_C$  is large. Numerical tests are made in the  $N_C = 3$  case, and it is found that these constraints are also well respected.

$$\begin{split} & \frac{d^2}{ds^2} T(\pi^0 \pi^0 \to \pi^0 \pi^0) [s,t] > 0 \,, \\ & \frac{d^2}{ds^2} T(\pi^+ \pi^0 \to \pi^+ \pi^0) [s,t] > 0 \,, \\ & \frac{d^2}{ds^2} T(\pi^+ \pi^+ \to \pi^+ \pi^+) [s,t] > 0 \,, \end{split}$$

[T. N. Pham, Tran. N. Truong, Phys. Rev. D31(1985)3027]

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# **Roy-Steiner equation**

## **Roy-Steiner Equation**

[Roy 1971]

- A coupled system of PWDRs.
- Analyticity, Unitarity, and Crossing symmetry.

Starting from the twice-subtracted fixed-t dispersion relation:

$$T(s,t,u) = \alpha(t) + s\beta(t) + \frac{s^2}{\pi} \int_{4m_{\pi}^2}^{\infty} \mathrm{d}s' \frac{\mathrm{Im}\,T(s',t,u')}{s'^2\,(s'-s)} + \frac{s^2}{\pi} \int_{-\infty}^{-t} \mathrm{d}s' \frac{\mathrm{Im}\,T(s',t,u')}{s'^2\,(s'-s)} \,\,, \tag{1}$$

A system of integral equations for the  $\pi\pi$  amplitudes:

$$\operatorname{Re} t_{J}^{I}(s) = k_{J}^{I}(s) + \sum_{I'} \sum_{J'} \int_{4m_{\pi}^{2}}^{\infty} ds' K_{JJ'}^{II'}(s', s) \operatorname{Im} t_{J'}^{I'}(s') \quad .$$
(2)

•  $K_{JJ'}^{II'}\left(s',s
ight)$  : Analytically calculable kinematic kernel functions.

• The only free parameters: S -wave scattering lengths  $a_0^0, \; a_0^2.$ 

# An important issue is the range of validity of the Roy equations.

# **Roy-Steiner equation**



# Shasanka Mohan Roy Frank Steiner

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## Lehmann ellipse constraints

$$\operatorname{Im}_{s} T(s', t) = 16\pi \sum_{l} (2l+1) \operatorname{Im}_{s} T_{l}(s') P_{l}(z(s', t)) , \qquad (3)$$

z: CMS scattering angle cosines.

The series of Legendre polynomials converges when z within the corresponding large Lehmann ellipses[Lehmann 1958].

#### Lehmann ellipses

- Focal points:  $z = \pm 1$ .
- Boundary: Touching the nearest singularity of  $\text{Im}_s T(s', t)$ .

Assuming that the scattering amplitude satisfies Mandelstam' s double spectral representation.



Figure: Domain of validity of the Roy Equation; black points denote the  $\sigma(f_0(500))$ .

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Similarly, for  $\pi K$  and  $\pi N$  scattering amplitudes:

Figure: Left: $\pi K$  systems, points denote the  $\kappa(K_0^*(700))$ ; Right: : $\pi N$  systems, points denote the  $N^*(890)$ 

Roy equation(fixed-t dispersion relation ): unfit to search for a wide resonance.

Image: A math a math

Using the hyperbolic dispersion relations, One can get the Roy-Steiner equations, which looks like:

$$\operatorname{Re} f_{l}(s) = N_{l}(s) + \sum_{l'} \int_{s_{th}} ds' K_{l,l'}(s,s') \operatorname{Im} f_{l}(s') + \sum_{J} \int_{t_{th}} dt' G_{l,J}(s,t') \operatorname{Im} g(t')_{J},$$

$$\operatorname{Re} g_{J}(t) = \tilde{N}_{J}(t) + \sum_{l'} \int_{s_{th}} ds' \tilde{K}_{J,l'}(t,s') \operatorname{Im} f_{l}(s') + \sum_{J'} \int_{t_{th}} dt' \tilde{G}_{J,J'}(t,t') \operatorname{Im} g(t')_{J}.$$
(4)

- $f_l(s)$ : s-channel PWAs;
- $g_J(t)$ : t-channel PWAs;
- N, K, G: Analytically calculable kinematic kernel functions.

