Two-Sided Bounds of Higgs Operators

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Based on [2309.15922], JHEP (2024)
QC, Ken Mimasu, Tong Arthur Wu, Guo-Dong Zhang and Shuang-Yong Zhou

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Outline

Dispersion Relation

Scattering process $ij \to kl$ with amplitude A_{ijkl}, and $i,\ j,\ k,\ l = 1,\ 2, \ldots, N$, particles in the low energy effective theory.

$$\boxed{A_{ijkl}(s,t) = \frac{1}{2\pi i}\oint_{\mathcal{C}} ds' \frac{A_{ijkl}(s',t)}{s'-s}}$$

- Analyticity
- Froissart-Martin bound (M. Froissart, 1961; A. Martin, 1963)

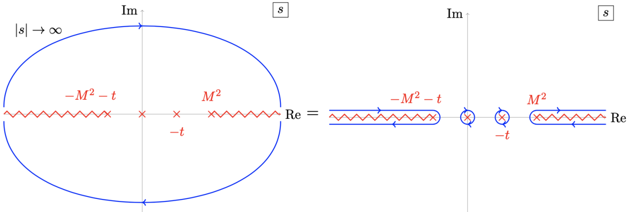


Figure from Caron-Huot & Duong, 2011.02957

$$A_{ijkl}(s,t) = \frac{\lambda_{ijkl}}{-s} + \frac{\lambda_{ijkl}}{-t} + \frac{\lambda_{ijkl}}{-u} + a^{(0)}_{ijkl}(t) + a^{(1)}_{ijkl}(t)s$$
$$+ \int_{\Lambda^2}^{\infty} \frac{d\mu}{\pi(\mu+\frac{t}{2})^2}\left[\frac{(s+\frac{t}{2})^2}{\mu-s}\mathrm{Im}\,A_{ijkl}(\mu,\,t) + \frac{(u+\frac{t}{2})^2}{\mu-u}\mathrm{Im}\,A_{ilkj}(\mu,\,t)\right]$$

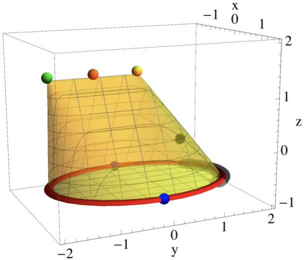
Unitarity and Positivity

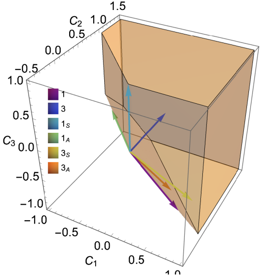
UV Unitarity of S-matrix: $S^\dagger S = \mathbf{1}$, $S = \mathbf{1} + iT$,

$$\text{Positivity}: \quad \boxed{2\mathrm{Im}T = T^\dagger T \succeq 0}$$

.

- The allowed space of Wilson coefficients is a convex cone.
  (C. Zhang & S.-Y. Zhou, 2005.03047; B. Bellazzini, L. Martucci & R. Torre, 1405.2960; C. Zhang, 2112.1665)

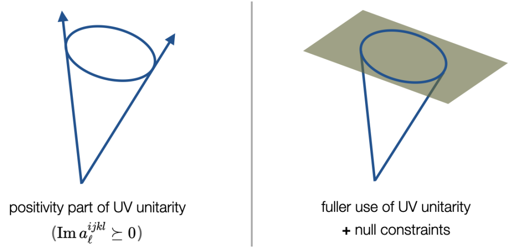




Unitarity and Positivity

Also inequalities from the relation $\boxed{2\mathrm{Im}T = T^\dagger T}$ itself: non-positivity part of UV unitarity conditions.

- The convex cone is capped from above and we have two-sided bounds.
  (QC, Ken Mimasu, Tong Arthur Wu, Guo-Dong Zhang & Shuang-Yong Zhou, 2309.15922)



Unitarity and Positivity

For a subspace of the whole external particle states space,

$$\boxed{\begin{aligned} &S^\dagger S \preceq \mathbf{1} \\ &(\mathbf{1} - \mathrm{Im}T)^2 + (\mathrm{Re}T)^2 \preceq \mathbf{1} \end{aligned}}$$

A weaker condition $(\mathbf{1} - \mathrm{Im}T)^2 \preceq \mathbf{1}$ leads to

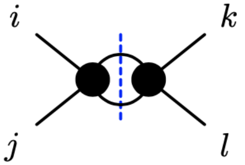
$$\boxed{\mathrm{Im}T \succeq 0\,, \quad \mathbf{2}\mathbf{1} - \mathrm{Im}T \succeq 0}$$

