Holographic Schwinger-Keldysh effective field theories including a non-hydrodynamic mode

Based on 2411.16306

Collaborated with: Yan Liu and Ya-Wen Sun

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Outline

Hydrodynamics

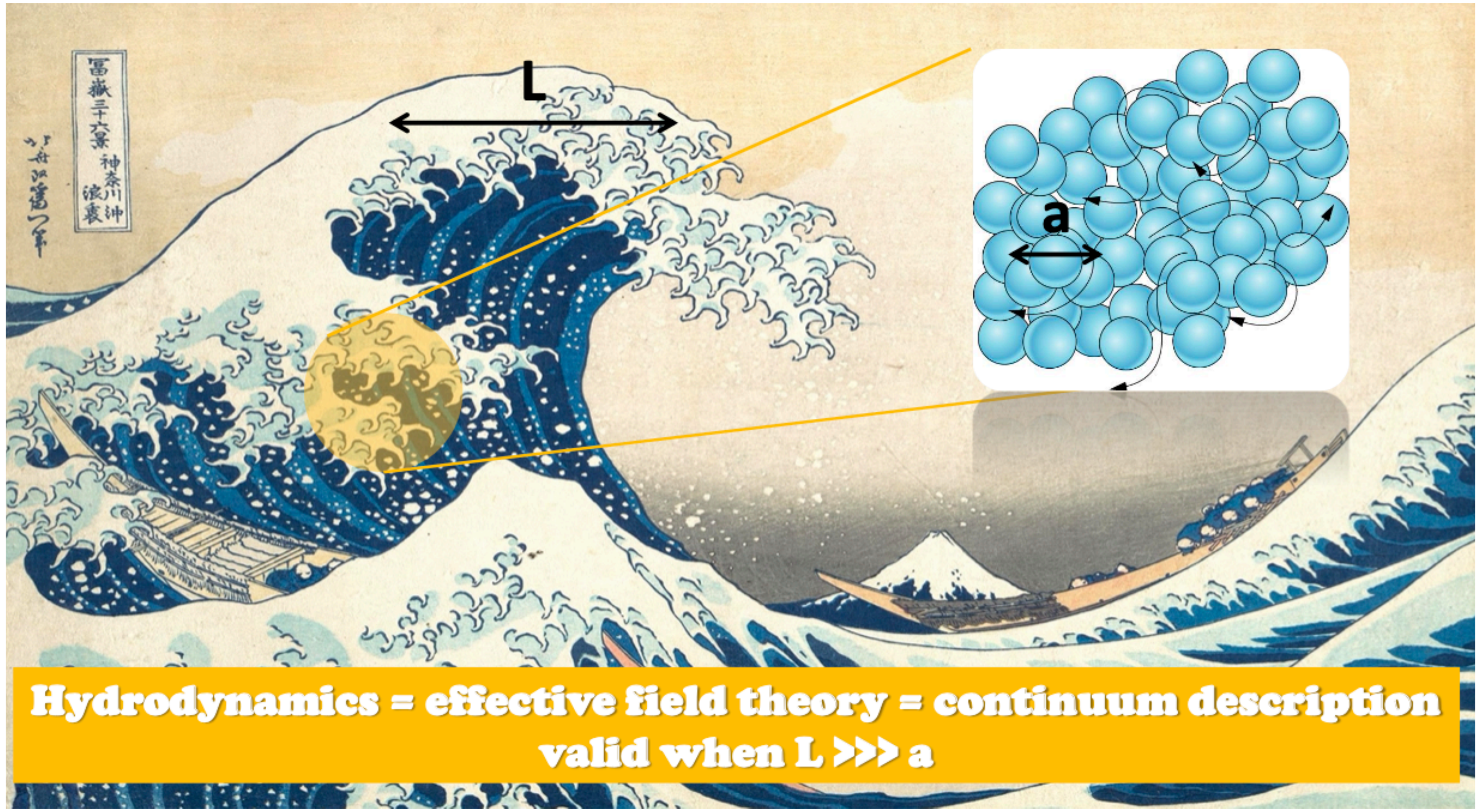
Schwinger-Keldysh for hydro

Quasi-Hydro

Schwinger-Keldysh for quasi-hydro

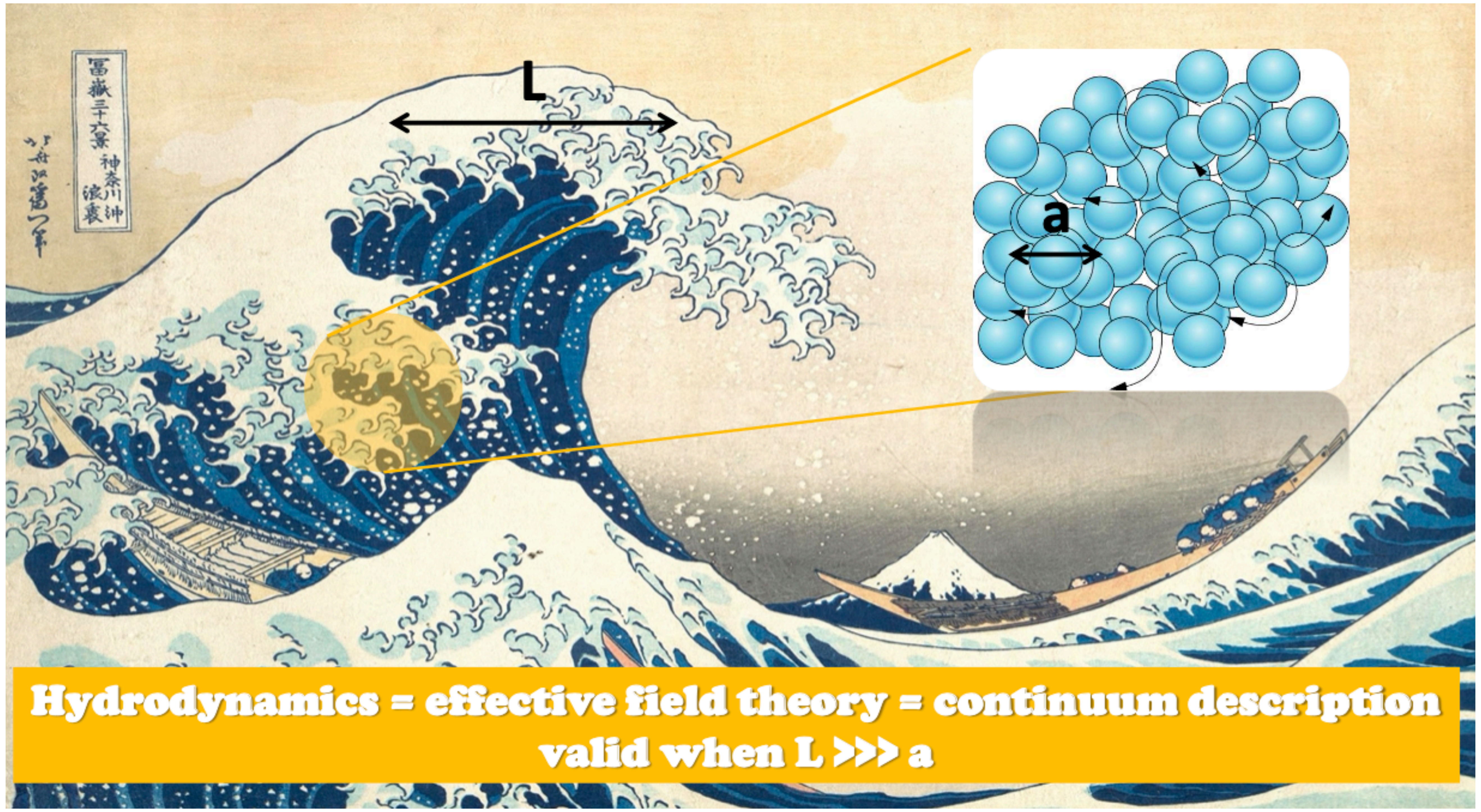
Summary

Hydrodynamics: basic ideas



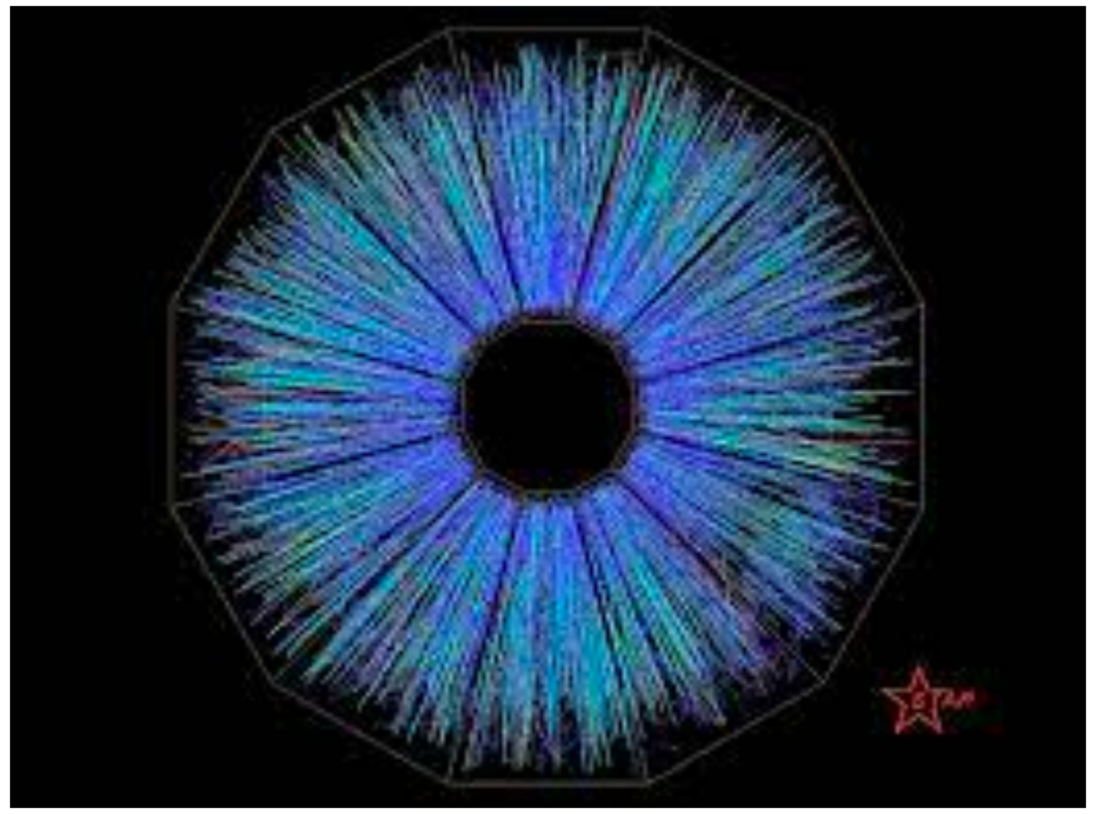
- long time, long distance
- excitations relax quickly
- local equilibrium

Hydrodynamics: basic ideas



- long time, long distance
- excitations relax quickly
- local equilibrium
- hydrodynamic variables
- conserved equations
- constitutive equations(dissipation)

Hydro and holography



QGP

QGP dissipates like a hdyro, whlie not a kinetic system

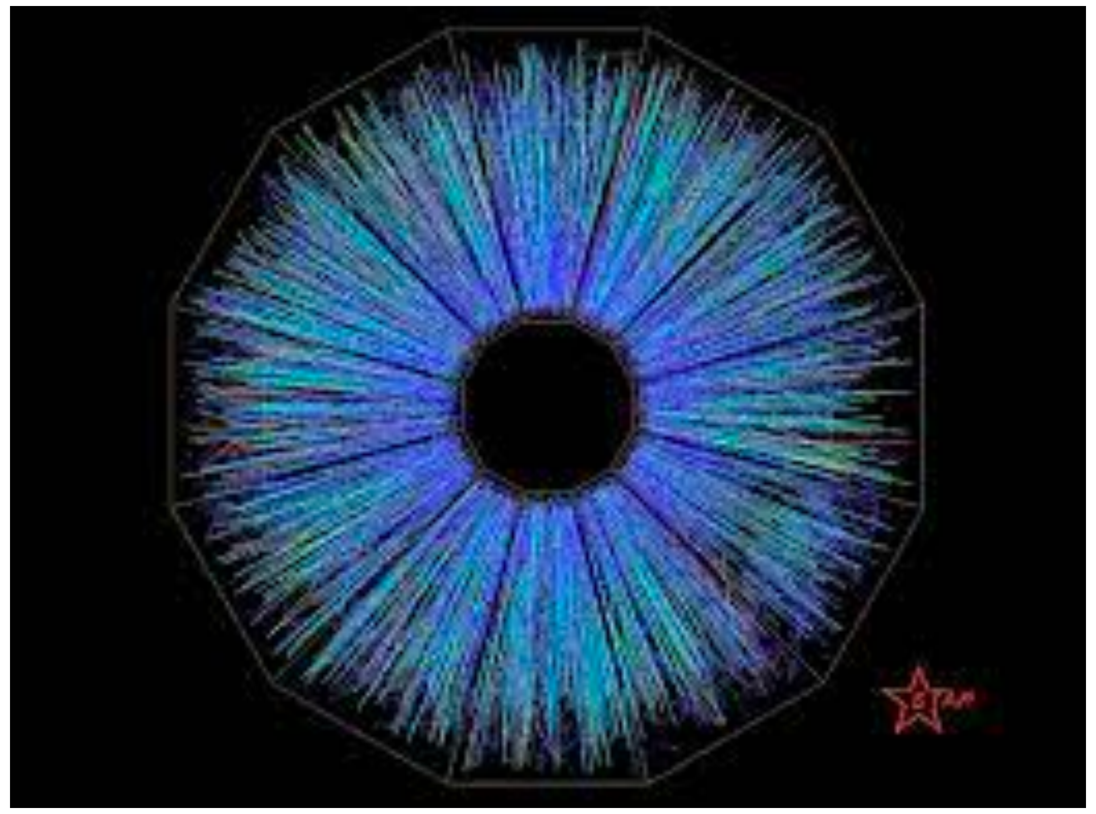
- minimal viscosity:

