QFT on the Worldline: Applications to the Schwinger Effect, Chirality Generation, and N-photon Scattering

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Sep. 11, 2024

Based on:

- PC, K. Fukushima, S. Pu, Phys. Rev. Lett. 121, 261602 (2018), [1807.04416]; PC, S. Pu, Int. J. Mod. Phys. A, 35, 28, 2030015 (2020), [2008.03635].
- PC, K. Hattori, D.L. Yang Phys. Rev.D 107, 5, 056016 (2023), [2208.12913].
- PC, J.P. Edwards, A. Ilderton, K. Rajeev, Phys.Rev.D 109, 6, 065003 (2024), [2311.14638]; (2024) [2405.07385].

Outline

The Worldline Formalism

- What exactly is the formalism? Best illustrated through the **One-Loop Effective Action**! Here for **QED**.
- Begin with QED partition function *in a background field*, A_μ–but without dynamical d.o.f.! ($D_\mu = \partial_\mu + ieA_\mu$)

  [C. Schubert, *Phys. Rept.* 10.101 (2001).]

$$\langle \text{out} | \text{in} \rangle = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{i\int d^4x \bar{\psi}(i\not{D}-m)\psi}$$

- Get Effective Action from *integrating out fermions:*

$$\langle \text{out} | \text{in} \rangle = \det(i\not{D} - m) = \frac{1}{2}\det(\not{D}^2 + m^2) = e^{i\Gamma[A]} \, .$$

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- Introduce **Schwinger propertime, T**:

$$\Gamma[A] = \frac{i}{2}\int_0^\infty \frac{dT}{T}\int d^4x \, \mathrm{tr}\langle x|e^{-i(\not{D}^2+m^2)T}|x\rangle \, ,$$

since $\ln \det = \mathrm{Tr} \ln$

- $\langle x|e^{-i(\not{D}^2+m^2)T}|x\rangle$ is just the *quantum mechanical-like* kernel, however here with *propertime* propagation! (Real-time, x^0, put on same footing as $\boldsymbol{x}$.)

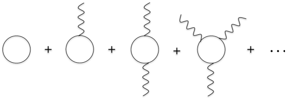
The Worldline Formalism

- Path integral form ($\not{D}^2 = D^2 + \frac{1}{2} e F_{\mu\nu}\sigma^{\mu\nu}$: scalar+spinor):

$$\langle x | e^{-i(\not{D}^2 + m^2)T} | y \rangle = \int \mathcal{D}x\, \mathcal{P} e^{i\int_0^T d\tau [-\frac{1}{4}\dot{x}^2 - A_\mu \dot{x}^\mu - \frac{1}{2} F_{\mu\nu}\sigma^{\mu\nu} - m^2]}$$

QFT in a **first quantized** formalism!

- *All orders in QED electromagnetic coupling to one-loop*!

$\Gamma[A] =$ 

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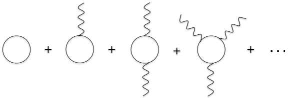


- Has appeared in many contexts [Z. Bern, D. Kosoer, *Nucl. Phys. B*, 379, 451—561 (1992); M. Strassler, *SLAC-PUB*-5757 (1992),;...];

PHYSICAL REVIEW VOLUME 80, NUMBER 3 NOVEMBER 1, 1950

Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction

R. P. Feynman*
Department of Physics, Cornell University, Ithaca, New York
(Received June 8, 1950)

The validity of the rules given in previous papers for the solution of problems in quantum electrodynamics is established. Starting with Fermi's formulation of the field as a set of harmonic oscillators, the effect of the oscillators is integrated out in the Lagrangian form of quantum mechanics. There results an expression for the effect of all virtual photons valid to all orders in $e^2/\hbar c$. It is shown that evaluation of this expression as a power series in $e^2/\hbar c$ gives just the terms expected by the aforementioned rules.

