Bosonic Fortuity from Structure of Loop Space at Finite N

Effective Field Theories, Gravity and Cosmology
Hangzhou Institute for Advnaced Study

Robert de Mello Koch

School of Science
Huzhou University

Based on: RdMK and Antal Jevicki, arXiv:2503.20097

April 26, 2025

Counting polynomials

How many degree d polynomials can be constructed using x and y?

$$\begin{array}{ccc} \text{degree } 0 & 1 & 1 \\ \text{degree } 1 & 2 & x,\ y \\ \text{degree } 2 & 3 & x^2,\ xy,\ y^2 \\ \text{degree } 3 & 4 & x^3,\ x^2y,\ xy^2,\ y^3 \\ \vdots & \vdots & \vdots \end{array}$$

We can generate all of these polynomials, exactly once, as follows:

$$\begin{aligned} \frac{1}{1-x}\frac{1}{1-y} &= (1+x+x^2+\cdots)(1+y+y^2+\cdots) \\ &= 1 \quad +x+y \quad +x^2+xy+y^2 \quad +\cdots \end{aligned}$$

To count set $x=t=y$ to obtain

$$\left(\frac{1}{1-t}\right)^2 = 1+2t+3t^2+4t^3+\cdots$$

The function

$$H(t) = \frac{1}{(1-t)^2} = \sum_{n=0}^{\infty} c_n t^n$$

Is called a *Hilbert series*. The integer c_n counts how many degree n polynomials can be freely generated from the 2 variables x and y.

More Counting

How many degree d polynomials can be constructed using x and y freely and using w^2 and v^3 each at most once? (i.e. we might not use either, or we might use w^2 or v^3, but we don't use both, or either more than once)

We can generate all of these polynomials, exactly once, as follows:

$$1 \times \frac{1}{1-x}\frac{1}{1-y} \; + \; w^2 \times \frac{1}{1-x}\frac{1}{1-y} \; + \; v^3 \times \frac{1}{1-x}\frac{1}{1-y}$$

Thus to count we can again set $x = y = w = v = t$ to obtain

$$\frac{1+t^2+t^3}{(1-t)^2} \; = \; 1 + 2t + 4t^2 + 7t^3 + \cdots$$

mathcing x, y degree 1, x^2, xy, y^2, w^2 degree 2 and $x^3, x^2y, xy^2, y^3, w^2x, w^2y, v^3$ degree 3.

The Hilbert series is now

$$H(t) = \frac{1+t^2+t^3}{(1-t)^2} = \sum_{n=0}^{\infty} c_n t^n$$

c_n again counts how many degree n polynomials can be freely generated from the 2 variables x and y, and using w^2 and v^3 each at most once.

The Basic Result

We will compute partition functions of free multi-matrix model quantum mechanics. Denote the matrices as ϕ^a, $a = 1, 2, \cdots, d$. They all take the form

$$Z(x) = \frac{1 + \sum_i c_i^s x^i}{\prod_j (1 - x^j)^{c_j^p}}.$$

Each factor in the denominator is associated to an operator of degree j

$$x^j \leftrightarrow P_A = \mathrm{Tr}(\phi^{a_1}\phi^{a_2}\cdots\phi^{a_j}) + \cdots$$

and is called a *primary invariant*. A takes $N_P = \sum_j c_j^p$ values.

Each monomial x^i in the numerator is associated with an operator of degree i

$$x^i \leftrightarrow S_B = \mathrm{Tr}(\phi^{a_1}\phi^{a_2}\cdots\phi^{a_i})$$

and is called a *secondary invariant*. B takes $N_S = 1 + \sum_i c_i^s$ values, where we set $S_1 = 1$.