$$\frac{\eta}{s} = \frac{1}{4\pi}\frac{\hbar}{k_B}$$

Policastro, Son, Starinets, hep-th/0104066

Davison, Schalm, Zaanen 1311.2451

Hydro and holography



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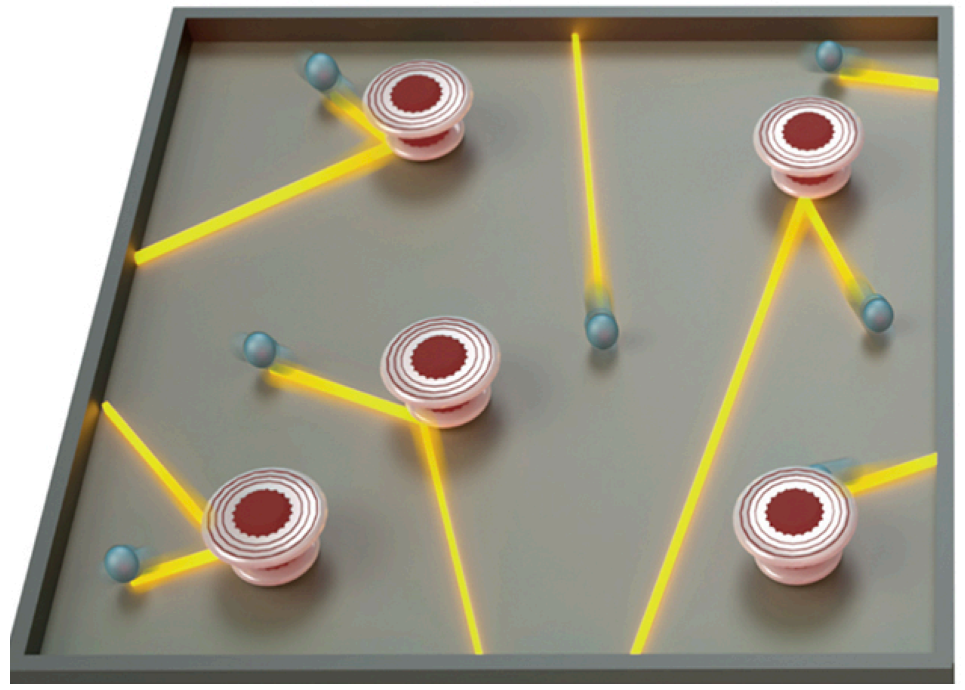

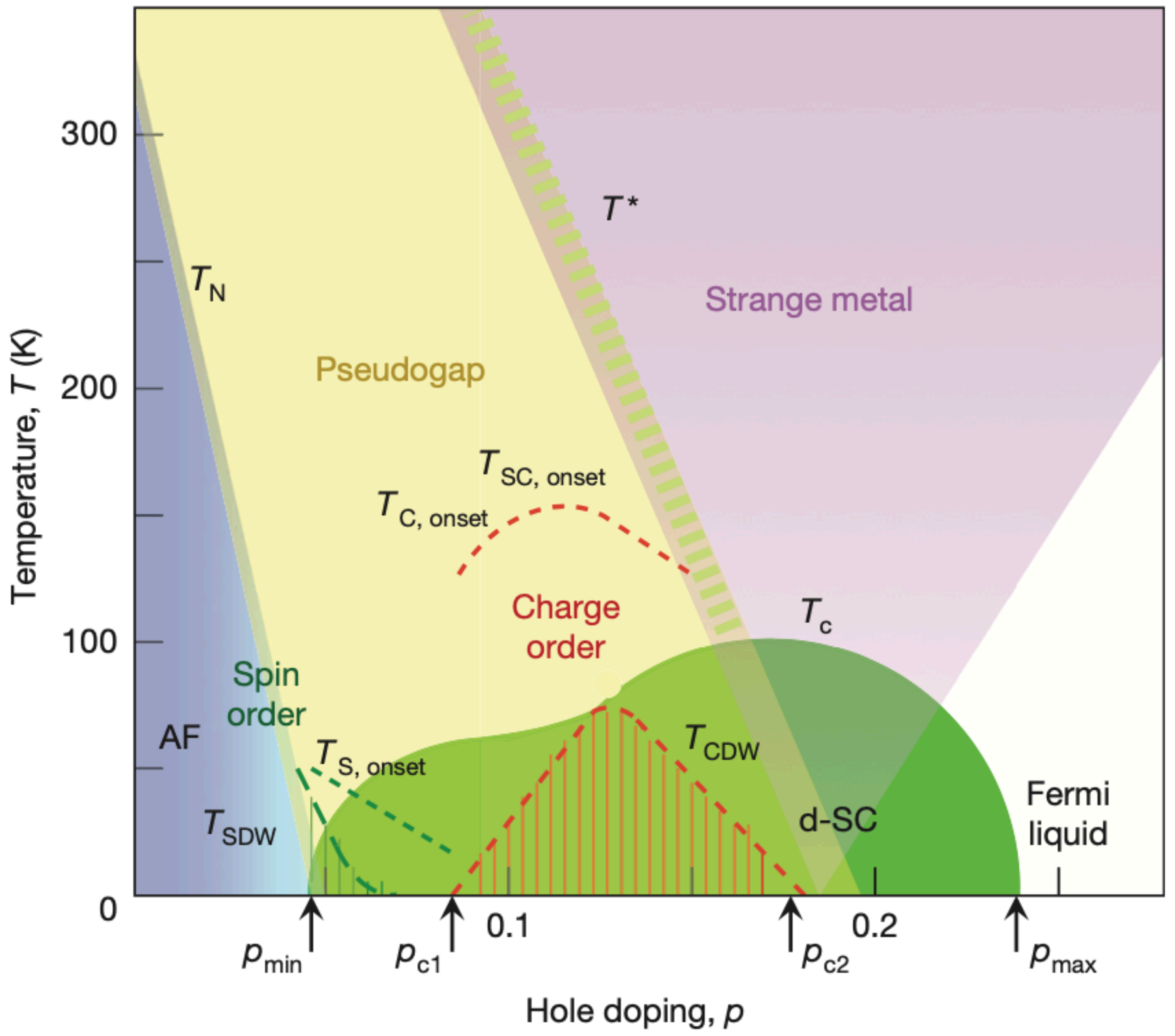

Strange metal phase <=> quantum critical region

Particle picture not works

- linear T resistivity

$$\rho_{FL} = a \cdot T^2 + b \cdot T^3 + \ldots$$

$$\rho_{SM} = a \cdot T$$

Policastro, Son, Starinets, hep-th/0104066

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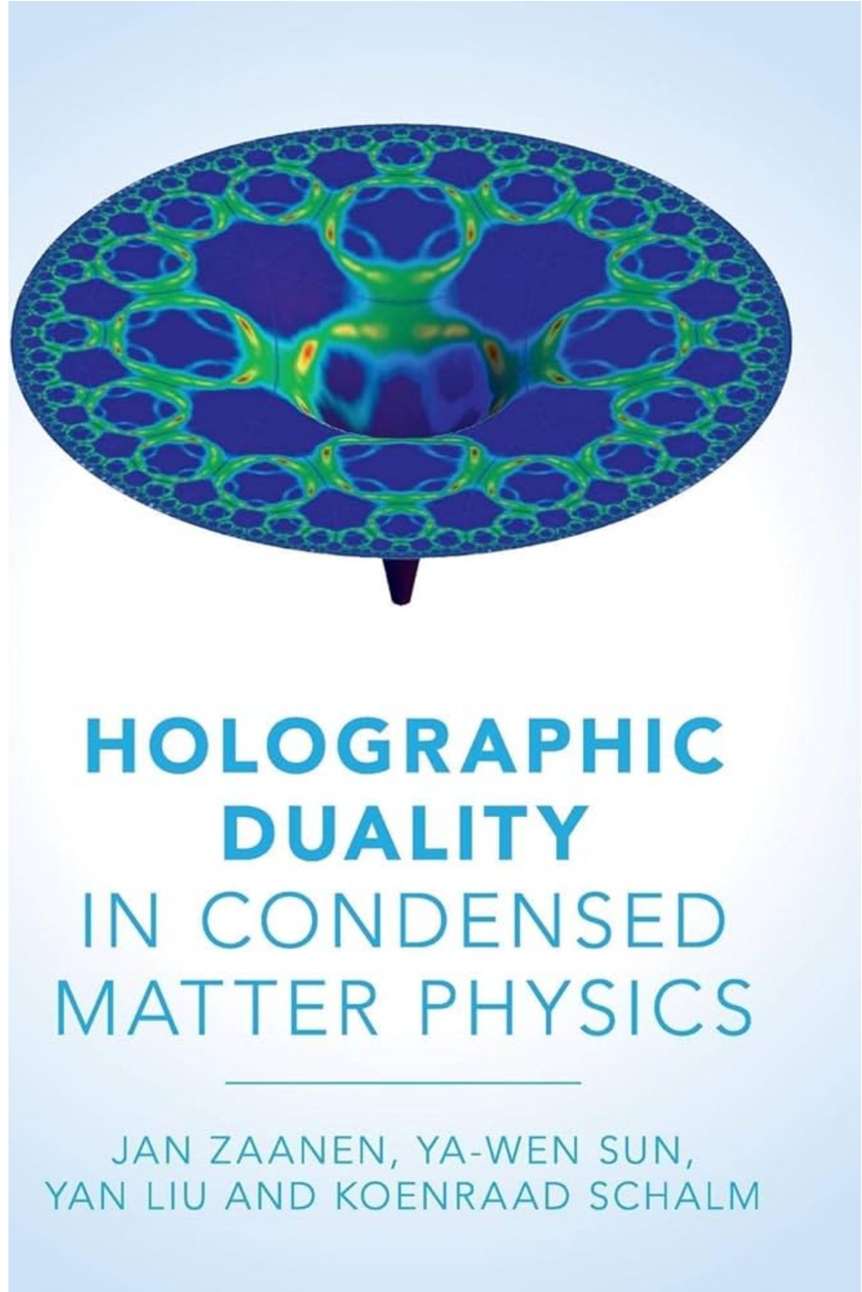
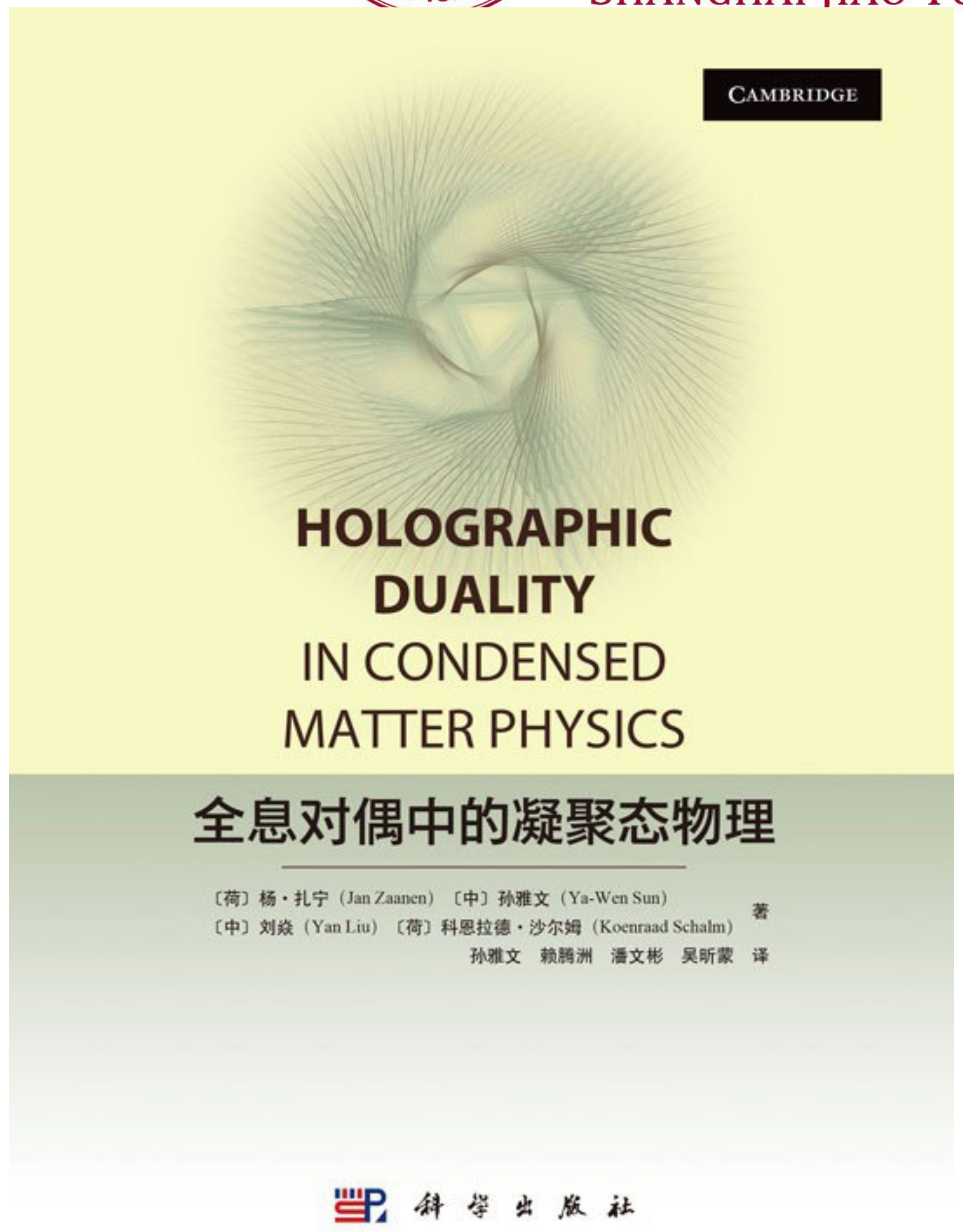
rapid hydrodynamization + minimal viscosity

