### **Effective Field Theories, Gravity and Cosmology**

## **Aspects of Ultra-slow-roll Inflation**

Institute of Theoretical Physics, Chinese Academy of Sciences



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### HIAS, April 25, 2025



## CONTENT

- Introduction: PBH, IGW, and USR
- Non-Gaussianity in USR
- NLO of gradient expansion in USR
- Application: Phenomenology in mHz and nHz GW













### "Source" of induced GWs



Domenech, 2402.17388, PBH book, Chapter 18

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### **PBH-IGW crosscheck**

Saito & Yokoyama 0812.4339; 0912.5317 Bugaev & Klimai 0908.0664; 1012.4697







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### **PBH-IGW crosscheck**

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### Including non-Gaussianity





### **Every step is linear/Gaussian:**

- (1) Linear Poisson equation.
- (2) Gaussian PDF  $\mathbb{P}(\mathscr{R})$  gives Gauss PDF  $\mathbb{P}(\delta_{\mathscr{P}})$ :  $\mathbb{P}(\mathcal{R})d\mathcal{R} = \mathbb{P}(\delta_{\ell})d\delta_{\ell}$
- (3) Critical density contrast  $\delta_{\ell,cr}$  given by HYK limit (Harada, Yoo, Kohri, 1309.4201).
- (4) Window function.

## (Simplest) Press-Schechter





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- (4) Window function.

## (Simplest) Press-Schechter





### **Non-Gaussianity must be taken into account:**

- (1) Use compaction function  $\mathscr{C}$  which nonlinearly depends on  $\mathscr{R}$ . (Harada et al 1503.03934; De Luca et al 1904.00970.)
- (2) Primordial non-Gaussianity of  $\mathscr{R}$ .
- (3)  $\mathscr{C}_{\rm cr}$  depends on profile. (Musco 1809.02127; Escrivà et al 1907.13311)

## Why non-Gaussianity?



- Instead, in peak theory, BBKS gives the profile of a local peak, from which the critical value of  $\overline{\mathscr{C}}_c$  can be calculated analytically. Then we transfer it to the critical value of the Laplacian of the curvature perturbation,  $\mu_2$ .
- The statistic quantities are  $\mu_2$  and its dispersion,  $\mu_4$ .
- The PBH mass function is then

$$\beta(M) = \int_{\mu_2 \ge \mu_{2,th}} d\mu_2 d\mu_4 \cdot n_{\text{peak}}(\mu_2(M,\mu_4),\mu_4) \left| \frac{d\ln M}{d\mu_2} \right|^{-1} M(\mu_2,\mu_4)$$

## Peak Theory

Kitajima et al, 2109.00791 SP, Sasaki, Takhistov, Jianing Wang, 2501.00295













Sasaki and Stewart, astro-ph/9507001 Lyth, Malik, Sasaki, astro-ph/0411220







Sasaki and Stewart, astro-ph/9507001 Lyth, Malik, Sasaki, astro-ph/0411220



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Cruces, SP, Sasaki, to appear



### **Gaussian Curvature Perturbation**



Stewart and Sasaki, astro-ph/9507001 Lyth and Roquigez, astro-ph/0504045



### **Gaussian Curvature Perturbation**



$$\begin{aligned} \mathcal{R} &= \delta N \approx N_{,\varphi} \delta \varphi + \frac{1}{2} N_{,\varphi\varphi} \delta \varphi^2 + \cdots \\ &= -H \frac{\delta \varphi}{\dot{\varphi}} + \frac{3}{5} f_{\rm NL} \left( -H \frac{\delta \varphi}{\dot{\varphi}} \right)^2 \cdot \end{aligned}$$

Stewart and Sasaki, astro-ph/9507001 Lyth and Roquigez, astro-ph/0504045 Maldacena, astro-ph/0210603



. . .