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## **RS** equation analysis of $\pi N$ scatterings

RS equation has been applied to study  $\pi N$  scatterings[Ditsche:2012,Hoferichter:2015].



FIG. 2: Validity domain of the fixed-b RS representation (a = 0). The blue and green lines correspond to the boundaries in the s' and t' integrals associated with  $\rho_{st}$ , respectively. The red line corresponds to the boundaries in the s' integral associated with  $\rho_{su}$ .

[Cao XH, Li QZ, HQZ, arXive:2207.09743].

- $S_{11}$ :  $\sqrt{s} = 919 \pm 4 (162 \pm 7)i$   $N^*(920)$ .
- $P_{33}$ :  $\sqrt{s} = 1213 \pm 2 (50 \pm 3)i \quad \Delta(1232).$

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FIG. 3: Phase shifts of the s-channel PWs from our solutions (solid line) and [III] (dashed line with error bands) in the low-energy region. The deviation of P<sub>33</sub> phase shift comes from the difference between values of GWU/SAID and [III] at the matching point (W<sub>m</sub> = 1.36 GeV), and they differ by about 2°.

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# Regge Trajectory



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 $\pi\pi$  scatterings and chiral symmetry breaking

# Ground State – Vacuum



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# The linear sigma model

The linear sigma model

$$\mathcal{L} = \mathcal{L}_{s} + c\sigma , \mathcal{L}_{s} = \frac{1}{2} [(\partial_{\mu}\sigma)^{2} + (\partial_{\mu}\pi)^{2}] + \frac{m^{2}}{2} [\sigma^{2} + \pi^{2}] - \frac{\lambda}{4} [\sigma^{2} + \pi^{2}]^{2} , (1)$$

when  $c \rightarrow 0,$  the lagrangian is invariant under  $SU_L(2) \times SU_R(2)$  chiral rotations

$$\begin{aligned} \vec{\pi} &\to \vec{\pi} + \vec{\alpha} \times \vec{\pi} - \vec{\beta}\sigma ,\\ \sigma &\to \sigma + \vec{\beta} \cdot \vec{\pi} \end{aligned} \tag{2}$$

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# To be or not to be, this is a question



Figure: IJ=00 channel  $\pi\pi$  scattering phase shift data from CERN-Munich and E865.

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# $\pi\pi$ phase shift at un-physical pion masses

#### [Cao XH et al., 2303.02596 [hep-ph]]

#### Modified Roy equations; Lattice data



FIG. 6.  $\pi\pi$  phase shifts at  $m_{\pi} = 391$  MeV from Roy equation solutions: S0, P and S2 stand for the results of the IJ = 00, 11, 20channels, respectively. For the sources of the shaded error bands, see the main text for details. The lattice data are taken from Refs. [24–26, 28].

$$a_0^0 = -(3.8^{+1.1}_{-1.2})$$
,  $a_0^2 = -(0.21^{+0.02}_{-0.03})$ ,  
 $\sqrt{s_{\sigma}} = 759^{+7}_{-16}$  MeV,  $|g_{\sigma\pi\pi}| = 493^{+27}_{-46}$  MeV.

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## $\pi\pi$ phase shift at un-physical pion masses

[Cao XH et al., 2303.02596 [hep-ph]]



FIG. 7. Validity domain of extended Roy equation for  $m_{\pi} = 391$  MeV. The dashed red boundary represents the validity domain by dropping the effects of the bound state  $\sigma$ , and the blue boundary corresponds to the complete validity domain within uncertainty from the location of the  $\sigma$ . The poles in the validity domain in the second RS are from left to right, as shown in Eq. (15).

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# $\pi\pi$ phase shift at un-physical pion masses

[Cao XH et al., 2303.02596 [hep-ph]]



FIG. 9. The qualitative trajectory of the  $\sigma$  pole on the second RS of the *s* plane with varying  $m_{\pi}$ . See the main text for the meaning of the labels 'VS-I,II'.

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# $O(N) \sigma$ model revisited at different pion masses

[Lyu YL et al., 2405.11313 [hep-ph]; 2402.19243 [hep-ph]]



Figure: Large N method. Pole position with respect to different  $m_{\pi}$ .



