Application of solving inverse scattering problem in holographic bulk reconstruction

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Based on:
Bo-Wen Fan（范博文） and Run-Qiu Yang, 2310.10419, 2505.xxxxx

Contents

- Background;
- Solving inverse scattering problem for scalar field;
- Solving inverse scattering problem for gauge field;
- Summary

Motivation

- The mathematical expression of a physical law is usually a set of equations
- It contains some parameters, and its solutions stand for the results which can be compared with observations.
- Deriving the "computed result" for a given set of parameters is called ``solving the direct problem''.
- Conversely, obtaining the set of the parameters from a given set of results is called "solving the inverse problem".

Examples of solving invers problems

- Using scattering data to find the masses, coupling constants of particles in Standard Model

$$L = \bar{\psi}\gamma^{\mu}(\partial_{\mu} - iqA_{\mu})\psi - m\bar{\psi}\psi - \frac{1}{4\pi}F_{\mu\nu}F^{\mu\nu}$$

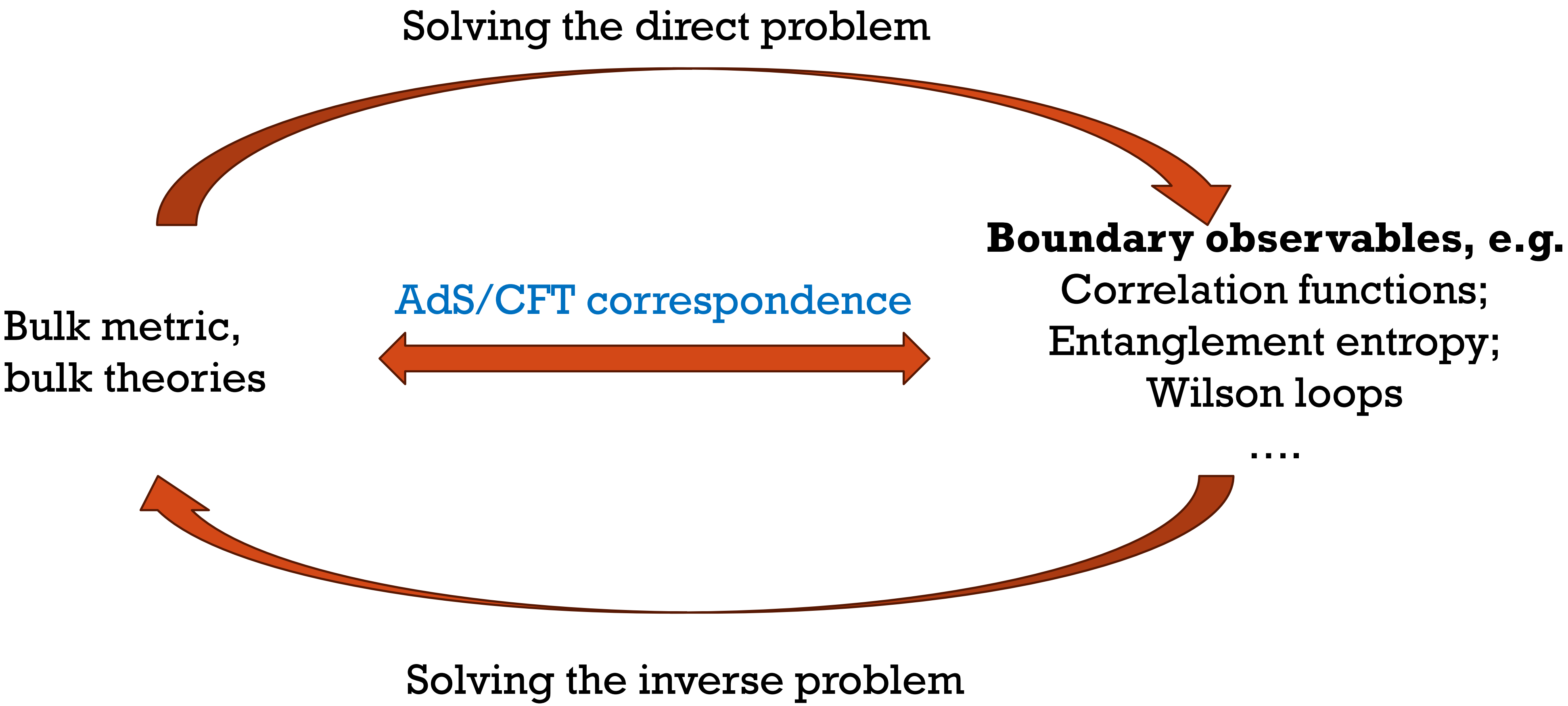
The values of q and m $\longleftarrow$ Scattering data