### Gaussian Curvature Perturbation



 $\begin{aligned} \mathcal{R} &= \delta N \approx N_{,\varphi} \delta \varphi + \frac{1}{2} N_{,\varphi\varphi} \delta \varphi^2 + \cdots \\ &= -H \frac{\delta \varphi}{\dot{\varphi}} + \frac{3}{5} f_{\rm NL} \left( -H \frac{\delta \varphi}{\dot{\varphi}} \right)^2 \cdots \end{aligned}$  $\mathcal{O}(\epsilon,\eta)$ 

Stewart and Sasaki, astro-ph/9507001 Lyth and Roquigez, astro-ph/0504045 Maldacena, astro-ph/0210603



## Ultra-slow-roll Inflation



Starobinski, JETP Lett. 55, 489 Byrnes, Cole, Patil, 1811.11158 Cole, Gow, Byrnes, Patil, 2204.07573 SP & Jianing Wang, 2209.14183





# Logarithmic Relation in the USR inflation

 $\mathcal{R} = \delta N = N_{,\varphi} \delta \varphi + \frac{1}{2} N_{,\varphi\varphi} \delta \varphi^2 + \cdots$ 

Φ



# Logarithmic Relation in the USR inflation

 $\begin{aligned} \mathscr{R} &= \delta N = N_{,\varphi} \delta \varphi + \frac{1}{2} N_{,\varphi\varphi} \delta \varphi^2 + \cdots \\ &+ N_{,\pi} \delta \pi + \frac{1}{2} N_{,\pi\pi} \delta \pi^2 + \cdots \end{aligned}$ 

(p)





## Logarithmic Relation in the USR inflation

$$\mathcal{R} = \delta N = N_{,\varphi}\delta\varphi + \frac{1}{2}N_{,\varphi\varphi}\delta\varphi^{2} + \cdots + N_{,\pi}\delta\pi + \frac{1}{2}N_{,\pi\pi}\delta\pi^{2} + \cdots$$
(For USR) 
$$= -\frac{1}{3}\ln\left(1 + \frac{3\delta\varphi}{\pi_{*}}\right).$$

$$\left(f_{\rm NL} = \frac{5}{2}, g_{\rm NL} = \frac{25}{3}, \cdots\right)$$

Namjoo, Firouzjahi, Sasaki, 1210.3692 Chen, Firouzjahi, Komatsu, Namjoo, Sasaki, 1308.5341 Cai, Chen, Namjoo, Sasaki, Wang, Wang, 1712.09998 Biagetti, Franciolini, Kehagias, Riotto, 1804.07124 Passaglia, Hu, Motohashi, 1812.08243 Also verified by stochastic approach, see e.g. Pattison et al 2101.05741



### Ultra-slow-roll inflation 0.100 $V(\varphi)$ Next-to-leading order in gradient expansion 0.001 $k^4$ 10<sup>-5</sup> $\pi$ 10<sup>-7</sup> $\epsilon \ll 1, \eta = -6$ 0.10 $10^{-11}$ Φ $\varphi_*$



# Ultra-slow-roll inflation $V(\varphi)$ ${\cal \pi}$ $\varphi$ $arphi_*$

$$\frac{\mathrm{d}^2 \varphi}{\mathrm{d}N^2} - 3 \frac{\mathrm{d}\varphi}{\mathrm{d}N} = 0 \qquad \qquad N = \int_{t_*}^t H \mathrm{d}t$$

$$\varphi(N) = \varphi_* + \frac{\pi_*}{3} \left( 1 - e^{3N} \right)$$
$$\pi(N) \equiv -\frac{\mathrm{d}\varphi}{\mathrm{d}N} = \pi_* e^{3N}$$

$$N = -\frac{1}{3}\ln\frac{\pi_*}{\pi}$$

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## Ultra-slow-roll inflation

In the "fiducial" patch

$$N = -\frac{1}{3} \ln \frac{\pi_*}{\pi}$$



## Ultra-slow-roll inflation

In the "fiducial" patch

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In a perturbed patch

$$\tilde{N} = -\frac{1}{3} \ln \frac{\tilde{\pi}_*}{\tilde{\pi}}$$

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## Ultra-slow-roll inflation

In the "fiducial" patch

$$N = -\frac{1}{3}\ln\frac{\pi_*}{\pi}$$

In a perturbed patch

$$\tilde{N} = -\frac{1}{3} \ln \frac{\tilde{\pi}_*}{\tilde{\pi}}$$

By  $\delta N$  formalism, the curvature perturbation is

$$\mathcal{R} = \delta N = \tilde{N} - N = \frac{1}{3} \ln \frac{\pi}{\pi} \frac{\pi_*}{\tilde{\pi}_*}$$
$$= \frac{1}{3} \ln \left( 1 - \frac{\delta \pi}{\pi} \right) - \frac{1}{3} \ln \left( 1 - \frac{\delta \pi_*}{\pi_*} \right)$$



