On-Shell EFT Matching through d-Dimensional Generalized Unitary Cuts

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2 One-loop Integral

3 d-Dimensional Generalized Unitary Method





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- 4 Example





Amplitudes and functional views for matching. [2011.02484]

Our on-shell method is based on the $\underline{\mbox{amplitude matching}}$ view. On-shell amplitudes:

independent of gauge complexness and unphysical freedoms

Systematic on-shell operator basis decomposition.

- UV amplitudes: $\mathcal{A}_{\mathrm{UV}}^{1L} = \mathcal{A}_{\mathrm{UV}}^{1L} \big|_{\mathsf{soft}} + \left| \mathcal{A}_{\mathrm{UV}}^{1L} \right|_{\mathsf{hard}}$.
- Regions:

hard region $l \sim M \gg m, p_i$, soft region $l \sim m, p_i \ll M.$

- EFT amplitudes: $\mathcal{A}_{EFT}^{1L} = c_i \mathcal{B}_i + (non-local) + (EFT 1-L).$
- \Rightarrow Matching: $c_i \mathcal{B}_i \xrightarrow{\text{at } M \text{ scale}} \mathcal{A}_{\text{UV}}^{1\text{L}} |_{\text{hard}}$.

* Hard region:

$$\frac{1}{(l+p_i)^2 - M^2} = \frac{1}{l^2 - M^2} \left[1 - \dots\right], \frac{1}{(l+p_i)^2 - m^2} = \frac{1}{l^2} \left[1 - \dots\right].$$

Using the identities, the integrand can be expanded as

$$A_{\rm UV}^{\rm 1L, integrand} = \frac{\mathcal{N}}{(l^2)^{\alpha} (l^2 - M^2)^{\beta}} + \mathcal{O}\left(\frac{1}{M^n}\right) \Rightarrow \left. \left. \mathcal{A}_{\rm UV}^{\rm 1L} \right|_{\rm hard} \right. = \sum \frac{g_k}{M^k} \mathcal{B}_k.$$

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General Structure of the One-Loop Integral

One-loop amplitudes can be decomposed as:

$$\mathcal{A}_N^{\mathsf{1L}} = \sum_i \left(C_4^{\{i\}} I_4^{\{i\}} + C_3^{\{i\}} I_3^{\{i\}} + C_2^{\{i\}} I_2^{\{i\}} + C_1^{\{i\}} I_1^{\{i\}} \right) + R.$$

- C: coefficients in d = 4. I_i : scalar integrals. R: the rational term.
- * Rational terms originate from setting $d = 4 2\epsilon$ and taking $\epsilon \to 0$ in the dimensional regularization scheme. $C^{(d)}I_2 = \left[C^{(4)} + \epsilon \times \mathcal{R} + \mathcal{O}(\epsilon^2)\right] \left[\frac{1}{\epsilon} + \dots\right] \xrightarrow{\epsilon \to 0} C^{(4)}I_2 + \mathcal{R}$
- ✓ Most scalar integrals have specific branch cuts: $I_2(s), I_3(s), I_4(s_1, s_2)$
- X The tadpole integral I_1 and the massive bubble integral $I_2(M^2)$ are independent of kinematics.

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5 Summary

Generalized Unitary Method

Optical Theorem. By using optical theorem, the discontinuity can be obtained from tree level on-shell amplitudes.

$$-i\operatorname{Disc} \mathcal{A}^{1\mathsf{L}}\Big|_{s_{\mathcal{I}}} = \sum_{\sigma_{1},\sigma_{2}} \int \frac{\mathrm{d}^{4}l_{1}}{(2\pi)^{2}} \times \delta^{+} \left(l_{1}^{2} - m_{1}^{2}\right) \delta^{+} \left(l_{2}^{2} - m_{2}^{2}\right) \\ \times \mathcal{A}_{L}^{\mathsf{Tree}} \left(l_{2}^{\sigma_{2}}, i \dots j, l_{1}^{\sigma_{1}}\right) \times \mathcal{A}_{R}^{\mathsf{Tree}} \left(-l_{1}^{-\sigma_{1}}, j+1, \dots, i-1, -l_{2}^{-\sigma_{2}}\right),$$



Recovering Propagators

Task

Extract coefficients of scalar integrals from **discontinuity**.