Hydro and holography

- Quantum criticality and the minimal viscosity
- Deriving the Navier–Stokes fluid from the bulk dynamical gravity (fluid/gravity)
- Hydrodynamics and quantum anomalies





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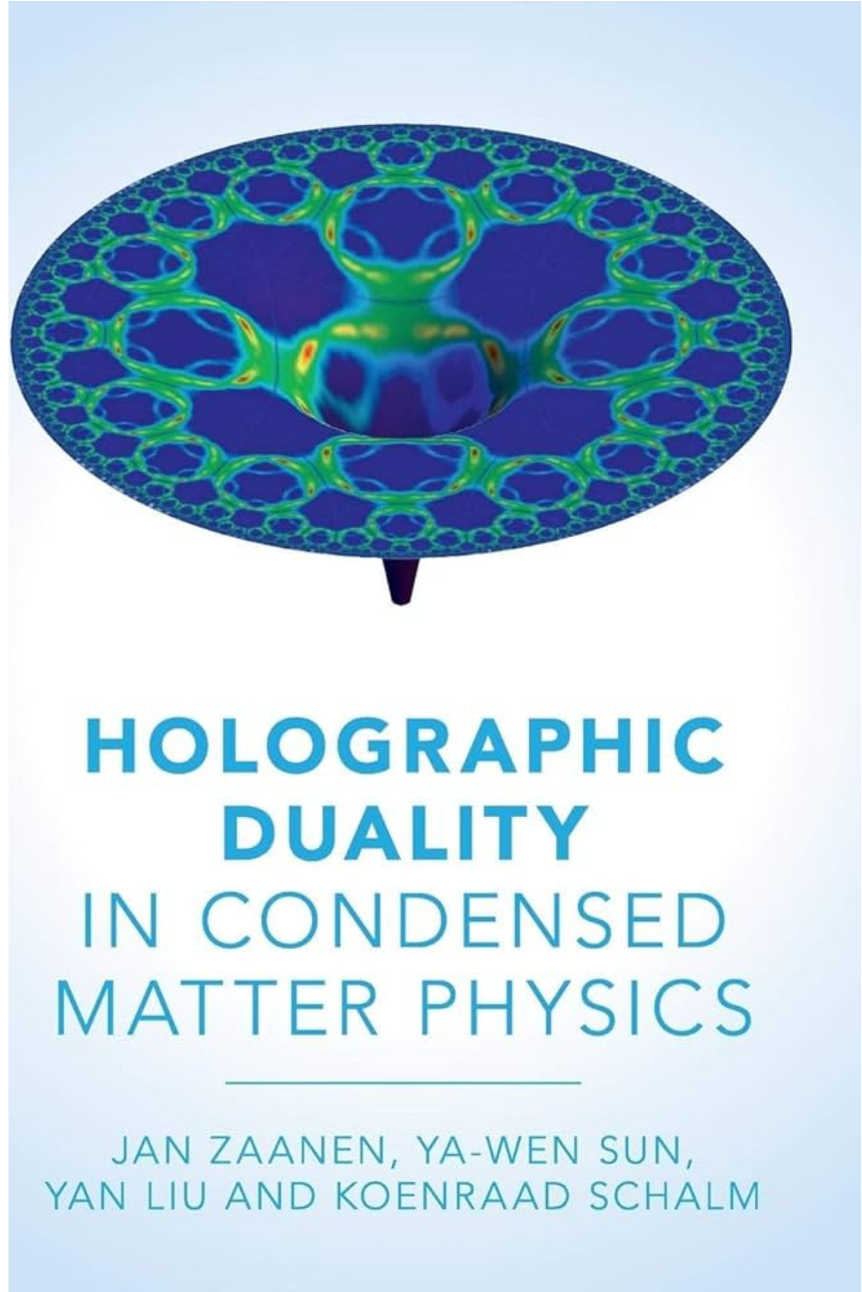
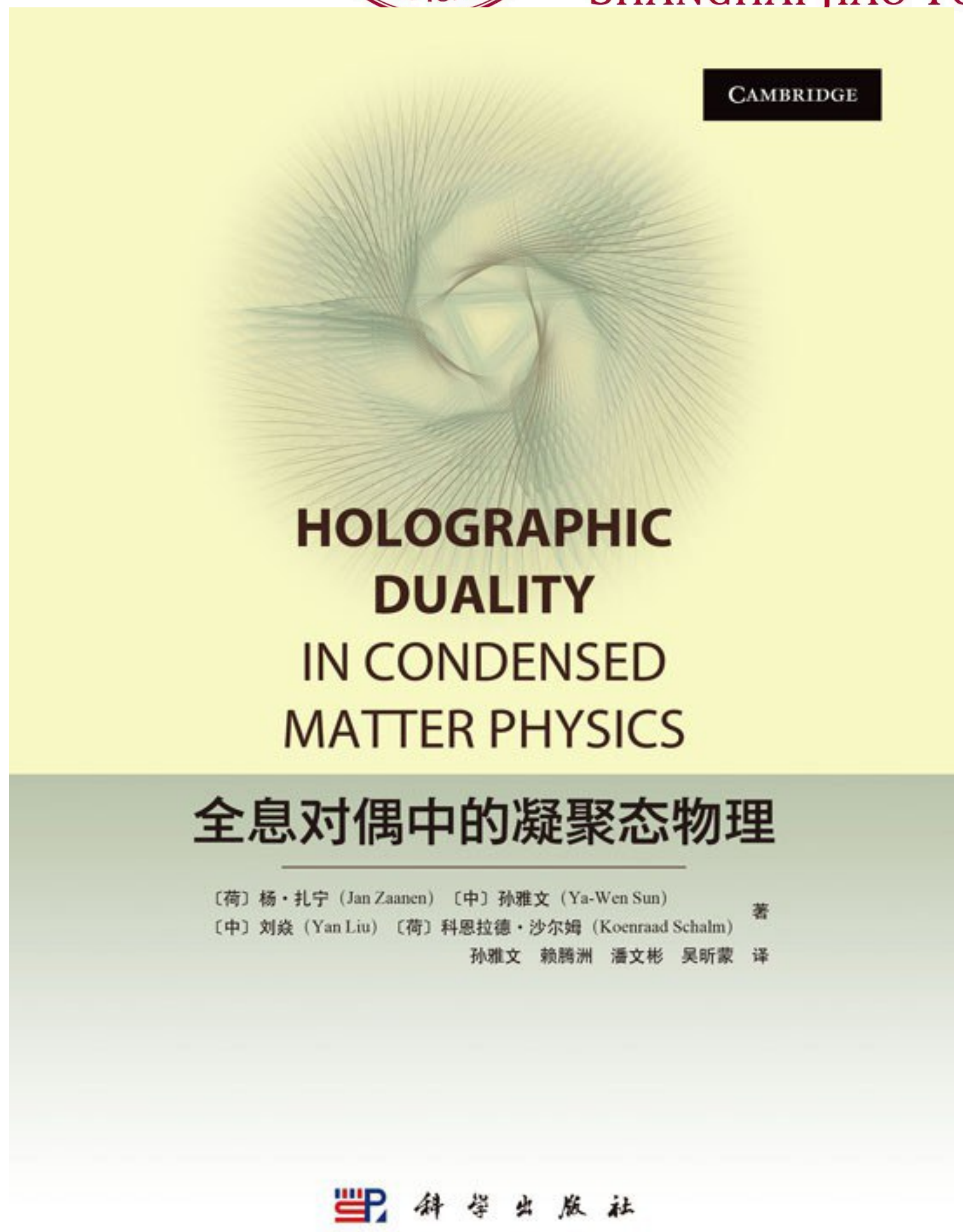
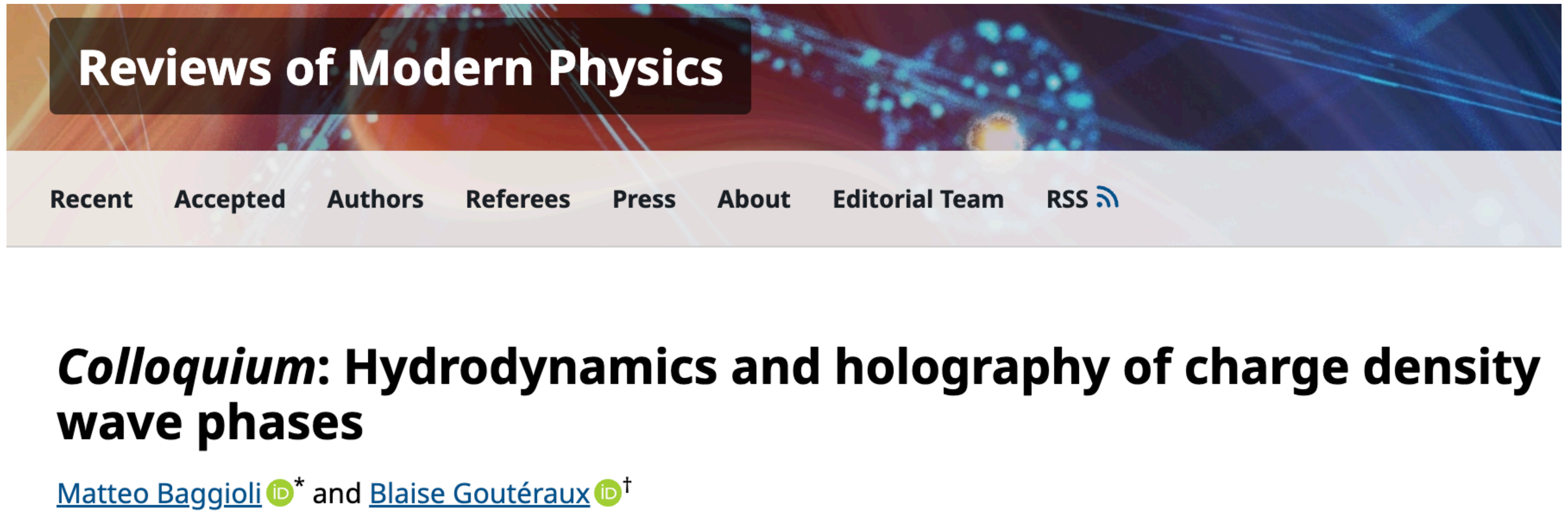
Many many others…





On exotic phases, e.g., charge density wave phases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***Colloquium*: Hydrodynamics and holography of charge density wave phases**

Matteo Baggioli* and Blaise Goutéraux†

Hydrodynamics: EOM is not enough

arXiv > hep-th > arXiv:1511.03646

High Energy Physics – Theory

[Submitted on 11 Nov 2015 (v1), last revised 27 Jan 2017 (this version, v3)]

Effective field theory of dissipative fluids

Michael Crossley, Paolo Glorioso, Hong Liu

D. Nickel, D. T. Son, J. de Boer, M. P. Heller, N. Pinzani-Fokeeva…….

Reasons

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Hydrodynamics is based on conserved equations + constitutive equations, it cannot capture effects of fluctuations, which is important in

- equilibrium time correlation functions(e.g. in liquid diffusion becomes long-time tail);
- dynamical critical phenomena;
- quantum phase transitions

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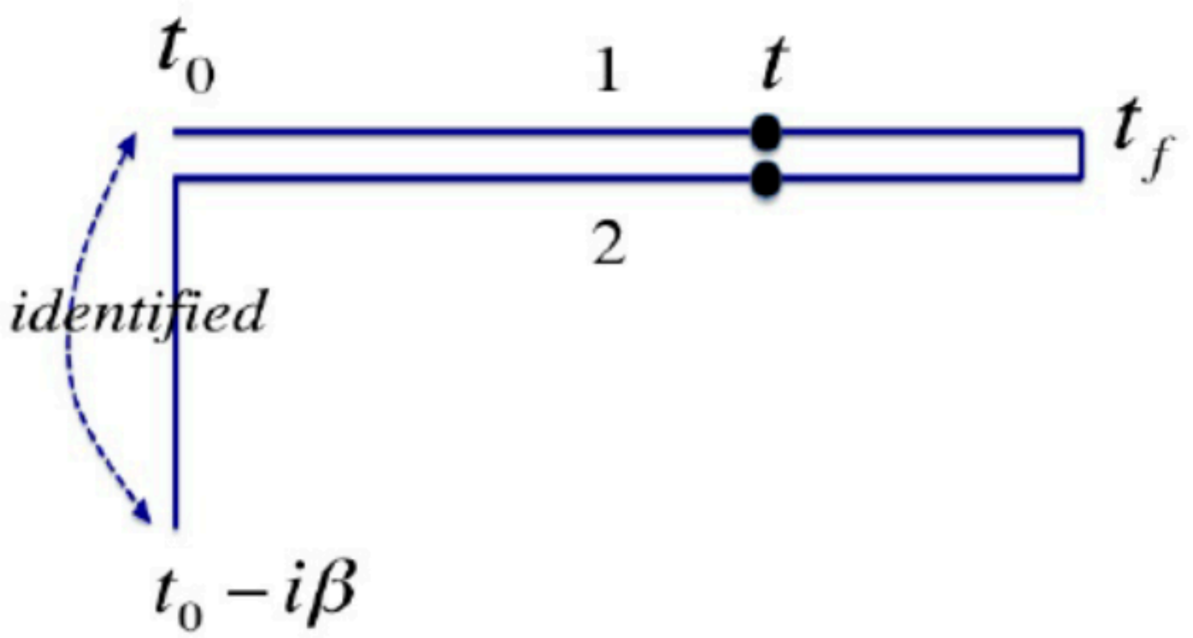
Currents (constitutive equations) are formulated phenomenologically, not from the generating functions.

