Next-to-leading order corrections to the conserved current three-point correlators in the large N expansion

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Based on ongoing work with Yongwei Guo and Ke Ren.

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1/N expansion of current 3pt correlators

1 Background: current three-point correlators in CFTs

- 2 Motivations: why do we care about the currents 3pt correlators?
- 3 The large N expansions of current 3pt correlators
- 4 Combining conformal symmetry constraints and Feynman loop diagrams



Background: current three-point correlators in CFTs

Conformal 2-pt correlators of currents J/T: fixed by conformal symmetry up to an overall coefficient $C_{J/T}$ (*H. Osborn and A. Petkos, 1993*)

$$\langle J_{\mu}(x)J_{\nu}(0)
angle = rac{C_J}{(x^2)^{d-1}}I_{\mu
u}(x), \ \langle T_{\mu
u}(x)T_{\rho\sigma}(0)
angle = rac{C_T}{(x^2)^d}\mathcal{T}_{\mu
u\rho\sigma}(x)$$

Conformal 3-pt correlators of currents J/T:

$$\langle J^{a}_{\mu}J^{b}_{\nu}J^{c}_{\rho}\rangle, \langle T_{\mu\nu}J^{a}_{\rho}J^{b}_{\sigma}\rangle, \langle T_{\mu\nu}T_{\rho\sigma}T_{\alpha\beta}\rangle$$

can be fixed by conformal symmetry up to few free parameters. Conformal Ward Identities associated with the conserved currents

 $\partial_{\mu}J_{\mu}(x) = 0, \ \partial_{\mu}T_{\mu\nu}(x) = 0, \ T^{\mu}_{\mu}(x) = 0,$ (Opeartor equation)

generates relations between 3-pt and 2-pt correlators, i.e.,

$$\partial_{\mu}\langle J^{a}_{\mu}(x)J^{b}_{\nu}(y)J^{c}_{\rho}(z)\rangle = f^{abe}\delta^{d}(x-y)\langle J^{e}_{\nu}(x)J^{c}_{\rho}(z)\rangle - f^{ace}\delta^{d}(x-z)\langle J^{e}_{\nu}(x)J^{b}_{\rho}(y)\rangle$$

	Free parameter	Ward Identity	Realized by
$\langle JJJ \rangle$	2	C_J, λ_{JJJ}	free boson/fermion
$\langle TJJ \rangle$	2	C_J, λ_{TJJ}	free boson/fermion
$\langle TTT \rangle_{3d}$	2	n _B , n _F	free boson/fermion
$\langle TTT \rangle_{4d}$	3	c _T , a, c	free boson/fermion/vector

- We only focus on the parity even sectors.
- The conformal Ward identities relate part of the 3-pt coefficients to the 2-pt coefficients $c_{T/J}$.
- The (TTT) has one less parameter in 3d than in 4d, and the free vector theory is non-conformal in 3d.

 $\langle TTT \rangle$ and anomalous traceless condition $T^{\mu}_{\mu} = 0$:

In 4d, put the theory on curved space background, there are local dimension 4 scalars and the trace has the form (see e.g. *Luty, Polchinski and Rattazzi, 2012*)

$$T^{\mu}_{\mu}=aE_4-cW^2$$

Euler density: $E_4 = R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2$ (Euler density); Weyl tensor square: $W_{\mu\nu\rho\sigma}^2 = R_{\mu\nu\rho\sigma}^2 - 2R_{\mu\nu}^2 + \frac{1}{3}R^2$.

a-theorem: $a_{UV} > a_{IR}$, alike the 2D c-theorem in a weaker version. (*Komargodski and Schwimmer, 2011*)

Consequence of a-theorem:

perturbative UV-IR RG flow has scale to conformal symmetry enhancement (*Luty, Polchinski and Rattazzi, 2012*)

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2. Why do we care about the current 3pt correlators?