- The gravitational wave detection is also a typical "solving inverse problem"

$$G_{ab} = 8\pi G T_{ab}$$

The values of M, a and Q $\longleftarrow$ Signals of detectors of gravitational wave

Solving direct and inverse problems in holography



“solving inverse problem” plays important role in holography

- Bulk reconstruction is the central question in understanding the emerging spacetime in holography ;

- It answers how the gravitational degrees of freedom emerge from field theory;

- It will help us to build holographic spacetime and theory more efficiently

Current methods of "solving inverse problem"

- Geodesic approximations of correlations

Geodesic length of two boundary points

Heavy mass limit $\Delta \gg d$ → Geodesic approximation: $\langle O(x)O(y)\rangle \sim \exp[-\Delta L(x,y)]$

This limit cannot be satisfied in materials

- Bulk-cone singularities

Two boundary points are connected by bulk null geodesics → Two boundary points have singular correlator

Cannot be applied into planar materials;
Locating all singularities of correlator is a challenge

Current methods of "solving inverse problem"

- Entanglement wedge reconstruction

  Uses boundary entanglement spectrum to reconstruct
  bulk geometry and local observables

  No explicitly method of reconstruction is given

- Some boundary quantities are conjectured to dual to some bulk geometric quantities

  holographic entanglement entropy, Wilson loop, complexity, ….

  It is still a challenge to experimentally measure them

Solving inverse scattering problem for scalar field

- We focus on the planar symmetric asymptotically AdS black brane

$$ds^2 = \frac{1}{z(\rho)^2}[h(\rho)(-dt^2 + d\rho^2) + d\vec{x}^2_{d-1}], \quad \rho \in \mathbb{R}^+$$

- To compute the retarded frequency 2-point correlation function of dual boundary scalar operator, we consider the bulk scalar field of a form

$$\Psi = \phi(\rho) z^{\frac{d-1}{2}} e^{-i(\omega t + kx)}$$

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}$$

- The Klein-Gordon equation then reads

$$-\frac{d^2\phi}{d\rho^2} + \left(V_k + \frac{4m^2 + d^2 - 1}{4\rho^2}\right)\phi = \omega^2\phi$$

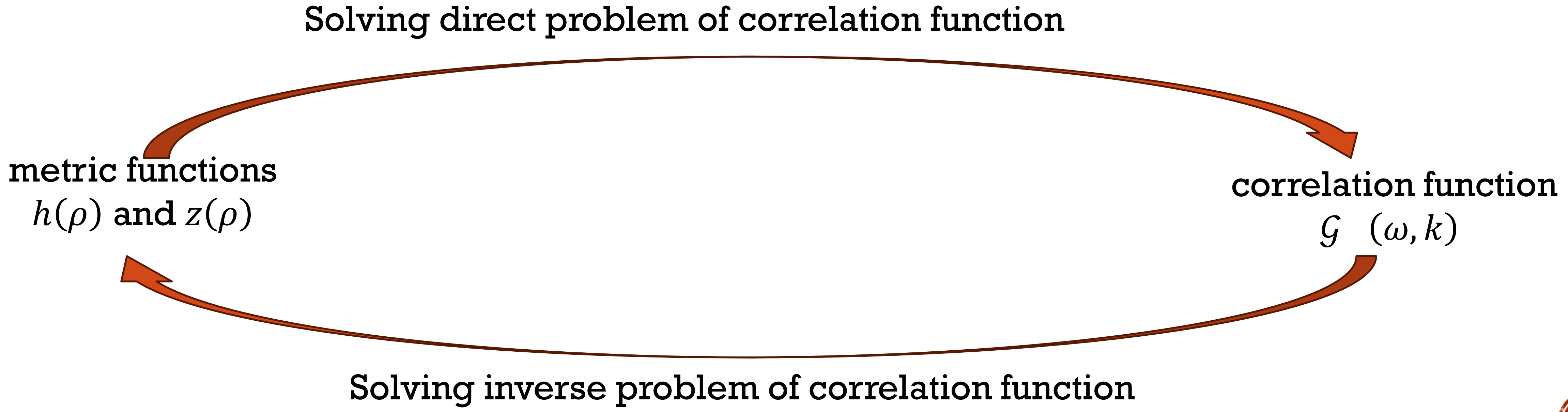
$$V_k = -\frac{(d-1)z''}{2z} + \frac{(d^2-1)}{4}\left(\frac{z'^2}{z^2} - \frac{1}{\rho^2}\right) + m^2\left(\frac{h}{z^2} - \frac{1}{\rho^2}\right) + hk^2$$