## Ultra-slow-roll inflation



$$\mathcal{R} \approx -\frac{1}{3} \ln \left( 1 - \frac{\delta \pi_*}{\pi_*} \right)$$
$$\left( f_{\rm NL} = \frac{5}{2}, \quad g_{\rm NL} = -\frac{25}{3}, \dots \right)$$

Namjoo, Firouzjahi, Sasaki, 1210.3692 Chen, Firouzjahi, Komatsu, Namjoo, Sasaki, 1308.5341 Cai, Chen, Namjoo, Sasaki, Wang, Wang, 1712.09998 Biagetti, Franciolini, Kehagias, Riotto, 1804.07124 Passaglia, Hu, Motohashi, 1812.08243 SP and Sasaki, 2211.13932 SP, 2404.06151 Relation with stochastic approach, see e.g. Jackson et al 2410.13683, Cruces et al 2410.17987



## Ultra-slow-roll inflation



$$\mathcal{R} = -\frac{1}{3} \ln\left(1 - \frac{\delta \pi_*}{\pi_*}\right)$$
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$$\ln\left(1+\frac{\delta\pi+\lambda_{\mp}\delta\varphi}{\pi+\lambda_{\mp}\varphi}\right)-\frac{1}{\lambda_{\pm}}\ln\left(1+\frac{\delta\pi_{\ast}}{\pi_{\ast}+\lambda_{\mp}\varphi_{\ast}}\right)$$

$$-\ln\left(1+\frac{\delta\pi_{*}}{\pi_{*}+\tilde{\lambda}_{\mp}(\varphi_{*}-\varphi_{m})}\right)-\frac{1}{\tilde{\lambda}_{\pm}}\ln\left(1+\frac{\delta\pi_{f}}{\pi_{f}+\tilde{\lambda}_{\mp}(\varphi_{f}-\varphi_{m})}\right)$$





## Application to USR $(\lambda_{-} = 0, \quad \lambda_{+} = 3)$ $(\tilde{\lambda}_{-} = \tilde{\eta}, \quad \tilde{\lambda}_{+} = 3 - \tilde{\eta})$



$$-\ln\left(1+\frac{\delta\pi_*}{\pi_*}\right) + \frac{1}{\tilde{\eta}}\ln\left(1+\frac{\delta\pi_*}{\pi_*+(3-\tilde{\eta})(\varphi_*-\varphi_m)}\right)$$

Smooth transition  $\tilde{\eta}(\varphi_* - \varphi_m) \ll \pi_*$ 

•  $\mathscr{R}$  is contributed by both the USR stage and the later slow-roll stage. The former is highly non-Gaussian (i.e. exp tail,  $f_{\rm NL} = 5/2$ ),

- The h factor ( $h \equiv -6\sqrt{\epsilon_V/\epsilon_*}$ ) defined in Cai et al 2017 is the ratio between the slow-roll velocity and the end-of-USR velocity.
- When  $h \sim \mathcal{O}(1)$ , both of the logarithms are of the same order.



# USR: Sharp end



## **USR: Smooth end**



SP and Sasaki, 2211.13932 SP, 2404.06151 c.f. Cai et al, 1712.09998



## USR

 $(\lambda_{-} = 0, \quad \lambda_{+} = 3)$  $(\tilde{\lambda}_{-} = \tilde{\eta}, \quad \tilde{\lambda}_{+} = 3 - \tilde{\eta})$ 

Pattison et al., 2101.05741 Ballesteros et al 2406.02417 Cruces, SP, Sasaki, in prep.

Sharp transition will make the separate universe approach (thus  $\delta N$  formalism) invalid transiently.

> Domenech et al., 2309.05750 Jackson et al., 2311.03281 Artigas, SP, Tanaka, in prep.





