The Two-Body Problem in Beyond-GR Theories: Precision Modeling in Scalar-Tensor and ESGB Gravity

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Based on [GLA and S-Y Zhou, PRD **110**, 124016 (2024)]

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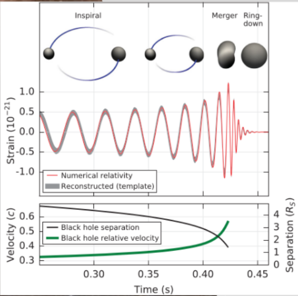
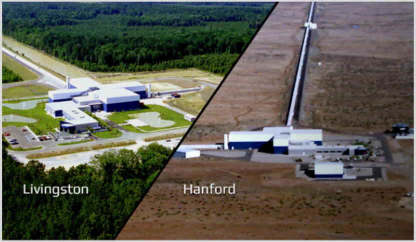
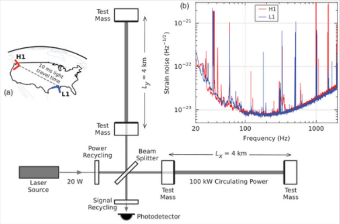
GW150914: First direct detection of gravitational waves







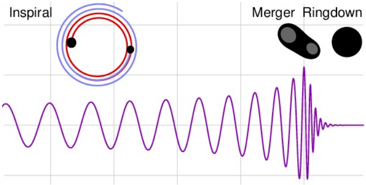
New window to observe the universe!

Today, 90 GW events have been observed by the LVK collaboration.

Phases of the Coalescence

Different methods are used to study each of these phases.

Inspiral: *Post-Newtonian approximation.*



In the adiabatic approximation to the circular motion

$$\begin{cases} h_+ = \dfrac{4}{D}(G_N\mathcal{M}_c)^{5/3}(\pi f_{\mathrm{gw}})^{2/3}\left(\dfrac{1+\cos^2\iota}{2}\right)\cos\left[\Phi(t)\right] \;, \\ h_\times = \dfrac{4}{D}(G_N\mathcal{M}_c)^{5/3}(\pi f_{\mathrm{gw}})^{2/3}\cos\iota\sin\left[\Phi(t)\right] \;. \end{cases}$$

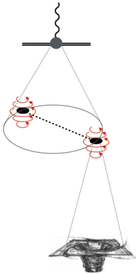
The **Gravitational-Wave Phase** $\Phi(t)$ can be modelled through

$$\Phi(t) = 2\int_{t_0}^{t} dt\,\omega(t) = -\frac{2}{G_N M}\int_{v(t_0)}^{v(t)} dv\,\frac{v^3}{P(v)}\,\frac{dE}{dv} \;.$$

Hierarchy of Scales

Hierarchy of scales and the **method of regions** for bound binary systems

[M. Beneke and V. A. Smirnov, Nucl. Phys. B **522**, 321-344 (1998)]



$$\text{Orbital scale:} \quad v^2 \sim \frac{G_N m}{r} \quad \Rightarrow \quad r_s \sim 2G_N m \sim r v^2$$

$$\text{GW scale:} \quad \lambda \sim \frac{r}{v}$$

In the nonrelativistic regime, $v \ll 1$, hierarchy of scales:

$$\boxed{r_s \ll r \ll \lambda}$$

Method of regions: $h_{\mu\nu} = \underbrace{H_{\mu\nu}}_{\text{potential modes}} + \underbrace{\bar{h}_{\mu\nu}}_{\text{radiative modes}}$

$$H_{\mu\nu} \;:\; \text{off-shell modes scaling as } (k^0, \mathbf{k}) \sim (v/r, 1/r)$$

$$\bar{h}_{\mu\nu} \;:\; \text{on-shell modes scaling as } (k^0, \mathbf{k}) \sim (v/r, v/r)$$

⇒ *Non-Relativistic General Relativity* construction of Goldberger and Rothstein.

[W. D. Goldberger and I. Z. Rothstein, Phys. Rev. D **73** (2006) 104029]

⇒ An nPN correction corresponds to $G^{n-k+1}(v^2)^k$, $0 \leq k \leq n+1$.

