Generic spinning binaries from the scattering amplitude perspective

Fei Teng

April 27, 2025

Bern, Kosmopoulos, Luna, Roiban, **FT**, 2203.06202
Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, Vines, 2308.14176
Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739

Gravitational wave: new window to probe our Universe

New physics!

- Probe dynamics of black holes
- Test general relativity
- Black hole formation
- ...

Future ground based observatories

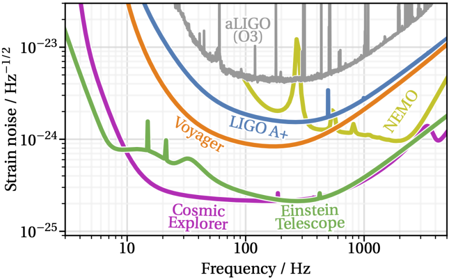
- Advanced LIGO
- Einstein Telescope
- Cosmic Explorer

Future space based observatories

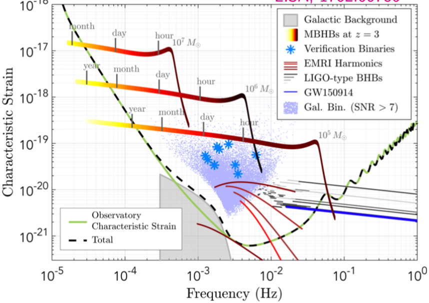
- LISA
- TaiJi
- TianQin

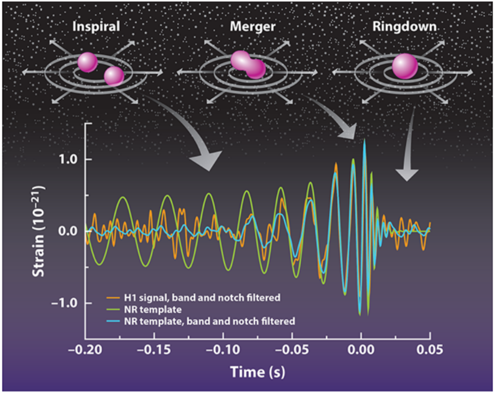
Require accurate theoretical prediction

cosmicexplorer.org



LISA, 1702.00786





Accurate theoretical prediction of the GW production puts challenges on the understanding of its source

$$H = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - \frac{G m_1 m_2}{|\mathbf{r}|} + \underbrace{\left[\# \frac{G^2 m_1 m_2 (m_1 + m_2)}{\mathbf{r}^2} + \dots \right]}_{\text{(corrections from Relativity)}}$$

I will focus on long range interactions [cf. Katsuki Aoki's talk]

How to organize perturbations?

- Post-Newtonian (PN) expansion

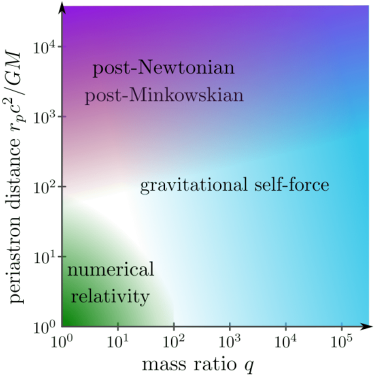
$$v^2 \sim \frac{Gm}{r} \ll 1$$

- Post-Minkowskian (PM) expansion

$$\frac{Gm}{r} \ll v^2 \sim 1$$

- Self-force expansion

$$\frac{Gm}{r} \sim v^2 \sim 1\,, \quad \frac{m_1}{m_2} \ll 1$$

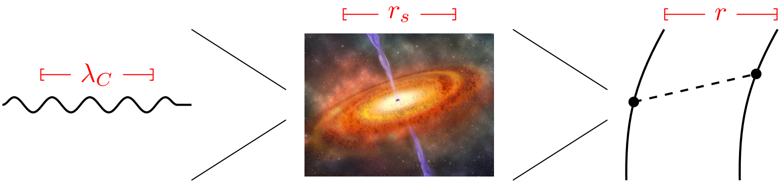


Khalil, Buonanno, Steinhoff, Vines, 2204.05047

Amplitude-based methods naturally lead to PM expansion

PM expansion is relevant to bound orbits with large eccentricity and scattering process

EFT matching using amplitudes Cheung, Rothstein, Solon, 1808.02489



Full theory: $$S_{\rm full} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R - \frac{1}{2}\int d^4x \sum_{i=1,2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - m_i^2\phi_i^2\right) + \mathcal{O}(R^2\phi^2)$$