Hydrodynamics: Effective action



U(1) diffusion as example

Use Schwinger-Keldysh formalism to define generating function around the equilibrium.

$$e^{iW[A_{1\mu},A_{2\mu}]} \equiv \mathrm{Tr}\left[\rho_0 \mathcal{P} e^{i\int d^dx\,(A_{1\mu}J_1^\mu - A_{2\mu}J_2^\mu)}\right]$$

For conserved current $\quad W[A_{1\mu}, A_{2\mu}] = W[A_{1\mu} + \partial_\mu\lambda_1, A_{2\mu} + \partial_\mu\lambda_2]$

arXiv > hep-th > arXiv:1805.09331

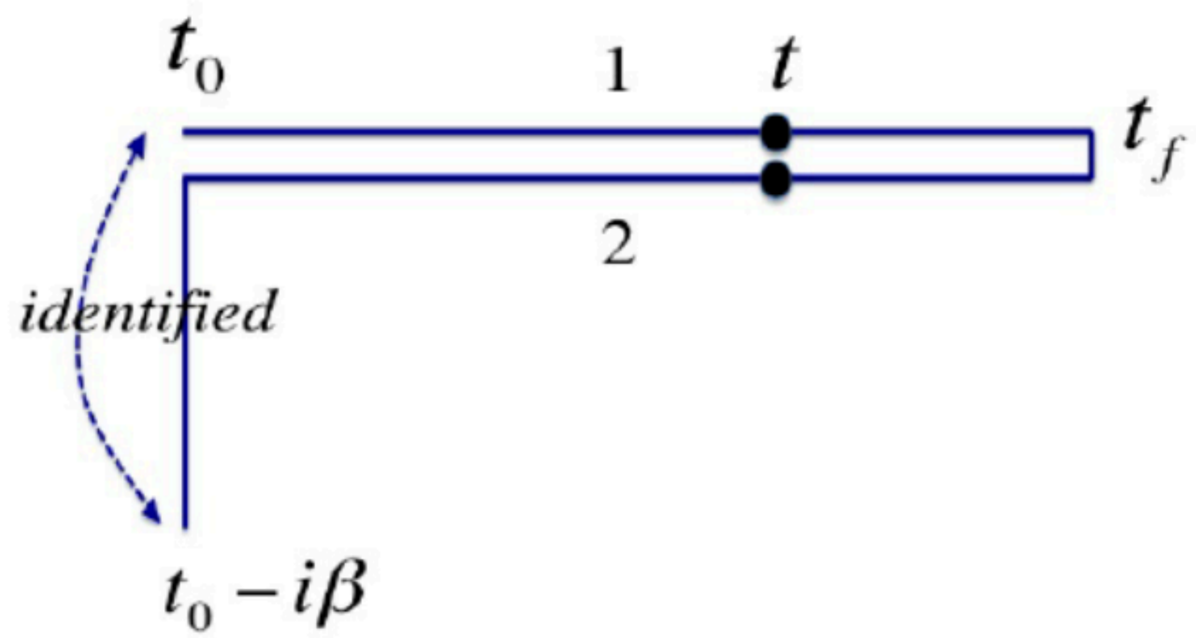
High Energy Physics – Theory

[Submitted on 23 May 2018]

Lectures on non-equilibrium effective field theories and fluctuating hydrodynamics

Paolo Glorioso, Hong Liu

Hydrodynamics: Effective action



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Integrate out the high-frequency variables, and only focus on low-energy variables, i.e. gapless mode in hydrodynamics.

$$e^{W[A_{1\mu},A_{2\mu}]} = \int D\varphi_1 D\varphi_2\, e^{iI_{\mathrm{EFT}}[B_{1\mu},B_{2\mu}]}$$

$$B_{1\mu} \equiv A_{1\mu} + \partial_\mu\varphi_1, \qquad B_{2\mu} \equiv A_{2\mu} + \partial_\mu\varphi_2$$

$$A_{1,2\mu} \to A_{1,2\mu} - \partial_\mu\lambda_{1,2}, \qquad \varphi_{1,2} \to \varphi_{1,2} + \lambda_{1,2}$$

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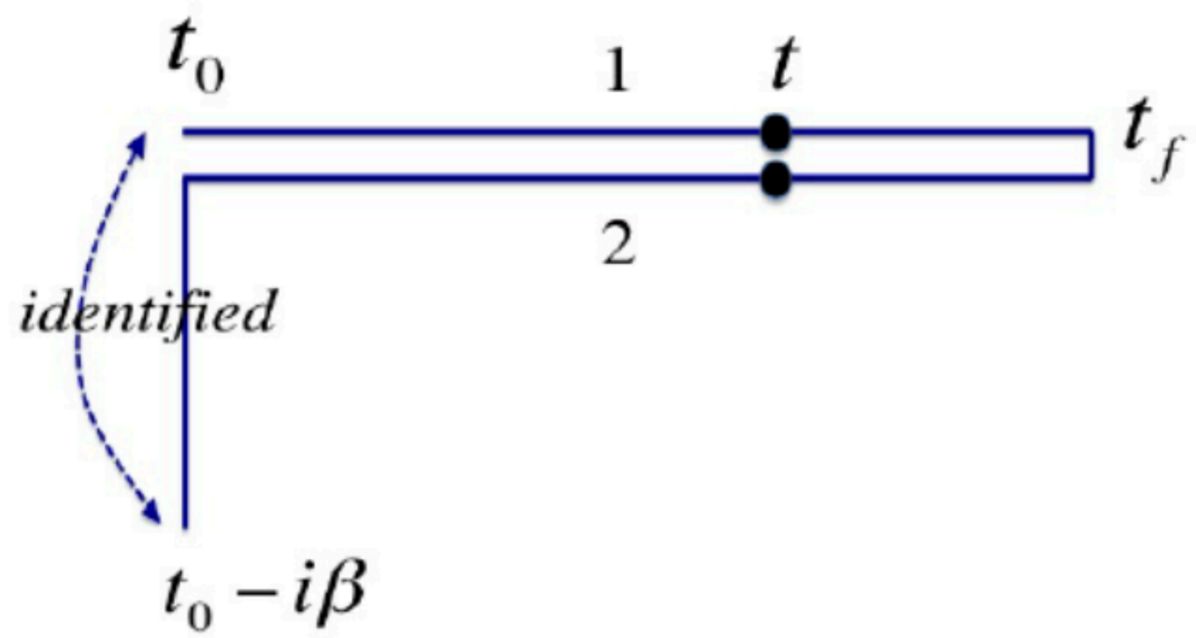

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$$B_{1\mu} \equiv A_{1\mu} + \partial_\mu\varphi_1, \qquad B_{2\mu} \equiv A_{2\mu} + \partial_\mu\varphi_2$$

Impose constraints in the effective action, such as symmetry, KMS, unitary and locality conditions.

$$A_{1,2\mu} \to A_{1,2\mu} - \partial_\mu\lambda_{1,2}, \qquad \varphi_{1,2} \to \varphi_{1,2} + \lambda_{1,2}$$

$$\frac{\delta I_{\mathrm{EFT}}}{\delta A_{1\mu}(x)} \equiv \hat{J}_1^\mu(x), \qquad \frac{\delta I_{\mathrm{EFT}}}{\delta A_{2\mu}(x)} \equiv -\hat{J}_2^\mu(x)$$

$$\frac{\delta I_{\mathrm{EFT}}}{\delta\varphi_{1,2}(x)} = -\partial_\mu \hat{J}^\mu_{1,2}(x) = 0$$

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Hydro EFT from holography

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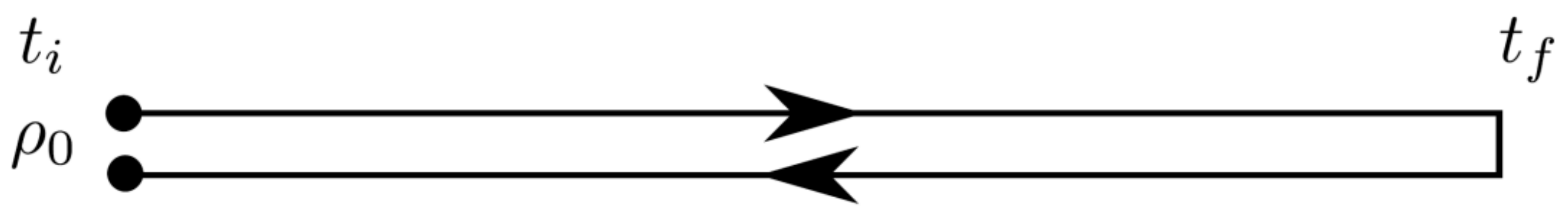

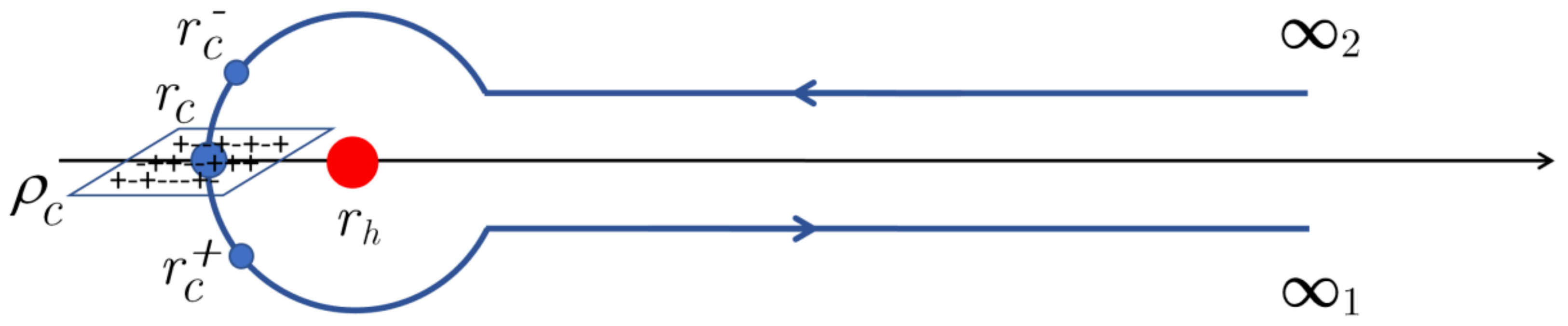

arXiv > hep-th > arXiv:1812.08785

High Energy Physics – Theory

[Submitted on 20 Dec 2018]

A prescription for holographic Schwinger-Keldysh contour in non-equilibrium systems

Paolo Glorioso, Michael Crossley, Hong Liu

Basic idea:

1. Complexify the radial coordinate in the bulk;
2. Compute the on-shell action along the closed loop.

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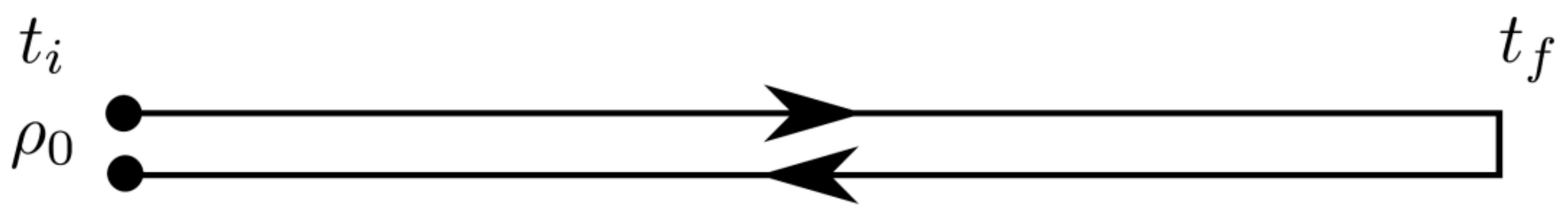