Partial Wave Positivity

Partial waves expansion: $\mathrm{Im}\, A_{ijkl}(\mu,\, t) = 16\pi \sum_{\ell=0}^{\infty}(2\ell+1)P_\ell\left(1+\frac{2t}{\mu}\right)\mathrm{Im}a_\ell^{ijkl}(\mu)$,

P_ℓ: Legendre Polynomial, $\rho_\ell^{ijkl} \equiv \mathrm{Im}a_\ell^{ijkl}$: partial wave spectral density

Unitarity: $\boxed{\mathrm{Im}\, a_\ell^{ijkl} = \sum a_\ell^{ij\to X}\left(a_\ell^{kl\to X}\right)^*}$



$$ii \to ii: \quad \mathrm{Im}a_\ell^{iiii} \geq \frac{1}{2}|a_\ell^{iiii}|^2 = \frac{1}{2}\left(\mathrm{Im}a_\ell^{iiii}\right)^2 + \frac{1}{2}\left(\mathrm{Re}a_\ell^{iiii}\right)^2 \Rightarrow \quad 0 \leq \mathrm{Im}a_\ell^{iiii} \leq 2,$$

$$ij \to ij: \quad \mathrm{Im}a_\ell^{ijij} \geq |a_\ell^{ijij}|^2 + |a_\ell^{ijji}|^2 \geq 2\big(\mathrm{Im}a_\ell^{ijij}\big)^2 \Rightarrow \quad 0 \leq \mathrm{Im}a_\ell^{ijij} \leq 1/2,$$

...

Null Constraints

The Mandelstam s, t, u crossing symmetry leads to a set of constraints on pole-subtracted amplitude $\tilde{A}_{ijkl} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{ijkl}^{m,n} \left(s + \frac{t}{2}\right)^m t^n$,

$$
\boxed{
\begin{aligned}
s \leftrightarrow u: &\quad \tilde{A}_{ijkl}(s,\, t) = \tilde{A}_{ilkj}(u,\, t) \\
s \leftrightarrow t: &\quad \tilde{A}_{ijkl}(s,\, t) = \tilde{A}_{ikjl}(t,\, s)
\end{aligned}
}
$$

Null constraints:

$$
s \leftrightarrow u: \quad \boxed{c_{ijkl}^{1,n}} + \boxed{c_{ilkj}^{1,n}} = 0
$$

$$
s \leftrightarrow t: \quad n_{ijkl}^{p,q} = \sum_{a=p}^{p+q} \frac{\Gamma(a+1) \boxed{c_{ijkl}^{a,p+q-a}}}{2^{a-p}\Gamma(p+1)\Gamma(a-p+1)} - \sum_{b=q}^{p+q} \frac{\Gamma(b+1) \boxed{c_{ikjl}^{b,p+q-b}}}{2^{b-q}\Gamma(p+1)\Gamma(b-p+1)} = 0
$$

Dispersion relation:

$$\text{L.H.S} \quad \color{red}{\text{IR}} \quad \boxed{c^{m,n}_{ijkl} \quad \doteq \quad \int \rho^{ijkl}_{\ell}} \quad \color{blue}{\text{UV}} \quad \text{R.H.S}$$

Positivity bound of dim-8 coefficient $c^{2,0}_{ijkl}$ by semi-definite program:

Variables

$$\rho^{ijkl}_{\ell}$$

Maximize/Minimize

$$c^{2,0}_{ijkl} = \sum_{\ell} 16(2\ell+1) \int_{\Lambda^2}^{\infty} \frac{d\mu}{\mu^3} \left[\rho^{ijkl}_{\ell}(\mu) + \rho^{ilkj}_{\ell}(\mu)\right] \qquad \text{(dispersion relation)}$$

Subject to Positivity inequalities of ρ^{ijkl}_{ℓ} (UV unitarity)

Null constraints of ρ^{ijkl}_{ℓ} (crossing symmetry)

Single-field Positivity Bound

A single real scalar odd under $\mathbf{Z}_2$, we only have one dim-8 coefficient $c^{2,0}$.

