Spin Dynamics from Algebra

Outline

- Introduction
- Formalism
- Discussion

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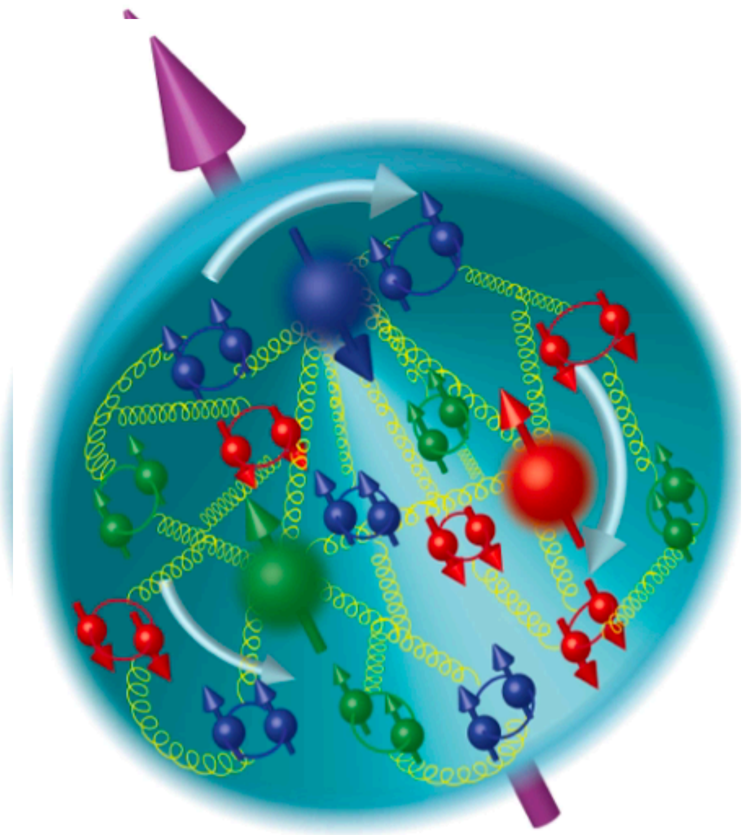
QCD Spin Structure