Implemented by method of regions
Beneke, Smirnov, hep-ph/9711391

Classical limit $(q, \ell, G) \to (\hbar q, \hbar \ell, \hbar^{-1} G)$

Integrate out soft gravitons

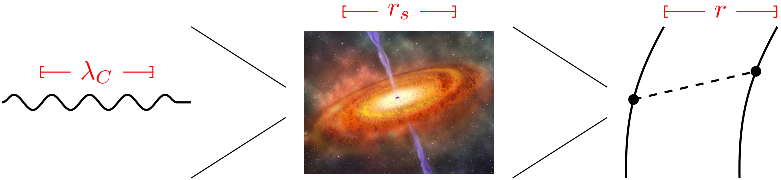
$$\mathcal{M}_{\rm QFT} = \mathcal{M}_{\rm EFT}$$

$V_{\rm PM}$ given by an ansatz

Solve $V_{\rm PM}$ by matching amplitudes

Effective theory: $$S_{\rm eff} = \int dt \left[m_1\sqrt{1-\mathbf{v}_1^2} + m_2\sqrt{1-\mathbf{v}_2^2} - V_{\rm PM}\right]$$

EFT matching using amplitudes Cheung, Rothstein, Solon, 1808.02489



Full theory: $S_{\text{full}} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R - \frac{1}{2}\int d^4x \sum_{i=1,2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - m_i^2\phi_i^2\right) + \mathcal{O}(R^2\phi^2)$

Implemented by method of regions
Beneke, Smirnov, hep-ph/9711391

Classical limit $(q, \ell, G) \to (\hbar q, \hbar\ell, \hbar^{-1}G)$

Integrate out soft gravitons

Observables $\xleftarrow[\text{Canxin Shi's talk}]{\text{Eikonal formula}}$ $\mathcal{M}_{\text{QFT}} = \mathcal{M}_{\text{EFT}}$

EOM

V_{PM} given by an ansatz

Solve V_{PM} by matching amplitudes

Effective theory: $S_{\text{eff}} = \int dt\left[m_1\sqrt{1-\mathbf{v}_1^2} + m_2\sqrt{1-\mathbf{v}_2^2} - V_{\text{PM}}\right]$

EFT matching Cheung, Rothstein, Solon, 1808.02489

- Full theory: Schwarzschild black hole $\Longrightarrow$ scalar field ϕ

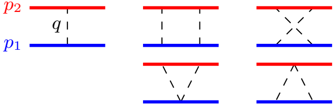
$$\mathcal{L} = \sqrt{-g}\left[-\frac{1}{16\pi G}R + \frac{1}{2}\sum_{i=1,2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - m_i^2\phi_i^2\right)\right] + \mathcal{O}(R^2\phi^2)$$

- Effective theory: potential $V(\mathbf{k},\mathbf{k}')$ given by an ansatz

$$L = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\sum_{i=1,2} a_i^\dagger(-\mathbf{k})\left(i\partial_t + \sqrt{\mathbf{k}^2+m_i^2}\right)a_i(\mathbf{k}) - \int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} V(\mathbf{k},\mathbf{k}')a_1^\dagger(\mathbf{k}')a_1(\mathbf{k})a_2^\dagger(-\mathbf{k}')a_2(\mathbf{k})\right]$$

- Solve the EFT potential by matching the full theory and EFT amplitudes order-by-order in G in the classical limit

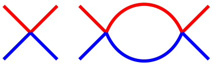
$$\mathcal{M}^{(L)} \xrightarrow{\text{classical limit}} \mathcal{M}^{(L)}_{\mathrm{EFT}}$$



$q \to \hbar q$

$\ell \to \hbar \ell$

$G \to \hbar^{-1} G$



no massless loops (quantum)
no massive contact vertices (delta-function potential)

$$\mathcal{M}^{(L)}_{\mathrm{classical}} \sim \frac{Gm^4}{\hbar^3 q^2}(Gmq)^L$$

Conservative Hamiltonian for hyperbolic trajectory

Cheung, Rothstein, Solon, 1808.02489
Bern, Cheung, Roiban, Solon, Shen, Zeng, 1901.04424
Bern, Parra-Martinez, Roiban, Ruf, Solon, Shen, Zeng, 2112.10750
Bern, Herrmann, Roiban, Ruf, Smirnov, 2406.01554

$$E_1 = \sqrt{\mathbf{p}^2 + m_1^2} \qquad E_2 = \sqrt{\mathbf{p}^2 + m_2^2}$$

$$\gamma = \frac{E_1 + E_2}{m_1 + m_2} \qquad \xi = \frac{E_1 E_2}{(E_1 + E_2)^2}$$

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \qquad \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