Variables

$\rho_{\ell,n}$

Maximize

$$c^{2,0} = \frac{1}{\Lambda^4} \sum_{\ell=0,2,\ldots,\ell_M;\ell_\infty} (2\ell+1) \sum_{n}^{N} \frac{1}{N} \frac{n}{N} 32 \rho_{\ell,n}$$

Subject to

$$0 \leq \rho_{\ell,n} \leq 2$$

$$\sum_{\ell=0,2,\ldots,\ell_M;\ell_\infty} (2\ell+1) \sum_{n}^{N} \frac{1}{N} \left(\frac{n}{N}\right)^3 \left(\ell^4 + 2\ell^3 - 7\ell^2 - 8\ell\right) \rho_{\ell,n} = 0$$

...

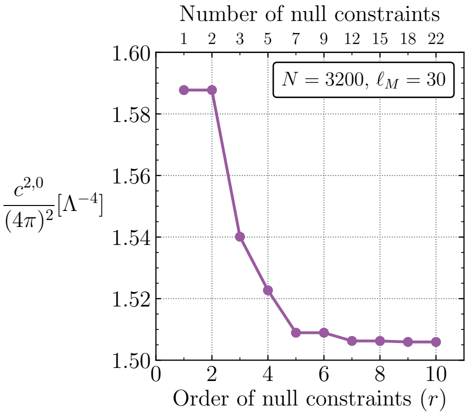
Single-field Positivity Bound

The upper bound of single scalar dim-8 coefficient is solved numerically by Linear Programming.

- The bound is

$$\boxed{0 \leq \frac{c^{2,0}}{(4\pi)^2} \leq \frac{1.506}{\Lambda^4}}$$

(QC, Ken Mimasu, Tong Arthur Wu, Guo-Dong Zhang & Shuang-Yong Zhou, 2309.15922)



Higgs EFT

The SM Higgs is an $SU(2)$ doublet complex scalar field $H = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2,\, \phi_3 + i\phi_4)^T$.
The dim-8 Higgs operators are

$$
\begin{aligned}
\mathcal{O}_{H^4}^{(1)} &= \left(D_\mu H^\dagger D_\nu H\right)\left(D^\nu H^\dagger D^\mu H\right) \\
\mathcal{O}_{H^4}^{(2)} &= \left(D_\mu H^\dagger D_\nu H\right)\left(D^\mu H^\dagger D^\nu H\right) \\
\mathcal{O}_{H^4}^{(3)} &= \left(D^\mu H^\dagger D_\mu H\right)\left(D^\nu H^\dagger D_\nu H\right)
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
C_1 &= -c_{1212}^{2,0} + 2c_{1313}^{2,0} \\
C_2 &= c_{1212}^{2,0} \\
C_3 &= c_{1111}^{2,0} - 2c_{1313}^{2,0}
\end{aligned}
}
$$

$$
\begin{aligned}
&2c_{1122}^{2,0} = c_{1111}^{2,0} - c_{1212}^{2,0} \\
&2c_{1133}^{2,0} = c_{1144}^{2,0} = c_{1111}^{2,0} - c_{1313}^{2,0} \\
&c_{1414}^{2,0} = c_{1313}^{2,0} \\
&2c_{1234}^{2,0} = c_{1212}^{2,0} - c_{1313}^{2,0} \\
&c_{1234}^{2,0} + c_{1243}^{2,0} = 0 \\
&c_{1324}^{2,0} = 0 \\
&c_{ijkl}^{2,0} = c_{ijkl}^{2,0}\Big|_{1\leftrightarrow 3,\, 2\leftrightarrow 4} \\
&c_{ijkl}^{2,0} = c_{ijkl}^{2,0}\Big|_{1\leftrightarrow 2,\, 3\leftrightarrow 4}
\end{aligned}
$$

Higgs Positivity Bounds

Semi-definite program for coefficients of Higgs dim-8 operators:

Variables

$$R^{1111}_{\ell,n},\ R^{11ii}_{\ell,n},\ R^{1i1i}_{\ell,n},\ \rho^{1jkl}_{\ell,n}\,, \qquad i,j,k,l \in \{2,3,4\}$$

Maximize/Minimize

$$\sum_I \alpha_I C_I\,, \qquad \text{where}$$

$$C_1 = \frac{1}{\Lambda^4}\sum_{\ell=0}^{\ell_M;\ell_\infty}(2\ell+1)\sum_{n=1}^{N}\frac{1}{N}\frac{n}{N}32(-R^{1212}_{\ell,n}+2R^{1313}_{\ell,n})\,,$$

$$C_2 = \frac{1}{\Lambda^4}\sum_{\ell=0}^{\ell_M;\ell_\infty}(2\ell+1)\sum_{n=1}^{N}\frac{1}{N}\frac{n}{N}32R^{1212}_{\ell,n}\,,$$

$$C_3 = \frac{1}{\Lambda^4}\sum_{\ell=0}^{\ell_M;\ell_\infty}(2\ell+1)\sum_{n=1}^{N}\frac{1}{N}\frac{n}{N}32(R^{1111}_{\ell,n}-2R^{1313}_{\ell,n})\,,$$