Extension to ST and ESGB Gravity Theories

Issues on both short and long scales motivates us to look for an alternative theory of gravity. Here we consider the **ESGB gravity**, described by

$$S = S_{\mathrm{EH}}[g_{\mu\nu}] + S_{\varphi}[\varphi, g_{\mu\nu}] + S_m[\Psi, \mathcal{A}^2(\varphi) g_{\mu\nu}]\,,$$

where S_{EH} and S_{φ} are given by

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}R\,,$$

$$S_{\varphi} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\left[-2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \alpha f(\varphi)\mathcal{G}\right]\,.$$

The quantity denoted by $\mathcal{G}$ is the **Gauss-Bonnet invariant** and is given by

$$\mathcal{G} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2\,.$$

Several are the motivations to consider this theory:

- Particular case of the Horndeski theories, it evades the no-hair theorem;
- This is important from the observational point of view since not only BHs are simpler objects than NSs, but also because most of the GW events detected so far correspond to binary BH coalescences;
- Gauss-Bonnet invariant is the only of such combination of quadratic curvature terms that yields second-order equations of motion;
- Naturally arises in the low-energy limit of heterotic string theory.

Point-particle description

- In the PN formalism, when scalar couplings are included, the "particles" acquire a field-dependent mass $m_A(\varphi)$ ["skeletonization" procedure (Eardley, 1975)].

$\Rightarrow$ This encodes information on the **compact objects' internal structure**

$$S_m \to S_m^{\rm pp}[g_{\mu\nu}, \varphi, \{x_A^\mu\}] = -\sum_A \int m_A(\varphi) d\tau_A \,.$$

Sensitivities (or "strong-field parameters") $\alpha_A^0, \beta_A^0, \beta_A'^0, \ldots$ $\Rightarrow$ Measure of the coupling strength of each body to the scalar field.

$$\alpha_A \equiv \frac{d\ln m_A(\varphi)}{d\varphi}\,, \qquad \beta_A \equiv \frac{d\alpha_A(\varphi)}{d\varphi}\,, \qquad \beta_A' \equiv \frac{d\beta_A(\varphi)}{d\varphi}\,.$$

Feynman rules are obtained from $S +$ (Gauge fixing) (Harmonic) by expanding

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad \varphi = \varphi_0 + \delta\varphi\,,$$

which, from the method of regions, are split into potential and radiation modes,

$$h_{\mu\nu} = H_{\mu\nu} + \bar{h}_{\mu\nu} \qquad \text{and} \qquad \delta\varphi = \Phi + \bar{\varphi}\,.$$

Based on theoretical and observational constraints $\to$ **small-α approximation**

$$\epsilon \equiv \frac{\alpha f'(\varphi_0)}{4m^2} \ll 1\,.$$

Spinning bodies in the EFT approach

In GR, minimal effective action can be fixed by **general covariance** and **invariance under worldline reparametrization**, taking the form:

$$S = \sum_{\rm a} \int d\lambda_{\rm a} \left(p^\mu_{\rm a} u^{\rm a}_\mu + \frac{1}{2} S^{\mu\nu}_{\rm a} \Omega^{\rm a}_{\mu\nu} \right) .$$

Variation of this action → **Mathisson-Papapetrou-Dixon equations**

$$\frac{Dp^\mu_{\rm a}}{d\lambda} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} u^\nu_{\rm a} S^{\alpha\beta}_{\rm a} , \quad \frac{DS^{\mu\nu}_{\rm a}}{d\lambda} = p^\mu_{\rm a} u^\nu_{\rm a} - p^\nu_{\rm a} u^\mu_{\rm a} .$$

Following [Porto, PRD 73, 104031 (2006)], a Routhian $\mathcal{R}$ can be associated to the above Lagrangian, which under the covariant supplementary spin condition (SSC),

$$p_\mu S^{\mu\nu} = 0 ,$$

reads

$$\mathcal{R} = \sum_{\rm a} \left(-m_{\rm a} \sqrt{-u^2_{\rm a}} + \frac{1}{2} S^{\rm a}_{ab} \omega^{ab}_\mu u^\mu_{\rm a} + \frac{1}{2m_{\rm a}} R_{abcd} S^{ab}_{\rm a} S^{ce}_{\rm a} \frac{u^d_{\rm a} u^{\rm a}_e}{\sqrt{-u^2_{\rm a}}} \right) + \dots .$$

In this framework, the EOMs for positions and spins are obtained from:

$$\frac{\delta}{\delta x^\mu} \int d\lambda \, \mathcal{R} = 0 \qquad \text{and} \qquad \frac{dS^{ab}}{d\lambda} = \{ S^{ab}, \mathcal{R} \} ,$$

with Poisson brackets given by the Lorentz algebra.

