Theoretical study of $J/\psi \to e^+e^-\pi^0$

Xiong-Hui Cao 曹雄辉

In collaboration with Feng-Kun Guo,
Christoph Hanhart and Bastian Kubis

ITP, CAS

2025年超级陶粲装置研讨会，湖南湘潭

2025.07.2-6

$J/\psi \to e^+e^-\pi^0$: motivation

- Sensitive to the transition form factor (TFF) of $J/\psi \to \pi^0\gamma^\star$

- The branching fractions of $J/\psi \to Pe^+e^-$ $(P = \pi^0, \eta, \eta')$
  BESIII, PRD 89, 092008 (2014)

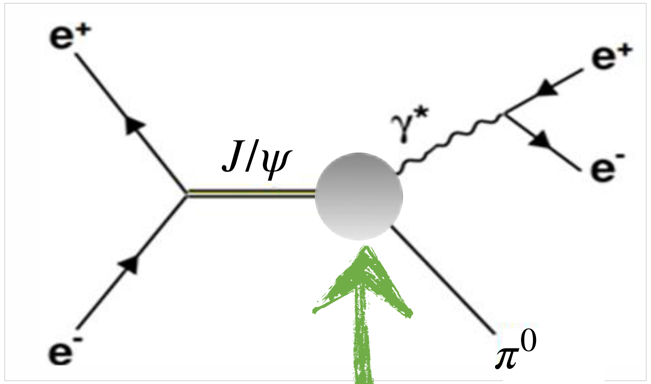
- Exp v.s. VMD

  η and η' $\surd$; π^0 ×

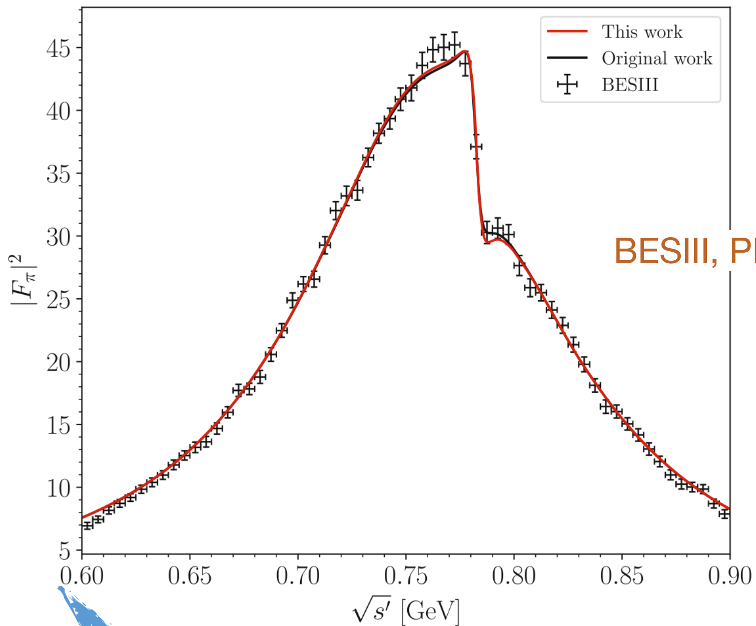
- VMD estimate:

  charmonium dominance,
  isospin-breaking transition

- Isospin-conserving manner, dominated by light-quark degrees of freedom ($2\pi\ldots$)



Transition form factor for $J/\psi \to \pi^0\gamma^\star$



BESIII, PLB 753 (2016)

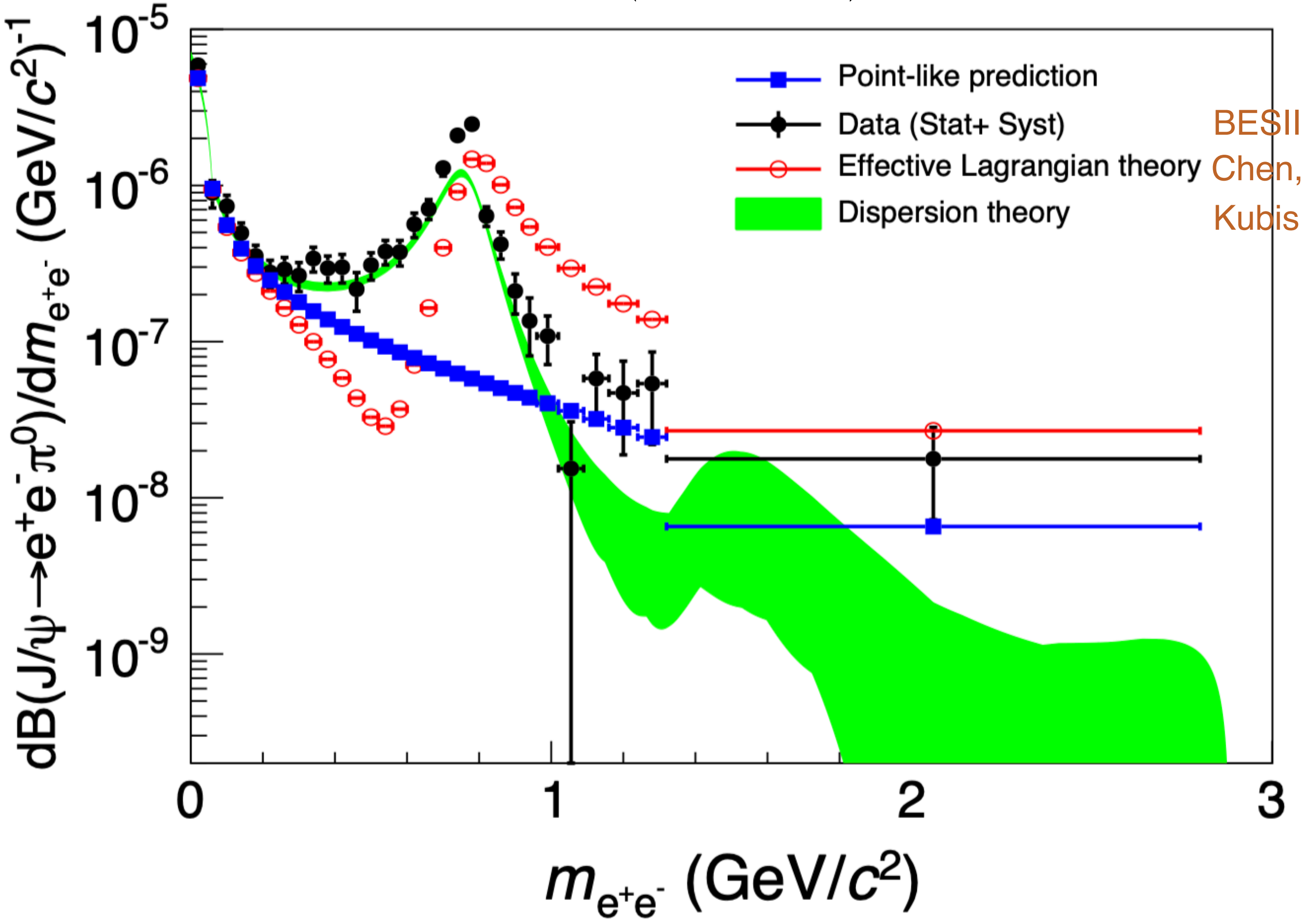
$\rho - \omega$ interference

BESIII result

BESIII, 2501.04344 (2025)

- A significant resonant structure corresponding to ρ, ω

$$\frac{\mathrm{dBR}_{\psi\to\pi^0\ell^+\ell^-}}{\mathrm{BR}_{\psi\to\pi^0\gamma}\ \mathrm{d}s} = \frac{16\alpha}{3\pi}\left(1+\frac{2m_\ell^2}{s}\right)\frac{q_\ell(s)q_{\psi\pi^0}^3(s)}{\left(M_\psi^2-M_{\pi^0}^2\right)^3}\left|F_{\psi\pi^0}(s)\right|^2 , \quad F_{\psi\pi^0}(s)=\frac{f_{\psi\pi^0}(s)}{f_{\psi\pi^0}(0)}$$



BESIII, 2501.04344
Chen, Guo and Zou, PRD (2015)
Kubis and Niecknig, PRD (2015)