- Take $\phi(\rho)$ to satisfies ingoing wave condition at horizon, we then have

$$\phi = \Phi^{(-)}(\omega,k)\rho^{\frac{1}{2}-v}(1+\cdots) + \Phi^{(+)}(\omega,k)\rho^{\frac{1}{2}+v}(1+\cdots) \qquad v = \Delta - \frac{d}{2}$$

- The frequency 2-point correlation function in standard quantization is defined as

$$\mathcal{G}(\omega,k) = \Phi^{(-)}(\omega,k)/\,\Phi^{(+)}(\omega,k)$$



Recover the potential

2310.10419

Step 1: For every given momentum k

$$\mathcal{G}(\omega,k) \Rightarrow |F_k(\omega)|^2 = \frac{\pi|\omega/2|^{2v}v}{\Gamma(v+1)^2\Im[\mathcal{G}(\omega,k)]}$$

Step 2: Using function $|F_k(\omega)|$ we then construct an integral kernel $\Omega_k(\rho,\sigma)$

$$\Omega_k(\rho,\sigma) = \int_0^\infty J_v(\omega\rho)J_v(\omega\sigma)\left(\frac{1}{|F_k(\omega|^2} - 1\right)\mathrm{d}\omega$$

Step 3: We then solve the Gel'fand-Levitan-Marchenko integral equation

$$K_k(\rho,\sigma) + \Omega_k(\rho,\sigma) + \int_0^\rho K_k(\rho,y)\Omega_k(y,\sigma)\mathrm{d}y = 0$$

Step 4: we will have following mathematical results to reconstruct effective potential

$$V_k = 2\frac{\mathrm{d}}{\mathrm{d}\rho}K_k(\rho,\rho)$$

$$V_k = -\frac{(d-1)z''}{2z} + \frac{(d^2-1)}{4}\left(\frac{z'^2}{z^2} - \frac{1}{\rho^2}\right) + m^2\left(\frac{h}{z^2} - \frac{1}{\rho^2}\right) + hk^2$$

- Take $\mathcal{G}(\omega, k_1)$ and $\mathcal{G}(\omega, k_2)$ to be correlation functions of arbitrary two different momentums.

- After we reconstruct their corresponding effective potentials V_{k_1} and V_{k_2}, the function $h(\rho)$ will be determined by

$$h(z) = \frac{V_{k_1} - V_{k_2}}{k_1^2 - k_2^2}$$

- The function $z(\rho)$ then is given according to following equation

$$V_k = -\frac{(d-1)z''}{2z} + \frac{(d^2-1)}{4}\left(\frac{z'^2}{z^2} - \frac{1}{\rho^2}\right) + m^2\left(\frac{h}{z^2} - \frac{1}{\rho^2}\right) + hk^2$$

Solve above equations and we will then obtain the metric components!