The space of all gauge invariant operators (loop space) takes the form

$$\mathcal{H} = \bigoplus_{B=1}^{N_S} \prod_{A=1}^{N_P} \sum_{\{n_A\}=0}^{\infty} (P_A)^{n_A} S_B$$

Matrix Quantum Mechanics; Loop Space

We study the matrix quantum mechanics of d $N \times N$ Hermittian matrices

$$H = \frac{1}{2}\sum_{a=1}^{d} \mathrm{Tr}(\Pi^a\Pi^a) + \frac{1}{2}\sum_{a=1}^{d} \mathrm{Tr}(\phi^a\phi^a)$$

$$[\Pi^a_{ij}(t), \phi^b_{kl}(t)] = -i\delta_{il}\delta_{jk}\delta^{ab}$$

The model has a $U(N)$ symmetry

$$\phi^a \to U^\dagger \phi^a U \qquad \Pi^a \to U^\dagger \Pi^a U$$

that we declare is a gauge symmetry.

The complete space of gauge invariant operators is given by traces of words constructed from the ϕ^a's as follows

$$\mathrm{Tr}(\phi^1\phi^2\phi^2\phi^1\phi^3\phi^6\cdots)$$

What is a complete description of this *loop space*?

Warm up: one matrix

We can diagonalize the Hamiltonian by introducing the creation and annihilation operators

$$A^\dagger \quad = \quad X - i\Pi$$

Up to a ground state energy

$$H \quad = \quad \mathrm{Tr}(A^\dagger A)$$

For a single matrix the partition function is

$$Z(x) \quad = \quad \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^N)}$$

Warm-up: one matrix

$$Z(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^N)}$$

What is the interpretation of this result? Each factor in the denominator corresponds to a specific generator; x^n corresponds to $\mathrm{Tr}(A^{\dagger n})$

Why can we stop at N matrices in the trace? For example, of $N=2$ we only have $\mathrm{Tr}(A^\dagger)$ and $\mathrm{Tr}(A^{\dagger\, 2})$. What about operators like $\mathrm{Tr}(A^{\dagger\, 3})$ and even higher powers?

These higher powers all follow from trace relations. For example, *any* 2×2 matrix obeys

$$\mathrm{Tr}(M)^3 - 3\mathrm{Tr}(M^2)\mathrm{Tr}(M) + 2\mathrm{Tr}(M^3) = 0$$

Such relations are explained by the fact that the above traces depend only on the eigenvalues of M. For 2×2 matrices there are only 2 eigenvalues, so once you know $\mathrm{Tr}(M)$ and $\mathrm{Tr}(M^2)$ you know everything!

$$\Rightarrow \qquad \mathrm{Tr}(A^{\dagger\, 3}) = \frac{3}{2}\mathrm{Tr}(A^{\dagger\, 2})\mathrm{Tr}(A^\dagger) - \frac{1}{2}\mathrm{Tr}(A^\dagger)^3$$

Molien-Weyl Formula

Count gauge invariant operators constructed from adjoint bosonic fields with energies E_i:

$$Z(\beta) = \sum_{n_1=0}^{\infty} x^{n_1 E_1} \sum_{n_2=0}^{\infty} x^{n_2 E_2} \cdots \times \#(n_1, n_2, ...)$$

where $x = e^{-1/T} = e^{-\beta}$ and $\#(n_1, n_2, ...)$ is the number of singlets in the tensor product $\mathrm{sym}^{\mathrm{n}_1}_{\mathrm{adj}} \otimes \mathrm{sym}^{\mathrm{n}_2}_{\mathrm{adj}} \otimes \cdots$. Number of singlets is an integral over $U(N)$ of the product of the characters of the reps being tensored:

$$Z(\beta) \quad = \quad \int_{U(N)} [DU] \prod_i \sum_{n_i=0}^{\infty} x^{n_i E_i} \chi_{\mathrm{sym}^{\mathrm{n_i}}_{\mathrm{adj}}}(U)$$

Reduce the matrix integral above to an integral over the eigenvalues of U

$$Z(x) = \frac{(Z_{N=1}(x))^N}{(2\pi i)^{N-1}} \oint_{|t_1|=1} \frac{dt_1}{t_1} \cdots \oint_{|t_{N-1}|=1} \frac{dt_{N-1}}{t_{N-1}} \prod_{1\le k\le r\le N-1} \frac{1-t_{k,r}}{f_{k,r}}$$

where $t_{k,r} = t_k t_{k+1} \cdots t_r$ and

$$Z_{N=1}(x) \quad = \quad \frac{1}{\prod_i (1-x^{E_i})} \qquad f_{k,r} \quad = \quad \prod_{i=1}^{d} (1-x^{E_i} t_{k,r})(1-x^{E_i} t_{k,r}^{-1})$$