$J/\psi \to \pi^0 \gamma^\star$ transition form factor

- The $J/\psi \to \pi^0 \gamma^\star$ transition form factor: Landsberg, Phys. Rept. (1985)

$$\left\langle \pi^0(p) \middle| j_\mu(0) \middle| \psi\left(p_V, \lambda\right) \right\rangle = -\, i\epsilon_{\mu\nu\alpha\beta}\epsilon^\nu\left(p_V, \lambda\right) p^\alpha q^\beta f_{\psi\pi^0}(s)$$

- (P-wave) $\pi\pi$ intermediate state:

$$\frac{\text{disc}}{2i} f_{\psi\pi^0}^{(2\pi)}(s) = \frac{s\sigma_\pi^3(s)}{96\pi} F_\pi^{V*}(s) f_1(s)$$

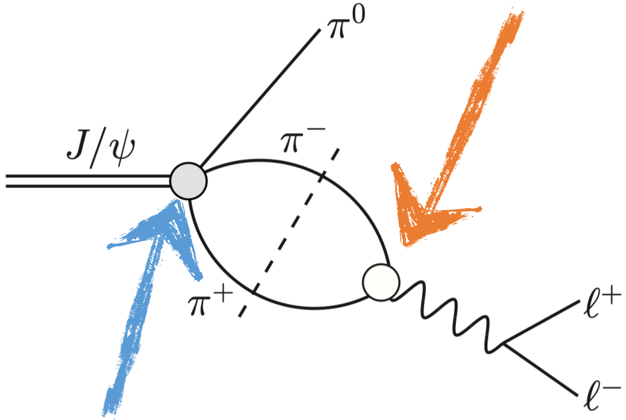
Koepp, PRD (1974)

- **Dispersion relation** (unsubtracted):

$$f_{\psi\pi^0}(s) = \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\text{disc}\, f_{\psi\pi^0}(s')}{s' - s - i\epsilon},$$

π vector form factor

$F_\pi^V(s)$



$f_1(s)$

P-wave $J/\psi \to 3\pi$ decay amplitude

$\rho - \omega$ mixing

- F_π^V depends on $\rho - \omega$ mixing
- Naive approach leads to a spectral function in which the double discontinuities of 2π and 3π intermediate states may no longer cancel

$$F_\pi^V(s) \to \left(1 + \epsilon_{\rho\omega}\frac{s}{M_\omega^2 - s - iM_\omega\Gamma_\omega}\right)F_\pi^V(s) \quad \Longrightarrow \quad \mathrm{Im}\left[\frac{\mathrm{disc}f_{\psi\pi^0}(s)\,|_{2\pi}}{2i}\right] \neq 0$$

- Two-potential formalism + Dispersion relation
  - Three channel system: $2\pi,\ 3\pi,\ \ell^+\ell^-$
  - Generalize to a dispersion relation including a consistent treatment $\rho - \omega$ mixing
  - P-wave $J/\psi \to 3\pi$ decay amplitude is construct from the Khuri-Treiman (KT) formalism Khuri and Treiman, Phys. Rev. (1960)…
  - Ensure unitarity, analyticity and crossing symmetry

Dispersive representation

- Modified dispersion relation (once sub.)

See also $\eta'\gamma^\star\gamma^\star$ TFF, Holz et.al., EPJC (2022)

$$f_{\psi\pi^0}^{(2\pi,3\pi)}(s) = f_{\psi\pi^0}^{(2\pi)}(0) + \frac{s}{96\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\sigma_\pi^3(s')F_\pi^{V*}(s')f_1(s')}{s'-s-i\epsilon}\left[1+\frac{\tilde{\epsilon}_{\rho\omega}s}{M_\omega^2-s-iM_\omega\Gamma_\omega}\right]$$

$$+\frac{w_{\psi\omega\pi}s}{M_\omega^2-s-iM_\omega\Gamma_\omega}\left[1+\frac{\tilde{\epsilon}_{\rho\omega}s}{48\pi^2g_{\omega\gamma}^2}\int_{4M_\pi^2}^{\infty}\mathrm{d}s'\frac{\sigma_\pi^3(s')|F_\pi^V(s')|^2}{s'(s'-s-i\epsilon)}\right]$$

$$+\frac{f_{\psi\pi^0}^{2\pi,3\pi}(0)w_{\psi\phi\pi}s}{M_\phi^2-s-iM_\phi\Gamma_\phi},$$

$I=1$ "$\rho-\omega$" mixing

$I=0$ "$\omega-\rho$" mixing

- $\rho-\omega$ mixing parameter $\tilde{\epsilon}_{\rho\omega}=\epsilon_{\rho\omega}-e^2g_{\omega\gamma}^2 = 1.99(2)\times10^{-3}-0.34(1)\times10^{-3}=1.65(2)\times10^{-3}$
- Weights $w_{\psi\omega(\phi)\pi}$
- Coupling $g_{\omega\gamma}=\sqrt{\dfrac{3\Gamma\left(\omega\to e^+e^-\right)}{4\pi\alpha^2M_\omega}}=0.0606(9)$ from VMD model
- P-wave $J/\psi\to3\pi$ decay amplitude $f_1(s)$

Dispersive representation

- Modified dispersion relation (once sub.)

See also $\eta'\gamma^\star\gamma^\star$ TFF, Holz et.al., EPJC (2022)

$$f^{(2\pi,3\pi)}_{\psi\pi^0}(s) = f^{(2\pi)}_{\psi\pi^0}(0) + \frac{s}{96\pi^2}\int_{4M_\pi^2}^{\infty} ds' \frac{\sigma_\pi^3(s')\,F_\pi^{V*}(s')f_1(s')}{s'-s-i\epsilon}\left[1+\frac{\tilde{\epsilon}_{\rho\omega}s}{M_\omega^2-s-iM_\omega\Gamma_\omega}\right] + \frac{w_{\psi\omega\pi}s}{M_\omega^2-s-iM_\omega\Gamma_\omega}\left[1+\frac{\tilde{\epsilon}_{\rho\omega}s}{48\pi^2 g_{\omega\gamma}^2}\int_{4M_\pi^2}^{\infty} ds' \frac{\sigma_\pi^3(s')\,|F_\pi^V(s')|^2}{s'(s'-s-i\epsilon)}\right] + \frac{f^{2\pi,3\pi}_{\psi\pi^0}(0)w_{\psi\phi\pi}s}{M_\phi^2-s-iM_\phi\Gamma_\phi},$$

$I=1$ “$\rho-\omega$” mixing

$I=0$ “$\omega-\rho$” mixing

This nontrivial consistency condition is indeed satisfied

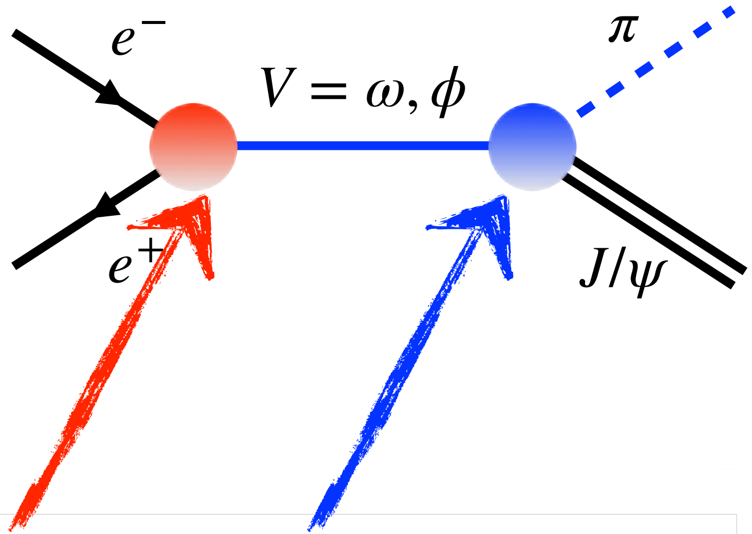
$$\mathrm{Im}\left[\frac{1}{2i}\,\mathrm{disc}\,f_{\psi\pi^0}(s)\Big|_{2\pi} + \frac{1}{2i}\,\mathrm{disc}\,f_{\psi\pi^0}(s)\Big|_{3\pi} + \frac{1}{2i}\,\mathrm{disc}\,f_{\psi\pi^0}(s)\Big|_{\gamma}\right] = 0$$

!