In addition, a relation is established between the amplitude for a given process in an arbitrary unquantized potential and in a quantum electrodynamical field. This relation permits a simple general statement of

Wealth of physics in compact expressions, e.g.:

- Euler-Heisenberg Lagrangian [W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936)]
- Sauter-Schwinger pair production [F. Sauter, Z. Phys. (1931); W. Heisenberg and H. Euler, Z. Phys. (1936); J. Schwinger, Phys. Rev. (1954)]
- Light by light scattering [H. Euler and B. Kockel, Naturwiss. (1935)]

Not limited to the effective action!

1. Feynman "dressed" propagators, QED in background field:

$$\langle \text{out}| T\{\psi(x)\bar{\psi}(y)\}|\text{in}\rangle = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \psi(x)\bar{\psi}(y) e^{i\int d^4x \bar{\psi}(i\not{D}-m)\psi}$$

$$= (i\not{D}+m)\int_0^\infty dT\, \langle x| e^{-i(\not{D}^2+m^2)T}|y\rangle$$

2. **In-in** or **Schwinger-Keldysh** worldline desciptions (to be discussed later!)
3. Scattering amplitudes!

Not limited to QED!

- **Non-Abelian Theories (Scalar Effective Action)**:

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x \, e^{-\frac{1}{4}\int_0^T d\tau \dot{x}^2} \mathrm{tr}\mathcal{P} e^{ig\int_0^T d\tau A_\mu \dot{x}^\mu}$$

- **Non-Abelian Theories (Gluon Effective Action)**:

$$\Gamma[A] = \frac{1}{2}\lim_{C\to\infty}\int_0^\infty \frac{dT}{T} e^{-CT(\frac{D}{2}-1)} \oint \mathcal{D}x\mathcal{D}\psi\mathcal{D}\bar{\psi}\,\mathrm{tr}\mathcal{P}$$
$$e^{-\int_0^T d\tau \left\{\frac{1}{4}\dot{x}^2 + igA_\mu\dot{x}^\mu + \bar{\psi}^\mu[\delta_{\mu\nu}(\frac{d}{d\tau}-C)-2igF_{\mu\nu}]\psi^\nu\right\}}$$

- **Even Gravity (Scalar Effective Action)!**

$$\Gamma[g] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x\sqrt{g}\, e^{-\int_0^T d\tau[\frac{1}{4}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \xi R(x)]}$$

- Many more QFTs ...

Outline

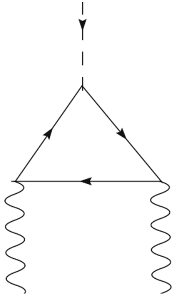
- Unlike the vector current the **axial (chiral) vector current**, j_5^μ, *is not conserved*.[S. L. Adler, *Phys. Rev.* 177, 2426 (1969); J. S. Bell, R. Jackiw, *Il Nuovo Cimento A* 60, 47.] Massless case:

$$\partial_\mu j_5^\mu = \frac{e^2}{2\pi^2}\vec{E}\cdot\vec{B}$$

- It was first thought that *classically* $\partial_\mu j_5^\mu = 0$. But, due to *quantum* effects, chiral symmetry broken.

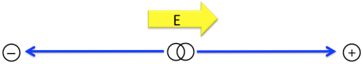
$$\frac{d(N_{5R} - N_{5L})}{dt} = \int d^3x \frac{e^2}{2\pi^2}\vec{E}\cdot\vec{B}$$

$N_{5(R/L)}$: # of right or left handed fermions (spin and momentum aligned)



D. E. Kharzeev, *Ann. of Phys.* 325, 205–218 (2010)

- Under a background electric field the quantum field theoretic vacuum is unstable against the production of particle anti-particle pairs: **Schwinger pair production** [J. Schwinger, *Phys. Rev.* 82, 664 (1951).]