Subject to

Positivity inequalities

Null constraints and gauge symmetry constraints

Higgs Positivity Bounds

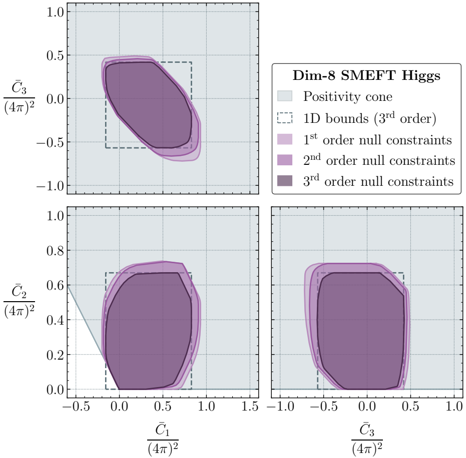


Figure: 2D bounds for Higgs dim-8 operators.(QC, Ken Mimasu, Tong Arthur Wu, Guo-Dong Zhang & Shuang-Yong Zhou, 2309.15922)

Experimental Comparison

The positivity bounds are complementary to experimental limits.

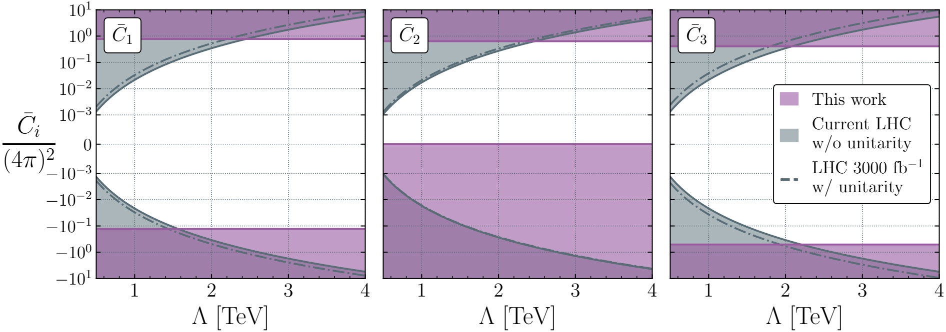


Figure: Bounds of Higgs dim-8 operators coefficients, compared to experimental limits.
(QC, Ken Mimasu, Tong Arthur Wu, Guo-Dong Zhang & Shuang-Yong Zhou, 2309.15922)

Higgs Positivity Bounds

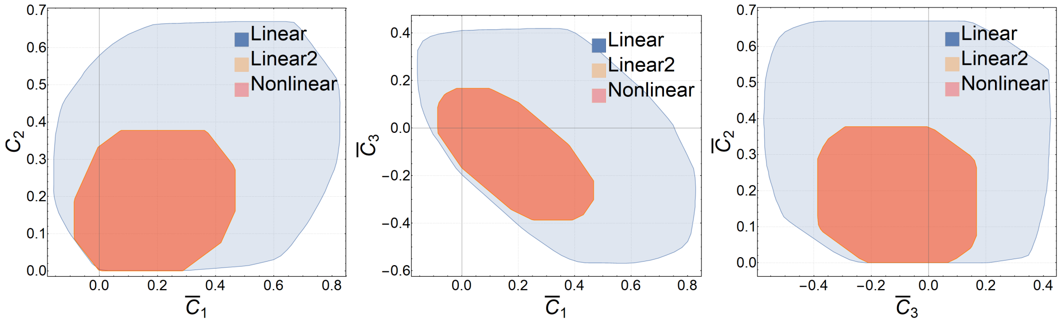


Figure: Bounds of Higgs dim-8 operators coefficients. Orange and yellow regions are the results of semi-definite program in complex basis (Dong-Yu Hong, Zhuo-Hui Wang & Shuang-Yong Zhou, 2404.04479) with nonlinear and linear unitarity conditions. Blue region is the result in real basis with more decision variables with linear unitarity conditions (QC, KM, TW, GDZ & SYZ, 2309.15922).

Summary

- UV unitarity conditions constrain low energy effective theory by "positivity" bounds, which are two-sided bounds.
- The upper bounds are of order $O(1)$ and are complementary to experimental limits.
- The two-sided bounds of other effective operators can be similarly computed, providing constraints of new physics.