Xiangdong Ji et al, Nature Review, 20

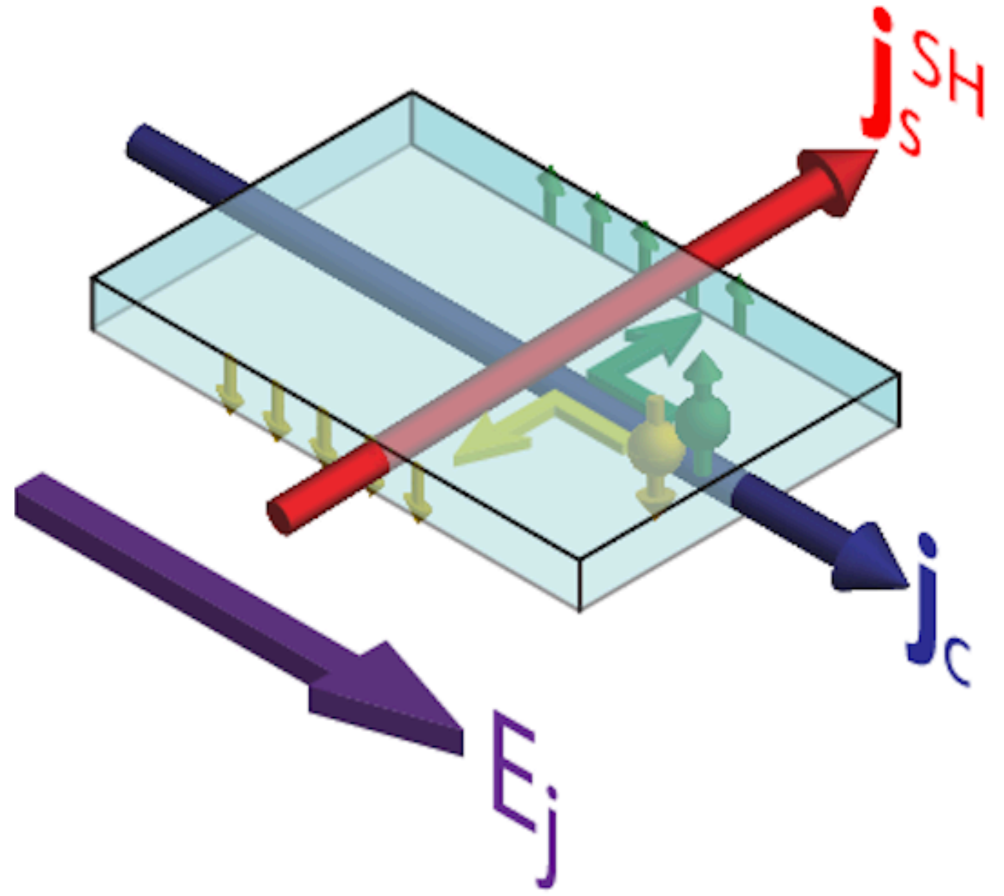


$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L$$

quark spin ($\approx 20\,\%$) *gluon spin* ($\sim 50\,\%$) *orbital ang. momentum*

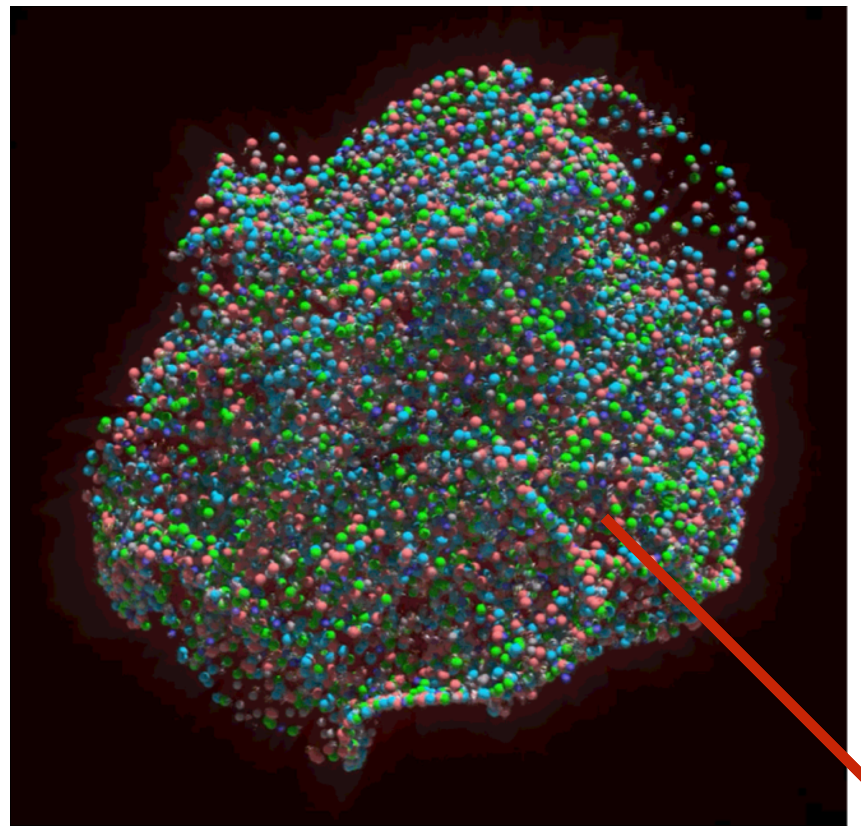
- Intriguing for proton (confined)
- Universal feature of quark-gluon spin interaction?
- Consider quark matter (deconfined)

Spin Hall Effect and Heavy-ion Collisions



Spin Hall material

$$\hat{s} \propto \vec{v} \times \vec{E}$$



Quark Matter

$$\hat{s} \propto \vec{v} \times \vec{\partial}$$

Gradient $\vec{\partial}$

- E-field generates spin-motion correlation
- Density gradient in quark matter polarizes spin

Rich Spin Effects in Quark Matter

$$s^i(t,\vec{x},\vec{v}) = \underbrace{c_\omega \omega^i}_{\text{vorticity}} + \underbrace{c_\sigma \epsilon^{ikj}\sigma^{jl}}_{\text{shear}} + \dots$$

$$\vec{\omega} = \nabla \times \vec{u}$$

$$\sigma_{ij} = (\partial_i u_j + \partial_j u_i)$$

Spin in phase space

- Rotation effect: long-predicted and confirmed experimentally

  Xin-Niang Wang&Zuo-Tang Liang PRL2005

- Shear-induced polarization (SIP): uncovered recently (inspired by TMD formalism)

  Shuai Liu & YY JHEP 21;Becattini et al, PLB 21

- Baryon density gradient polarizes spin

  Shuai Liu &YY, PRD 20; Fu et al, 2022

- New window into spin structure of deconfined QCD

Next?