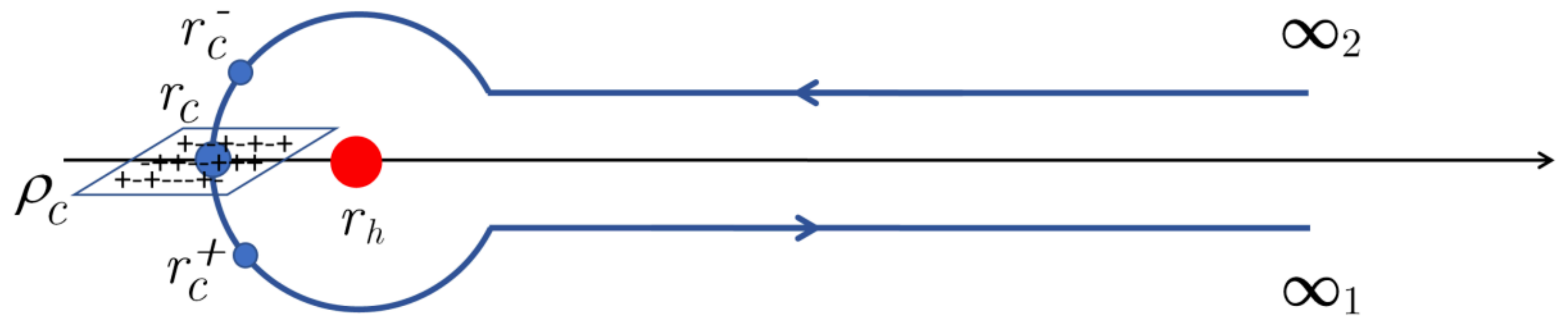

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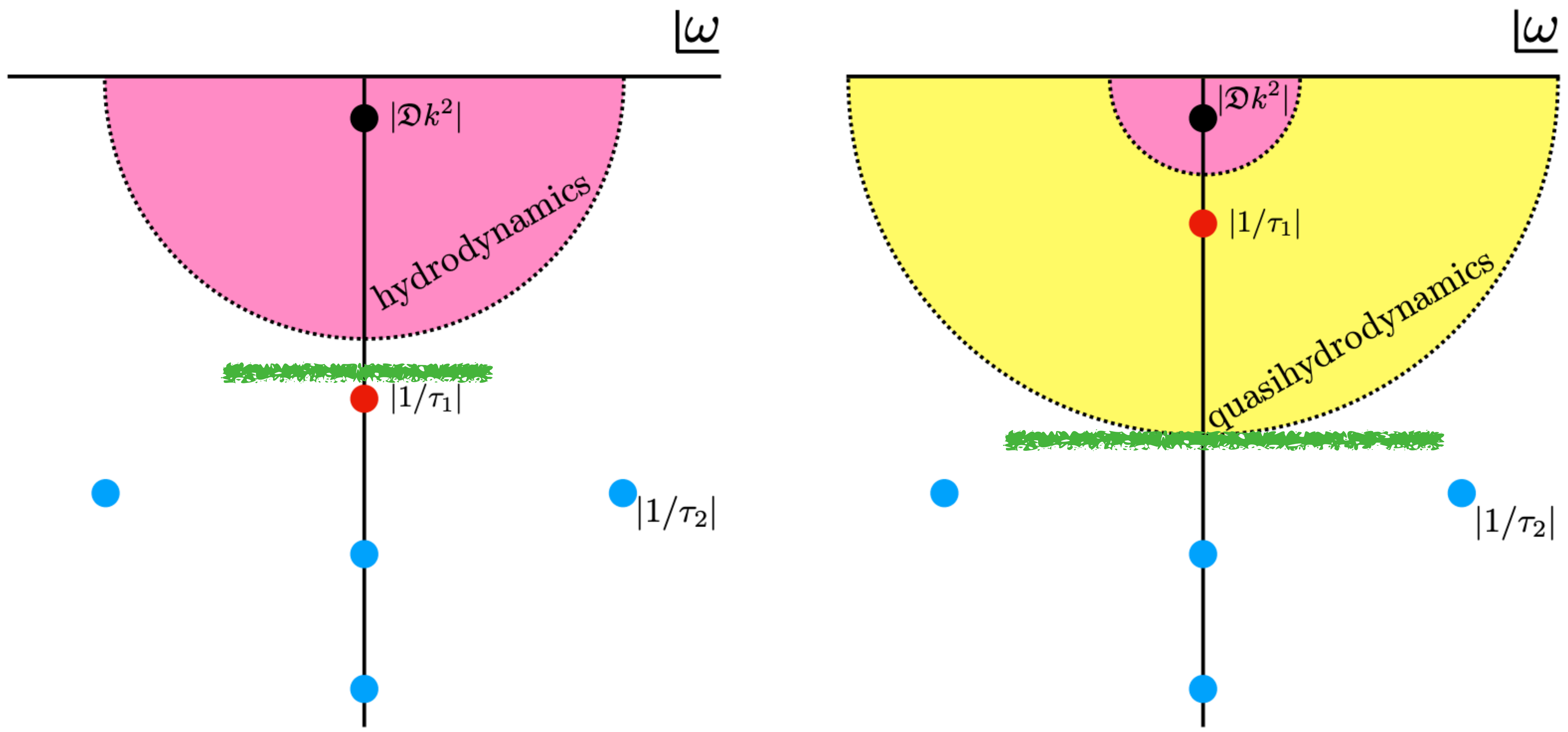
2. Compute the on-shell action along the closed loop.

It has been used in many physics, (pseudo-)Goldstone(e.g 2304.14173), chiral anomalous MHD(2412.02361), and generalized to open systems with non-Gaussian interactions(e.g. 2004.02888…)

Quasi-hydro: beyond hydrodynamics

Quasi-hydrodynamics picture



Symmetries are weakly broken

Examples:

MHD(charge relaxation)

Charge density wave(weak momentum relaxation)

Elasticity(higher form current)

Negative magneto-resistivity(Axial U(1))

……..

Grozdanov, Lucas, Poovuttikul, 1810.10016

Our question: EFT of quasi-hydro?

Properties we expect

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Properties we expect

- SK action: Unitary + Local + KMS
- The action generates the correct conserved equation + constitutive equation
- Include two modes, one is the gapless and another is gapped, from retarded Green's function

Difficulty

Our question: EFT of quasi-hydro?

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Properties we expect

- SK action: Unitary + Local + KMS
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Difficulty

- How to introduce a gapped mode, if we do not include more matter field?

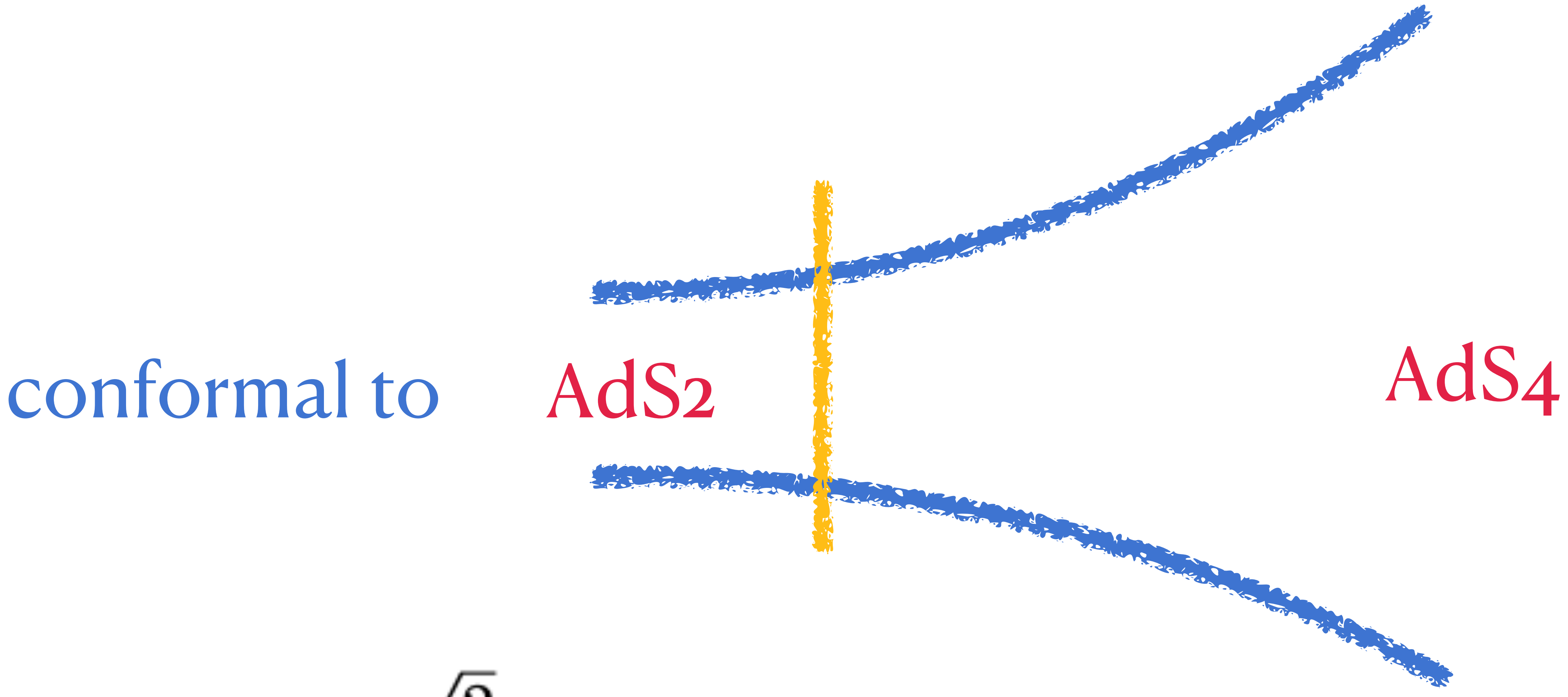
U(1) diffusion in conformal to AdS2

Our Model

$$S=\int d^4x\sqrt{-g}\left(R-\frac{3}{2}(\partial\phi)^2+\frac{6}{L^2}\cosh\phi-\frac{1}{2}\sum_{I=1}^{2}(\partial\psi_I)^2\right)$$

U(1) diffusion in conformal to AdS2

Our Model



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$$r=\frac{\sqrt{2}}{m\zeta^2}$$

$$(t,\ \zeta,\ x,\ y)\to(\lambda t,\ \lambda\zeta,\ x,\ y)$$

$$ds^2\to\lambda^{-1}ds^2$$

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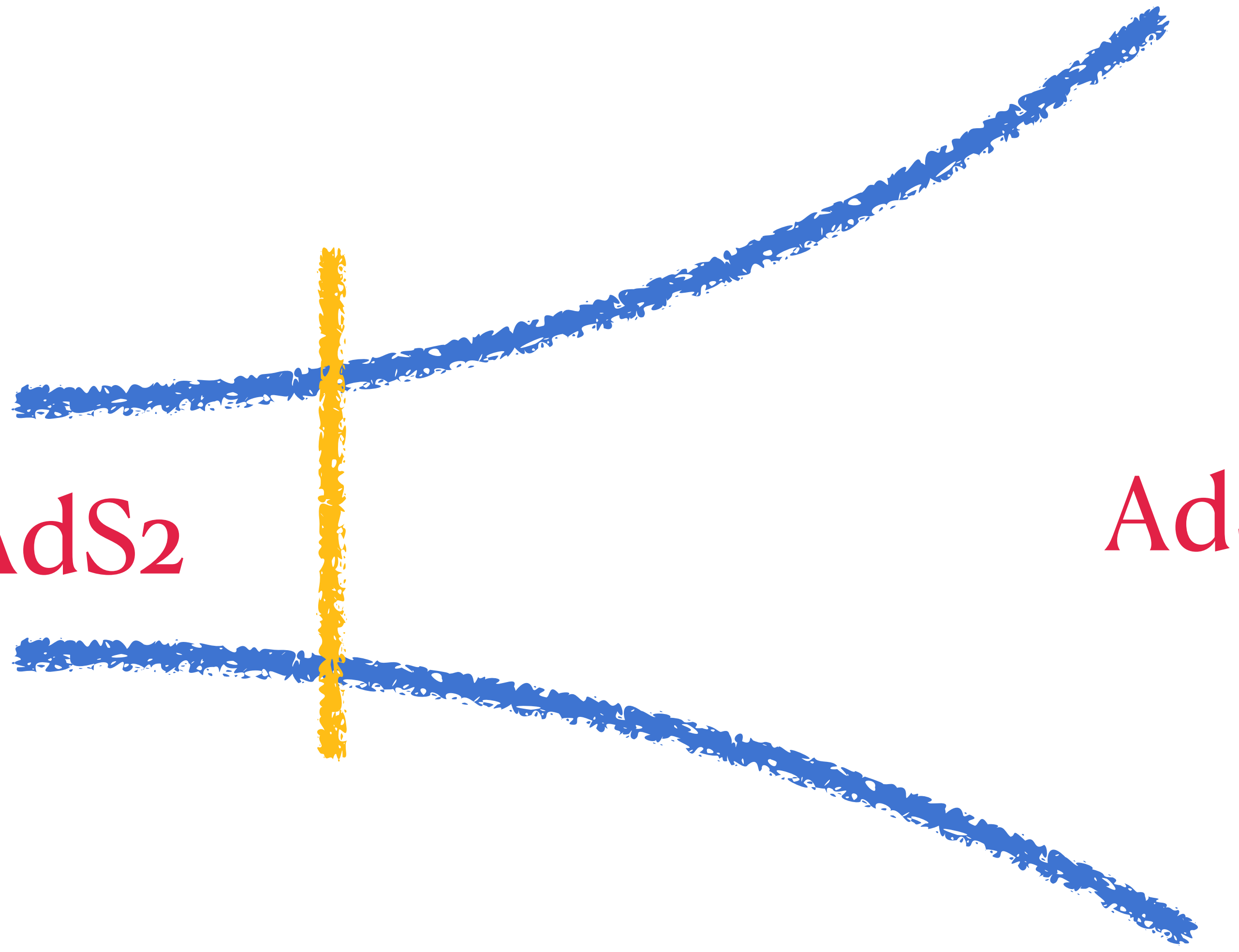

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$$ds^2 \to \lambda^{-1} ds^2$$