Hamiltonian: $H = E_1 + E_2 + \sum_{n=1}^{\infty} \frac{G^n}{|\mathbf{r}|^n} c_{n\mathrm{PM}}(\mathbf{p}^2)$

$$c_{1\mathrm{PM}} = -\frac{\nu^2 (m_1 + m_2)^2}{\gamma^2 \xi}(2\sigma^2 - 1)$$

$$c_{2\mathrm{PM}} = -\frac{\nu^2 (m_1 + m_2)^3}{\gamma^2 \xi}\left[\frac{3(5\sigma^2 - 1)}{4} - \frac{4\nu\sigma(2\sigma^2 - 1)}{\gamma\xi} + \frac{\nu^2(1-\xi)(2\sigma^2 - 1)^2}{2\gamma^3\xi^2}\right]$$

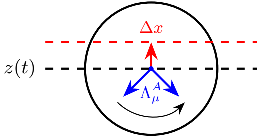
State-of-the-art: $c_{3\mathrm{PM}}$, $c_{4\mathrm{PM}}^{\mathrm{hyp}}$ and $c_{5\mathrm{PM}}^{\mathrm{hyp}\ 1\mathrm{SF}}$

- $c_{3\mathrm{PM}}$ is not known to general relativists before computed this way
- $c_{5\mathrm{PM}}$ for GR is obtained using the amplitude-worldline hybrid method

Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch, 2403.07781

See Zhengwen Liu's talk for world-line methods

World-line description of spin



Consider a rigid spinning body

$$L = -p_\mu \dot{z}^\mu + \frac{1}{2}\mathsf{S}^{\mu\nu}\Lambda_{A\mu}\frac{D\Lambda^A{}_\nu}{D\lambda} + \frac{\xi}{2}\left(p^2 - m^2 + \frac{C_2}{4}R_{\mu\nu\rho\sigma}\hat{p}^\mu S^\nu \hat{p}^\rho S^\sigma + \mathcal{O}(S^3)\right)$$
$$+ \chi_\mu \mathsf{S}^{\mu\nu}\hat{p}_\nu + \zeta_\mu(\Lambda_0^\mu - \hat{p}^\mu)$$

- We decompose the spin tensor into the rotation and boost components

$$\mathsf{S}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\hat{p}_\rho S_\sigma + (\hat{p}^\mu \mathrm{K}^\nu - \hat{p}^\nu \mathrm{K}^\mu)$$

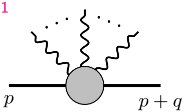
- The covariant spin supplementary condition (SSC) sets $\mathrm{K}^\mu = 0$
- Spin gauge freedom: the freedom to choose the time direction of the body-fixed frame, which also corresponds to the choice of worldline
- SSC fixes the spin gauge freedom
- Non-minimal interactions: one independent spin-induced multipole moment per order in spin

$C_2 R_{pSpS}$
$C_3 \nabla_S \tilde{R}_{pSpS}$
$C_4 \nabla_S \nabla_S R_{pSpS}\ \ldots$

Porto, gr-qc/0511061
Porto, Rothstein, gr-qc/0604099
Levi, Steinhoff, 1501.04956
Vines, Kunst, Steinhoff, Hinderer, 1601.07529

On-shell description of spin

Bern, Luna, Roiban, Shen, Zeng, 2005.03071



- On-shell spin-s states are symmetric traceless and transverse

$$\varepsilon_{a_1 a_2 \dots a_s} = \varepsilon_{(a_1 a_2 \dots a_s)} \qquad p^{a_1}\varepsilon_{a_1 a_2 \dots a_s} = \eta^{a_1 a_2}\varepsilon_{a_1 a_2 \dots a_s} = 0$$

- Classical limit $\Longrightarrow$ spin coherent state $\varepsilon^s_{a_1 a_2 \dots a_s} = \varepsilon^+_{a_1}\varepsilon^+_{a_2}\dots\varepsilon^+_{a_s}$ with large s

$$\varepsilon^s_p \cdot M^{ab} \cdot \varepsilon^s_{p+q} \sim S^{ab} \qquad (M^{ab})_{c(s)}{}^{d(s)} = -2is\delta^{[a}_{(c_1}\eta^{b](d_1}\delta^{d_2}_{c_2}\dots\delta^{d_s)}_{c_s)}$$

$$\varepsilon^s_p \cdot \{M^{ab}M^{cd}\} \cdot \varepsilon^s_{p+q} \sim S^{ab}S^{cd} \qquad S^{ab} = (1/m)\varepsilon^{abcd}p_c S_d$$

- The spin tensor satisfy covariant spin supplementary condition (SSC)

$$S^{ab}p_b = 0 \qquad (S^{ab} \text{ is boosted from rest frame } S^{ij})$$

- Transversality and covariant SSC are related
- Spin magnitude is conserved: $S^{ab}S_{ab} \sim S^a S_a \sim \boldsymbol{S}^2 = \text{const}$

How to describe interactions?