Example of BTZ black hole

- By setting the inverse horizon $z_h = 1$, the BTZ black hole is given by $z(\rho) = \tanh \rho$ and $h(\rho) = 1/\cosh^2(\rho)$

- The boundary 2-point correlation function is given by

$$G(k,\omega) = \frac{\Gamma(-\nu)\Gamma(\frac{1+\nu-i\omega+ik}{2})\Gamma(\frac{1+\nu-i\omega-ik}{2})}{\Gamma(\nu)\Gamma(\frac{1-\nu-i\omega+ik}{2})\Gamma(\frac{1-\nu-i\omega-ik}{2})}$$

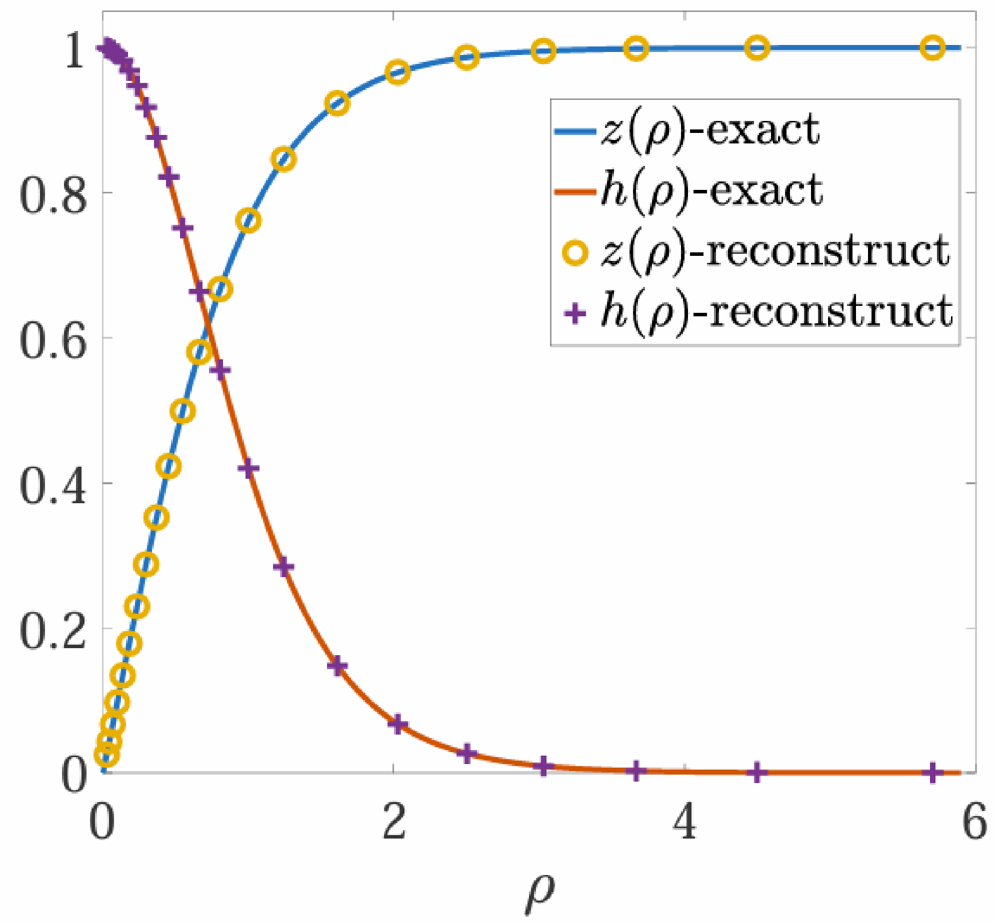


Figure 1. Comparison between the reconstructed metric components and their exact values with our method. Here it uses two-point functions of $k_1 = 0$ and $k_2 = 1$ to recover the metric.

A little more discussion

- The requirement that bulk field satisfies the Klein-Gorden equation can be relaxed.

$$L = -(\partial\Psi)^2 - m^2\Psi - W\left(X_i, R_{\mu\nu\rho\sigma}, g_{\mu\nu}, \Psi\right), \qquad \frac{\partial W}{\partial\Psi|_{\Psi=0}} = 0$$

- We obtain a similar equation of motion

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}\rho^2} + \left(V_k + \frac{4m^2 + d^2 - 1}{4\rho^2} + \tilde{V}\right)\phi = \omega^2\phi \qquad \tilde{V} = \frac{h}{z^2}\frac{\partial^2 W}{\partial\Psi^2}\bigg|_{\Psi=0}.$$

- We then can reconstruct the metric components for and potential $\tilde{V}$.

- It can even be used to reconstruct the bulk theory of scalar field!