Schwinger Mechanism
Background

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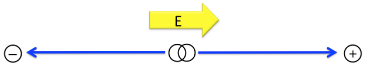
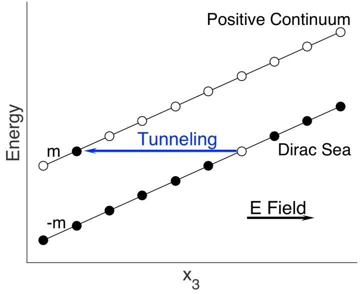


- Imaginary part of effective action, $\langle \mathrm{out}|\mathrm{in}\rangle$, characterizes the *vacuum non-persistence*.

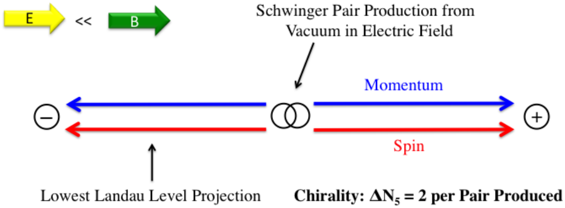
$$\langle \mathrm{out}|\mathrm{in}\rangle = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{i\int d^4x \bar{\psi}(i\not{D}-m)\psi}$$

$$\text{Num. of pairs} = 1 - |\langle \mathrm{out}|\mathrm{in}\rangle|^2$$

$$\approx \exp\left(-\frac{\pi m^2}{eE}\right)$$



- Add in a strong background magnetic field:



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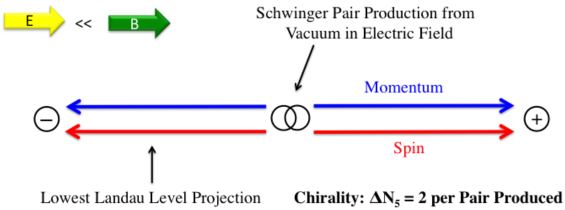

- Predicts the chiral anomaly [K. Fukushima, D.E. Kharzeev, and H.J. Warringa, *Phys. Rev. Lett.* 104, 212001 (2010).]!

$$\text{Num. of pairs} \approx \frac{e^2 EB}{4\pi^2} \coth\left(\frac{B}{E}\pi\right) \exp\left(-\frac{\pi m^2}{eE}\right)$$

$$\xrightarrow{B \gg E} \frac{e^2 EB}{4\pi^2} \exp\left(-\frac{\pi m^2}{eE}\right) = \frac{1}{2}\langle \partial_0 n_5 \rangle$$

$n_5 = \bar{\psi}\gamma^0\gamma^5\psi$ is the chiral density.

Expectation Values and Setup

Background

The **axial Ward identity** [K. Fujikawa, *PRL* 10.1103 (1979).],

$$\partial_0 n_5 = 2im\bar{\psi}\gamma^5\psi + \frac{e^2}{2\pi^2}\vec{E}\cdot\vec{B},$$

for fermion mass, m, is exact and well-known at the **operator** level.

- **Vacuum Expectation Value** behavior?
- Schwinger mechanism in parity violating fields $\rightarrow \langle in| \neq \langle out|$!

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Setup and Methods

Parallel electric, E, and magnetic, B, fields (in x_3 direction)

1. Schwinger proper time
2. Worldline path integral techniques
3. In-In (Schwinger-Keldysh real-time) formalism

Standard QFT treatment of well-known in-out propagator for homogeneous fields is

$$S^c(x,y) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, i\psi(x)\bar{\psi}(y)e^{i\int d^4x\bar{\psi}(i\not{D}-m)\psi}$$
$$= (i\not{D}+m)\int_0^\infty dT\, g(x,y,T)$$

with kernel, g, in proper time, s,

$$g(x,y,T) = i\,\langle x|\exp(-i(\hat{\not{D}}^2+m^2)T)\,|y\rangle\,.$$

Kernel may be cast into worldline path integral representation.

- Indispensible for Schwinger pair production.
- Automatically regulates traces of γ^5.