## Logarithmic Duality

 $(f_{NL} = -\frac{5}{6}\lambda_{-})$   $\mathscr{R} = -H\frac{\delta\varphi}{\dot{\phi}} + \frac{3}{5}f_{NL}\left(-H\frac{\delta\varphi}{\dot{\phi}}\right)^{2}$ 

**Slow-roll** inflation Stewart and Sasaki, 1995 Lyth and Roquigez, 2005

$$\mathscr{R} = -\mu \ln \left( 1 - \frac{\mathscr{R}_g}{\mu} \right)$$

Constant-roll Atal, Garriga, Marcos-Caballero, 1905.13202 Atal, Cid, Escrivà, Garriga, 1908.11357 Escrivà, Atal, Garriga, 2306.09990 Inui, Motohashi, SP, et al, 2409.13500 Ma, Wang, et al, in prep

 $\mathscr{R}(\delta\varphi,\delta\pi)$ 

Ultra-slow-roll Namjoo, Firouzjahi, Sasaki, 1210.3692 Cai, Chen, et al 1712.09998 Biagetti et al 1804.07124 Passaglia et al 1812.08243

SP and Sasaki, 2211.13932

$$\mathcal{R} = \frac{1}{\lambda_{\pm}} \ln \left( 1 + \frac{\delta \pi + \lambda_{\mp} \delta \varphi}{\pi + \lambda_{\mp} \varphi} \right) - \frac{1}{\lambda_{\pm}} \ln \left( 1 + \frac{\delta \pi_{*}}{\pi_{*} + \lambda_{\mp} \varphi_{*}} \right) + \cdots$$

 $\mathscr{R} = -\frac{1}{2}\ln\left(f(\mathscr{R}_{G})\right)$ Extensions,

Kawaguchi et al, 2305.18140 SP and Yokoyama, in prep

$$\mathscr{R} = \frac{2}{3}\ln\left(1+\delta\right)$$

Curvaton scenario, SP and Sasaki, 2112.12680 Ferrante et al, 2211.01728 Hooper et al. 2308.00756





## EFT and Logarithmic Duality

Lyth and Roquigez, 2005 Maldacena, astro-ph/0210603 Koyama, 1002.0600

 $\mathscr{R}(\delta \varphi, \delta \pi)$ 

This is not enough to recover the consistency relation, unless the contribution from the bulk is considered.

SP and Sasaki, 2211.13932

## We need the EFT of inflation!

Frouzjahi et al., 2502.09481,10287





SP, Sasaki, PRD 108, L101301



## Example 2: Curvaton

 $\mathcal{A}$ 

$$e^{4\zeta} - \frac{4r}{3+r} \left(1 + \frac{\delta\chi}{\chi}\right)^2 e^{\zeta} + \frac{3r-3}{3+r} = 0$$
  
$$\zeta = \zeta(\delta\chi/\chi) \longrightarrow \begin{cases} \frac{r}{3} \left[2\frac{\delta\chi}{\chi} + \left(\frac{\delta\chi}{\chi}\right)^2\right] & \text{when } r \ll 1 \\ \frac{2}{3}\ln\left|1 + \frac{\delta\chi}{\chi}\right| & \text{when } r \sim 1 \end{cases}$$



•  $\zeta(\delta\chi)$  degenerates to logarithmic when the curvaton dominates.



SP, Sasaki, PRD 108, L101301



# Example 2: Curvaton

 $\mathcal{A}$ 

•  $\zeta(\delta\chi)$  degenerates to a logarithmic relation  $(f_{\rm NL} = -5/4)$  when the curvaton dominates.





# **Probability Distribution Function**

For the simplest single-logarithm case:  $\Re \equiv \delta N = \frac{1}{\lambda}$ 



$$\ln\left(1 + \frac{\delta\pi + \lambda_{+}\delta\varphi}{\pi + \lambda_{+}\varphi}\right)$$
  

$$= P(\mathcal{R})d\mathcal{R} = P(\delta\varphi)d\delta\varphi$$
  

$$= P(\delta\varphi)d\delta\varphi$$
  

$$= \left[-\frac{\varphi^{2}}{2\sigma_{\delta\varphi}^{2}}\left(e^{\lambda_{-}\mathcal{R}} - 1\right)^{2}\right]$$





# **Probability Distribution Function**

For the simplest single-logarithm case:



exponential tail



## PBH abundance







exponential tail  $P(\mathcal{R}) \sim e^{-3\mathcal{R}}$ 

## Gumbel tail $P(\mathscr{R}) \sim \exp\left(-c^2 e^{2\lambda_- \mathscr{R}}\right)$

credit: Ao Wang



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# Separate Universe



Leach, Sasaki, Wands, Liddle, astro-ph/0101406 SP and Jianing Wang, 2209.14183

Domenech et al., 2309.05750 Jackson et al., 2311.03281

## Ultra-slow-roll inflation 0.100 $V(\varphi)$ Next-to-leading order in gradient expansion 0.001 $k^4$ 10<sup>-5</sup> $10^{-7}$ $\pi$ $\epsilon \ll 1, \eta = -6$ 0.10 $10^{-11}$ Φ $\varphi_*$