A little more discussion

$$|F_k(\omega)|^2 = \frac{\pi|\omega/2|^{2v}v}{\Gamma(v+1)^2\Im[\mathcal{G}(\omega,k)]}$$

- In solving inverse problem, we seemingly only use information of imaginary part of correlation function;

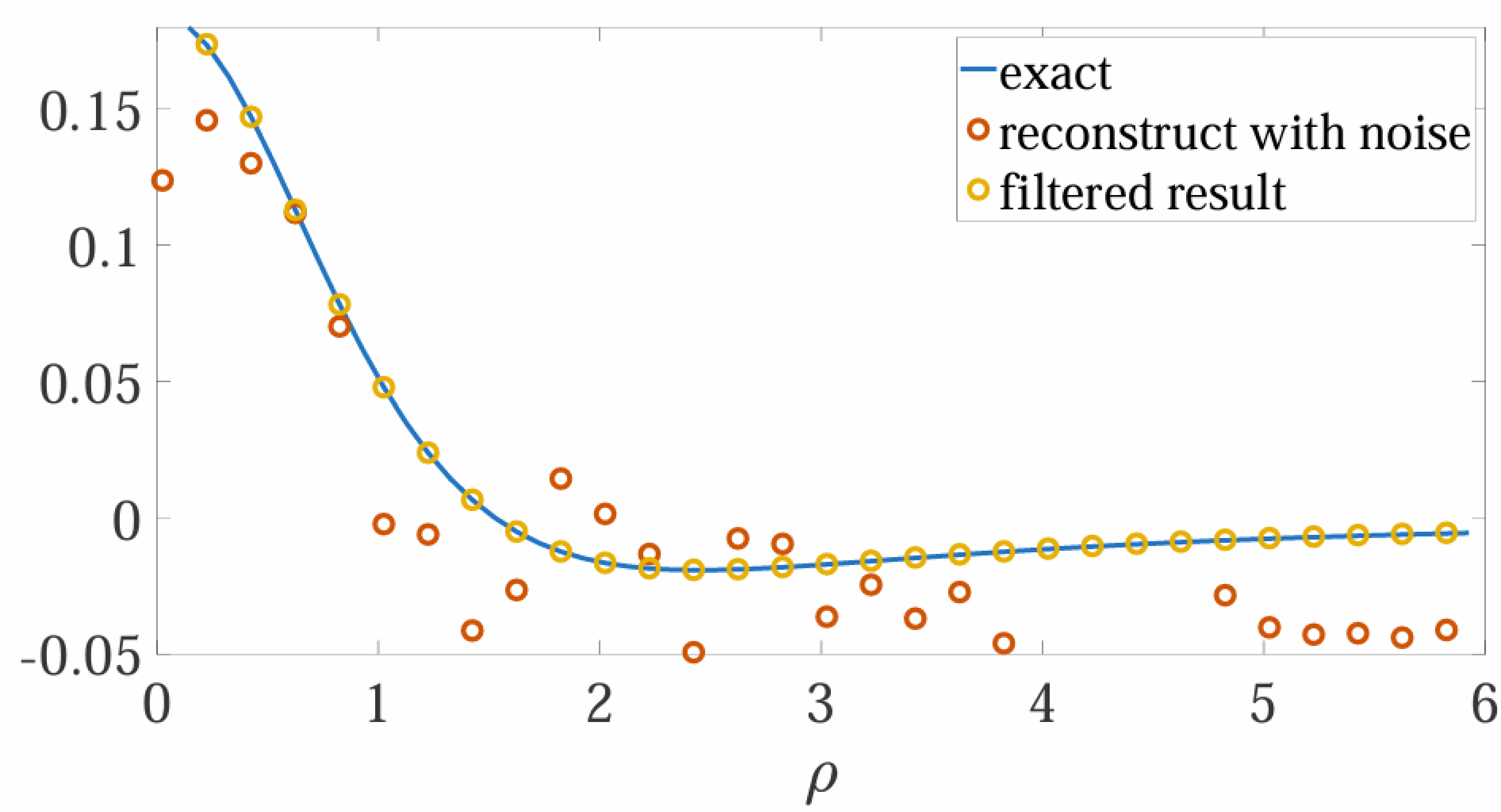
- This not true! In fact, the real and imaginary parts are not independent! They are related according to Hilbert transformation:

$$\Re[\chi(\omega)] = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\Im[\chi(\omega')]}{\omega'-\omega}d\omega', \qquad \Im[\chi(\omega)] = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\Re[\chi(\omega')]}{\omega'-\omega}d\omega'$$