$$\nabla_\mu \phi_s = \partial_\mu \phi_s + (i/2)\omega_{\mu ab} M^{ab} \phi_s$$
$$\mathbb{S}^a = (-i/2m)\epsilon^{abcd} M_{cd} \nabla_b$$

Higher spin quantum field theory ($\phi_s \equiv \phi_{a_1 a_2 \ldots a_s}$)

$$\mathcal{L} = -\frac{1}{2}\phi_s(\nabla^2 + m^2)\phi_s + \frac{1}{8} R_{abcd}\phi_s M^{ab} M^{cd} \phi_s - \frac{C_2}{2m^2} R_{af_1bf_2} \nabla^a \phi_s \mathbb{S}^{(f_1}\mathbb{S}^{f_2)} \nabla^b \phi_s$$
$$+ \frac{D_2}{2m^2} R_{abcd} \nabla_i \phi_s \{M^{ai} M^{cd}\} \phi_s + \frac{E_2 - 2D_2}{2m^4} R_{abcd} \nabla^{(a}\nabla^{i)} \phi_s \{M^b{}_i M^d{}_j\} \nabla^{(c}\nabla^{j)} \phi_s + \mathcal{O}(M_{ab}^3)$$

We prefer to use a formalism that is uniform in s:

- Contractions of ϕ_s facilitated by M^{ab} only
- Propagator uniform in s: $i\delta^{b(s)}_{a(s)}/(p^2 - m^2)$
- Classical and large spin limit is straightforward
- There are additional lower spin ($s' < s$) states in the spectrum

While problematic for a quantum description, these additional states do not produce inconsistency in the classical limit: We seem to get a more generic spinning object

Bern, Luna, Roiban, Shen, Zeng, 2005.03071
Bern, Kosmopoulos, Luna, Roiban, **FT**, 2203.06202
Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739
See Alex Ochirov's talk for alternative higher spin formalisms

Generalized spin coherent state

Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, Vines, 2308.14176

The external state now contains lower spin components. Consider the coherent sum

$$\mathcal{E}_{\mu_1\ldots\mu_s} = \varepsilon^{(s)}_{\mu_1\ldots\mu_s} + u_{(\mu_1}\varepsilon^{(s-1)}_{\mu_2\ldots\mu_s)} + \ldots$$

Similar coherent sum was also considered in Aoude, Ochirov, 2108.01649, etc

Classical limit

$$\mathcal{E}_p \cdot M^{ab} \cdot \mathcal{E}_{p+q} \sim \mathsf{S}^{ab} \qquad \mathsf{S}^{ab} = S^{ab} + (i/m)(p^a K^b - p^b K^b)$$

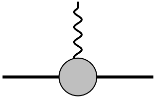
$$\mathcal{E}_p \cdot \{M^{ab} M^{cd}\} \cdot \mathcal{E}_{p+q} \sim \mathsf{S}^{ab}\mathsf{S}^{cd}$$

where K^a is identified as the boost generator, and $S^{ab} p_b = K^a p_a = 0$

- K^a emerges from the transition between spin s and lower spin states
- Consequently, $\mathsf{S}^{ab}\mathsf{S}_{ab} \sim \boldsymbol{S}^2 - \boldsymbol{K}^2$ is a still constant but $\boldsymbol{S}^2$ is not

Classical Compton amplitudes

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739



Three-point amplitude (metric):

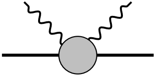
$$\mathcal{M}_3 = -(\varepsilon_1 \cdot p)^2 + \frac{(\varepsilon_1 \cdot p)\tilde{f}_1^{\mu\nu} p_\mu S_\nu}{m} - \frac{(1+C_2)(\varepsilon_1 \cdot p)^2 (k_1 \cdot S)^2}{2m^2}$$
$$- \frac{D_2(k_1 \cdot K)(\varepsilon_1 \cdot p)\tilde{f}_1^{\mu\nu} p_\mu S_\nu}{m^2} - \frac{E_2 (k_1 \cdot K)^2 (\varepsilon_1 \cdot p)^2}{2m^2}$$

- A stationary metric source by K-dependent multipole moments
- The presence of K does NOT modify the spin-induced dipole contribution
- LO matched to Rasheed-Larsen black hole Rasheed, hep-th/9505038, Larsen, hep-th/9909102
- K drops out of the amplitude when $D_2 = E_2 = 0$

$S^{\mu\nu} = (1/m)\epsilon^{\mu\nu\rho\sigma} p_\rho S_\sigma$

$f_{\mu\nu} = k_\mu \varepsilon_\nu - k_\nu \varepsilon_\mu$ and $\tilde{f}^{\mu\nu} = (i/2)\epsilon^{\mu\nu\rho\sigma} f_{\rho\sigma}$