- **Worldline Path Integral:** [C. Schubert, *Phys. Rept.* 10.101 (2001).]

$$g(x, y, T) = i \int \mathcal{D}x \, \mathcal{P} \exp\Big\{ i \int_0^T d\tau [-\frac{1}{4}\dot{x}^2 - A_\mu \dot{x}^\mu - \frac{1}{2} F_{\mu\nu}\sigma^{\mu\nu} - m^2] \Big\}$$

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- Spin factor diagonalizes in homogeneous fields and path integral portion reduces to one of a boson.
- Evaluate through steepest descents
- Reproduce exact result: [J. Schwinger, *Phys. Rev.* 82, 664 (1951).]

$$g(x,y,T) = \frac{e^2 EB}{(4\pi)^2}\sinh^{-1}(eET)\sin^{-1}(eBT)$$
$$\times \exp\left[-i\left(\tfrac{1}{2}eF^{\mu\nu}\sigma_{\mu\nu} + m^2\right)T + i\varphi(x,y,T)\right]$$

$$\varphi = \tfrac{1}{2}x_\mu eF^{\mu\nu}y_\nu + \tfrac{1}{4}\left[(z_3^2 - z_0^2)eE\coth(eET) + (z_1^2 + z_2^2)eB\cot(eBT)\right]$$

In-out pseudoscalar [Schwinger, *Phys. Rev.* 82, 664 (1951)] and the axial Ward identity [W. Dittrich, H. Gies *Probing the Quantum Vacuum* (2000)] can be found to one-loop:

$$\langle out|\,\bar{\psi}i\gamma^5\psi\,|in\rangle = -\,\mathrm{tr}\,\gamma^5 S^c(x,x) = -\frac{e^2EB}{4m\pi^2}$$

$$\langle out|\,\partial_0 n_5\,|in\rangle = 0$$

for any mass (even $m \to 0$)!

- **The anomaly has vanished!**
- Naïve treatment of massless limit would lead to an incorrect result! $\to$ Limit should be taken last.

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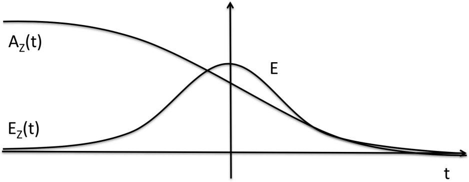
Inconsistency? Not quite.

- What's the problem $\to$**Inequivalent Vacuum States!**
- Need out-of-equilibrium contruction to see produced pairs.

But in-out expectation values perfectly valid calculations.
$\to$ **Vacuum Polarization**.

Inequivalent asymptotic $t \to \pm\infty$ **states**, $\langle \text{in}| \neq \langle \text{out}|$;
e.g., Sauter profile:



In-out propagator, S^c, misses out-of-equilibrium phenomena.

- **Solution is provided by using in-in vacuum states!**
- An in-in prescription is the same as a **Schwinger-Keldysh or real-time** [J. Schwinger, *J. Math. Phys.* 407 (1961).] formalism.
- However, direct application of the Schwinger-Keldysh formalism is challenging and few exact results are known...

Look for application of the worldline proper time formalism!

- Start from in-in normalization and insert complete set of out states (also SK generating functional without sources):

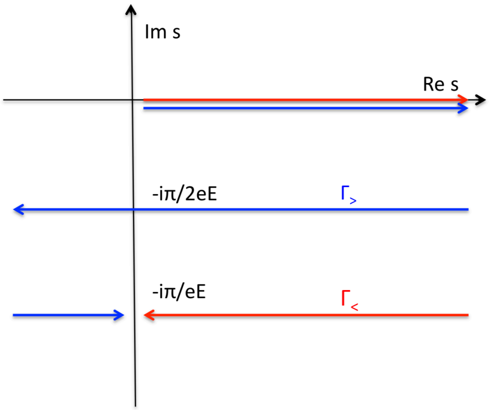
$$\langle \text{in}|\text{in}\rangle = 1 = \sum_{\alpha} \langle \text{in}|\alpha, \text{out}\rangle \langle \alpha, \text{out}|\text{in}\rangle$$
$$= \left|\langle \text{out}|\text{in}\rangle\right|^2 + \sum_{n,m} \langle \text{in}|\, b_n^{\dagger\, out} a_m^{\dagger\, out} \,|\text{out}\rangle \langle \text{out}|\, a_m^{out} b_n^{out} \,|\text{in}\rangle + \ldots$$