Recovering:

$$2\pi\delta^+(l^2-m^2) \to -\frac{i}{l^2-m^2+i\epsilon} = \frac{-i}{D_{l,m}}$$

Phase space integrals $|\Rightarrow$

Projection on integrals via discontinuity

$$\int d\Pi_{\text{LIPS}} \mathcal{A}_L \mathcal{A}_R = \mathcal{P}_X \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{i}{D_{l_1,m_1}} \frac{i}{D_{l_2,m_2}} \mathcal{A}_L \mathcal{A}_R,$$

 $X = (l_1 + l_2)^2$. \mathcal{P}_X extracts the X-channel discontinuity.

E.g. $\mathcal{P}_s[I_2(s) + I_2(t) + I_4(s,t)] = I_2(s) + I_4(s,t)$

$$A_{\mathsf{full}}^{1\mathrm{L}} = \sum_{k=1}^{n_{\mathcal{C}}} \prod_{j=1}^{k-1} \left(1 - \mathcal{P}_{s_j} \right) \mathcal{P}_{s_k} \left[A^{1\mathrm{L}} \Big|_{s_k}^{\mathrm{cuts}} \right]$$

Using the sewed integrand, we could recover the full amplitude by following steps:



d-dimensional loop momentum:

$$\overline{l} = l + \widetilde{\mu}, \quad \widetilde{\mu}^2 = g_{-2\epsilon}^{\mu\nu} l^{\mu} l^{\nu}.$$

- **d-dim** on-shell conditions: $\bar{l}^2 = m^2 \leftrightarrow \left| l^2 = m^2 + \tilde{\mu}^2 \right|$.
- Massive spinors: masses shift as $m^2 \rightarrow m^2 + \tilde{\mu}^2$.
- Massless spinors: *d*-dimensional equations of motion:

$$ec{l}\left|l
ight] = \widetilde{\mu}\left|l
ight|, \; ec{l}\left|l
ight
angle = \widetilde{\mu}\left|l
ight
angle.$$

• Integral: $\int d^d \bar{l} = \int d^4 l \ d^{-2\epsilon} \tilde{\mu}$. $\tilde{\mu}^2$, $\langle a \tilde{\mu} b \rangle$, ... e.g. $I_3[\tilde{\mu}^2] = -1/2$, $I_4[\tilde{\mu}^4] = -1/6$.

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Simple Example: Scalar QED $\gamma^+\gamma^-\gamma^+\gamma^-$ 1-loop

$$\begin{split} A^{1\mathrm{L}} \Big|_{s_{k}}^{\mathrm{cuts}} &\text{ is from sewing tree level amplitudes in d dimension.} \\ \mathcal{A}^{1\mathrm{L}} \Big|_{s}^{\mathrm{cuts}} = & \int \frac{d^{4-2\epsilon}\bar{l}}{(2\pi)^{4-2\epsilon}} \mathcal{A}_{\gamma+\gamma-\phi\phi^{\dagger}}^{\mathrm{Tree}}(1,2,\bar{l}_{s},\bar{l}) \mathcal{A}_{\gamma+\gamma-\phi\phi^{\dagger}}^{\mathrm{Tree}}(3,4,-\bar{l},-\bar{l}_{s}) \frac{i}{\bar{l}^{2}} \frac{i}{\bar{l}^{2}} \\ &= \int \frac{d^{4-2\epsilon}\bar{l}}{(2\pi)^{4-2\epsilon}} \frac{4(\langle 2\bar{l}_{s}1]\langle 4\bar{l}3])^{2}}{t^{2}D_{\bar{l}}D_{\bar{l}_{s}}}, \quad \left(\langle 2l1] + \langle 2\tilde{\mu}1]\right) \\ &\times \left(\frac{1}{D_{p_{1}+\bar{l}}} + \frac{1}{D_{p_{1}+\bar{l}_{s}}}\right) \left(\frac{1}{D_{p_{3}-\bar{l}}} + \frac{1}{D_{p_{3}-\bar{l}_{s}}}\right) \\ &= \int [\mathrm{d}\bar{l}] \; \frac{4\;\mathrm{Tr}_{-}\left[2\bar{l}_{s}13\bar{l}4\right]^{2}}{\langle 13\rangle^{2}[24]^{2}t^{2}}(\ldots) \\ \mathrm{Tr}_{\pm}\left[p_{1}p_{2}\ldots\right] = \mathrm{Tr}\left[P_{L/R} \not\!\!p_{1}P_{R/L} \not\!\!p_{2}\ldots\right]. \end{split}$$