Isoscalar-vector pole contribution

- Weights $w_{\psi\omega(\phi)\pi}$ are determined by the pole dominance model
- Matching the expressions of the $e^+e^- \to J/\psi\pi^0$ cross section at isoscalar-vector poles

$$f_{\psi\pi}(s)\Big|_{s\to M_V^2} = \frac{w_{\psi V\pi}\, s}{M_V^2 - s - iM_V\Gamma_V}$$

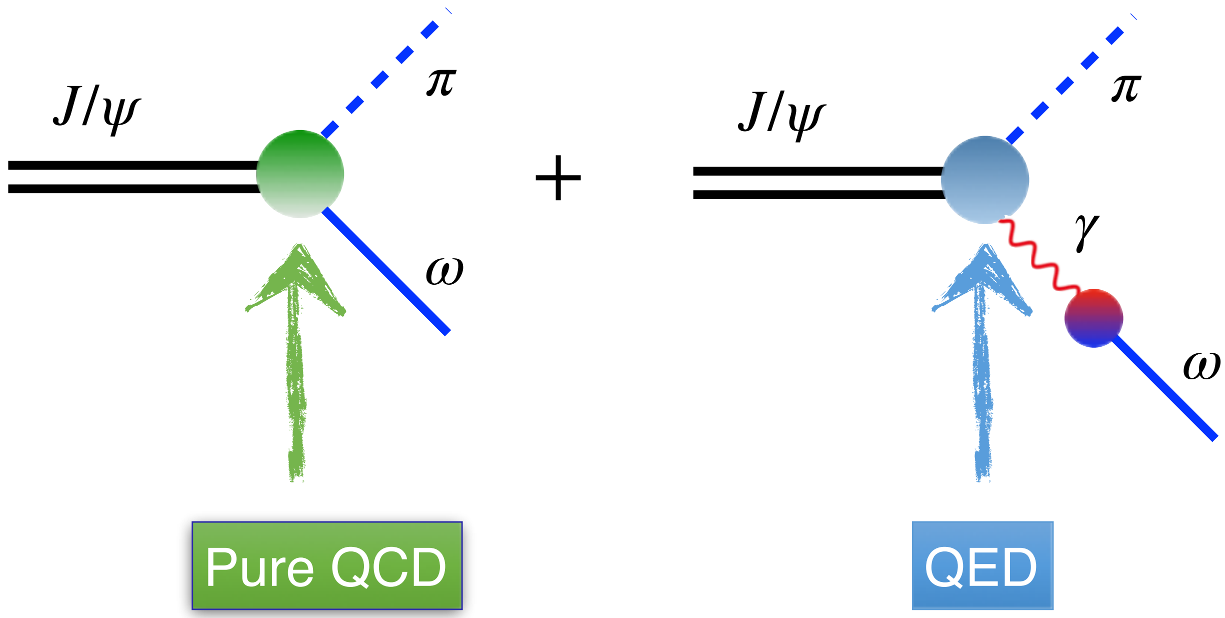


$$|w_{\psi V\pi}| = \sqrt{\frac{72 M_\psi^3 \Gamma_\psi \Gamma_\omega \mathcal{BR}(V\to e^+e^-)\mathcal{BR}(J/\psi \to V\pi^0)}{\alpha^2 M_V \lambda^{\frac{3}{2}}\left(M_\psi^2, M_V^2, M_\pi^2\right)}}$$

$$= \begin{cases} 4.38(24)\times 10^{-5}\ \mathrm{GeV}^{-1}, & \mathrm{V}=\omega \\ 4.71(3)\times 10^{-6}\ \mathrm{GeV}^{-1} \text{ or } 8.61(5)\times 10^{-7}\ \mathrm{GeV}^{-1}, & \mathrm{V}=\phi \end{cases}$$

QED effects

- Weights $w_{\psi\omega(\phi)\pi}$ are complex due to the intermediate γ



- Phases of weights $w_{\psi\omega(\phi)\pi}$ are from the interference between $\omega(\phi)$ and γ

Khuri-Treiman representation

- Decay amplitude for $V(p_V) \to \pi^+(p_+)\,\pi^-(p_-)\,\pi^0(p_0)$:

$$\mathcal{M}(s,t,u) = i\epsilon_{\mu\nu\alpha\beta}\epsilon^\mu p_+^\nu p_-^\alpha p_0^\beta \mathcal{F}(s,t,u)$$

- s-channel partial-wave expansion of the amplitude

$$\mathcal{F}(s,t,u) = \sum_{J=0}^{\infty} (q_{\pi\pi}(s)q_{\psi\pi}(s))^{J-1} P'_J(z_s)\, f_J(s)$$

- KT decomposition of the amplitude via *reconstruction theorem*

Stern, Sazdjian and Fuchs, PRD (1993), for $\pi\pi$

$$\mathcal{F}(s,t,u) = \sum_{J=0}^{J_{\max}} (q_{\pi\pi}(s)q_{\psi\pi}(s))^{J-1} P'_J(z_s)\, \mathcal{F}_J(s) + (s \leftrightarrow t) + (s \leftrightarrow u)$$

- Consider only P-wave: $\mathcal{F}(s,t,u) = \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u), \quad \mathcal{F} \equiv \mathcal{F}_1$

several complex variables → single complex variables

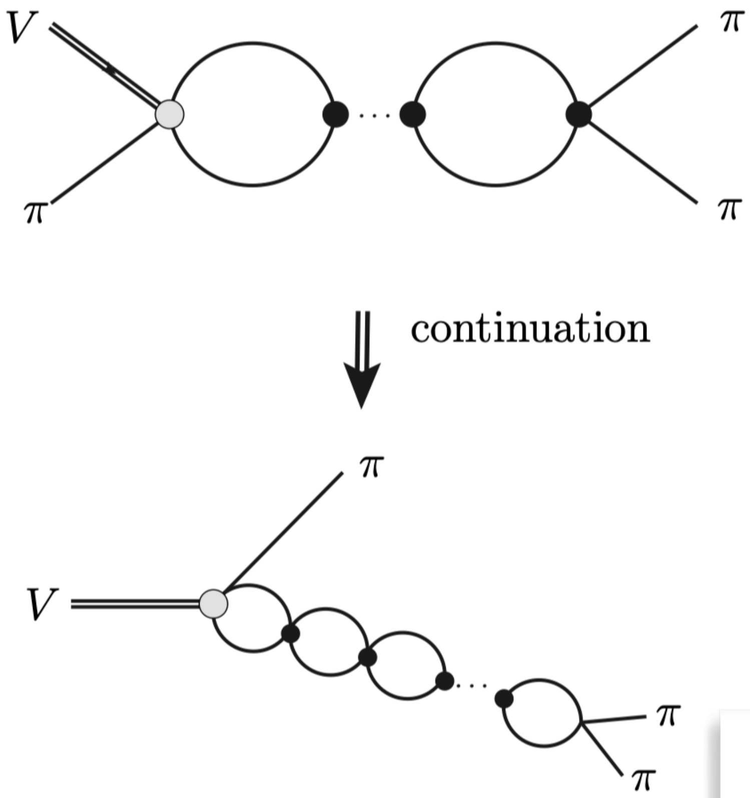
- Partial wave projection of the KT decomposition Khuri and Treiman, Phys. Rev. (1960)…

$$f_1(s) = \mathcal{F}(s) + \hat{\mathcal{F}}(s), \quad \hat{\mathcal{F}}(s) = 3\int_{-1}^{1} \frac{dz_s}{2}\left(1 - z_s^2\right) \mathcal{F}\left(t\left(s, z_s\right)\right) \equiv 3\left\langle (1 - z_s^2)\mathcal{F} \right\rangle$$