Spinning bodies in ESGB gravity

Now, promoting the dynamical quantities p^μ and $S^{\mu\nu}$ to functions φ,

$$p^\mu \to p^\mu(\varphi) \qquad \text{and} \qquad S^{\mu\nu} \to S^{\mu\nu}(\varphi)\,.$$

Then, following the same derivation as in GR, under the same SSC, we obtain

$$\mathcal{R} = \sum_{\mathrm{a}} \left(-m_{\mathrm{a}}(\varphi)\sqrt{-u_{\mathrm{a}}^2} + \frac{1}{2} S^{\mathrm{a}}_{ab}(\varphi)\omega_\mu^{ab} u^\mu_{\mathrm{a}} + \frac{1}{2m_{\mathrm{a}}} R_{abcd} S^{ab}_{\mathrm{a}}(\varphi) S^{ce}_{\mathrm{a}}(\varphi) \frac{u^d_{\mathrm{a}} u^{\mathrm{a}}_e}{\sqrt{-u_{\mathrm{a}}^2}} \right) + \dots\,.$$

In addition this, we must also consider **rotation-induced finite-size couplings**:

$$S^h_{\mathrm{fs}} = -\frac{1}{2}\int d\lambda\, Q^{ab}_E(\varphi)\frac{E_{ab}}{\sqrt{-u^2}}\,, \qquad \text{with} \qquad Q^{ab}_E = \frac{C_{ES^2}(\varphi)}{m(\varphi)} S^a{}_c S^{bc}\,,$$

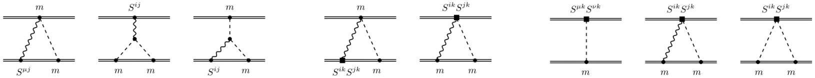
for the **gravitational field**, and

$$S^\varphi_{\mathrm{fs}} = -\frac{1}{2}\int d\lambda \sqrt{-u^2}\, Q^{ab}_\varphi(\varphi)\nabla^\perp_a \nabla^\perp_b \varphi\,, \qquad \text{with} \qquad Q^{ab}_\varphi = \frac{C_{\varphi S^2}(\varphi)}{m(\varphi)} S^a{}_c S^{bc}\,,$$

for the **scalar field**, where $\nabla^\perp_a \equiv (\delta^b_a - \frac{u^b u_a}{u^2})\nabla_b$.
$\Rightarrow$ Orthogonatily property of $\nabla^\perp_a$ and E_{ab} is crucial here since it guarantees that the SSC will still be preserved after the inclusion of the above finite-size coupling.

Spin correction to the 3PN order in ESGB gravity

Computation



Spin contributions:

$$V_0^{3\mathrm{PN}} = \frac{G^2 m_2^0 \alpha_2^0}{r^3} n^i \left[m_1^0 \alpha_1^0 \left(S_1^{0i} - S_1^{ij}(v_1^j - 2v_2^j) \right) - \frac{1}{2} m_2^0 \alpha_2^0 S_1^{ij}(v_1^j - v_2^j) \right] + (1 \leftrightarrow 2)\,,$$

$$V_{h,\mathrm{fs}}^{3\mathrm{PN}} = -\frac{G^2 m_2^0 \alpha_2^0}{2r^4} \left(\alpha_1^0 C^{(0)}_{ES^2,1} + \frac{m_2^0}{m_1^0} C^{(1)}_{ES^2,1} \right) S_1^{ik} S_1^{jk} (\delta_{ij} - 3n_i n_j) + (1 \leftrightarrow 2)\,,$$