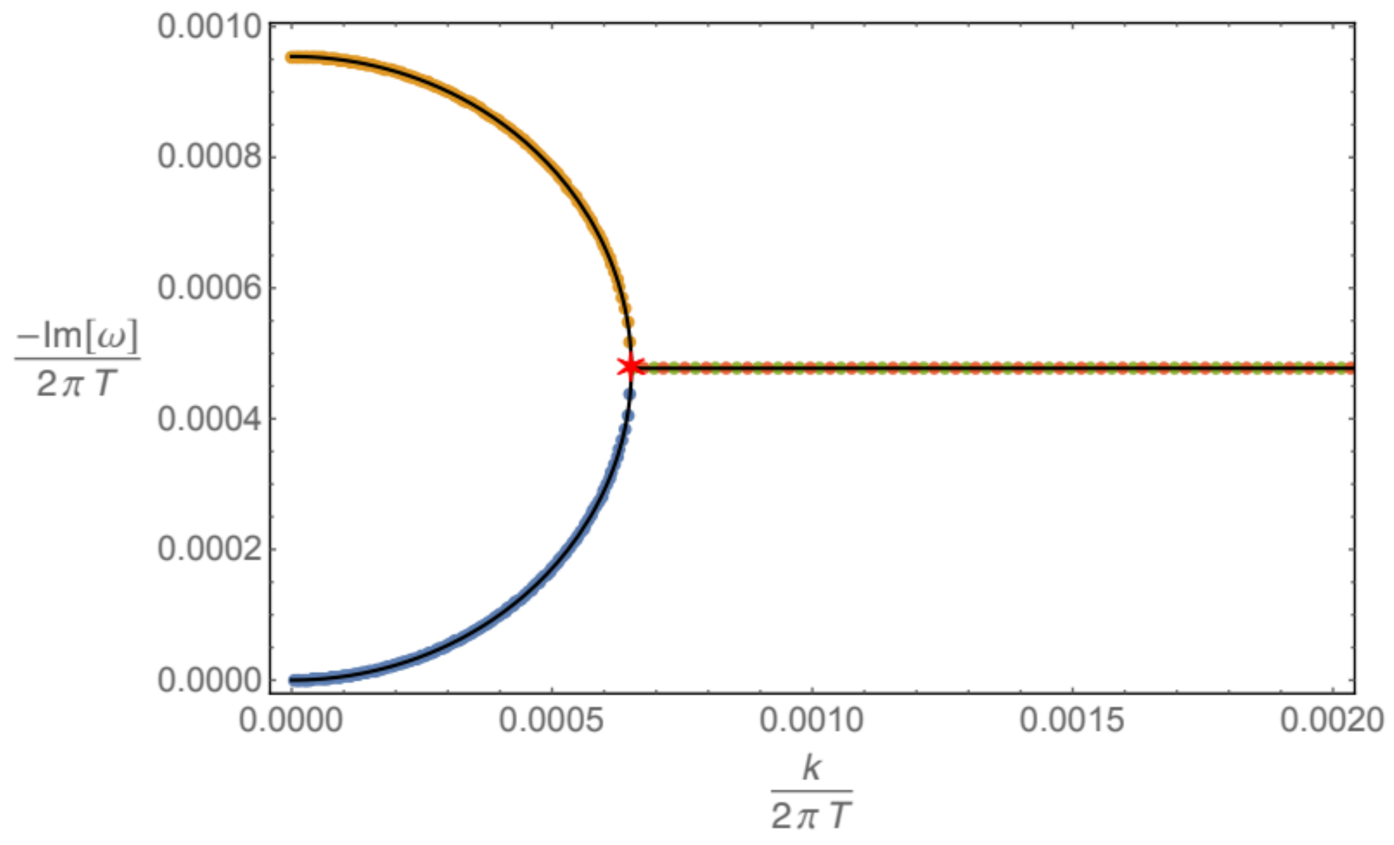
This is a famous model to study holographic strange metal, together with the linear-T resistivity, see for example, 1311.2451, 2211.05492

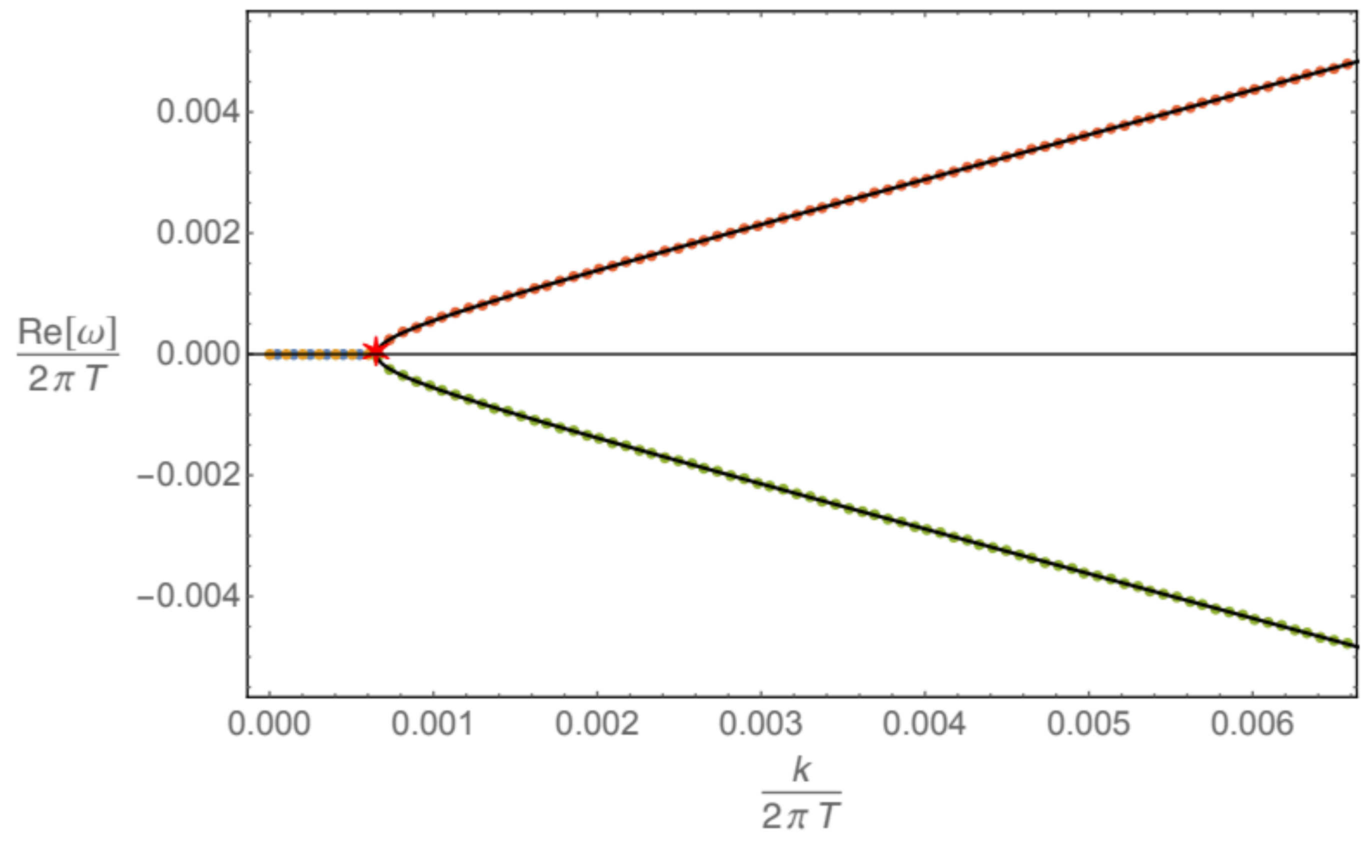
U(1) diffusion in conformal to AdS2

$\omega \ll T \ll m$

$$S = -\frac{1}{4}\int d^4x\sqrt{-g}\, e^{\alpha\phi} F_{mn}F^{mn}$$

Dispersion of gapless and gapped modes





$$\omega^2 + \frac{i}{\tau}\omega - \frac{D_c}{\tau}k^2 = 0$$

$$D_c = \int_{r_0}^{\infty} dr\, \frac{e^{\alpha\phi_0}}{f e^{\alpha\phi}}$$

Diffusion-sound crossover

$$\tau = \int_{r_0}^{\infty} dr \left(\frac{e^{\alpha\phi_0}}{u e^{\alpha\phi}} - \frac{1}{4\pi T}\,\frac{1}{r - r_0}\right)$$

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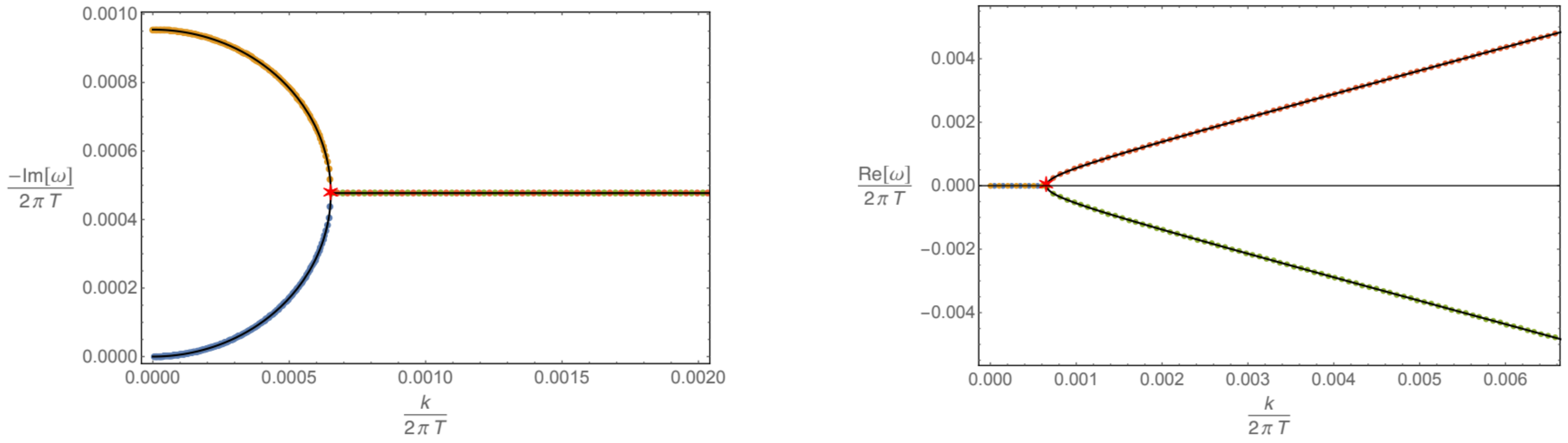

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Diffusion-sound crossover

The charge diffusive mode is natural, because it comes from the conservation law of the U(1) charge.
However, the gapped mode means it is a relaxed mode, relaxation of what?

Maxwell-Cattaneo model

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U(1) diffusion with a relaxed spatial current

Thermodynamics

$$d\epsilon = Tds + \mu dn + \frac{\chi_v}{2} dv^2$$

v_i

Constitutive equations

$$J^\mu = nu^\mu + \alpha_v v^\mu \qquad \alpha_v v^\mu = -\sigma \Delta^{\mu\nu} \left(T\partial_\nu(\mu/T) + \frac{\chi_v}{\alpha_v} \partial_t v_\nu \right)$$

0th order 1st order

relax with time

Conserved equations

$$\partial_t j^t + \partial_i j^i = 0\,, \qquad \partial_t j^i + \frac{D}{\tau}\partial_i j^t = -\frac{1}{\tau} j^i$$

Dispersion

$$i\omega(1 - i\omega\tau) - Dk^2 = 0$$

Akash Jain and Pavel Kovtun 2309.00511

MC model from holography: idea

To obtain the SK action, we compute the on-shell action along the contour in the complex radial coordinate.

$$
\begin{aligned}
e^{W[A_{1\mu},A_{2\mu}]} &= \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{iI_{\text{eff}}[B_1,B_2]} \\
&= \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \mathcal{D}b_{1\mu} \mathcal{D}b_{2\mu} e^{iS_{\text{eff}}[B_1,B_2,b_{1\mu},b_{2\mu}]}
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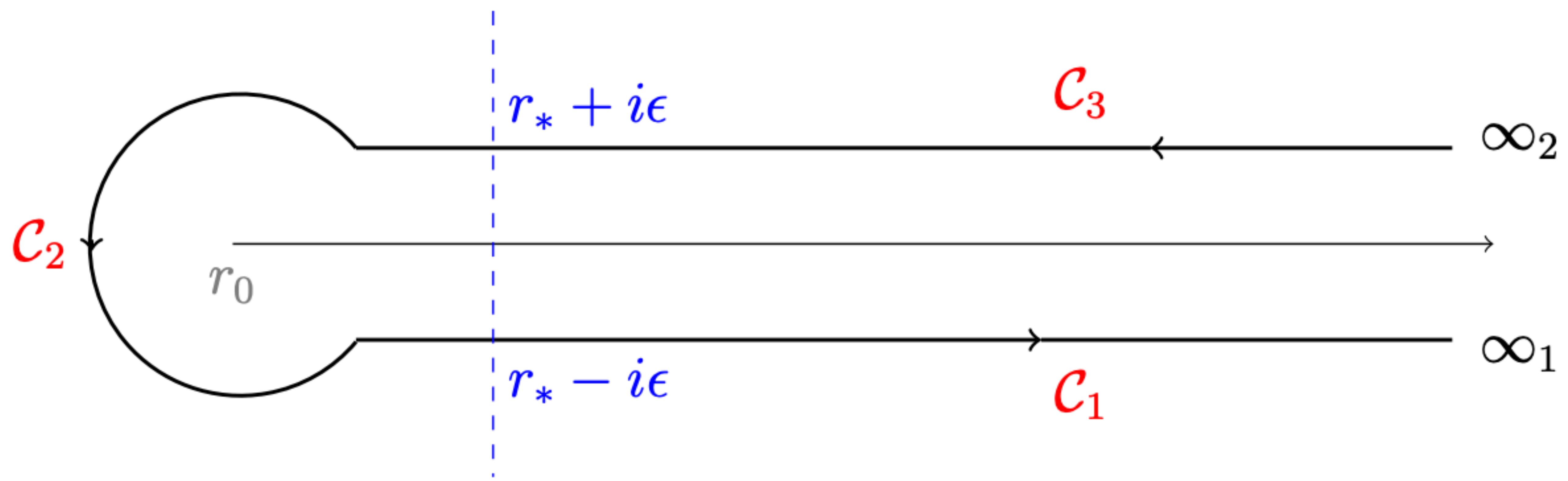