- This also gives us a method to filter the noise of experimental measurements.

$$\begin{cases} R_0 & \tilde{R}_0 \\ I_0 & \tilde{I}_0 \end{cases} \longrightarrow \begin{cases} R_1 = \dfrac{R_0+\tilde{R}_0}{2} & \tilde{R}_1 \\ I_1 = \dfrac{I_0+\tilde{I}_0}{2} & \tilde{I}_1 \end{cases} \longrightarrow \begin{cases} R_2 = \dfrac{R_1+\tilde{R}_1}{2} \\ I_2 = \dfrac{I_1+\tilde{I}_1}{2} \end{cases} \longrightarrow \cdots\cdots$$

$$\begin{cases} R_0 \\ I_0 \end{cases} \rightarrow \tilde{R}_0,\ \tilde{I}_0 \quad \Rightarrow \quad \begin{cases} R_1 = \dfrac{R_0 + \tilde{R}_0}{2} \\ I_1 = \dfrac{I_0 + \tilde{I}_0}{2} \end{cases} \rightarrow \tilde{R}_1,\ \tilde{I}_1 \quad \Rightarrow \quad \begin{cases} R_2 = \dfrac{R_1 + \tilde{R}_1}{2} \\ I_2 = \dfrac{I_1 + \tilde{I}_1}{2} \end{cases} \quad \Rightarrow \quad \cdots\cdots$$



A little more discussion

- The correlation functions belong to the scattering matrix of bulk theory.

$$(M, g_{\mu\nu}) \qquad \Psi = \Phi_\Delta^{(+)}(t,\vec{x}) z^\Delta (1 + \cdots) + \Phi_\Delta^{(-)}(t,\vec{x}) z^{\mathrm{d}-\Delta}(1+\cdots) \qquad G(t,\vec{x}) = \frac{\Phi_\Delta^{(+)}(t,\vec{x})}{\Phi_\Delta^{(-)}(t,\vec{x})}$$

- Euclidean asymptotically AdS manifold and its metric are determined by the scattering matrix

$$\left(M_1, g_{\mu\nu}^{(1)}\right) \Rightarrow G^{(1)}(t,x) \qquad \left(M_2, g_{\mu\nu}^{(2)}\right) \Rightarrow G^{(2)}(t,x)$$

$$\left(M_1, g_{\mu\nu}^{(1)}\right) = \left(M_2, g_{\mu\nu}^{(2)}\right) \Leftrightarrow G^{(1)}(t,x) = G^{(2)}(t,x)$$

The explicit method of reconstruction is still unknown

- This is still an active topic, and we hope that some useful progresses will be made in the future.

Solving inverse scattering problem for scalar field

- Gauge field is dual to boundary conserved current and 2-point correlation can be obtained from the conductivity of materials

$$\sigma(\omega, k) = \frac{i\mathcal{G}(\omega, k)}{\omega}$$

Correlation-induced superconductivity dynamically stabilized and enhanced by laser irradiation