Classical Compton amplitudes



Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739

The same property holds at four points

$$s = 2p\cdot k_1,\; t = 2k_1\cdot k_2,\; u = 2p\cdot k_2$$

$$\mathcal{M}_4 = \frac{4}{stu}\Big[\alpha^2 - \alpha\,\mathcal{O}_{(1)} + \frac{1}{2}\mathcal{O}_{(1)}^2 + C_2\,\alpha\,\mathcal{O}_{(2)} + D_2\,\alpha\left(\mathcal{O}_{(1)}\frac{(k_1+k_2)\cdot K}{m} - \mathcal{K}_{(1,1)}\right) + E_2\left(\alpha\,\mathcal{O}_{(2)}\Big|_{S\to K}\right)\Big]$$

$$\alpha = p\cdot f_1\cdot f_2\cdot p$$

$$\mathcal{O}_{(1)} = \frac{1}{m}\Big[f_2(p,k_1)\tilde{f}_1(p,S) + \frac{s}{2}\,\tilde{f}_{12}(p,S) + (1\leftrightarrow 2)\Big]$$

$$\mathcal{O}_{(2)} = \frac{1}{2m^2}\Big[t f_1(p,S) f_2(p,S) + \alpha(k_1\cdot S + k_2\cdot S)^2\Big]$$

$$\mathcal{K}_{(1,1)} = \frac{t}{2m^2}\Big[f_2(p,K)\tilde{f}_1(p,S) + f_1(p,K)\tilde{f}_2(p,S)\Big]$$

When $D_2 = E_2 = 0$, the additional dynamical freedom drops out automatically

Non-minimal interactions up to $\mathcal{O}(M_{ab}^2)$

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739

$$\nabla_\mu \phi_s = \partial_\mu \phi_s + (i/2)\omega_{\mu ab} M^{ab}\phi_s$$
$$\mathbb{S}^a = (-i/2m)\epsilon^{abcd} M_{cd}\nabla_b$$

$$\mathcal{L} = -\frac{1}{2}\phi_s(\nabla^2 + m^2)\phi_s + \frac{1}{8}R_{abcd}\phi_s M^{ab}M^{cd}\phi_s - \frac{C_2}{2m^2}R_{af_1bf_2}\nabla^a\phi_s\mathbb{S}^{(f_1}\mathbb{S}^{f_2)}\nabla^b\phi_s$$
$$+ \frac{D_2}{2m^2}R_{abcd}\nabla_i\phi_s\{M^{ai}M^{cd}\}\phi_s + \frac{E_2 - 2D_2}{2m^4}R_{abcd}\nabla^{(a}\nabla^{i)}\phi_s\{M^b{}_i M^d{}_j\}\nabla^{(c}\nabla^{j)}\phi_s$$

- The C_2-operator has an origin in the world-line formalism for neutron stars
  Porto, 0511061; Levi, Steinhoff, 1501.04956
- It is the only independent operator assuming that rest frame spin is the only dynamical degree of freedom
- The D_2- and E_2-operators supply additional $\mathcal{O}(SK)$ and $\mathcal{O}(K^2)$ interactions

$D_2 = E_2 = 0 \quad \Longrightarrow \quad$ Conventional compact object described by $H(\mathbf{r}, \mathbf{p}, \mathbf{S})$

$C_2 = D_2 = E_2 = 0 \quad \Longrightarrow \quad$ Kerr black hole

Generic values: generic compact object described by $H(\mathbf{r}, \mathbf{p}, \mathbf{S}, \mathbf{K})$

World-line Lagrangian with K

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739

The above Compton amplitudes can be reproduced by the following world-line model

$$L = -p_\mu \dot{z}^\mu + \frac{1}{2}\mathsf{S}^{\mu\nu}\Lambda_{A\mu}\frac{D\Lambda^A{}_\nu}{D\lambda} + \frac{\xi}{2}(p^2 - \mathrm{M}^2) \qquad \mathsf{S}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\hat{p}_\rho S_\sigma + \hat{p}^\mu \mathrm{K}^\nu - \hat{p}^\nu \mathrm{K}^\mu$$

$$\mathrm{M}^2 = m^2 + \left[\frac{1+C_2}{4}R_{\hat{p}S\hat{p}S} + \frac{1+D_2}{2}\widetilde{R}_{\hat{p}S\hat{p}\mathrm{K}} + \frac{1+E_2}{4}R_{\hat{p}\mathrm{K}\hat{p}\mathrm{K}} + \mathcal{O}(\mathsf{S}^3)\right] \qquad \mathrm{K}^\mu = -\mathsf{S}^{\mu\nu}\hat{p}_\nu$$