Two matrix model, $N = 2$

Graded partition function: (call ϕ^1 as X and ϕ^2 as Y)

$$Z(x, y) = \frac{1}{(1-x)(1-y)(1-x^2)(1-xy)(1-y^2)}.$$

Generators:

$$\begin{aligned} m_1 &= \mathrm{Tr}(X), \qquad m_2 = \mathrm{Tr}(Y), \\ m_3 &= \mathrm{Tr}(X^2), \quad m_4 = \mathrm{Tr}(XY), \quad m_5 = \mathrm{Tr}(Y^2), \end{aligned}$$

Complete set of trace relations from the Cayley-Hamilton theorem. For $N = 2$, $T_2(A, B, C) = 0$, where

$$\begin{aligned} T_2(A, B, C) &= \mathrm{Tr}(A)\mathrm{Tr}(B)\mathrm{Tr}(C) - \mathrm{Tr}(AB)\mathrm{Tr}(C) - \mathrm{Tr}(AC)\mathrm{Tr}(B) \\ &\quad - \mathrm{Tr}(A)\mathrm{Tr}(BC) + \mathrm{Tr}(ABC) + \mathrm{Tr}(ACB), \end{aligned}$$

and A, B and C are any words constructed using X, Y as letters.

Words of length 3

Consider single trace operators constructed using n X fields and m Y fields.

For $m+n=3$, each choice of m and n gives a single trace relation and there is only one gauge-invariant single trace operator we can define. Thus all operators are determined in terms of our generating set.

For instance, choosing $m=2$ and $n=1$, we obtain

$$T_2(X,Y,Y)=\mathrm{Tr}(X)\mathrm{Tr}(Y)^2-2\mathrm{Tr}(XY)\mathrm{Tr}(Y)-\mathrm{Tr}(X)\mathrm{Tr}(XY)+2\mathrm{Tr}(XY^2)=0,$$

which implies

$$\mathrm{Tr}(XY^2)=\frac{1}{2}\Big(2m_2m_4+m_1m_5-m_1m_2^2\Big).$$

Swapping X and Y gives

$$\mathrm{Tr}(YX^2)=\frac{1}{2}\Big(2m_1m_4+m_2m_3-m_1^2m_2\Big).$$

Words of length 4

As $m+n$ increases, the number of distinct gauge-invariant operators that can be constructed increases.

Consider $m=2=n$: two independent operators, $\mathrm{Tr}(X^2Y^2)$ and $\mathrm{Tr}(XYXY)$, can be constructed.

$T_2(Y^2, X, X) = 0$ gives

$$\mathrm{Tr}(X^2Y^2) = \frac{1}{2}\Big(m_3m_5 + 2m_1m_2m_4 + m_1^2m_5 - m_1^2m_2^2 - m_1^2m_5\Big).$$

$T_2(XY, X, Y) = 0$ implies

$$\mathrm{Tr}(XYXY) = \frac{1}{2}\Big(m_2^2m_3 + 2m_4^2 - m_3m_5 + m_1^2m_5 - m_1^2m_2^2\Big).$$

Crucially, the growth in the number of independent operators is matched by the emergence of additional trace relations.

All Operators

We will prove that all gauge invariant operators can be written in terms of our generating set. Proof proceeds by induction.