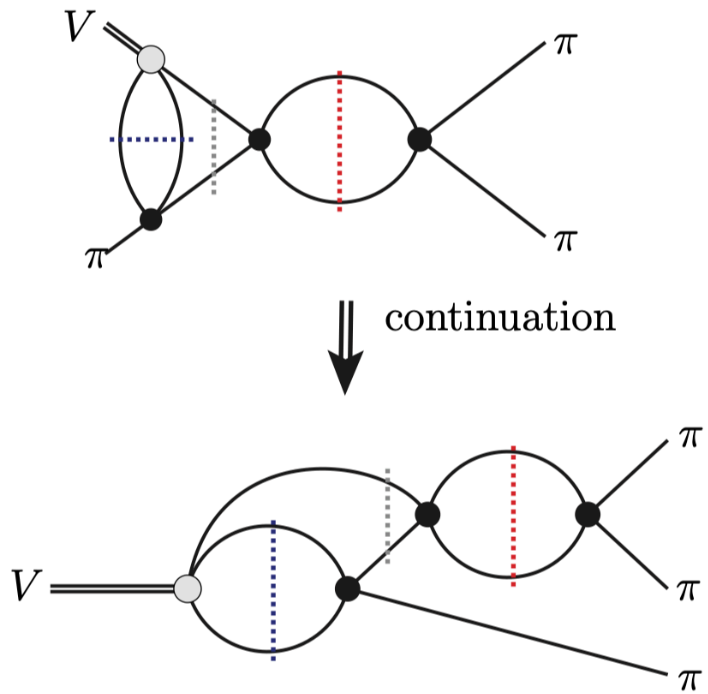
  - $\mathcal{F}(s)$: right-hand cut (RHC)
  - $\hat{\mathcal{F}}(s)$: left-hand cut (given by the RHC of the crossed channels $\mathcal{F}(t)$, $\mathcal{F}(u)$)

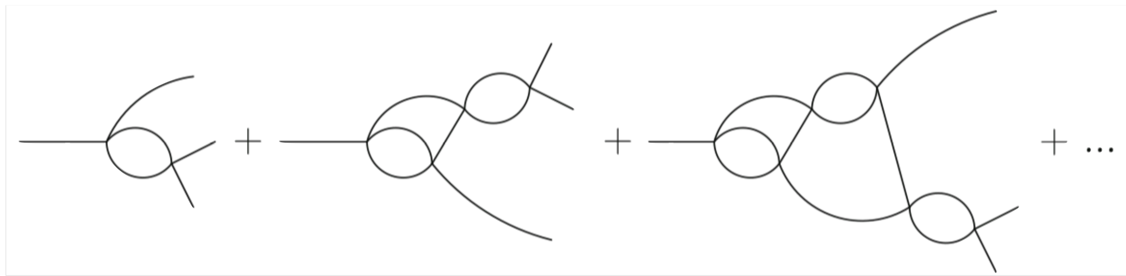
Three-particle decay dynamics

- In many decay processes one wants to take into account unitarity/FSI in the three possible channels



- Khuri-Treiman equations:
  - Include full (direct+crossed) rescattering effects



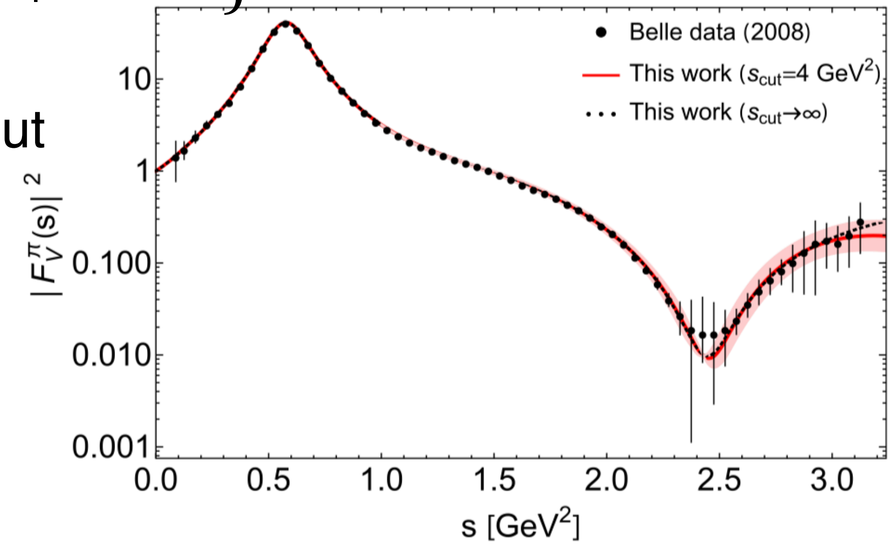


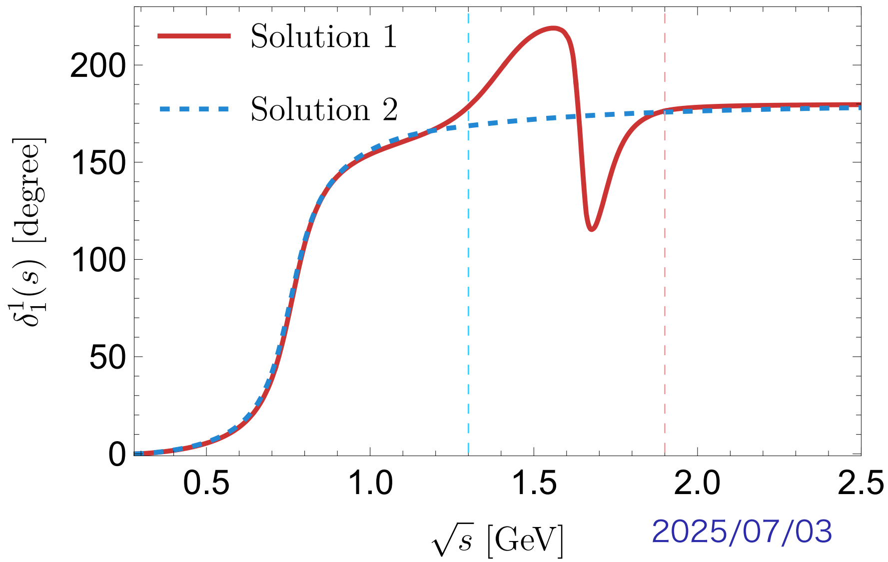
KT equation: phase shift

- KT-type representation

$$\mathscr{F}(s) = a\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'}\frac{\sin\delta(s')\hat{\mathscr{F}}(s')}{\left|\Omega(s')\right|(s'-s)}\right\}, \quad \hat{\mathscr{F}}(s) = 3\left\langle(1-z_s^2)\mathscr{F}\right\rangle$$

- $\pi\pi$ P-wave phase-shift δ taken as input
  - Solution 1: Schneider, Kubis and Niecknig, PRD (2012)
  - Solution 2: Pelaez, Raban and Ruiz de Elvira, PRD (2025) (fit-1)
  - The deviation in these solutions: **theoretical uncertainty**





$J/\psi \to \pi^0\gamma^\star$ TFF: 2π

- Dispersive representation (unsubtracted)

$$f^{(2\pi)}_{\psi\pi^0}(s) = \frac{1}{48\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}x \frac{x\sigma_\pi^3(x)F_\pi^{V*}(x)f_1(x)}{x-s}$$

$$F_\pi^V(s) = \Omega(s)$$

- Sum rule

$$f^{(2\pi)}_{\psi\pi^0}(0) = \frac{1}{48\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}x\ \sigma_\pi^3(x)F_\pi^{V*}(x)f_1(x)$$

- sum rule results in $\left|f^{2\pi}_{\psi\pi^0}(0)\right| = (4.8 \pm 0.2)\times 10^{-4}\mathrm{GeV}^{-1}$
- $\left|f_{\psi\pi^0}(0)\right|$ from real photon width, $\Gamma_{\psi\to\pi^0\gamma} = \frac{\alpha\left(M_\psi^2 - M_{\pi^0}^2\right)^3}{24M_\psi^3}\left|f_{\psi\pi^0}(0)\right|^2$

$$\left|f_{\psi\pi^0}(0)\right| = (6.0 \pm 0.3)\times 10^{-4}\mathrm{GeV}^{-1}$$

- two-pion intermediate state alone saturates the sum rule for the TFF normalization to about **80%**

$J/\psi \to \pi^0\gamma^\star$ TFF: 4π (ρ')