- Rich data across different systems and energies: interesting polarization pattern seen
- No single framework describe all data quantitatively assuming spin-medium interaction is weak and/or full-equilibration
- Open questions
  - What drives the spin dynamics?
  - What does polarization pattern reveal?

Kinetic Equation with Spin

- Powerful tool to describe spin dynamics
- Significant progress in formulation from microscopic theory

e.g. Jianhua Gao@Shandong, Xu-Guang Huang@Fudan, Koichi Hattori@ZJU, Defu Hou@CCNU, Shu Lin@中山, Qun Wang, Shuo Fang,Shi Pu@USTC, Pengfei Zhuang, Shuzhe Shi et al @Tsinghua,

- Eqns are complicated in form, making interpretation less transparent
- This talk: re-formulation with algebra *Zonglin Mo,YY in preparation*
  - Getting insight into open questions
  - Developing methods that might be useful for other systems

Chiral Perturbation Theory

- Pion fields $\pi_A(x)$: conjugate to symmetry generator

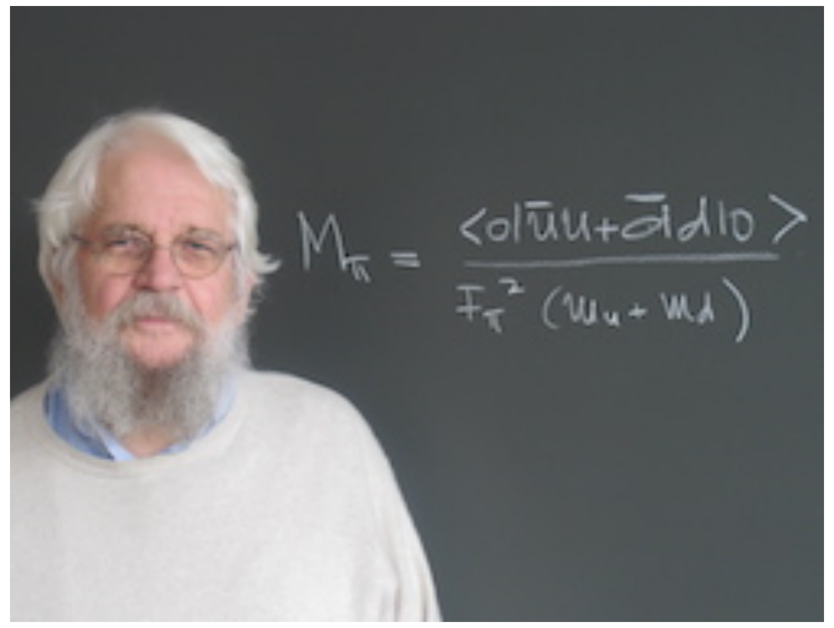
$$\text{SU(2)} \quad \Sigma \sim e^{i\pi_A \rho_A}$$

- Dynamics fixed by current algebra
- Algebra method has broad application

[Submitted on 25 Sep 1996]

Phonons as Goldstone Bosons

H. Leutwyler (University of Bern and CERN)



Leutwyler

Symmetry, Algebra and Dynamics

$$e^{i\int_x \theta_a(x)\,\hat{Q}_a}$$

Identifying symmetry

$\longrightarrow$

$$\langle \left[\hat{H}[Q], Q_a\right] \rangle \to \partial_t \theta_a = \ldots$$

E.o.M from algebra

- Elegant, non-linear, robust w.r.t coupling
  - Spin hydro. for superfluid Helium — *Dzyaloshinskii, Volovick 1978*
  - Hydro. of nuclear matter — *Son PRL2000*
  - Neutral charm mesons near threshold — *Bratten-Hammer PRL2022*
- Applying similar idea to kinetic theory

Boltzmann Equation

"My memory for figures, … , always lets me down when I am counting beer glasses.” Ludwig Boltzmann