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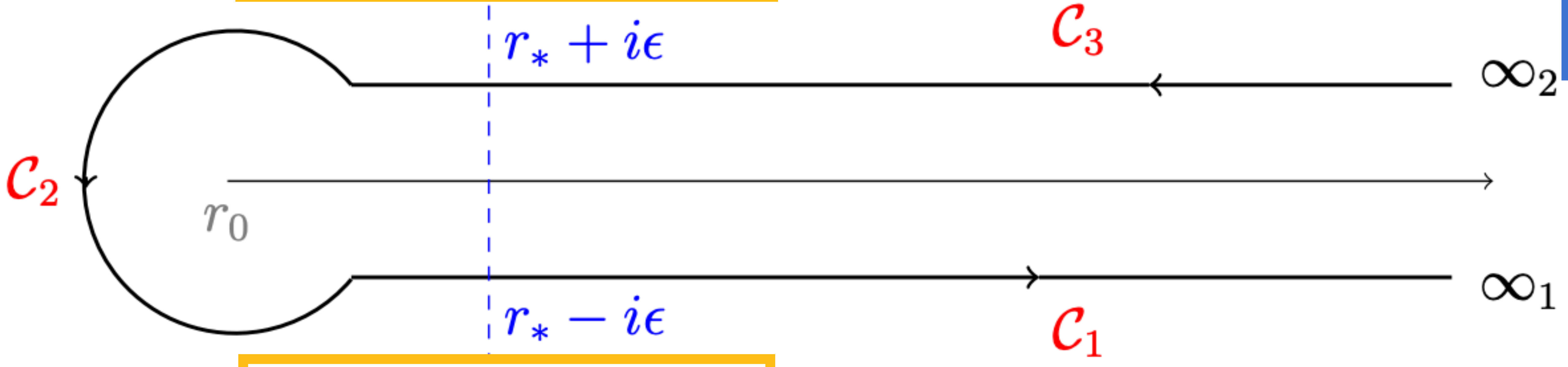
$$

Diffusive d.o.f

$$\boxed{C_\mu(r \to r_{*2}) = b_{2\mu}}$$

$$\boxed{C_\mu(r \to \infty_2) = B_{2\mu} = A_{2\mu} + \partial_\mu \varphi_2}$$



$$\boxed{C_\mu(r \to \infty_1) = B_{1\mu} = A_{1\mu} + \partial_\mu \varphi_1}$$

$$\boxed{C_\mu(r \to r_{*1}) = b_{1\mu}}$$

Spatial vector d.o.f

$$S_{\text{eff}}[B_{1\mu}, B_{2\mu}, b_{1\mu}, b_{2\mu}] = S_{\mathcal{C}_1 \cup \mathcal{C}_3}[b_{1\mu}, B_{1\mu}, b_{2\mu}, B_{2\mu}] + S_{\mathcal{C}_2}[b_{1\mu}, b_{2\mu}]$$

Our result

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The effective action
(up to 1st order derivative in time)

$$S = \int dv dx dy \Big[\chi B_{a0} B_{r0} + i\sigma T\, b_{ai} b_{ai} - \sigma\, b_{ai} \partial_0 b_{ri} - c_1 (B_{ai} - b_{ai})(B_{ri} - b_{ri})\Big]$$

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Recover Maxwell-Cattaneo

$$b_{ai} = B_{ai} + v_{ai}\,, \qquad b_{ri} = B_{ri} + v_{ri}$$

$$\mathcal{L} = \chi B_{a0} B_{r0} + i\sigma T (B_{ai} + v_{ai})(B_{ai} + v_{ai}) - c_1 v_{ai} v_{ri} - \sigma (B_{ai} + v_{ai}) \partial_0 (B_{ri} + v_{ri}),$$

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Recover Maxwell-Cattaneo

$$\mathcal{L} = \chi B_{a0} B_{r0} + i\sigma T (B_{ai} + v_{ai})(B_{ai} + v_{ai}) - c_1 v_{ai} v_{ri} - \sigma (B_{ai} + v_{ai}) \partial_0 (B_{ri} + v_{ri}),$$

$$b_{ai} = B_{ai} + v_{ai}\,, \qquad b_{ri} = B_{ri} + v_{ri}$$

Green's functions

$$G^R_{xx}(\omega, k) = \frac{-i\,\delta^2 W}{\delta A_{ax}(-\omega, -k)\,\delta A_{rx}(\omega, k)} = \frac{-\omega^2 \sigma}{i\omega(1 - i\omega\tau) - Dk^2}\,,$$

$$G^R_{yy}(\omega, k) = \frac{-i\,\delta^2 W}{\delta A_{ay}(-\omega, -k)\,\delta A_{ry}(\omega, k)} = \frac{i\omega\sigma}{1 - i\omega\tau}\,,$$

KMS $\quad G^S_{\mu\nu}(\omega, k) = i \coth\left(\frac{\beta\omega}{2}\right) \mathrm{Im} G^R_{\mu\nu}(\omega, k)$

Summary

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We construct SK EFT for a quasi-hydrodynamics, which recovers MC model
In parallel, we construct SK EFT for a semi-holographic system(I didn't show).

We go further from the holographic SK method, by introducing an IR cutoff in the contour and assign a IR vector field there.

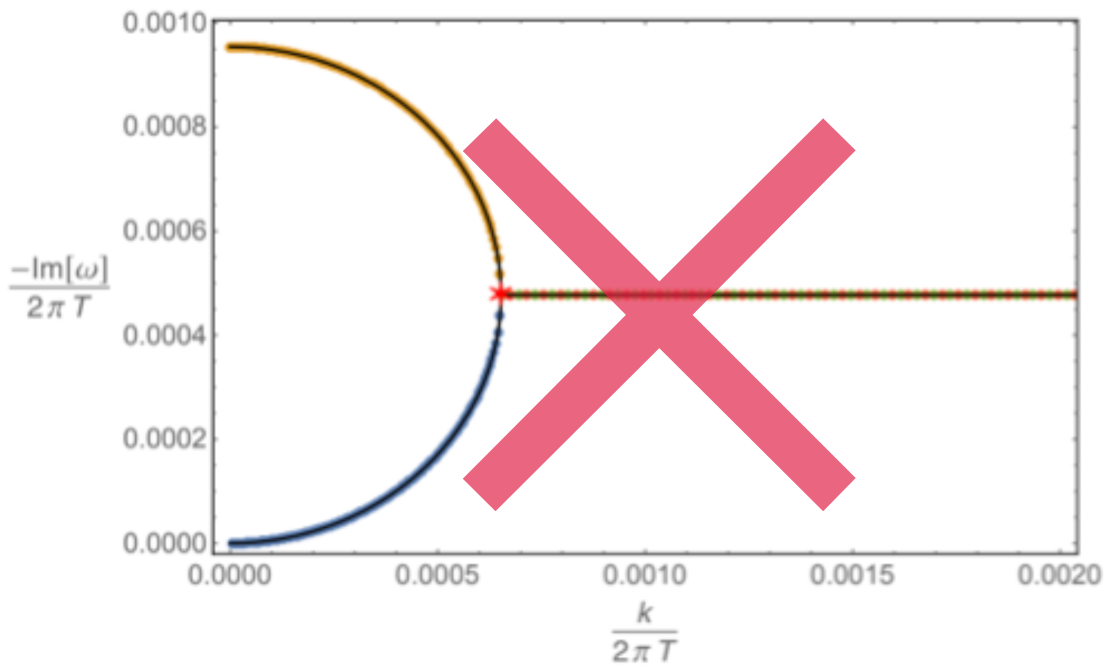
We expect this approach is universal to construct SK EFT beyond hydro limit.

Thank you!

Semi-holographic case

$$\omega \sim T \ll m$$

Gapless mode + gapped mode from IR CFT



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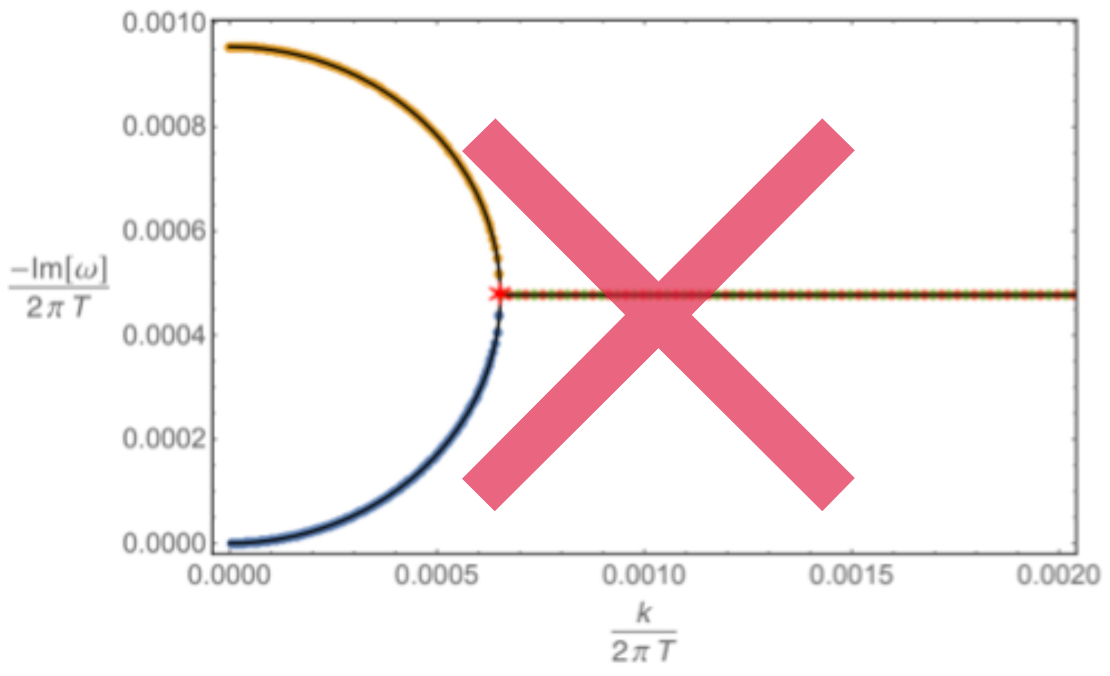
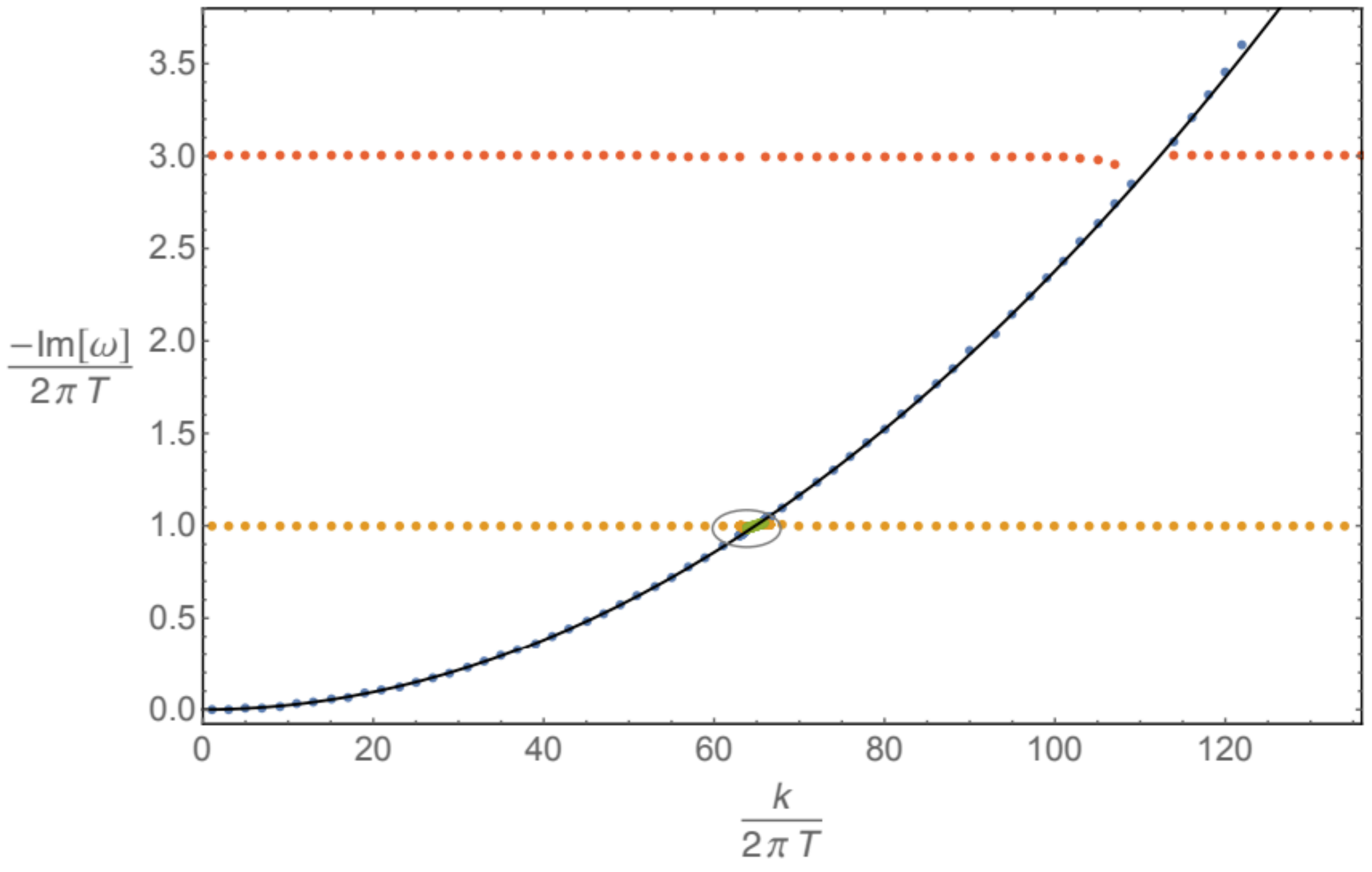
Semi-holographic case