- The branching fractions of the J/ψ into multipion are actually larger
- From data on $e^+e^- \to$ [hadrons] $_{I=1}$, the most important inelastic intermediate state of isospin $I=1$ ought to be 4π, which is approximate to effective ρ' pole

$$\frac{\text{disc}}{2i} f^{(\rho')}_{\psi\pi^0}(s) = \frac{M^2_{\rho'} g_{J/\psi\to\rho'\pi} g_{\rho'\gamma}\sqrt{s}\Gamma_{\rho'}(s)}{\left(M^2_{\rho'} - s\right)^2 + s\Gamma^2_{\rho'}(s)}$$

Zanke, Hoferichter and Kubis, JHEP (2021)

$$\Gamma^{(4\pi)}_{\rho'}(s) = \theta\left(s - 16M^2_\pi\right)\frac{\gamma_{\rho'\to4\pi}(s)}{\gamma_{\rho'\to4\pi}\left(M^2_{\rho'}\right)}\Gamma_{\rho'}, \quad \gamma_{\rho'\to4\pi}(s) = \frac{\left(s - 16M^2_\pi\right)^{\frac{9}{2}}}{s^2}$$

- $\left|g_{\rho'\gamma}\right| = 0.0752$ from VMD estimate Zanke, Hoferichter and Kubis, JHEP (2021)
- Effective coupling:

$\text{BR}(J/\psi \to \rho'\pi)\text{BR}(\rho' \to 2\pi) = 2.2(1.2) \times 10^{-4}$ PDG

$\text{BR}(\rho' \to 2\pi) = 6\,\%$

Zanke, Hoferichter and Kubis, JHEP (2021)

→

$\text{BR}(J/\psi \to \rho'\pi) = 3.7(1.8) \times 10^{-3}$

$\left|g_{J/\psi\to\rho'\pi}\right| = 2.73^{+0.61}_{-0.80} \times 10^{-3}\ \text{GeV}^{-1}$

$$\left|f^{(\rho')}_{\psi\pi^0}(0)\right| = 1.3^{+0.3}_{-0.4} \times 10^{-4}\text{GeV}^{-1}$$

$J/\psi \to \pi^0\gamma^\star$ TFF: charmonium

- We adopt the simple monopole ansatz Fu et al., Mod.Phys.Lett.A (2012)

$$f^{c\bar{c}}_{\psi\pi^0}(s) = \frac{\left| f^{(c\bar{c})}_{\psi\pi^0}(0) \right| e^{i\delta^{c\bar{c}}}}{1 - s/\Lambda^2}$$

- The effective pole mass Λ is is fixed to the mass of the lowest 1^{--} charmonium J/ψ

- Constraint: $\left| f_{\psi\pi^0}(0) \right| = (6.0 \pm 0.3) \times 10^{-4}\ \mathrm{GeV}^{-1}$

$$\left| f^{(c\bar{c})}_{\psi\pi^0}(0) \right| = -f^{(2\pi,\rho')}_{\psi\pi^0}(0)\cos\delta^{c\bar{c}} + \sqrt{\left| f_{\psi\pi^0}(0) \right|^2 - \left(f^{(2\pi,\rho')}_{\psi\pi^0}(0) \right)^2 \sin^2\delta^{c\bar{c}}}$$

Bin averaged fit

- We fix the sign of the weight factor for ϕ

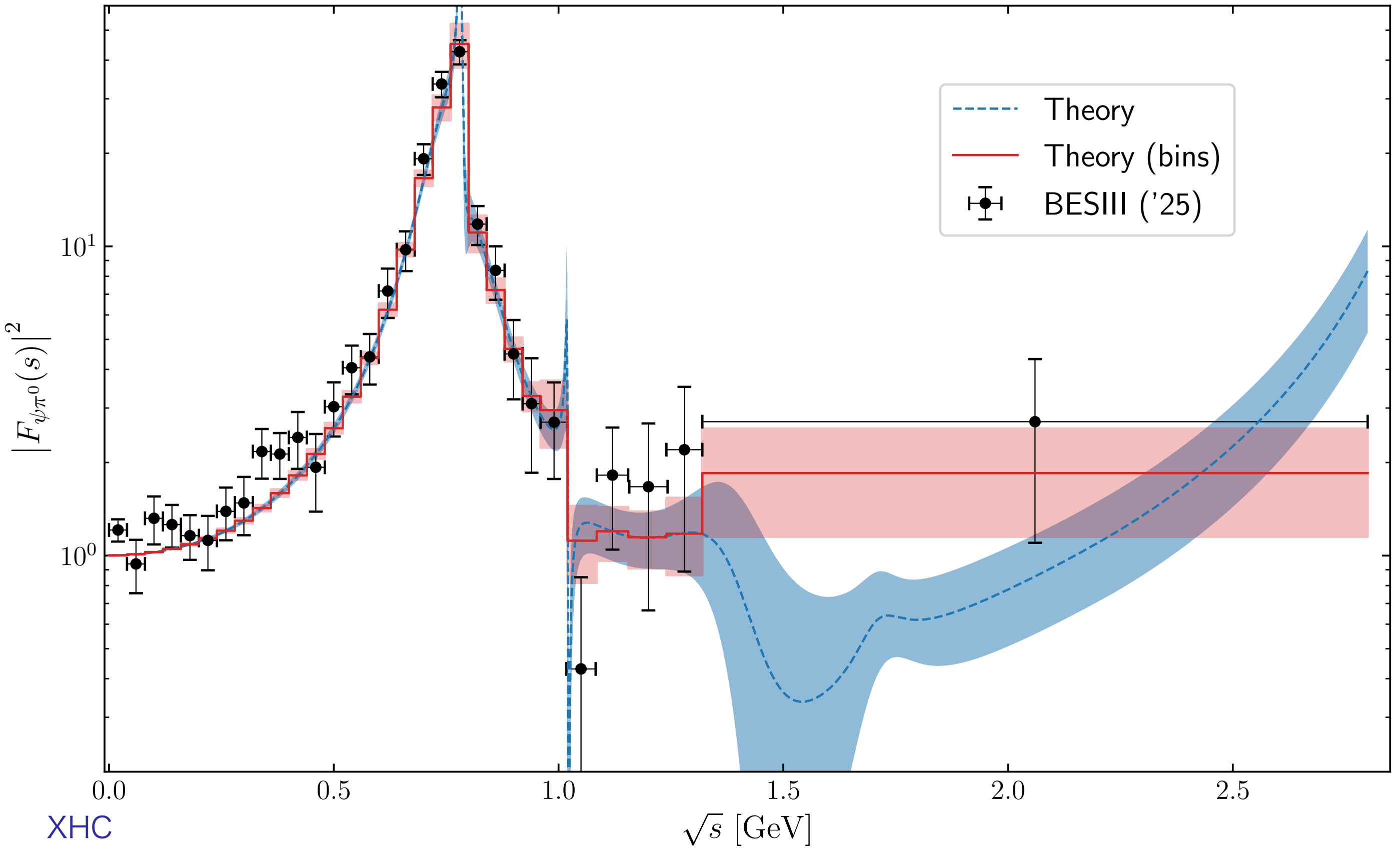
$$w_{\psi\phi\pi} = -\left|w_{\psi\phi\pi}\right|$$

- Subtracted constant $f^{(2\pi)}_{\psi\pi^0}(0)$ is fixed by the sum rule

$$f^{(2\pi)}_{\psi\pi^0}(0) = \frac{1}{96\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}s'\sigma_\pi^3(s')\,F_\pi^{V*}(s')\,f_1(s')$$

- Only four fit parameters!