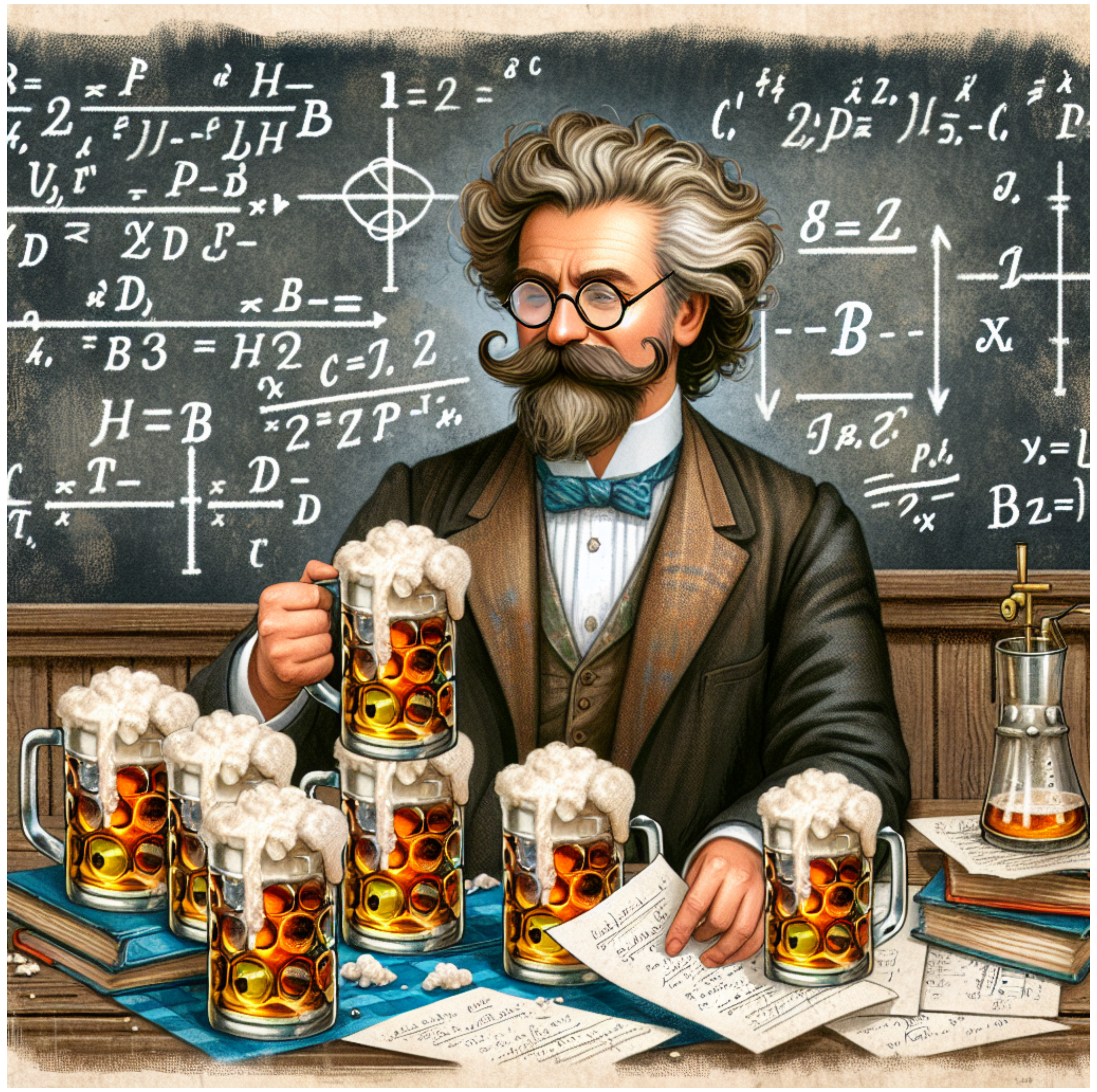


Image by DeepSeek

- Describing the evolution of a cloud of particles in phase space

Canonical Transformation (C.T.)