- Conductivity/transportation near quantum critical points
- Q Current correlators in conformal bootstrap
- Onformal collider bounds
- Graviton non-gaussianities during inflation
- S Large N and large gap in holographic theories

Current correlators in condensed matter physics

At the quantum critical point, the current has OPE

$$J \times J \sim C_J \mathbf{1} + \lambda_{JJJ} S + \lambda_{JJT} T + \cdots,$$

conductivity $\sigma(\omega)$ near the quantum critical point at finite temperature

$$\sigma(\omega_n) \propto \frac{1}{32} C_J + \lambda_{JJS} (\frac{T}{\omega_n})^{\Delta_S} + \lambda_{JJT} (\frac{T}{\omega_n})^3 + \cdots$$

- S is the leading singlet scalar in the theory. If S is irrelevant, then the λ_{JJT} is the subleading term of the conductivity. (Subir Sachdev et al. 2014)
- Leading order results on λ_{JJJ} and λ_{TJJ} have been computed (*Maldacena and Pimentel 2011, Subir Sachdev et al. 2012, Bzowski et al. 2013*)
- Hard to compute the 3pt structure constants analytically, usually using Monte Carlo simulation to study the conductivity.
 (Witczak-Krempa et al. 2013, Lucas et al. 2016, Wang et al. 2020).

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Current correlators in conformal bootstrap

New surprising bootstrap Island for 3d Ising (Chang et al., 2024)



New ingredient: the stress tensor current T correlators. New results: $(\Delta_{\sigma}, \Delta_{\epsilon}) = (0.518148806(24), 1.41262528(29))$, the size of the island is just 1/100 of the previous result without T!

Current correlators for bootstrapping gauge theories

Goal: bootstrap strongly coupled gauge theories, e.g., $N_f = 4 \text{ QED}_3$ (*S. Albayrak, R. Erramilli, ZL, D. Poland, X. Yuan, 2021*)



• Much smaller size of island with current correlators?

Substantial applications in solving the strongly coupled gauge theories.

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1/N expansion of current 3pt correlators

Coefficients in the current correlators $\langle TJJ \rangle$, $\langle TTT \rangle$ are strongly bounded! (*Hofman and Maldacena 2008, Buchel et al. 2009, Chowdhury et al. 2012*)

•
$$\langle TJJ \rangle$$
: $\frac{(d-2)\Gamma(\frac{d}{2}+1)}{2(d-1)\pi^{\frac{d}{2}}} \leq \frac{\lambda_{JJT}}{C_J} \leq \frac{\Gamma[\frac{d}{2}+1]}{2\pi^{\frac{d}{2}}}$
• $\langle TTT \rangle_{3d}$: $\theta \equiv \tan^{-1}(\frac{n_F}{n_B}) \in [0, \frac{\pi}{2}]$;
• $\langle TTT \rangle_{4d}$: $\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}$.

Proved using analytical bootstrap method:

Crossing symmetry + *unitarity* + *lightcone limit*

(D. Hofman, Daliang Li, D. Meltzer, D. Poland, F. Rejon-Barrera, 2016)

Important applications in QCD, holography, etc.

Graviton non-gaussianities during inflation

Graviton non-gaussianties: the graviton three-point function $\langle ggg \rangle$ in de Sitter space. (*Maldacena and Pimentel, 2011*)

- The graviton non-gaussianties relate to higher order corrections to the gravity action.
- Due to the de Sitter isometry, the three-point function only has two parity even possible shapes.
- The two shapes correspond to the two independent coefficients of the $\langle {\cal TTT} \rangle$ in 3d CFT

 $\langle ggg
angle_{dS_4} \sim \langle TTT
angle_{CFT_3}$

The current 3pt correlator $\langle TTT \rangle$ in 3d CFT could help to study the Graviton non-gaussianties, and possible higher order terms in the gravity action, though it could be weak in the CMB obervations. (*See e.g., talks by Profs. Masahide Yamaguchi, Shi Pi, Zhong-zhi Xiangyu et al.*)

How strong the graviton non-gaussianity could be?

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Current correlators in holography: large N and large gap

Graviton non-gaussianity: corrections to the Einstein-Hilbert action.

Gravity corrections evluated from conformal $\langle TTT \rangle_{4d}$ (*Camanho, Edelstein, Maldacena and Zhiboedov 2014*):

- Einstein gravity: a = c;
- Constraints from causality: $\left|\frac{a-c}{c}\right| \leq \frac{1}{\Delta_{pap}^2}$; Δ_{gap} : single-trace, $\ell > 2$.

Conclusion: if $\Delta_{gap} \gg 1$, $\Rightarrow a \simeq c$, \Rightarrow the gravity is Einstein-like.

Swampland conjecture: (Obied, Ooguri, Spodyneiko and Vafa, 2018)

- Einstein-like quantum gravity in AdS should be supersymmetric
- Non-susy quantum gravity in AdS can NOT be Einstein-like.