Assume single-trace loops containing at most k matrices are determined by the trace relations. (established for $k \leq 4$)

Consider the loop $\mathrm{Tr}(X^{n_1} Y^{m_1})$ with $n_1 + m_1 = k + 1$ for $k \geq 4$. At least one of n_1 or m_1 must be greater than 1. Without loss of generality, assume $n_1 > 1$. The trace relation for $A = X$, $B = X^{n_1 - 1}$, and $C = Y^{m_1}$ is

$$2\mathrm{Tr}(X^{n_1} Y^{m_1}) - \mathrm{Tr}(X)\mathrm{Tr}(X^{n_1-1} Y^{m_1}) - \mathrm{Tr}(X^{n_1})\mathrm{Tr}(Y^{m_1})$$

$$-\mathrm{Tr}(XY^{m_1})\mathrm{Tr}(X^{n_1-1}) + \mathrm{Tr}(X)\mathrm{Tr}(X^{n_1-1})\mathrm{Tr}(Y^{m_1}) = 0.$$

By the induction hypothesis, every term in this equation except the first contains at most k matrices in the trace and is thus expressible in terms of our generating set.

This establishes that $\mathrm{Tr}(X^{n_1} Y^{m_1})$ can also be expressed in terms of these variables. The same argument applies, with trivial changes, in the case where $m_1 > 1$.

All Operators

Consider $\mathrm{Tr}(X^{n_1}Y^{m_1}\cdots X^{n_q}Y^{m_q})$, with

$$n_1+m_1+\cdots+n_q+m_q=k+1.$$

Call invariants, with q alternating $X^{\#}Y^{\#}$ blocks, type-q invariants.

Trace relation obtained from $A=X^{n_1}$, $B=Y^{m_1}$, and $C=X^{n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q}$, is

$$\mathrm{Tr}(X^{n_1}Y^{m_1}\cdots X^{n_q}Y^{m_q})+\mathrm{Tr}(X^{n_1+n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q+m_1})$$

$$-\mathrm{Tr}(X^{n_1})\mathrm{Tr}(X^{n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q+m_1})-\mathrm{Tr}(Y^{m_1})\mathrm{Tr}(X^{n_1+n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q})$$

$$-\mathrm{Tr}(X^{n_1}Y^{m_1})\mathrm{Tr}(X^{n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q})+\mathrm{Tr}(X^{n_1})\mathrm{Tr}(Y^{m_1})\mathrm{Tr}(X^{n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q})=0.$$

Second and third lines contain at most k matrices in a trace. By the induction hypothesis they are expressible in terms of our generating set.

First term is type-q invariant. Second term is type-$(q-1)$ invariant. We established type-1 invariant $\mathrm{Tr}(X^{n_1}Y^{m_1})$ can be expressed in terms of our generating set $\Rightarrow$ the type-2 invariant $\mathrm{Tr}(X^{n_1}Y^{m_1}X^{n_2}Y^{m_2})$ can also be expressed in terms of these variables.

This reasoning extends recursively, proving that all type-q invariants can be determined in terms of our generating.

Three matrix model, $N = 2$

$$Z(x,y,z) = \frac{1+xyz}{(1-x)(1-y)(1-z)(1-x^2)(1-y^2)(1-z^2)(1-xy)(1-xz)(1-yz)}.$$

Terms in the denominator correspond to the set of *primary* invariants

$$\begin{aligned}
m_1 &= \mathrm{Tr}(X), \qquad m_2 = \mathrm{Tr}(Y), \qquad m_3 = \mathrm{Tr}(Z), \\
m_4 &= \mathrm{Tr}(X^2), \qquad m_5 = \mathrm{Tr}(Y^2), \qquad m_6 = \mathrm{Tr}(Z^2), \\
m_7 &= \mathrm{Tr}(XY), \qquad m_8 = \mathrm{Tr}(YZ), \qquad m_9 = \mathrm{Tr}(ZX),
\end{aligned}$$

The term in the numerator correspond to the *secondary* invariant

$$s = \mathrm{Tr}(XYZ)$$

Primary invariants act freely - they can be raised to any power. The secondary invariants are *quadratically reducible* and appear at most linearly, if at all.

Proof that s is quadratically reducible

When constructing the complete space of gauge invariant observables, we should not allow s to act more than linearly.