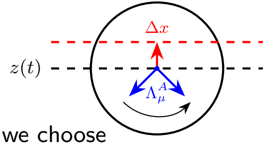
- Notably, NO SSC (for example, $\mathsf{S}^{\mu\nu}p_\nu = 0$) is imposed
- The classical Compton amplitude is identified as the ratio between the amplitude of the outgoing spherical wave and incoming plane wave Saketh, Vines, 2208.03170

$$h^{\mu\nu} = e^{-ik\cdot x}\varepsilon^\mu\varepsilon^\nu + \frac{e^{ikr-i\omega t}}{4\pi r}\mathcal{M}^{\mu\nu,\rho\sigma}_{\mathrm{Comp}}\varepsilon_\rho\varepsilon_\sigma$$

- The matching requires an identification $iK^a \equiv \mathrm{K}^a$
- Self-consistent world-line theory involving both S^{ab} and K^a exists

d'Ambrosi, Kumar, van Holten, 1501.04879

Generic spinning body with K



- The vector K^a is the displacement between the world-line we choose (center-of-spin) and the actual center-of-mass

$$J^{\mu\nu} = z^\mu p^\nu - z^\nu p^\mu + \mathsf{S}^{\mu\nu} = (z^\mu - \mathrm{K}^\mu/|p|)p^\nu - (z^\nu - \mathrm{K}^\nu/|p|) + \epsilon^{\mu\nu\rho\sigma}\hat{p}_\rho S_\sigma$$

- When $D_2 = E_2 = 0$, K drops out of the EOM under the redefinition of world-line

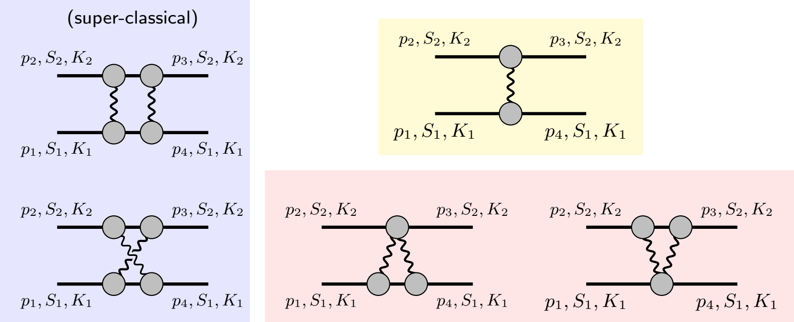
$$z'^\mu = z^\mu - \mathrm{K}^\mu/|p|$$

- One can show that the EOM is the same as that with the covariant SSC

$$L = -p_\mu \dot{z}^\mu + \frac{1}{2}\mathsf{S}^{\mu\nu}\Lambda_{A\mu}\frac{D\Lambda^A{}_\nu}{D\lambda} + \frac{\xi}{2}\left(p^2 - m^2 + \frac{C_2}{4}R_{\mu\nu\rho\sigma}\hat{p}^\mu S^\nu \hat{p}^\rho S^\sigma\right) + \chi_\mu \mathsf{S}^{\mu\nu}\hat{p}_\nu + \zeta_\mu(\Lambda^\mu_0 - \hat{p}^\mu)$$

- Emergence of spin gauge symmetry when $D_2 = E_2 = 0$
- For generic D_2 and E_2, K^μ is a genuine dynamical variable that contributes at the quadrupole level

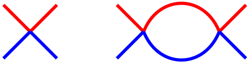
Two-body amplitudes



$$
\begin{aligned}
\mathcal{M}^{2\ \text{body}} = {} & A_0 + A_1 \boldsymbol{L}\cdot\boldsymbol{S} + A_{2,1}\boldsymbol{S}^2 + A_{2,2}\boldsymbol{K}^2 + A_{2,3}\boldsymbol{S}\cdot\boldsymbol{K} + A_{2,4}(\boldsymbol{b}\cdot\boldsymbol{S})^2 \\
& + A_{2,5}(\boldsymbol{p}\cdot\boldsymbol{S})^2 + A_{2,6}(\boldsymbol{b}\cdot\boldsymbol{K})^2 + A_{2,7}(\boldsymbol{p}\cdot\boldsymbol{K})^2 + A_{2,8}(\boldsymbol{b}\cdot\boldsymbol{S})(\boldsymbol{p}\cdot\boldsymbol{S}) \\
& + A_{2,9}(\boldsymbol{L}\cdot\boldsymbol{S})(\boldsymbol{b}\cdot\boldsymbol{K}) + A_{2,10}(\boldsymbol{L}\cdot\boldsymbol{S})(\boldsymbol{p}\cdot\boldsymbol{K}) + A_{2,11}(\boldsymbol{b}\cdot\boldsymbol{S})(\boldsymbol{L}\cdot\boldsymbol{K}) \\
& + A_{2,12}(\boldsymbol{p}\cdot\boldsymbol{S})(\boldsymbol{L}\cdot\boldsymbol{K}) + A_{2,13}(\boldsymbol{b}\cdot\boldsymbol{K})(\boldsymbol{p}\cdot\boldsymbol{K})
\end{aligned}
$$