- First term: Probability that the vacuum stays the vacuum (no pairs of particles in out state).
- Second term: Probability for Schwinger pair production of single pair.
- Apply to **in-in propagator** $\rightarrow$

$$S^c_{in}(x, y) = i \langle \text{in}|\, \mathcal{T}\{\psi(x)\bar{\psi}(y)\} \,|\text{in}\rangle$$

[E. Fradkin, G. Gitman, and S. Shvartsman, *Quantum Electrodynamics in Unstable Vacuum* 1991] demonstrated that the in-in propagator, is expressible entirely in terms of the worldline kernel, $z = x - y$:

$$S^c_{in}(x,y) = (i\not{D} + m)\left[\theta(z_3)\int_{\Gamma_>} + \theta(-z_3)\int_{\Gamma_<}\right] ds\, g(x,y,s).$$



In-In Pseudoscalar and Axial Ward Identity:

$$\langle \text{in} | \bar{\psi} i\gamma^5 \psi | \text{in} \rangle = -\operatorname{tr} \gamma^5 S^c_{in}(x,x)$$
$$= -\frac{e^2 EB}{4m\pi^2}\left[1 - \exp\left(-\frac{m^2\pi}{eE}\right)\right]$$

And using the axial Ward identity:

$$\langle \text{in} | \partial_0 n_5 | \text{in} \rangle = \frac{e^2 EB}{2\pi^2} \exp\left(-\frac{m^2\pi}{eE}\right)$$

- **Chirality is spontaneously generated from the vacuum through the Schwinger mechanism!**
- Mass effects for the axial Ward identity.
- Generation of chirality agrees with physical heuristic picture!
- Only the lowest Landau level contributes.
- Calculation confirmation of in-in chiral density [H. J. Warringa, *PRD*86, 085029 (2012).]:

$$\langle \text{in} | n_5 | \text{in} \rangle = \frac{e^2 EB}{2\pi^2}\, t \exp\left(-\frac{m^2\pi}{eE}\right)$$

Outline

Non-Abelian Schwinger Pair Production

- Schwinger pair production from **SU**(2) *scalar* effective action

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint \mathcal{D}x \, e^{-\frac{1}{4}\int_0^T d\tau \dot{x}^\mu \dot{x}_\mu} \mathcal{W}$$

$$\mathcal{W} := \mathrm{tr}\mathcal{P} e^{ig\int_0^T d\tau A_\mu \dot{x}^\mu}$$

- Recall: $\mathrm{Im}\Gamma$ for pair production.

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- Recall: $\mathrm{Im}\Gamma$ for pair production.
- Use *non-perturbative* **Worldline Instanton** method! [I.K. Affleck, O. Alvarez, and N.S. Manton, *Nucl. Phys.* B197, 509 (1982); G.V. Dunne and C. Schubert, *PRD*, 105004 (2005).]
  1. Find *classical* e.o.m. This is $\ddot{x}_\mu = -i\frac{g}{m}F_{\mu\nu}\dot{x}^\nu$ for U(1).
  2. Periodic solutions are *worldline instantons*!
  3. Plug classical solutions into Γ

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  2. Periodic solutions are *worldline instantons*!
  3. Plug classical solutions into Γ
- But how to treat: *Matrix weighted action?* and *Path ordering?*