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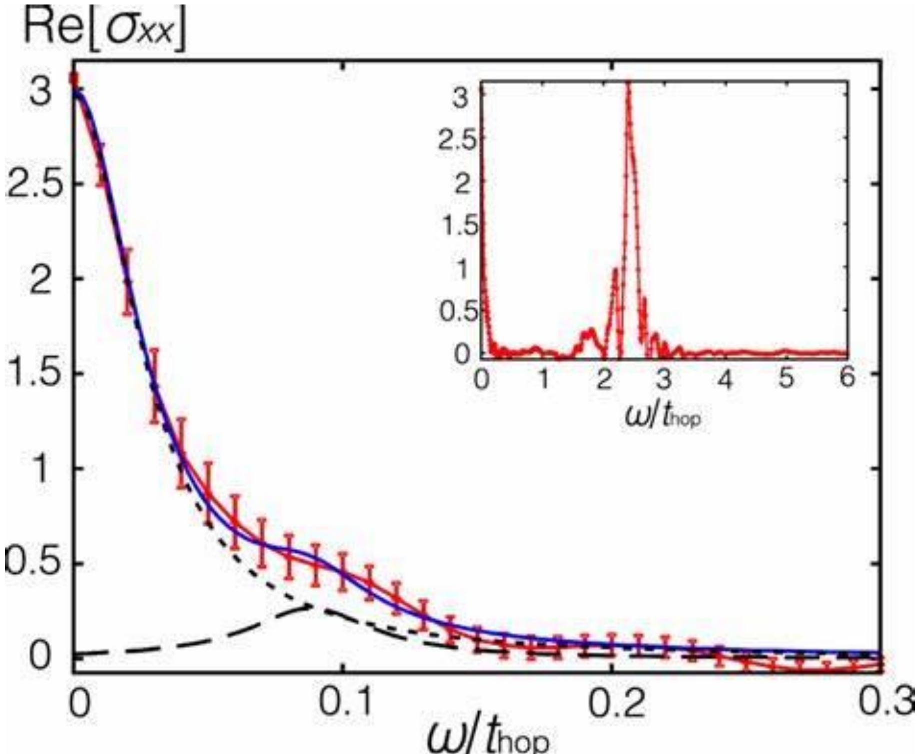

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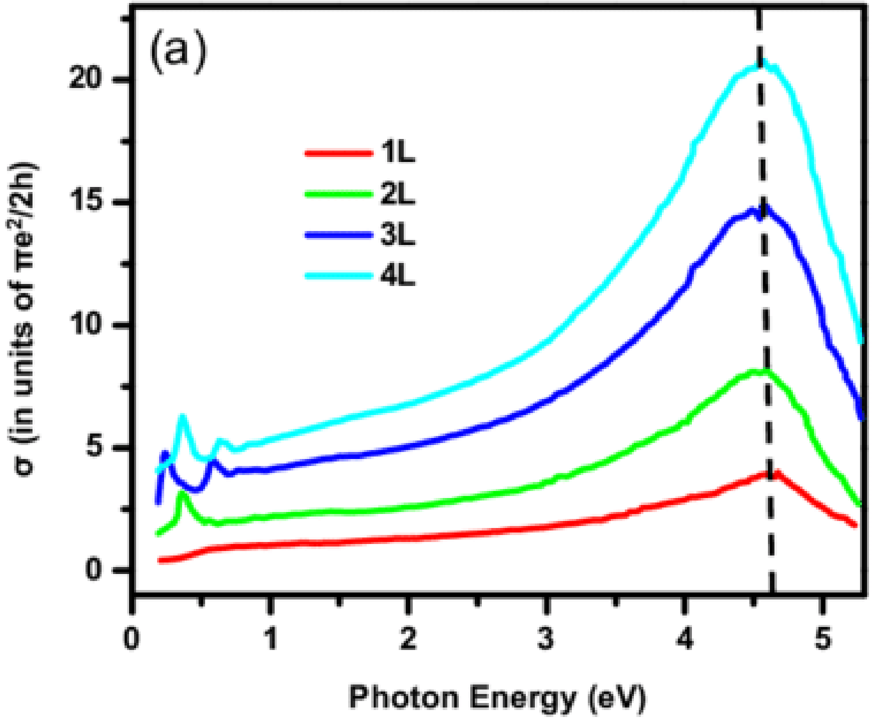

Experimental measurement of the optical conductivity for single-and few-layer graphene