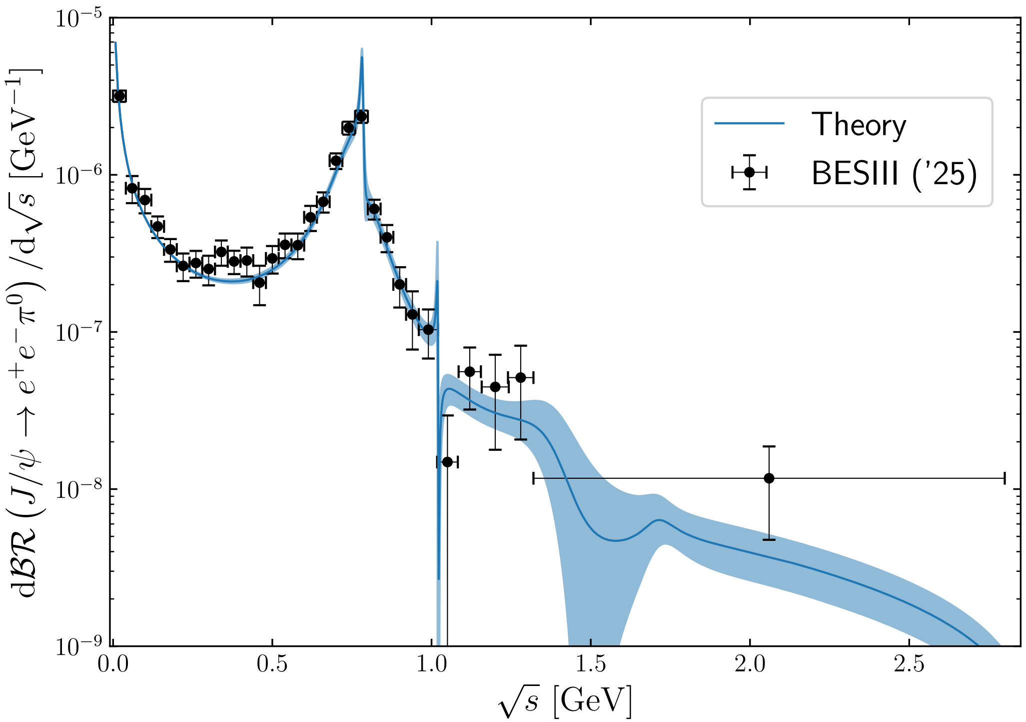
Parameters	Fit	Theory [Eq. (46)]
$\lvert w_{\psi\omega\pi}\rvert$ [$10^{-5}\mathrm{GeV}^{-1}$]	4.14(52)	4.38(24)
$\lvert w_{\psi\phi\pi}\rvert$ [$10^{-6}\mathrm{GeV}^{-1}$]	6.68(3.13)	4.71(3)
$\delta_{\psi\omega\pi}$	0.86(31)	-
$\delta^{c\bar{c}}$	0.88(20)	-
χ^2/ d.o.f.	$27.61/(30-4) = 1.06$	

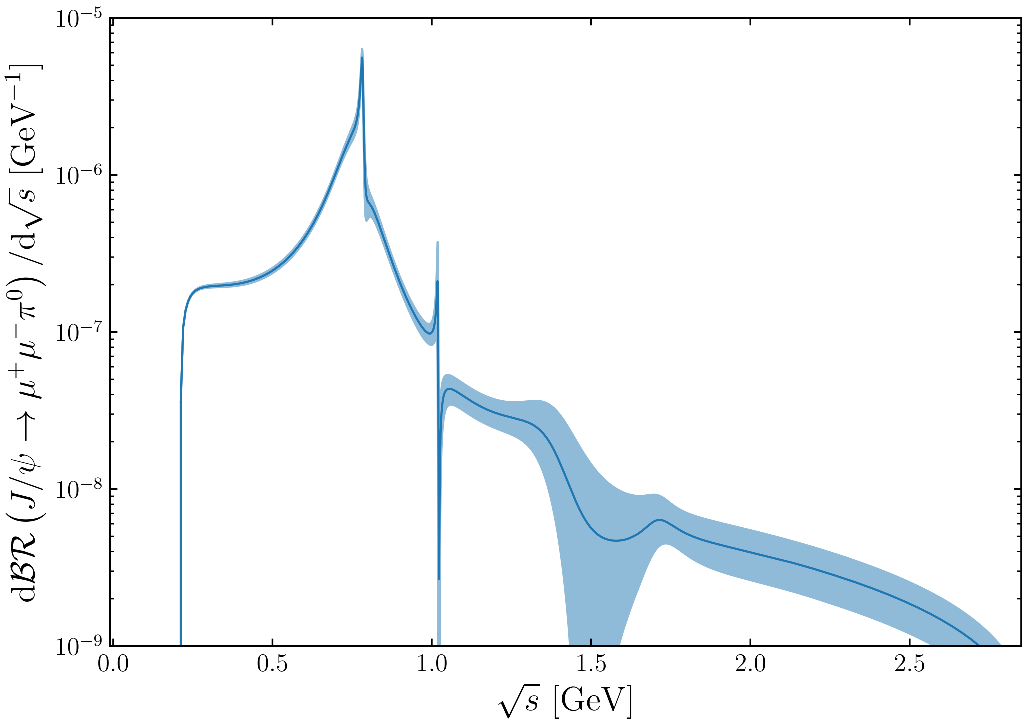


- Significant cusp structure corresponding to ω, ϕ
- Nontrivial dip structure corresponding to $\rho'(1450)$

Branching fractions

- Differential branching fractions





- Branching fractions ($\times 10^{-4}$)

	Exp [41]	This Work	DR [13]	RChT [53]	RChT [54]	VMD [49]
$J/\psi \to \pi^0 e^+e^-$	$8.06 \pm 0.31(\text{stat}) \pm 0.38(\text{syst})$	$7.01^{+0.62}_{-0.58}$	$(5.5\ldots6.4)$	12.94 ± 0.44	11.91 ± 1.38	$3.89^{+0.37}_{-0.33}$
$J/\psi \to \pi^0 \mu^+\mu^-$	-	$4.71^{+0.57}_{-0.53}$	$(2.7\ldots3.3)$	3.04 ± 0.10	2.80 ± 0.32	$1.01^{+0.10}_{-0.09}$

Summary and outlook

- We update the predictions for $J/\psi \to e^+e^-\pi^0$ by incorporating $\rho - \omega$ interference with the dispersive method ensuring unitarity, analyticity and crossing symmetry

- $\mathrm{BR}\left(J/\psi \to \pi^0 e^+ e^-\right) = 7.01^{+0.62}_{-0.58} \times 10^{-7}, \quad \mathrm{BR}\left(J/\psi \to \pi^0 \mu^+ \mu^-\right) = 4.71^{+0.57}_{-0.53} \times 10^{-7}$

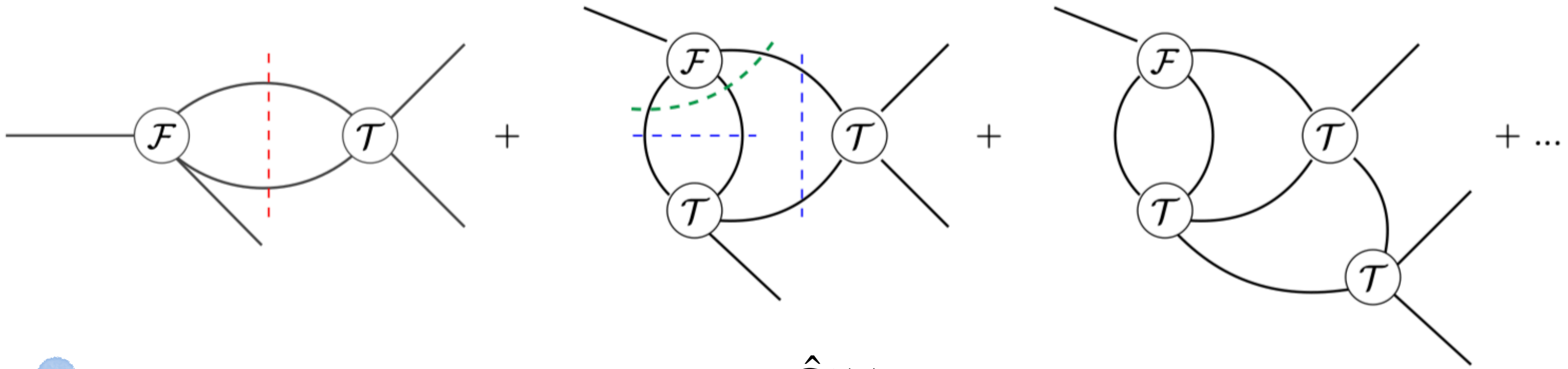
- Extend to $\omega, \phi \to e^+e^-\pi^0$, where the $\rho - \omega$ mixing effect was ignored in previous dispersive analyses JPAC, 2505.15309

Thank you for your attention!