Effective Hamiltonian through matching

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739



Consider canonical spin in the COM frame

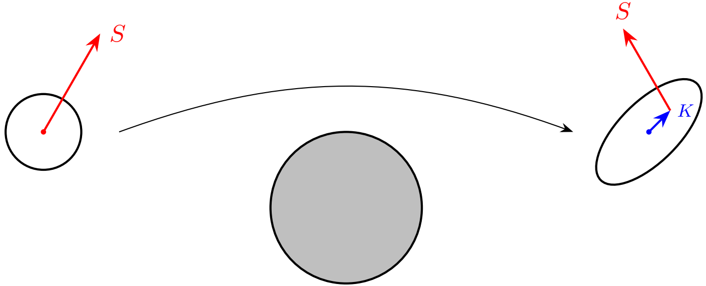
$$H=\sqrt{\mathbf{p}^2+m_1^2}+\sqrt{\mathbf{p}^2+m_2^2}+\sum_a\sum_{n=1}^{\infty}\left(\frac{G}{|\mathbf{r}|}\right)^n c_n^a(\mathbf{p}^2)\Sigma_a$$

where the operators Σ_a takes value in

$$\begin{array}{ccc}
\mathbf{1} & \left((\mathbf{r}\times\mathbf{p})\cdot\mathbf{S}\right)/\mathbf{r}^2 & \left(\mathbf{r}\cdot\mathbf{K}\right)/\mathbf{r}^2 \\
\left(\mathbf{r}\cdot\mathbf{S}\right)^2/\mathbf{r}^4 & \left(\mathbf{r}\cdot\mathbf{K}\right)\left((\mathbf{r}\times\mathbf{p})\cdot\mathbf{S}\right)/\mathbf{r}^4 & \left(\mathbf{r}\cdot\mathbf{K}\right)^2/\mathbf{r}^4 \\
\mathbf{S}^2/\mathbf{r}^2 & \left(\mathbf{K}\cdot(\mathbf{p}\times\mathbf{S})\right)/\mathbf{r}^2 & \mathbf{K}^2/\mathbf{r}^2 \\
\left(\mathbf{p}\cdot\mathbf{S}\right)^2/\mathbf{r}^2 & \left(\mathbf{r}\cdot\mathbf{S}\right)\left((\mathbf{r}\times\mathbf{K})\cdot\mathbf{p}\right)/\mathbf{r}^4 & \left(\mathbf{p}\cdot\mathbf{K}\right)^2/\mathbf{r}^2
\end{array}$$

- c_0^a matches to the tree level amplitude at $\mathcal{O}(G)$
- Iteration of c_0^a should agree exactly with the super-classical box coefficients at $\mathcal{O}(G^2)$
- c_1^a matches to the triangle coefficients at $\mathcal{O}(G^2)$ order of V
- The coefficient of $\left(\mathbf{r}\cdot\mathbf{K}\right)/\mathbf{r}^2$ vanishes identically
- All the c_n^a coefficients are local in $\mathbf{p}^2$

Generic spinning body with K

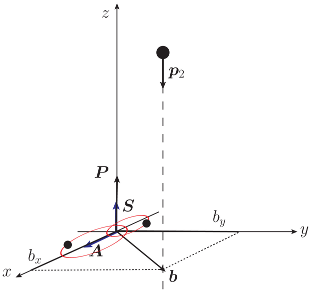


An additional conservative gapless degree of freedom

Scattering off a Newtonian bound state

$$
\begin{aligned}
H &= \frac{\boldsymbol{p}_2^2}{2m_2} + \frac{\boldsymbol{P}^2}{2m_1} + H_0(\boldsymbol{p},\boldsymbol{r}) - \frac{G m_{B1} m_2}{|\boldsymbol{R} + \frac{m_{B2}}{m_1}\boldsymbol{r}|} - \frac{G m_{B2} m_2}{|\boldsymbol{R} - \frac{m_{B1}}{m_1}\boldsymbol{r}|} \\
&= \frac{\boldsymbol{p}_2^2}{2m_2} + \frac{\boldsymbol{P}^2}{2m_1} + H_0(\boldsymbol{p},\boldsymbol{r}) - \frac{G m_1 m_2}{|\boldsymbol{R}|} - \frac{3G\mu_B m_2}{2|\boldsymbol{R}|^5} \underbrace{\left((\boldsymbol{r}\cdot\boldsymbol{R})^2 - \frac{1}{3}|\boldsymbol{r}|^2|\boldsymbol{R}|^2 \right)}_{Q_{ij}(\boldsymbol{r})Q^{ij}(\boldsymbol{R})} + \dots
\end{aligned}
$$