Any action of s^2 can be replaced by an action of the primary invariants and terms with s appearing at most linearly, thanks to the constraint

$$
\begin{aligned}
&s^2 + s\,(m_1 m_2 m_3 - m_1 m_8 - m_2 m_9 - m_3 m_7) + \frac{1}{4} m_1^2 m_2^2 m_3^2 - \frac{1}{2} m_1^2 m_2 m_3 m_8 \\
&-\frac{1}{4} m_1^2 m_5 m_6 + \frac{m_1^2 m_8^2}{2} - \frac{1}{2} m_1 m_2^2 m_3 m_9 - \frac{1}{2} m_1 m_2 m_3^2 m_7 + \frac{1}{2} m_1 m_2 m_6 m_7 \\
&+\frac{1}{2} m_1 m_3 m_5 m_9 - \frac{1}{4} m_2^2 m_4 m_6 + \frac{m_2^2 m_9^2}{2} + \frac{1}{2} m_2 m_3 m_4 m_8 - \frac{1}{4} m_3^2 m_4 m_5 + \frac{m_3^2 m_7^2}{2} \\
&+\frac{m_4 m_5 m_6}{2} - \frac{m_4 m_8^2}{2} - \frac{m_5 m_9^2}{2} - \frac{m_6 m_7^2}{2} + m_7 m_8 m_9 = 0.
\end{aligned}
$$

Complete space of gauge invariant operators

Complete space of gauge invariant operators is thus given by the direct sum of the space

$$m_1^{n_1} m_2^{n_2} m_3^{n_3} m_4^{n_4} m_5^{n_5} m_6^{n_6} m_7^{n_7} m_8^{n_8} m_9^{n_9} \times 1$$

and the space

$$m_1^{n_1} m_2^{n_2} m_3^{n_3} m_4^{n_4} m_5^{n_5} m_6^{n_6} m_7^{n_7} m_8^{n_8} m_9^{n_9} \times s$$

It is natural to interpret the first space above as perturbative excitations of the vacuum state, created by the identity.

It is natural to interpret the second space above as perturbative excitations of the non-trivial state, created by the secondary invariant s.

Thus the primary invariants play the role of perturbative degrees of freedom. Acting with the primary invariants is creating perturbative excitations. The secondary invariants play the role of non-trivial states (like a soliton). The state created by the secondary invariant can support perturbative excitations.

Hironaka Decomposition

All partition functions we compute take the form

$$Z(x) = \frac{1 + \sum_i c_i^s x^i}{\prod_j (1 - x^j)^{c_j^m}}.$$

This is the Hilbert series of the invariant ring $C_{N,d}$ of $GL(N)$ invariants of d $N \times N$ matrices. It matches the structure of the Hironaka decomposition.

The denominator encodes *primary* invariants, while the numerator encodes *secondary* invariants.

The number of primary invariants equals the number of denominator factors and gives the Krull dimension of the ring: $(d-1)N^2 + 1$.

That our partition functions all take the Hironaka form is key to our analysis. The Hochster-Roberts theorem ensures that $C_{N,d}$ is Cohen-Macaulay, since $GL(N)$ is a linearly reductive group over a field of characteristic zero. Thus, the ring admits a Hironaka decomposition, i.e., it is a free module over a polynomial subalgebra.

Comment on Primary Invariants

Number of primary invariants $M = (d-1)N^2 + 1$.

Single-trace operators with at most N matrices are generating invariants: these can't be eliminated as trace relations only begin there are $N+1$ matrices in the trace.

Length L is the number of matrices in the trace. The number of distinct single-trace operators of length L is

$$N_{\mathrm{op}}(L) \approx \frac{d^L}{L}.$$

Captures leading behavior, but systematically underestimates the actual count.

For modest value of N, total number of single trace operators of length N ($\sim e^N$) vastly exceeds number of primary invariants ($= (d-1)N^2 + 1$). Although single-trace operators with $\leq N$ matrices are included, tiny fraction are actually primary invariants.

Example: Two matrix model at $N = 20$. Total number of single-trace operators with $\leq N$ matrices is 111,321. Number of primary invariants is $M = N^2 + 1 = 401$.

Counting Invariants

For a matrix model with two matrices, the number of primary and secondary invariants counted as a function of N are given below.

N	Primary Invariants	Secondary Invariants
2	5	0
3	10	1
4	17	63
5	26	15,423
6	37	312,606,719
7	50	21,739,438,196,735

Table: Growth in the number of invariants as N increases.