Backup

Unitarity and analyticity

- Unitarity relation for the single variable amplitude $\mathscr{F}(s)$



- Complications: integration contour for $\hat{\mathscr{F}}(s)$
- **"Pinocchio" method** (**triangle topology** continuation) Bronzan and Kacser, Phys. Rev. (1963)
  - deform path of angular integral to avoid crossing branch cuts
- Gasser-Rusetsky method Gasser and Rusetsky, EPJC (2018)
  - deform path of dispersion integral
- Pasquier inversion R. Pasquier and J.Y. Pasquier, Phys. Rev. (1968, 1969)
  - interchange of the order of integrations to obtain integral equations in one variable

Analytic continuation of the inhomogeneities

- $i\epsilon$ prescription: give the vector-meson mass an infinitesimal positive imaginary part, $M_V^2 + i\epsilon$ if $M_V > 3M_\pi$ Bronzan and Kacser, Phys. Rev. (1963)
- Angular average integral ($s + t + u = 3M_\pi^2 + M_V^2 \equiv = 3s_0$)

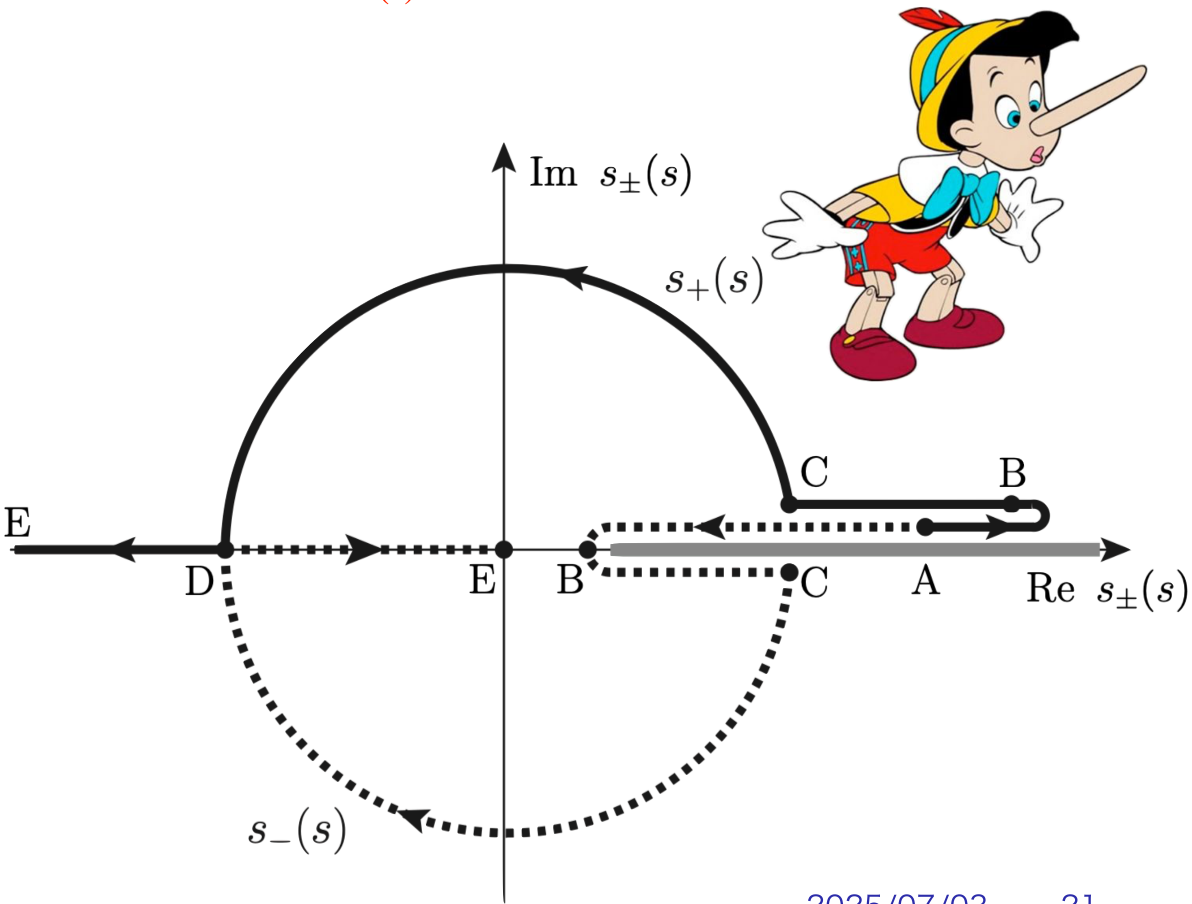
$$\langle z^n \mathcal{F} \rangle \equiv \frac{1}{2}\int_{-1}^{1} z^n \mathcal{F}\left(\frac{3s_0 - s + z\kappa(s)}{2}\right) \mathrm{d}z = \frac{1}{\kappa(s)}\int_{s_-(s)}^{s_+(s)} \mathrm{d}s' \left(\frac{2s' - 3s_0 + s}{\kappa(s)}\right)^n \mathcal{F}(s')$$

- The trajectories of $s_\pm(s)$

$$2s_+(s) = \begin{cases} 3s_0 - s + |\kappa(s)| + i\epsilon, & s \in \left[4M_\pi^2, M_-^2\right], \\ 3s_0 - s + i|\kappa(s)|, & s \in [M_-^2, M_+^2], \\ 3s_0 - s - |\kappa(s)|, & s \in [M_+^2, \infty), \end{cases}$$

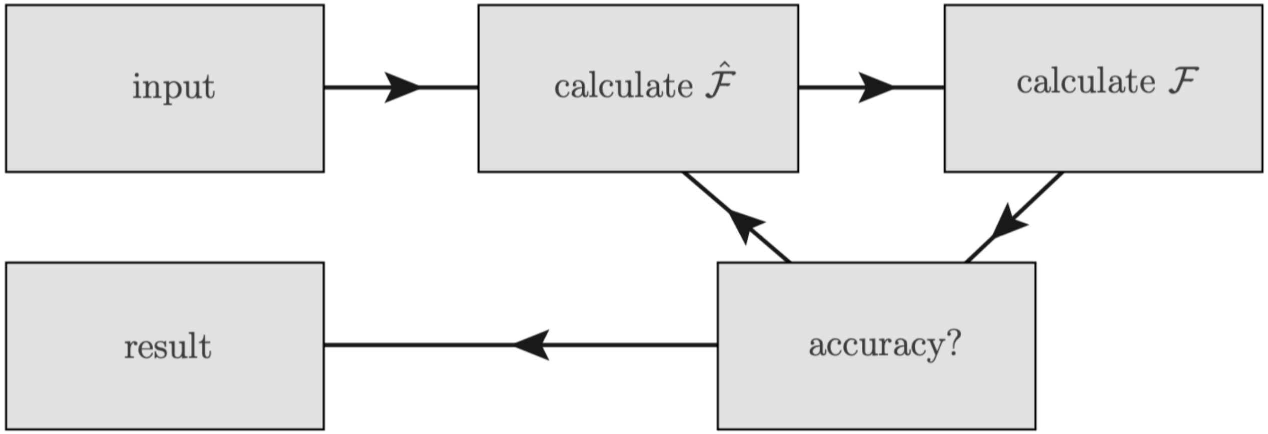
$$2s_-(s) = \begin{cases} 3s_0 - s - |\kappa(s)| + i\epsilon, & s \in \left[4M_\pi^2, \frac{M_V^2 - M_\pi^2}{2}\right], \\ 3s_0 - s - |\kappa(s)| - i\epsilon, & s \in \left[\frac{M_V^2 - M_\pi^2}{2}, M_-^2\right], \\ 3s_0 - s - i|\kappa(s)|, & s \in [M_-^2, M_+^2], \\ 3s_0 - s + |\kappa(s)|, & s \in [M_+^2, \infty), \end{cases}$$