- Symmetry of (collision-less) kinetic eqn.: covariance under the relabeling of $(\vec{x}, \vec{p})$ that preserving the phase space volume

$$\vec{x} \to \vec{x} - \epsilon \nabla_{\vec{p}} G\,, \qquad \vec{p} \to \vec{p} + \epsilon \nabla_{\vec{x}} G\,,$$

- Generator

Delacrétaz et. al (U. Chicago group), 2022

$$\hat{T}_0(\vec{x}, \vec{p}) \equiv \int_{\vec{y}} \hat{\psi}_S^\dagger(\vec{x} + \vec{y}/2) \hat{\psi}_S(\vec{x} - \vec{y}/2)\, e^{i\vec{y}\cdot\vec{p}}$$

- Poisson brackets from the algebra of $\hat{T}_0$ in classical limit

$$[\hat{T}_0(\vec{x}, \vec{p}), \hat{T}_0(\ldots)] \to \{\ldots, \ldots\}_{P.B.}$$

Modern Formulation

Delacrétaz et. al (U. Chicago group), 2022

- Distribution $f(t, \vec{x}, \vec{p})$ is related to the conjugate variable of $\hat{T}_0$

- Kinetic eqn:

$$\partial_t f = \{\epsilon_p, f\}_{P.B.}$$

$$\epsilon_{\vec{p}}[f] = \frac{\delta H[f]}{\delta \phi} = E_{\vec{p}} + \int_{\vec{p}'} F(\vec{p}, \vec{p}')\delta f(\vec{p})$$

in-medium *off-equili.*

F : kernel parametrized by EFT coefficients (c.f. Landau parameters)

- Vector Wigner function is the Noether current of C.T.

$$\langle \psi^\dagger \gamma^i \psi \rangle \sim \vec{v} f$$

Spin-dependent Canonical Transformation

- Generator of spin-dependent C.T. (rotation of spin in phase space)

proposed by Delacrétaz et. al, 2022

$$\hat{T}_i(\vec{x}, \vec{p}) \sim \hat{\psi}^\dagger_S \sigma_i \hat{\psi}_S$$

- Conjugate field $\alpha^i(t, \vec{x}, \vec{p}) \sim$ spin chemical potential in phase space
- E.O.M from the algebra among $\hat{T}_0, \hat{T}_i$
- Noether current determines spin distribution

Spin Dynamics from Algebra

$$\partial_t f = \{f, \epsilon_p\} + \{\alpha^l, \mathscr{B}^l\}$$

$\epsilon_{\vec{p}} = \frac{\delta H}{\delta f}$

$\propto \partial(\text{mag. moment potential})$

$$\partial_t \alpha^i = \{\alpha^i, \epsilon_p\} + \{f, \mathscr{B}^i\} - \epsilon^{ijk}\alpha_i \mathscr{B}_j$$

Torque on mag. moment

$\mathscr{B} = \frac{\delta H}{\delta \alpha^i}$

Mag. field functional

$$\mathscr{B} = (B^i + B^i_{ind}[\delta f]) + (\text{E-field}) \times \vec{v}_p$$

c.f. $\Delta E \propto \hat{S} \cdot \hat{B}$

- Off-equilibrium distribution drives the spin dynamics

After complicated yet straightforward calculations, we finally derive the AKE with self-energy corrections,

$$\square^{(n)}_{a,\mu}\mathcal{A}^{<}+\tilde{q}_\mu\square^{(n)}_{q}\mathcal{A}^{<}+\tilde{m}\square^{(n)}_{m,\mu}\mathcal{A}^{<} = \mathcal{C}^{(n)}_{1,\mu}+\hbar(\tilde{q}_\mu\mathcal{C}^{(n),q}_{2,\mu}+\tilde{m}\mathcal{C}^{(n),m}_{2,\mu}). \tag{101}$$

The free-streaming part can be divided in three parts [87, 88], which are proportional to $a_\rho f_{\rm A}$, $\tilde{q}_\mu$ and $\tilde{m}$ separately,

$$\begin{aligned}\square^{(n)}_{a,\mu}\mathcal{A}^{<} &= \delta(\tilde{q}^2-\tilde{m}^2)\Big\{\tilde{q}\cdot\tilde{\nabla}(a_\mu f^<_{\rm A})+Qe\overline{F}_{\mu\nu}(a^\nu f^<_{\rm A})+2\epsilon_{\mu\nu\alpha\beta}\frac{\overline{\Sigma}^\nu_{\rm A}}{\hbar}\tilde{q}^\alpha a^\beta f^<_{\rm A}\\ &\quad+\frac{\tilde{q}\cdot\nabla\overline{\Sigma}_{\rm F}}{\tilde{m}}(a_\mu f^<_{\rm A})-\tilde{q}_\mu\frac{\nabla^\nu\overline{\Sigma}_{\rm F}}{\tilde{m}}(a_\nu f^<_{\rm A})\Big\},\end{aligned} \tag{102}$$