Merging

Weighted projection \Leftrightarrow Remove redundancies



Recovering Rational Terms: Obvious case in *t*-channel.

$$\begin{split} A_{\gamma_{+}\phi\phi^{\dagger}}^{\text{Tree}}(p_{1},p_{3},\bar{l},\bar{l}_{t}), & A_{\gamma_{-}\phi\phi^{\dagger}}^{\text{Tree}}(p_{2},p_{4},-\bar{l}_{t},-\bar{l}) \text{ needed.} \\ \mathcal{A}_{\gamma_{+}\phi\phi^{\dagger}}^{\text{Tree}} = -\frac{2i\,[13]^{2}\,\tilde{\mu}^{2}}{t} \left(\frac{1}{D_{\bar{l}+p_{1}}} + \frac{1}{D_{\bar{l}_{t}+p_{1}}}\right) \stackrel{\underline{\tilde{\mu}}\rightarrow 0}{=} 0 \\ \mathcal{P}_{t}^{\text{weighted}} \left[\mathcal{A}\Big|_{t}^{\text{cuts}}\right] = \frac{8i\,[13]^{2}\,\langle 24\rangle^{2}}{(4\pi)^{2}t^{2}} \left(\frac{I_{4}[\tilde{\mu}^{4}](s,t)}{2} + s \leftrightarrow u\right) \\ = 0 + \mathcal{R}_{t} \\ \mathcal{R}_{t} = \frac{-4i\,[13]^{2}}{3(4\pi)^{2}\,[24]^{2}}. \\ \mathcal{A}_{\text{merged}}^{1\text{L}} = \mathcal{P}_{t}^{\text{weighted}} \left[\mathcal{A}\Big|_{t}^{\text{cuts}}\right] + \left\{\mathcal{P}_{s}^{\text{weighted}}\left[\mathcal{A}\Big|_{s}^{\text{cuts}}\right] + (s \leftrightarrow u)\right. \end{split}$$

$$\mathcal{L}_{\rm int} = -g \, \phi A_{\mu} A^{\mu} - g_1 \phi \, \bar{\psi} \psi - g_2 \phi \, \bar{\chi} \chi,$$

Integrating out the massive fields gives the Wilson coefficients for $\langle 12 \rangle \langle 34 \rangle$, representing $\psi_1^- \bar{\psi}_2^- \chi_3^- \bar{\chi}_4^-$. Using our procedure, the one-loop amplitude is:

$$\mathcal{A}^{1L} = \frac{g^2 g_1 g_2 \langle 12 \rangle \langle 34 \rangle \left(-12M^4 + 4M^2 s - s^2 \right)}{64\pi^2 M^4 (s - M^2)^2} I_2(s; M^2, M^2) + \mathcal{R}.$$

$$\begin{array}{ll} \mbox{Rational term:} & \mathcal{R} = \frac{2ig^2g_1g_2 \ \langle 12\rangle \ \langle 34\rangle}{(4\pi)^2(s-M^2)^2}. \\ \\ I(s,M^2,M^2) = c_0 \log\left(\frac{\mu^2}{M^2}\right) + c_1\frac{s}{M^2} + \dots \end{array}$$

The next-leading-order contribution from this loop amplitude is

$$c_{\frac{1}{M^4}}^{\text{NLO}}\left\langle 12\right\rangle \left\langle 34\right\rangle = \frac{2g^2g_1g_2}{(4\pi)^2M^4}\left\langle 12\right\rangle \left\langle 34\right\rangle + \frac{1}{(4\pi)^2M^4}\left\langle 34\right\rangle + \frac{1}{(4$$

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- The on-shell method avoids complexness from gauge factors and unphysical modes.
- ✓ On-shell $\mathcal{A}^{\mathsf{Tree}}$ with $\tilde{\mu}$ for -2ϵ contribution is needed.
- Sew in *d*-dim and recover propagators can recover integrals with correct discontinuity.
- Rational terms can be recovered.
- ✓ Rational terms can contribute to Wilson coefficients.
 - ? Generalize to high loops.

Thanks for your attention!