Figure: N/D improved. Pole position with respect to different  $m_{\pi \equiv b}$   $(\equiv b)$ 

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## $O(N) \sigma$ model revisited at different temperatures

- The leading order effective potential at finite temperature T can be obtained with imaginary time formalism [M. L. Bellac, *Thermal Field Theory*, CUP 2011] by Wick rotation to Euclidean space and substituting the momentum integral with a sum over Matsubara frequencies  $\omega_n = 2\pi n T$ , i.e.  $\int \frac{d^4k}{(2\pi)^4} f(k_0, \mathbf{k}) \rightarrow iT \sum_n \int \frac{d^4k}{(2\pi)^3} f(k_0 = i\omega_n, \mathbf{k})$ .
- By minimizing the effective potential, we can obtain the gap equations for v and m<sup>2</sup><sub>π</sub> as functions of temperature [J. O. Andersen, D. Boer, and H. J. Warringa, PRD 2004] :

$$\begin{aligned} v^{2}(T) &= f_{\pi}^{2} + \frac{N}{16\pi^{2}} \left( m_{\pi}^{2} \log \frac{m_{\pi}^{2}}{M^{2}} - m_{\pi}^{2}(T) \log \frac{m_{\pi}^{2}(T)}{M^{2}} \right) - NA^{T \neq 0} \left( m_{\pi}^{2}(T), T \right) \\ \alpha &= v(T)m_{\pi}^{2}(T) \end{aligned}$$

where  $v(0) = f_{\pi}$  and  $m_{\pi}(0) = m_{\pi}$  are set to zero-temperature values.



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#### $O(N) \sigma$ model revisited at different temperatures

• The I = J = 0 channel scattering amplitude at LO 1/N expansion with finite temperature is obtained as,

$$\mathcal{T}_{00}^{T}(s) = -\frac{1}{32\pi} \frac{s - m_{\pi}^{2}(T)}{(s - m_{\pi}^{2}(T)) B^{T}(s, m_{\pi}(T), M) - v^{2}(T)/N}$$

where the finite temperature effect introduced by loop diagrams can be standardly calculated as

$$\begin{split} B^T(s, m_{\pi}(T), M) &= B\left(s, m_{\pi}(T), M\right) + B^{T \neq 0}\left(s, m_{\pi}(T), T\right) \,, \\ B^{T \neq 0}\left(s, m_{\pi}(T), T\right) &= \int_0^\infty \frac{\mathrm{d}k\,k^2}{8\pi^2\omega_k^2} n_B(\omega_k) \left(\frac{1}{E + 2\omega_k} - \frac{1}{E - 2\omega_k}\right) \,, \end{split}$$

with  $\beta = 1/T$ ,  $\omega_k = \sqrt{k^2 + m_{\pi}^2(T)}$  and  $n_B(\omega_k) = (e^{\beta \omega_k} - 1)^{-1}$  is the Bose-Einstein distribution.  $B^{T \neq 0}$  is evaluated in the CM frame and  $s = E^2$ .



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# N/D improved calculations



FIG. 16. The thermal  $\sigma$  pole trajectory obtained in N/D modified O(N) model. The zero-temperature pion mass is set as  $m_{\pi}(0) = 139$  MeV. When the temperature increases, similar to the leading order result [18],  $\sigma$  turns into two virtual states (VS III) and then becomes a bound state (BS) after VS I moves towards and across the threshold to the first Riemann sheet (RS). The left-hand cut branch point extends to  $s_L = 4m_{\pi}^2(T) - m_{\sigma,tu}^2$  due to the  $\sigma$  exchange in crossed channels. Additionally, the third virtual state pole (VS III) generated close to  $s_L$  will meet VS II on the real axis, then becoming a pair of subthreshold (Sub.) poles and going into the complex plane. Finally, the pair of subthreshold poles tends to  $s_A = m_{\pi}^2(T)$  as  $T \gg T_c$ .

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# Temperature properties of $N^*(920)$





Figure 3. Groundstate masses in the positive- and negative-parity channels at *all* temperatures, assuming the exponential decay of Eq. (4.2), in the N (left) and  $\Omega$  (right) channels.

 $T_c = 185 {
m MeV};~{
m [G.~Aarts~et~al.,~JHEP~06~(2017)]}$   $N^*(920)?!$ 

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# **Conclusions and Perspectives**

- **1** There exists a spin one half, negative parity nucleon pole.
- **②**  $\sigma$  becomes a bound state pole at large  $m_{\pi}$  and large temperature.
- Output the second se

- Thanks for patience!