The second and third terms in the r.h.s of Eq. (101) read,

$$\begin{aligned}\square^{(n)}_{q}\mathcal{A}^{<} &= \delta(\tilde{q}^2-\tilde{m}^2)\Big\{2\tilde{q}^\alpha\frac{\overline{\Sigma}_{{\rm T},\alpha\beta}}{\hbar\tilde{m}}a^\beta f^<_{\rm A}+\frac{2\overline{\Sigma}_{{\rm T},\alpha\beta}}{\tilde{m}}\tilde{q}^\alpha S^{\beta\rho}_n\nabla_\rho f^<_{\rm V}+\epsilon^{\rho\sigma\alpha\beta}\frac{\overline{\Sigma}_{{\rm T},\alpha\beta}}{2\tilde{m}}\tilde{q}_\sigma\nabla_\rho f^<_{\rm V}\\ &\quad-\tilde{q}_\lambda[\overline{\Sigma}^\lambda_{\rm A}f^<_{\rm V}]^F_{\rm P.B.}+\frac{\hbar}{2}S_{n,\rho\sigma}[(Qe\overline{F}^{\rho\sigma})f^<_{\rm V}]^F_{\rm P.B.}-\frac{(\tilde{q}\cdot n)\overline{\Sigma}_{\rm A}\cdot\nabla f^<_{\rm V}}{(\tilde{q}\cdot n+\tilde{m})}\\ &\quad+\hbar\left(\frac{\epsilon_{\rho\nu\alpha\beta}\tilde{q}^\alpha\tilde{\nabla}^\rho n^\beta}{2(\tilde{q}\cdot n+\tilde{m})}-\frac{\tilde{\nabla}^\rho(\tilde{q}\cdot n+\tilde{m})}{\tilde{q}\cdot n+\tilde{m}}S_{n,\rho\nu}\right)\tilde{\nabla}^\nu f^<_{\rm V}\Big\}\\ &\quad+\delta'(\tilde{q}^2-\tilde{m}^2)(\tilde{q}\cdot\tilde{\nabla}f^<_{\rm V})\left[\hbar\frac{\tilde{q}^\rho n^\sigma Qe\tilde{\overline{F}}_{\rho\sigma}}{(\tilde{q}\cdot n+\tilde{m})}-2(\overline{\Sigma}_{\rm A}\cdot\tilde{q})\right],\end{aligned} \tag{103}$$