Thank You!

Linear unitarity conditions

$$0 \le \rho_\ell^{iiii} \le 2, \qquad \left|\rho_\ell^{iijj}\right| \le 1 - \left|1 - \frac{\rho_\ell^{iiii} + \rho_\ell^{jjjj}}{2}\right|,$$

$$0 \le \rho_\ell^{ijij} \le \frac{1}{2}, \quad \left|\rho_\ell^{ijkl}\right| \le \frac{1}{4} - \left|\frac{1}{4} - \frac{\rho_\ell^{ijij} + \rho_\ell^{klkl}}{2}\right|, \qquad (i \ne j \ne k \ne l)$$

$$\left|(\rho_\ell^{iijj} + \rho_\ell^{kkll}) \pm (\rho_\ell^{iikk} + \rho_\ell^{jjll})\right| \le 2.$$

R-variable Definitions

$$R^{1111}_{\ell,n} \equiv \frac{\rho^{1111}_{\ell,n} + \rho^{2222}_{\ell,n} + \rho^{3333}_{\ell,n} + \rho^{4444}_{\ell,n}}{4},$$

$$R^{1122}_{\ell,n} \equiv \frac{\rho^{1122}_{\ell,n} + \rho^{3344}_{\ell,n}}{2}, \qquad R^{1212}_{\ell,n} \equiv \frac{\rho^{1212}_{\ell,n} + \rho^{3434}_{\ell,n}}{2},$$

$$R^{1133}_{\ell,n} \equiv \frac{\rho^{1133}_{\ell,n} + \rho^{2244}_{\ell,n}}{2}, \qquad R^{1313}_{\ell,n} \equiv \frac{\rho^{1313}_{\ell,n} + \rho^{2424}_{\ell,n}}{2},$$

$$R^{1144}_{\ell,n} \equiv \frac{\rho^{1144}_{\ell,n} + \rho^{2233}_{\ell,n}}{2}, \qquad R^{1414}_{\ell,n} \equiv \frac{\rho^{1414}_{\ell,n} + \rho^{2323}_{\ell,n}}{2}.$$

Complex Higgs Basis

More convenient to work in the complex basis to match the effective operators by considering $HH^* \to HH^*$ scattering.

$$\boxed{A_{i\bar{j}k\bar{l}}(s,\, t) = \delta_{i\bar{j}}\delta_{k\bar{l}} f(s,\, t) + \delta_{i\bar{l}}\delta_{\bar{j}k} f(u,\, t)}$$

where $H^* \equiv \frac{1}{\sqrt{2}}(H^*_{\bar{1}}, H^*_{\bar{2}})^T$, $i,\, j,\, k,\, l = 1, 2$ and $\bar{i},\, \bar{j},\, \bar{k},\, \bar{l} = \bar{1}, \bar{2}$.

Complex Higgs Basis Program

Variables

$$\rho^s_{\ell,\,n}, \quad \rho^t_{\ell,\,n}, \quad \rho^u_{\ell,\,n}$$

Maximize/Minimize

$$c_1 = \frac{1}{\Lambda^4} \sum_{\ell=0}^{\ell_M, \ell_\infty} 16(2\ell+1) \sum_n^N \frac{1}{N} \frac{n}{N} \left[\left(1-(-1)^\ell\right) \left(\rho^s_{\ell,\,n} + \rho^t_{\ell,\,n}\right) + \left(1+(-1)^\ell\right) \rho^u_{\ell,\,n} \right]$$

$$c_2 = \frac{1}{\Lambda^4} \sum_{\ell=0}^{\ell_M, \ell_\infty} 16(2\ell+1) \sum_n^N \frac{1}{N} \frac{n}{N} \left[\left(-1+(-1)^\ell\right) \left(\rho^s_{\ell,\,n} + \rho^u_{\ell,\,n}\right) + \left(1+(-1)^\ell\right) \rho^t_{\ell,\,n} \right]$$

$$c_3 = \frac{1}{\Lambda^4} \sum_{\ell=0}^{\ell_M, \ell_\infty} 16(2\ell+1) \sum_n^N \frac{1}{N} \frac{n}{N} \left[\left(1-(-1)^\ell\right) \left(\rho^u_{\ell,\,n} - \rho^t_{\ell,\,n}\right) + \left(1+(-1)^\ell\right) \rho^s_{\ell,\,n} \right]$$

Subject to

Positivity inequalities

Null constraints