Critical question: in any holographic CFT, can we turn on a large gap in the single trace spectrum without SUSY?

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3. The large N expansion of current 3pt correlators

Consider a general CFT₀ in *d* Euclidean space, with a single trace operator \mathcal{O} of dimension $\Delta_{\mathcal{O}} < d/2$, its double-trace deformation can generate a new IR fixed point (*Igor Klebanov et al., 2016*)

$$S_{\lambda} = S_{CFT_0} + \lambda \int d^d x \mathcal{O}^2(x).$$

Using Hubbard- Stratonovich transformation, the action can be rewritten

$$S_{\lambda} = S_{CFT_0} + \int d^d x \sigma \mathcal{O} - rac{1}{4\lambda} d^d x \sigma^2.$$

The second term is irrelevant in the UV and one obtains an effective propagator of the auxiliary field σ through the cubic interaction

$$\langle \sigma({\it p})\sigma(-{\it p})
angle \propto 1/({\it p}^2)^{d/2-2+\delta},$$

we use δ to regularize the divergence in loop integrals.

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Large N expansion of the o(n) vector model

The large N expansion of O(N) vector model starts with the Lagrangian

$$S_{O(N)} = \int d^d x \left((\partial \phi^i)^2 + \frac{1}{\sqrt{N}} \sigma \phi^i \phi^i \right),$$

with propagators

$$\langle \phi^i(p)\phi^j(-p)=\delta^{ij}/p^2,\ \langle \sigma(p)\sigma(-p)
angle \propto 1/(p^2)^{d/2-2+\delta}.$$

Then we use the classcial renormalization scheme $\phi_p = \phi Z_{\phi}^{1/2}, \sigma_p = \sigma Z_{\sigma}^{1/2}$. What is new: $T_p = T * Z_T, \ Z_T = 1 + \frac{1}{N}(Z_{T,1}/\delta + Z_{T,2})$.

- Renormalization of ϕ, σ are fixed from the 1-loop diagram of $\langle \phi \phi \rangle, \langle \sigma \sigma \rangle$.
- Renormalization of T is fixed from the loop diagrams of $\langle T\phi\phi\rangle$ and conformal Ward identity.

Feynman diagrams in the large N expansion

Feynman diagrams for the subleading term in large N expansion of $\langle TJJ \rangle$ in QED₃



Challenging part: 3-loop 3-leg integrals with δ -regularization for $\langle TJJ \rangle$. In $\langle TTT \rangle$, we even have 4-loop 3-leg integrals!

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1/N expansion of current 3pt correlators

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Conformal symmetry constraints and Feynman diagrams

Question: how to analytically evaluate the 3-loop/4-loop 3-leg integrals with δ -regularization in general d?

- For loop integrals in QFT, doable using IBP+differential equations...
- While for CFTs, our common wisdom indicates, the conformal symmetry can make significant contributions!

Key observation: simplify loop integrals using conformal symmetry!

- The conformal symmetry already fixes the current 3pt correlators up to few free parameters.
- So we only need to extract the free parameters from the Feynman loop-integrals instead of evaluating the whole integrals.
- The free parameters in current 3pt correlators can be obtained by taking special limits of external momentum.
- Cross check the results using Ward identities of conserved currents.

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Simplify loop integrals using conformal symmetry

Solve current 3pt functions using free boson/fermion theories:

- The loop integrals, after renormalization, should produce the exact conformal 3pt functions of currents.
- The conformal symmetry fixes the conformal 3pt functions of currents functions, given by free boson/fermion/vector theories.
- So evaluate the current 3pt functions in free theories are enough to solve the general current 3pt functions, up to free parameters.

Fun fact: the general conformal 3pt functions are simple in position space, but are remarkably complicated in momentum space! Though they are connected through Fourier transformations, up to contact terms. (*Bzowski et al. 2013, Corian et al. 2013, Baumann et al. 2020, ...*)

We can further simplify the 3pt functions of free field theories by taking special limit of external momentum.

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Goal: extract corrections to the free parameters in current 3pt functions from the loop integrals.

The general 3pt functions $\langle JJJ \rangle$, $\langle TJJ \rangle$, $\langle TTT \rangle$ contain two independent external momentum, e.g., $\langle T(q)J(p)J(-p-q) \rangle$.