Artigas, SP, Tanaka, 2408.09964

















Artigas, SP, Tanaka, 2408.09964



## CONTENT

- Introduction
- PBH formation and observational constraints
- Non-Gaussianity and their impact on PBHs
- Prediction in mHz and nHz



- Primordial NG must be taken into account when calculating PBH abundance
- When fixing PBH abundance, NG impact on SGWB is mild
- LISA/Taiji/TianQin can probe the induced GW when PBH=DM

# PBH as DM

## Cai, SP and Sasaki, 1810.11000 SP, 2404.06151

- Quadratic: Cai, SP, Sasaki 1810.11000; Unal 1811.09151 •
- Higher orders: *f*<sub>NI</sub> : Adshead+ 2105.01659; Abe+ 2209.13891; Garcia-Saenz+ 2207.14267. g<sub>NL</sub>: Yuan+, 2308.07155; Li+, 2309.07792. *i*<sub>NI</sub> : Perna+, 2403.06962
- When fixing PBH abundance, NG impact on SGWB is mild.







 $\Omega_{GW,peak}h^2 \gtrsim 8.5 \times 10^{-11} > LISA, Taiji, TianQin, BBO, DECIGO, \cdots$ 

When PBH are all the dark matter, LISA/Taiji/TianQin/BBO/DECIGO can probe the induced GW signal, which is relatively robust against non-Gaussianity.



## PBH as DM





## PBH as DM

Inui, Joana, Motohashi, SP, Tada, Yokoyama, 2411.07647





# Detectability





Luo et al., 2502.20138 Hong, Kuroyanagi, Pi, Wang, Zhang, to appear



# **Application: nHz SGWB**



NANOGrav, 2306.16219



# **Application: nHz SGWB**





## **Crosscheck by PBH and IGW**












### IGW as nHz SGWB



 $\mathscr{P}_{\mathscr{R}} = A\delta(\ln k - \ln k_*)$ 

monochromatic



## IGW as nHz SGWB

### How to solve the PBH overproduction

- (1) Use more conservative method to calculate. (Inomata et al 2306.17834; Iovino et al 2406.20089)
- (2) Suppress PBH abundance by increase the threshold, usually by changing the equation-of-state. (Domenech and SP, 2010.03976; Domenech, SP, et al, 2402.18965)
- (3) Suppress PBH abundance by negative non-Gaussianity, where curvaton scenario is the only known model. (SP and Sasaki 2112.12680; Franciolini et al. 2306.17149)















## NG and nHz GW

### SP and Sasaki, 2112.12680 Ferrante et al, 2211.01728



$$= \zeta(\delta\chi/\chi) \longrightarrow \begin{cases} \frac{r}{3} \left[ 2\frac{\delta\chi}{\chi} + \left(\frac{\delta\chi}{\chi}\right)^2 \right] & \text{when} \\ \frac{2}{3} \ln \left| 1 + \frac{\delta\chi}{\chi} \right| & \text{when} \end{cases}$$

## **PBH** in logarithmic $\mathcal{R}$





## PBH in logarithmic *R*







## PBH in logarithmic *R*







# **PTA** implication



- PBH overproduction is a serious problem, mainly because we use state-od-art peak theory.
- Considering bubble channel, more PBH will form.
  - Negative non-Gaussianity up to  $\gamma \ge -4$  can not help.
  - Considering higher-order contribution to induced GW, the required  $\mathscr{A}_{g}$  is smaller, which may alleviate the tension. However we do not have a systematic method of calculating.

Inui, Joana, Motohashi, SP, Tada, Yokoyama, 2411.07647

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### Conclusion

- Ultra-slow-roll inflation is effective to enhance the power spectrum.
- which is important in calculating the PBH abundance.
- encoded in the extended  $\delta N$  formalism.

• By using the  $\delta N$  formalism, we can derive the nonlinear curvature perturbation,

• The infrared scaling, "steepest growth", reflects the gradient terms, which can be