$$\omega \sim T \ll m$$

Gapless mode + gapped mode from IR CFT

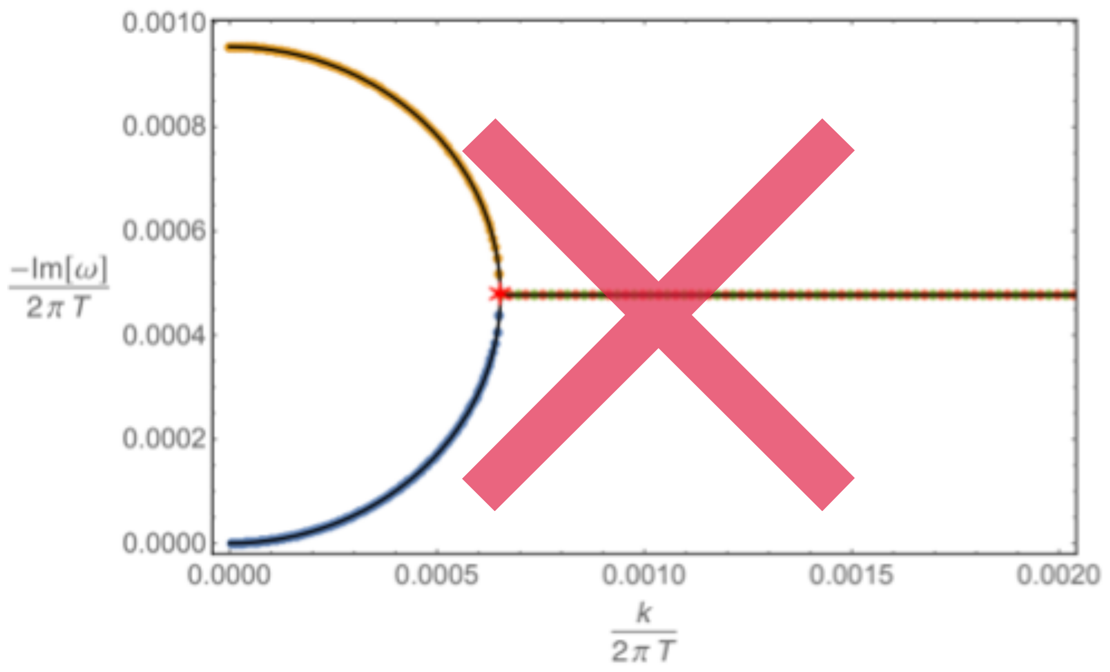


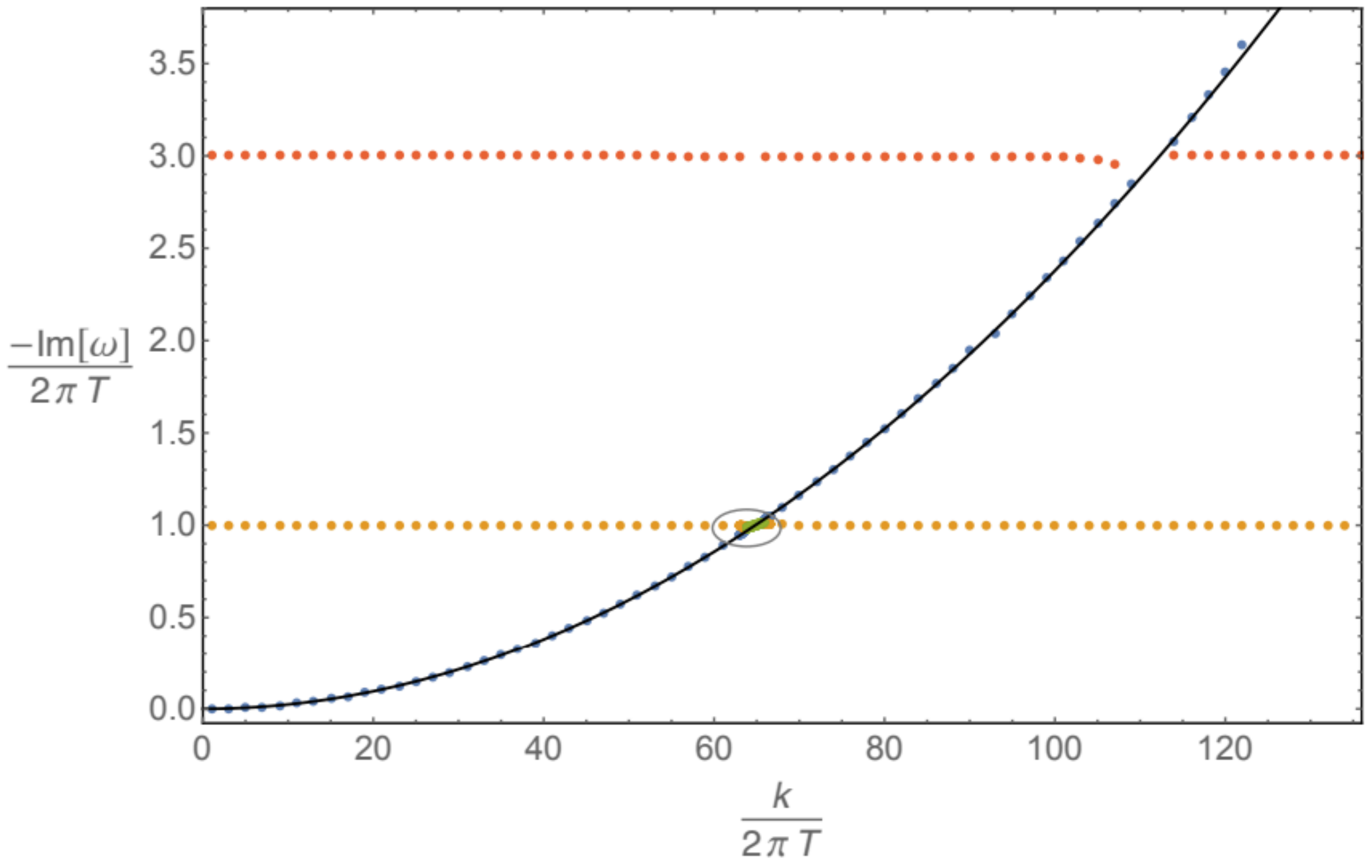
Energy diffusion Quantum Critical Points, 2011.12301;
Crystal diffusion, 2102.05810…

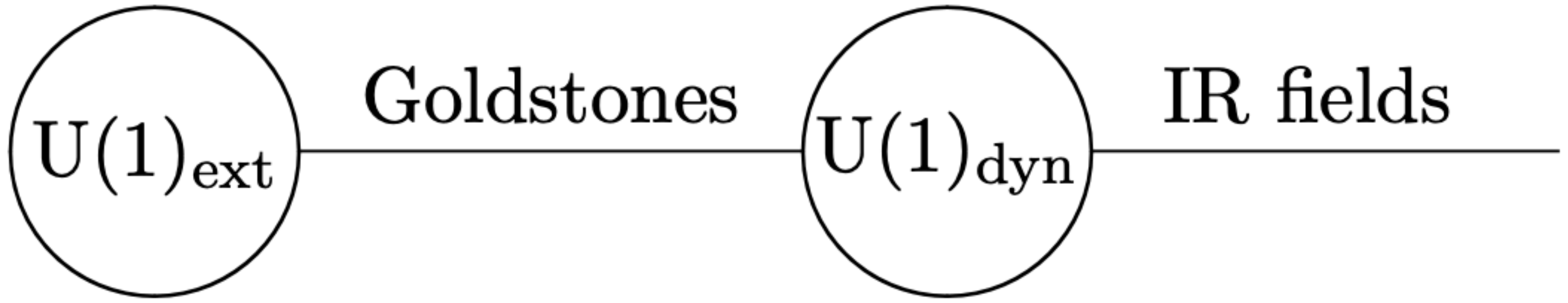
Semi-holographic case

$$\omega \sim T \ll m$$

Gapless mode + gapped mode from IR CFT







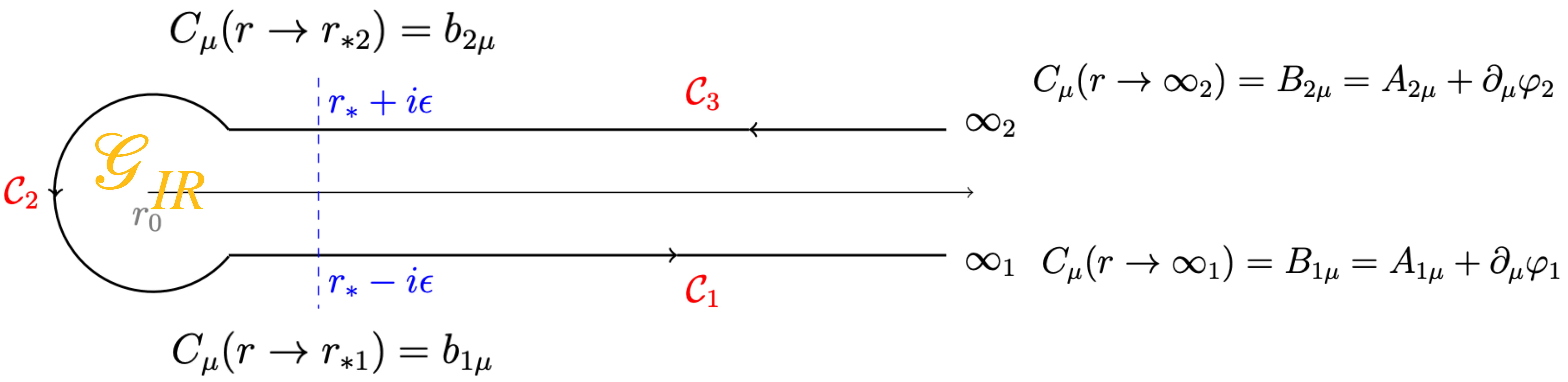
$$G_R(\omega,k) = K\frac{b_+^{(0)} + \omega b_+^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega)\left(b_-^{(0)} + \omega b_-^{(1)} + O(\omega^2)\right)}{a_+^{(0)} + \omega a_+^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega)\left(a_-^{(0)} + \omega a_-^{(1)} + O(\omega^2)\right)}$$

Energy diffusion Quantum Critical Points, 2011.12301;
Crystal diffusion, 2102.05810…

0907.2694, 1009.3094, 1001.5049

Semi-holographic case: method

Action, dispersion, KMS



$$
\begin{aligned}
S = \int dv dx dy \Big[\chi B_{a0} B_{r0} - c_1 (B_{ai} - b_{ai})(B_{ri} - b_{ri}) + c_2 (b_{ai}\partial_0 B_{ri} - B_{ai}\partial_0 b_{ri})\Big] + \\
+ \int \frac{d\omega dk_x dk_y}{(2\pi)^3} \Big[- 2\pi T^\alpha \sigma \coth\left(\frac{\beta\omega}{2}\right) \mathcal{G}_{\rm IR}(\omega, k, T)\, b_{ai} b_{ai} - 4\pi T^\alpha \sigma\, \mathcal{G}_{\rm IR}(\omega, k, T)\, b_{ai} b_{ri} \Big].
\end{aligned}
\tag{3.81}
$$

Semi-holographic case: Result

Green's functions & dispersions

$$G^S_{tt} = \frac{-i\,\delta^2 W}{\delta A_{at}(-\omega,-k)\,\delta A_{at}(\omega,k)} = \frac{-4i\pi T^\alpha \sigma k^2\omega^2 \coth(\frac{\beta\omega}{2})\,\mathrm{Im}(\mathcal{G}_{\mathrm{IR}})}{|(-\omega^2+4\pi T^\alpha D k^2\mathcal{G}_{\mathrm{IR}})-\omega^2 4\pi T^\alpha\tilde{\sigma}\mathcal{G}_{\mathrm{IR}}|^2},$$

$$G^S_{xx} = \frac{-i\,\delta^2 W}{\delta A_{ax}(-\omega,-k)\,\delta A_{ax}(\omega,k)} = \frac{-4i\pi T^\alpha \sigma \omega^4 \coth(\frac{\beta\omega}{2})\,\mathrm{Im}(\mathcal{G}_{\mathrm{IR}})}{|(-\omega^2+4\pi T^\alpha D k^2\mathcal{G}_{\mathrm{IR}})-\omega^2 4\pi T^\alpha\tilde{\sigma}\mathcal{G}_{\mathrm{IR}}|^2}, \quad (3.84)$$

$$G^S_{yy} = \frac{-i\,\delta^2 W}{\delta A_{ay}(-\omega,-k)\,\delta A_{ay}(\omega,k)} = \frac{-4i\pi T^\alpha \sigma \omega^4 \coth(\frac{\beta\omega}{2})\,\mathrm{Im}(\mathcal{G}_{\mathrm{IR}})}{|1+4\pi T^\alpha\tilde{\sigma}\mathcal{G}_{\mathrm{IR}}|^2},$$

$$G^R_{tt} = \frac{-i\,\delta^2 W}{\delta A_{at}(-\omega,-k)\,\delta A_{rt}(\omega,k)} = \frac{4\pi T^\alpha \sigma \mathcal{G}_{\mathrm{IR}} k^2}{(-\omega^2+4\pi T^\alpha D k^2\mathcal{G}_{\mathrm{IR}})-\omega^2 4\pi T^\alpha\tilde{\sigma}\mathcal{G}_{\mathrm{IR}}},$$

$$G^R_{xx} = \frac{-i\,\delta^2 W}{\delta A_{ax}(-\omega,k)\,\delta A_{rx}(\omega,k)} = \frac{\omega^2 4\pi T^\alpha \sigma \mathcal{G}_{\mathrm{IR}}}{(-\omega^2+4\pi T^\alpha D k^2\mathcal{G}_{\mathrm{IR}})-\omega^2 4\pi T^\alpha\tilde{\sigma}\mathcal{G}_{\mathrm{IR}}}, \quad (3.85)$$

$$G^R_{yy} = \frac{-i\,\delta^2 W}{\delta A_{ay}(-\omega,-k)\,\delta A_{ry}(\omega,k)} = \frac{-4\pi T^\alpha \sigma \tilde{\mathcal{G}}_{\mathrm{IR}}}{1+4\pi T^\alpha\tilde{\sigma}\tilde{\mathcal{G}}_{\mathrm{IR}}},$$

KMS

$$G^S_{\mu\nu}(\omega,k) = i\coth\left(\frac{\beta\omega}{2}\right)\mathrm{Im}G^R_{\mu\nu}(\omega,k)$$