To extract the corrections to the free parameters, it suffices to take a special limit of external momentum $q \rightarrow 0$, and keep the leading and subleading terms in the small q expansion.

The 3-loop 3-leg integrals essentially reduce to 2-loop 2-leg integrals.

Cautious: the limit $q \rightarrow 0$ and the loop integrals may NOT commute! **Solutions**: expansions by regions or subgraphs.

Example: $\langle JJJ \rangle$ in O(N) vector model

Results of Feynman diagrams in the small q limit:

$$\begin{split} D_1 &= \eta_{O(N)} D_0 \left(\left(\frac{3d^3 - 30d^2 + 76d - 32}{2(d-4)(d-2)d} - C_{O(N)} \right) - \left(\frac{1}{\delta} - \log\left(\frac{p^2}{\mu^2} \right) \right) \right) \\ D_2 &= \eta_{O(N)} D_0 \left(\left(\frac{5d^3 - 46d^2 + 100d - 32}{2(d-4)(d-2)d} - C_{O(N)} \right) - \left(\frac{1}{\delta} - \log\left(\frac{p^2}{\mu^2} \right) \right) \right); \\ D_3 &= \eta_{O(N)} D_0 \left(\left(C_{O(N)} - \frac{3d^3 - 25d^2 + 52d - 16}{(d-4)(d-2)d} \right) + \left(\frac{1}{\delta} - \log\left(\frac{p^2}{\mu^2} \right) \right) \right); \\ D_4 &= \eta_{O(N)} D_0 \left(\left(C_{O(N)} + \frac{3d^2 - 12d + 16}{(d-4)(d-2)d} \right) + \left(\frac{1}{\delta} - \log\left(\frac{p^2}{\mu^2} \right) \right) \right); \\ D_5 &= \eta_{O(N)} D_0 \left(\left(C_{O(N)} + \frac{7d^2 - 28d + 16}{(d-4)(d-2)d} \right) + \left(\frac{1}{\delta} - \log\left(\frac{p^2}{\mu^2} \right) \right) \right); \\ D_6 &= \eta_{O(N)} D_0 \left(\left(-C_{O(N)} - \frac{d^3 + d^2 - 20d + 16}{(d-4)(d-2)d} \right) - \left(\frac{1}{\delta} - \log\left(\frac{p^2}{\mu^2} \right) \right) \right); \end{split}$$

This leads to the subleading correction to the λ_{JJJ} :

$$\lambda_{JJJ}^{(1)} = -\frac{8\sin\left(\frac{\pi d}{2}\right)\Gamma(d-2)}{\pi(d-2)\Gamma\left(\frac{d}{2}-2\right)\Gamma\left(\frac{d}{2}+1\right)}\frac{1}{N}.$$

Example: $\langle JJJ \rangle$ in Gross-Neveu model

Using the same method, we obtain the subleading correction to the λ_{JJJ} in the Gross-Neveu model λ_{JJJ} :

$$\lambda_{JJJ}^{(1)} = \frac{2^d \sin\left(\frac{\pi d}{2}\right) \Gamma\left(\frac{d-1}{2}\right)}{\pi^{3/2} \Gamma\left(\frac{d}{2}+1\right)} \frac{1}{N}.$$

The subleading order corrections of structure constants in $\langle TJJ \rangle$ and $\langle TTT \rangle$ can be computed similarly.

The results will be employed to study the **conductivity near quantum critical point** and the **conformal bootstrap**.

Results for the 3D conformal gauge theories can be used to study the 3d duality web.

The subleading correction to $\langle TTT \rangle$ at the Banks-Zaks fixed point can be applied for the conformal collider physics.

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- The current three-point functions have broad applications in condensed matter physics, conformal bootstrap, cosmology, and quantum gravity;
- New method for the 1/N expansion of the current 3pt functions: conformal symmetry constraints + Feynman loop diagrams.
 Conformal symmetry fixes the general three-point functions, loop corrections to the parameters extracted from Feynman diagrams.
- We take a special limit of the external mometum, which can simplify the 3-loop 3-leg integrals to 2-loop 2-leg integrals.
- This method can be applied for a large class of CFTs, which will be of interest for condensed matter physics and bootstrap studies.
- Relation between conformal symmetry and Feynman loop diagrams could be generalized to 4-pt correlators, with deeper connections to the analytical conformal bootstrap.

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