$$\begin{aligned} V_{\varphi,\mathrm{fs}}^{3\mathrm{PN}} = {} & \frac{G m_2^0 \alpha_2^0}{2 m_1^0 r^3} C^{(0)}_{\varphi S^2,1} \left[\left(1 - \frac{v_1^2}{2} - \frac{v_2^2}{2} \right) S_1^{ik} S_1^{jk} - (S_1^{i0} S_1^{j0} + 2 S_1^{0k} S_1^{ik} v_2^j) \right] (\delta_{ij} - 3n_i n_j) \\ & + \frac{G m_2^0 \alpha_2^0}{m_1^0 r^3} C^{(0)}_{\varphi S^2,1} S_1^{ik} S_1^{jk} v_1^i [v_1^j - 3(\mathbf{v}_1 \cdot \hat{\mathbf{r}}) n^j] \\ & - \frac{G^2 (m_2^0)^2 \alpha_2^0}{m_1^0 r^4} \left(2C^{(0)}_{\varphi S^2,1} + \frac{\alpha_2^0}{2} C^{(1)}_{\varphi S^2,1} \right) S_1^{ik} S_1^{jk} (\delta_{ij} - 3n_i n_j) + (1 \leftrightarrow 2)\,, \end{aligned}$$

Final result:

$$V_{\mathrm{eff}} = V^{3\mathrm{PN}}_{\mathrm{ESGB}} + V^{3\mathrm{PN}}_{\mathrm{spin-GR}} + V_0^{3\mathrm{PN}} + V^{3\mathrm{PN}}_{h,\mathrm{fs}} + V^{3\mathrm{PN}}_{\varphi,\mathrm{fs}}\,.$$

Rotating BH solutions in ESGB gravity

General ansatz for stationary, axially symmetric spacetimes:

$$ds^2 = -n\sigma^2(1+2h)dt^2 + n^{-1}(1+2M)dr^2 + r^2(1+2k)[d\theta^2 + \sin^2\theta\,(d\phi - \hat{\omega}dt)^2]\,.$$

⇒ $n(r), \sigma(r), \varphi(r)$ are expanded in a Taylor series in ϵ around Schwarzschild.
⇒ $h(r,\theta), M(r,\theta), k(r,\theta), \hat{\omega}(r,\theta)$ can be expanded in a complete basis and χ:

$$\hat{\omega}(r,\theta) = \sum_{n=1,3,5,\dots} \sum_{l=1,3,5,\dots}^{n} \chi^n \omega_l^{(n)}(r) S_l(\theta)\,,$$

$$h(r,\theta) = \sum_{n=2,4,\dots} \sum_{l=0,2,4,\dots}^{n} \chi^n h_l^{(n)}(r) P_l(\cos\theta)\,,$$

and likewise for $M(r,\theta)$ and $k(r,\theta)$. For the scalar field, we similarly have

$$\varphi(r,\theta) = \varphi_{\text{static}}(r) + \sum_{n=2,4,\dots}^{N_\chi - p} \sum_{l=0,2,4,\dots}^{n} \chi^n \varphi_l^{(n)}(r) P_l(\cos\theta)\,.$$

⇒ **Field eqs.** reduce to a set of ODEs in r, which can be solved iteratively in ϵ.
⇒ **Unique solutions**: asymptotically flatness and regularity of the scalar field.

Properties of the slowly-rotating solutions

$\Rightarrow$ Solutions were obtained up to $\mathcal{O}(\epsilon^7, \chi^5)$.

Asymptotic behavior

$$g_{tt} = -1 + \frac{2m}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) , \quad \varphi = \varphi_0 + \frac{D}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) , \quad \omega = \frac{2J}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right) .$$

Computation of $D = D(m, \varphi_0) \Rightarrow$ Hairy BH solutions of secondary type.

Geometrical properties

- BH event horizon r_h: $g_{\phi\phi} g_{tt} - g_{t\phi}^2 = 0$
- Angular velocity of the BH horizon: $\Omega_\mathrm{h} = -\lim_{r \to r_\mathrm{h}} g_{t\phi}/g_{\phi\phi}$
- BH moment of inertia: $I = J/\Omega_\mathrm{h}$
- Quadrupole moments of the gravitational Q_h and scalar Q_φ fields.