Quantum-critical conductivity of the Dirac fluid in graphene

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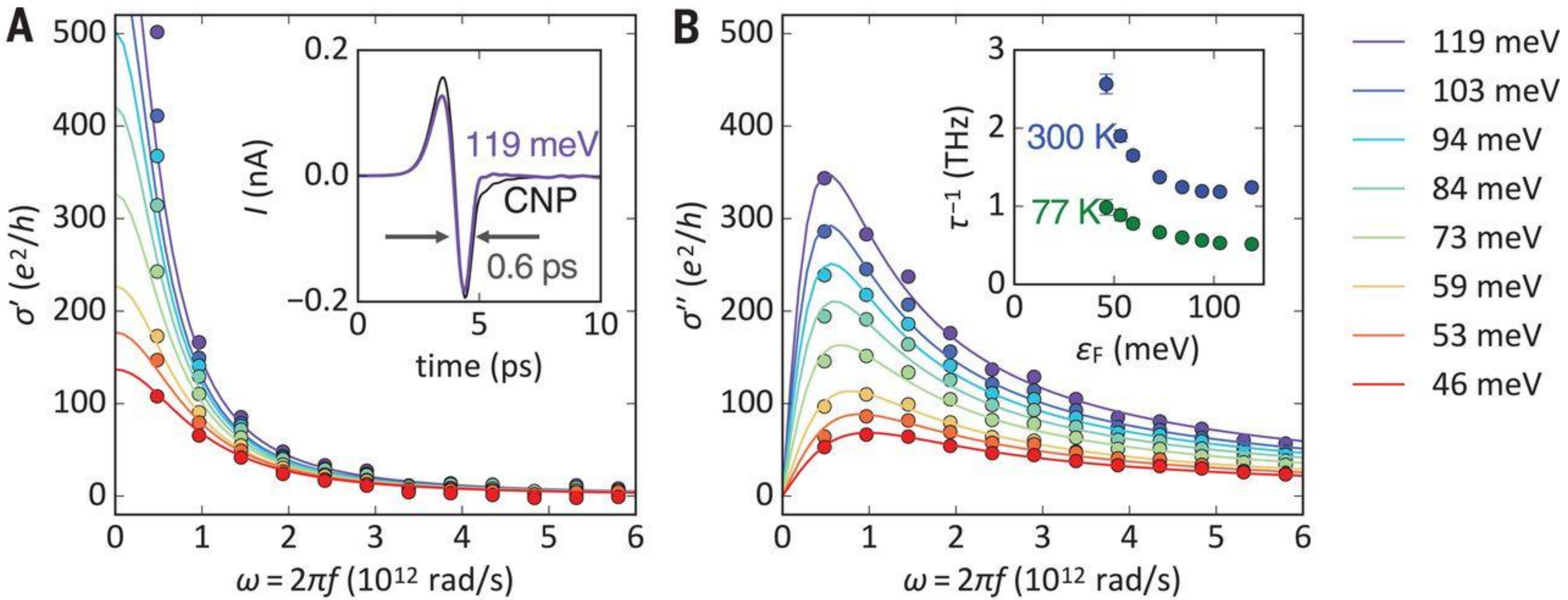

(**A** and **B**) Real (A) and imaginary (B) parts of the extracted optical conductivity for several Fermi energies between 46 and 119 meV (electron doping) at 77 K

Correlation function of gauge field in holography

- As a preliminary investigation, let us assume the dual gauge field in will be

$$L = \mathcal{F}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

- In weak field limit, it reduces into Maxwell theory

- To find the relationship between 2-point correlation function and bulk metric, we consider the ansatz

$$A_x = A_x(\rho) z^{\frac{d-3}{2}} e^{-i(\omega t + k y)}$$

$$\mathrm{d}s^2 = \frac{1}{z(\rho)^2}[h(\rho)(-\mathrm{d}t^2 + \mathrm{d}\rho^2) + \mathrm{d}\vec{x}^2_{d-1}], \quad \rho \in \mathbb{R}^+$$

- The Maxwell equations then read

$$-\frac{\mathrm{d}^2 A_x}{\mathrm{d}\rho^2} + \left(V_k + \frac{d^2 - 4d + 3}{4\rho^2}\right) A_x = \omega^2 A_x \qquad V_x = -\frac{d-3}{2z} z'' + \frac{d^2 - 4d + 3}{4}\left(\frac{z''}{z} - \frac{1}{\rho^2}\right) + k^2 h$$

- The inverse problem of gauge field is mathematically same as the inverse problem of scalar field.

Summary

- This talk considers the bulk reconstruction of metric from the viewpoint the ``solving inverse problem'' of correlation functions;

- It shows that a planar symmetric static asymptotically anti-de Sitter black brane can be reconstructed from 2-point correlation functions of single scalar operators and transverse conductivity;

Pure algebraic reconstruction

- This reconstruction solves an integral equation;

- It proposes an explicit method to ``directly measure'' the corresponding holographic spacetime for a material that has holographic dual;

- It also provides a powerful approach to check if a material has holographic dual.