where $m_1 = m_{B1} + m_{B2}$ and $\mu_B = m_{B1} m_{B2}/m_1$



Scattering off a Newtonian bound state

$$\begin{aligned}\mathcal{A}_{\mathrm{i}\to\mathrm{f}} &= \int_{-\infty}^{+\infty} dt e^{i(\mathsf{E}_{\mathrm{f}}^B - \mathsf{E}_{\mathrm{i}}^B)t} \left\langle \mathrm{i} \left| \frac{3G\mu_B m_2}{2|\boldsymbol{R}|^5} \left((\boldsymbol{r}\cdot\boldsymbol{R})^2 - \frac{1}{3}|\boldsymbol{r}|^2|\boldsymbol{R}|^2 \right) \right| \mathrm{f} \right\rangle \\ &= \frac{3G\mu_B m_2 r_{\mathrm{cl},n}^2}{2|\boldsymbol{b}|^2 v_0} \left[\frac{2(\boldsymbol{b}\cdot\boldsymbol{A})^2}{|\boldsymbol{b}|^2} - |\boldsymbol{A}|^2 \right]\end{aligned}$$

- Trajectory: $\boldsymbol{R} = (b_x, b_y, -v_0 t)$
- Initial and final state have the same energy; otherwise exponentially suppressed
- Use elliptical orbit coherent state with $bv_0^2 \gg r_{\mathrm{cl},n}$ Bhaumik, Dutta-Roy, Ghosh, 1986

$$\begin{aligned}\langle\alpha|x|\alpha\rangle &= r_{\mathrm{cl},n}\Big[\cos(2\omega_{\mathrm{cl}}t) + \sin(2\chi)\Big] \\ \langle\alpha|y|\alpha\rangle &= r_{\mathrm{cl},n}\sin(2\omega_{\mathrm{cl}}t)\cos(2\chi) \\ \langle\alpha|z|\alpha\rangle &= 0\end{aligned}$$

- Laplace-Runge-Lenz vector $\boldsymbol{A} = \sin(2\chi)\hat{\boldsymbol{x}}$

Scattering off a Newtonian bound state

$$\mathcal{M}^{2\text{ body}} \sim A_{2,1}\boldsymbol{K}^2 + A_{2,6}(\boldsymbol{b}\cdot\boldsymbol{K})^2$$

Match to the field theory amplitude:

- Spin $\Leftrightarrow$ bound system total orbital angular momentum
- Due to the geometric configuration, the spin does not appear in $\mathcal{A}_{\mathrm{i}\to\mathrm{f}}$
- $\boldsymbol{K}$-vector $\Leftrightarrow$ Laplace-Runge-Lenz vector

$$\boldsymbol{K} = i\,Gm_1^2\frac{\mu_B}{m_1}\sqrt{\frac{\mu_B}{2|\mathsf{E}_{\mathrm{i}}^B|}}\,\boldsymbol{A}$$

- Wilson coefficient

$$E_2^{\text{bound 2-body}} = \frac{3|\mathsf{E}_{\mathrm{i}}^B|m_1}{\mu_B^2}(m_1 r_{\mathrm{cl},n})^2$$

$$\mathcal{L} \sim \frac{E_2}{2m^4}R_{abcd}\nabla^{(a}\nabla^{i)}\phi_s\{M^b{}_iM^d{}_j\}\nabla^{(c}\nabla^{j)}\phi_s$$

Summary

- Framework for effective description of generic spinning binaries
  - Field theory: transition between fields with different s
  - World-line: introduce additional dynamical variables
  - Allow more Wilson coefficients compared to the conventional formalism
- Equivalence of the field-theory and world-line description
  - Consider a world-line model involving spin S and another dynamical DOF K
  - Demonstrate by matching classical Compton amplitudes
  - Field theory and world-line agree at $\mathcal{O}(S^3)$
- The presence of K does not affect spin-induced dipole moment
- K drops out when additional Wilson coefficients take special values
  - No constraints needed [can use naive kinetic term $\phi_s(\nabla^2 + m^2)\phi_s$ for classical physics]
  - Simplify calculation [propagators and vertices uniform in s; straightforward large s limit]

Discussion

- Effective Hamiltonian at two-loop $\mathcal{O}(S^3)$ and beyond
- Efficient organization of loop integrands involving spin
- Better understanding of tidal operators at S^4 and beyond
- Phenomenology of generic compact bodies?

Thanks for listening!