Thermodynamic properties

- Surface gravity and temperature: $\kappa^2 \equiv -\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)\Big|_{r_\mathrm{h}} , \qquad T \equiv \frac{\kappa}{2\pi} .$
- Wald entropy $S_\mathrm{w} = \frac{1}{4} \int_{r_\mathrm{h}} d\Sigma_\mathrm{h} \left[1 + 2\alpha f(\varphi) \tilde{R}\right] ,$

$\Rightarrow$ Explicit verification of the first law of BH thermodynamics:

$$dm = T dS_\mathrm{w} + \Omega_\mathrm{h} dJ - D d\varphi_0 .$$

The matching

⇒ From isolated bodies at length scales $r_s \sim Gm$ to point particles at scales r.
⇒ Coefficients to be matched: $m(\varphi), S^{ab}(\varphi), C_{ES^2}(\varphi)$, and $C_{\varphi S^2}(\varphi)$.

Matching → compare exterior fields in the large distance limit. From EFT side:

$$\langle h_{\mu\nu}(x)\rangle \qquad \text{and} \qquad \langle \varphi(x)\rangle$$

We obtain the following **matching conditions**:

$$m_{\rm a}(\varphi_0) = m\,, \qquad m'_{\rm a}(\varphi_0) = -D\,, \qquad S_{\rm a}(\varphi_0) = J\,.$$

This ensures point-particle action to produce same BH solution at all $\mathcal{O}(1/r)$.

⇒ The matching conditions admit solution, obtained perturbatively as

$$m_{\rm a}(\varphi) = \mu_{\rm a}\left[1 + \sum_{n=1}^{7}\sum_{l=0,2,4}\frac{\alpha^n S_{\rm a}^l F_l^{(n)}(\varphi)}{\mu_{\rm a}^{2n+2l}}\right]\,, \qquad S_{\rm a}(\varphi) = S_{\rm a} = \text{const.}$$

Inverting this expression, we get

$$M_{\rm irr}^2 = \frac{S_{\rm w}}{4\pi} = \frac{1}{2}\left[\mu_{\rm a}^2 + (\mu_{\rm a}^2 - S_{\rm a}^2)^{1/2}\right]\,.$$

The matching conditions ⇒ $\delta S_{\rm w} = 0$: Thus, $\mu_{\rm a}$ and $S_{\rm a}$ are good parameters.

State-of-the-art results

⇒ Two-body Lagrangian in ST theories:

- 1PN order: Damour and Esposito-Faresè [1992]
- 2PN order:
  - Mirshekari and Will [2013]
  - Also GLA [2024] (via EFT methods)
- 3PN order: Bernard [2018] [Missing coefficients!]

⇒ Generalization to ESGB gravity to $\mathcal{O}$(3PN):

- Julié and Berti [2019]
- Also GLA [2024] (via EFT methods)

⇒ Conservative dynamics included within an EOB Hamiltonian:

- 2PN: Julié and Duruelle [2017]
- 3PN: Julié *et. al.* [2023]

⇒ Waveform modeling:

- Inspiral waveform at 2PN order [2022]
- Full IMR waveform at 3PN order by Julié, Pompili, Buonanno [2024]

In all the above cases, no spin corrections had been considered!

Conclusions

In this presentation, we have

- Explored the PN dynamics of spinning BH binaries in ESGB gravity;
- Exployed an EFT approach and extended the Routhian formalism to incorporate spin effects in scalar interactions;
- Derived for the first time the $V_{\rm eff}$ for spinning BHs in generic ESGB gravity;

Besides this, we have shown how

- Sensitivities are matched with analytic BH solutions accurate to $\mathcal{O}(\epsilon^7, \chi^5)$;
- Thermodynamic properties give insight into the inspiral phase evolution.

For the future...

- Investigate radiative processes in binaries in ESGB gravity.

$\Rightarrow$ Surprisingly, little progress has been made in theories beyond GR in this direction. No radiative effects have been explored in the PN nor PM frameworks.

Thank you!