and

$$\begin{aligned}\square^{(n)}_{m,\mu}\mathcal{A}^{<} &= \delta(\tilde{q}^2-\tilde{m}^2)\Big\{2\overline{\Sigma}_{{\rm T},\mu\nu}\frac{a^\nu f^<_{\rm A}}{\hbar}+[\overline{\Sigma}_{\rm F}(a_\mu f^<_{\rm A})]^F_{\rm P.B.}+\tilde{m}[\overline{\Sigma}_{{\rm A},\mu}f^<_{\rm V}]^F_{\rm P.B.}\\ &\quad-\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\left((\nabla^\nu f^<_{\rm V})\overline{\Sigma}^{\alpha\beta}_{\rm T}+\tilde{q}^\nu[\overline{\Sigma}^{\alpha\beta}_{\rm T}f^<_{\rm V}]^F_{\rm P.B.}\right)+2\overline{\Sigma}_{{\rm T},\mu\nu}S^{\nu\alpha}_n\nabla_\alpha f^<_{\rm V}\\ &\quad-\frac{(\tilde{q}\cdot\overline{\Sigma}_{\rm A})\nabla_\mu f^<_{\rm V}+\tilde{m}(\overline{\Sigma}_{\rm A}\cdot n)\nabla_\mu f^<_{\rm V}-\tilde{m}n_\mu(\overline{\Sigma}_{\rm A}\cdot\nabla f^<_{\rm V})}{(\tilde{q}\cdot n+\tilde{m})}\\ &\quad+\hbar[\overline{\Sigma}_{\rm F}S_{n,\mu\rho}]^F_{\rm P.B.}\nabla^\rho f^<_{\rm V}-\hbar S_{n,\mu\nu}[(\nabla^\nu\overline{\Sigma}_{\rm F})f^<_{\rm V}]^F_{\rm P.B.}+\hbar\tilde{\nabla}^\rho S_{n,\rho\mu}[\overline{\Sigma}_{\rm F}f^<_{\rm V}]^F_{\rm P.B.}\\ &\quad+\hbar\frac{(\tilde{m}n^\nu+\tilde{q}^\nu)}{2(\tilde{q}\cdot n+\tilde{m})}\left(\epsilon_{\mu\nu\beta\gamma}\frac{\tilde{\nabla}^\beta(\tilde{q}\cdot n+\tilde{m})}{(\tilde{q}\cdot n+\tilde{m})}\tilde{\nabla}^\gamma f^<_{\rm V}-[(Qe\tilde{\overline{F}}_{\mu\nu})f^<_{\rm V}]^F_{\rm P.B.}\right)\\ &\quad-\hbar\epsilon_{\mu\nu\alpha\beta}\frac{n^\alpha\nabla^\beta\overline{\Sigma}_{\rm F}+\tilde{m}\tilde{\nabla}^\beta n^\alpha}{2(\tilde{q}\cdot n+\tilde{m})}\tilde{\nabla}^\nu f^<_{\rm V}\Big\}\\ &\quad+\delta'(\tilde{q}^2-\tilde{m}^2)\Big\{(\tilde{q}\cdot\tilde{\nabla}f^<_{\rm V})I_\mu+2q_\mu f^<_{\rm V}Qe\tilde{F}_{\alpha\beta}\overline{\Sigma}^{\alpha\beta}_{\rm T}\\ &\quad-\hbar[\overline{\Sigma}_{\rm F}f^<_{\rm V}]^F_{\rm P.B.}\left(\tilde{q}^\alpha Qe\tilde{F}_{\mu\alpha}+S_{n,\mu\nu}2\tilde{q}_\alpha QeF^{\nu\alpha}\right)\Big\},\end{aligned} \tag{104}$$

where

$$I_\mu \equiv -\hbar\frac{Qe\tilde{\overline{F}}_{\mu\alpha}(\tilde{m}n^\alpha+\tilde{q}^\alpha)}{(\tilde{q}\cdot n+\tilde{m})}-2\hbar S_{n,\mu\nu}\nabla^\nu\overline{\Sigma}_{\rm F}-\epsilon_{\mu\rho\alpha\beta}\tilde{q}^\rho\overline{\Sigma}^{\alpha\beta}_{\rm T}+2\tilde{m}\overline{\Sigma}_{{\rm A},\mu}. \tag{105}$$

state of art spin kinetic eqn. Fang Shuo, Shi Pu, Di-lun Yang,

check in progress

$$\partial_t\alpha^i = \{\alpha^i,\epsilon_p\}+\{f,\mathscr{B}^i\}-\epsilon^{ijk}\alpha_i\mathscr{B}_j$$

Spin Distribution

- Noether current from E.o.M

$$\partial_t \alpha^i = \partial_l \tau^{li} + \dots$$

Spin current tensor $\tau^{li} = v^l s^i$

Spin distribution $s^i(t, \vec{x}, \vec{p}) \propto (\alpha^i + f_0' \mathscr{B}^i) + \dots$

Off-equil. distribution f generates spin distribution

$$s^i \propto B^i_{ind}[\delta f]$$

Polarization measurement probes quark-medium interaction

- Algebra formalism can be matched to field theory cal.
- Fermion self-energy Σ_μ plays the role of vector potential

See also works by Shuo Fang, Shu Lin, She Pu, Naoki Yamamoto, and Di-Lun Yang

$$s^i \propto \mathscr{B}^i_{ind} = \epsilon^{ijk}\partial_j \Sigma_k[\delta f]$$

$\Sigma[\delta f] \sim$ spin carrier-medium coupling

$$\sim g^2 \int_{p'} D(p-p')\,\delta f_{\text{quark}} + \dots \ .$$

Gluon distribution

Unique opportunity to measure quark self-energy in medium

Summary

- Spin observables probe in-medium properties of quarks/gluons
- Algebra method is powerful in studying spin dynamics
- (adding dissipative effect would be interesting)
- Wigner function as Noether current of symmetry operation (application in other QCD systems)

Back-up