$$M_\pm^2 = (M_V \pm M_\pi)^2$$

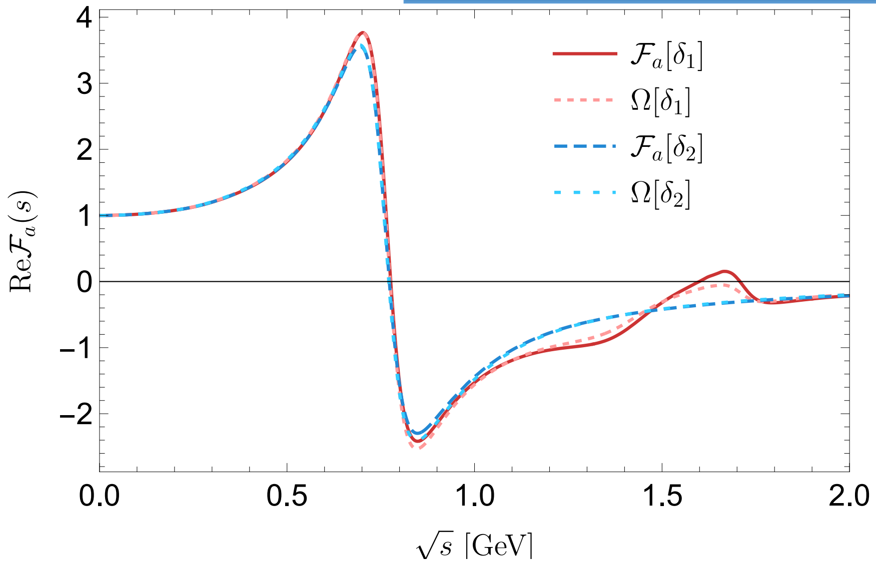


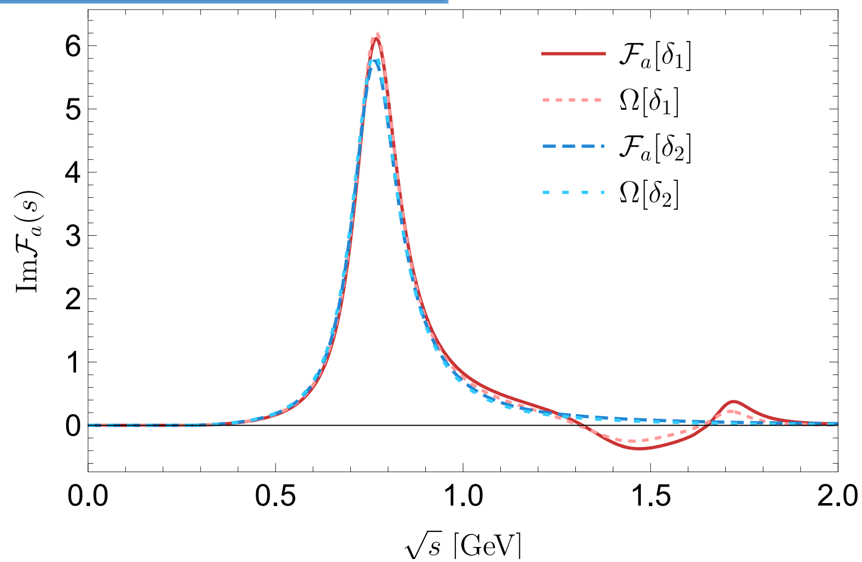
KT equations: solutions

- Solution by numerical **iteration**
- Initial input: $\mathscr{F}(s) = \Omega(s)$



$f_1 = a(\mathscr{F}_a + \hat{\mathscr{F}}_a) \equiv a\bar{\mathscr{F}}_a,$ KT basis function





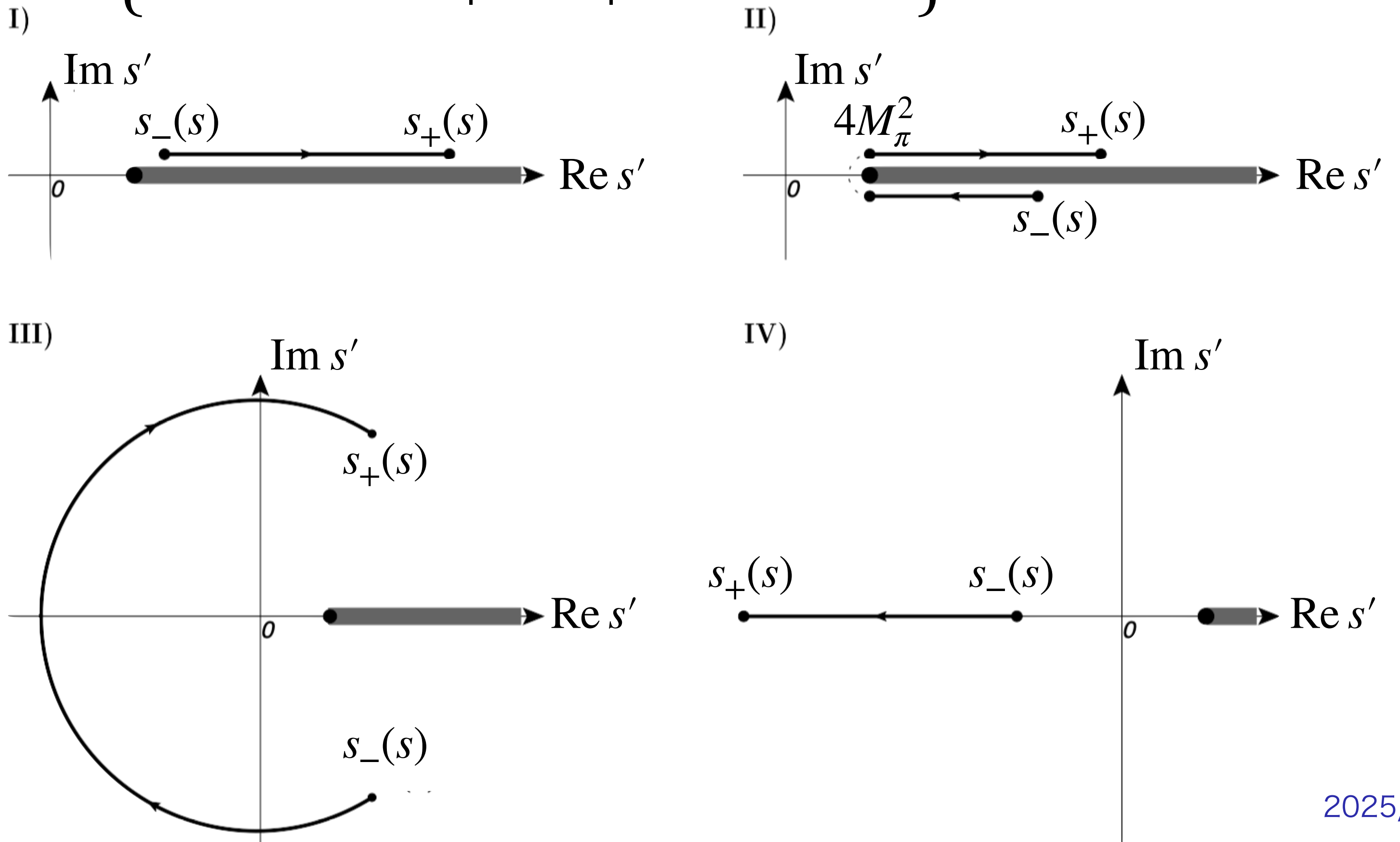
Analytic continuation of the inhomogeneities

$$\langle z^n \mathcal{F}\rangle \equiv \frac{1}{2}\int_{-1}^{1} z^n \mathcal{F}\left(\frac{3s_0 - s + z\kappa(s)}{2}\right) \mathrm{d}z = \frac{1}{\kappa(s)}\int_{s_-(s)}^{s_+(s)} \mathrm{d}s' \left(\frac{2s' - 3s_0 + s}{\kappa(s)}\right)^n \mathcal{F}(s')$$

a singularity-free function $\tilde{\mathcal{F}}(s)$

$$\tilde{\mathcal{F}}(s) \equiv \kappa^3(s)\hat{\mathcal{F}}(s) = 3\int_{s_-(s)}^{s_+(s)} ds' \left(\kappa^2(s) - \left(2s' - 3s_0 + s\right)^2\right) \mathcal{F}(s')$$

$$\mathcal{F}(s) = a\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} \frac{\mathrm{d}s'}{s'} \frac{\sin\delta(s')\,\tilde{\mathcal{F}}(s')}{\left|\Omega(s')\right| \kappa^3(s')(s'-s)}\right\}$$



TFF

BESIII, 2501.04344 (2025)