Growth in the number of primary invariants is $N^2 + 1$.

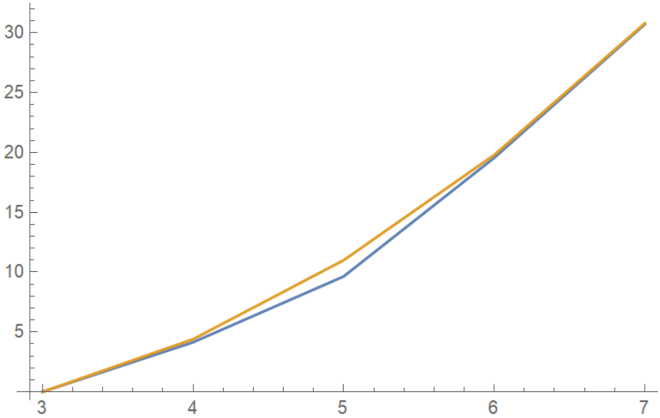
Growth in secondary invariants is much more rapid than a power.

Counting Invariants

Make the ansatz $N_{\text{Secondary}} = e^{1.1N^2 - 3.3N}$.

Above curve is orange. Blue is data. Suggests secondary invariants are black hole microstates.

$$S_{BH} = \log(N_{\text{states}}) = O(N^2)$$



Fortuity

$\frac{1}{16}$ black hole microstates in $\mathrm{AdS}_5 \times \mathrm{S}^5$ are represented by operators in $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$, that obey the equation: $QO_{BH} = 0$.

Recently a distinction between two types of solns was made:

- Solve $QO_{BH} = 0$ for all N - monotone operators.
- Solve $QO_{BH} = 0$ for $N = N_*$ - fortuitous operators. [Chang, Lin 2402.10129]

Most heavy operators are fortuitious $\Rightarrow$ the black hole entropy is explained by fortuitous operators.

Different chaos properties: fortuitous operators are more chaotic than monotone operators.

Black hole microstates are constructed using $O(N^2)$ fields $\Rightarrow$ black hole microstate in the $N = 10$ theory is not a microstate in $N = 100$ theory.

Makes us question if black holes are present at $N = \infty$ or rather at large but finite N?

These seem like general lessons that should extend beyond considerations of susy. In this talk I will argue that indeed they are.

Bosonic Fortuity

Recall that operators that are dual to black hole microstates at some N where they obey $QO = 0$ are not dual to black hole microstates at a larger values of N where $QO \neq 0$.

An operator that represents a black hole microstate in the $N = 10$ theory will not represent a black hole microstate in the $N = 100$ theory.

As N increases we find that more and more secondary invariants change their character and transition to become primary invariants.

This is a purely bosonic analog of the fortuity mechanism.

Infinite N

In the infinite N limit, there are no trace relations for any operator you look at, since these only kick in when you have $N+1$ matrices in the trace.

Thus, the large N limit is freely generated and the partition function takes the form

$$Z(x) = \frac{1}{\prod_i (1-x^i)^{n_i}} \tag{1}$$

with n_i =the number of single trace operators constructed from n_i matrices.

All secondary invariants have now been promoted to primary invariants. There are no "black hole microstates" at $N=\infty$.

Summary

The space of gauge invariant operators that can be constructed in the multi-matrix model quantum mechanics of d matrices is generated from primary and secondary invariants

$$\mathcal{H} \quad = \quad \bigoplus_{B=1}^{N_S} \prod_{A=1}^{N_P} \sum_{\{n_A\}=0}^{\infty} (P_A)^{n_A} S_B$$

The number of primary invariants is $(d-1)N^2+1$. The primary invariants generate a Fock space structure and represent perturbative degrees of freedom.

The number of secondary invariants grows as $\sim e^{N^2}$. We conjecture that black hole microstates are represented as secondary invariants.

As N is increased the character of invariants changes and secondary invariants transition to become primary invariants. This is a bosonic analog of the fortuity mechanism.

Thanks for your attention!