Use **Coherent state formalism** [W.-M. Zhang, D.H. Feng, and R. Gilmore, *Rev. Mod. Phys.* 62, 867 (1990).] to cast Wilson loop into path integral:

$$\mathcal{W} = \oint \mathcal{D} \exp u\left\{\frac{ig}{2}\int_0^T d\tau\, \mathrm{tr}\left[\sigma_3\left(uA_\mu \dot{x}^\mu u^{-1} - \frac{i}{g}\dot{u}u^{-1}\right)\right]\right\}$$

Wong's Equations and Non-Abelian Worldline Instantons

- Take steepest descents in proper time, T, and absorb, $\tau \to T\tau$, to find for the worldline action:

$$S = m\sqrt{\int_0^1 d\tau \dot{x}^\mu \dot{x}_\mu} - \frac{ig}{2}\int_0^1 d\tau\, tr[\sigma_3(uA_\mu \dot{x}^\mu u^{-1} + \frac{i}{g}u\dot{u}^{-1})]$$

- Equations of motion are **Wong's equations** [W. Wong *Nuovo Cimento* A 65, 689 (1970).]

  1. Isospin equation

$$I = \frac{1}{2}u^{-1}\sigma_3 u$$
$$\dot{I} = [igA_\mu \dot{x}^\mu, I]$$

  2. Lorentz force equation (non-Abelian equivalent)

$$\ddot{x}_\mu = -\frac{ig\sqrt{\dot{x}^2}}{m}\mathrm{tr}[IG_{\mu\nu}]\dot{x}^\nu$$

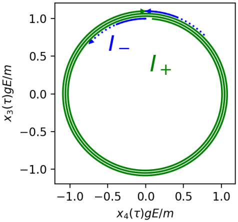
Periodic solutions to Wong's equations are **worldline instantons**

Homogeneous Background

- Abelianized homogeneous fields.

  $$G_{12} = B\sigma_3 \qquad G_{34} = iE\sigma_3$$

  Worldline instantons with radius m/gE (same as QED)
- Exponential suppression of $\frac{\pi n m^2}{gE}$.



Non-Abelian Pair Production in BPST Background

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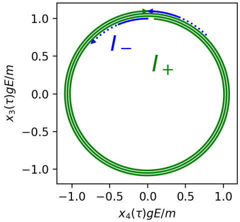

BPST Instanton Background

- BPST instanton [A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin *Phys. Rev. B* 59, 85 (1975)] is hermitian, and we confirm there can be **no** imaginary part in Γ.
- All real fields in *Euclidean* spacetime **(It's like we have all magnetic fields and no electric fields for pair production)**.
- Extend $SU(2) \to SL(2,\mathbb{C})$ and conceive of **complex BPST instanton** → pair production!

Outline

Axial Gauge Field Background

Saw how the Schwinger effect in worldline formalism

1. *gives rise to chirality and the axial anomaly!*
2. can be studied in non-Abelian fields too.

May also study exotic couplings: QED + **axial gauge field, A_5^μ!**

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1. *gives rise to chirality and the axial anomaly!*
2. can be studied in non-Abelian fields too.

May also study exotic couplings: QED + **axial gauge field, A_5^μ!**

- Study the augmented kernel

$$\hat{\not{\Pi}}_5 \equiv i\gamma^\mu(D_\mu + i\gamma_5 A_{5\mu})$$

  with constant A_5^μ. (e.g., $A_5^0 = \mu_5$ chiral chemical potential)

- *In worldline formalism (also Schwinger propertime) can use other first-quantized methods, not just path integral!*
- Express the effective action for augmented Dirac operator as a **sum over its eigenvalues**, $\hat{\not{\Pi}}_5 \psi_N = \lambda_N \psi_N$:

$$\Gamma[A, A_5] = -i\mathrm{tr}\int d^4x \langle x| \ln[\hat{\not{\Pi}}_5 - m]|x\rangle$$

$$= \frac{i}{2}\int_0^\infty \frac{dT}{T}\sum_N e^{-i(-\lambda_N^2 + m^2)T}$$

Axial Gauge Field Background

Two combinations admit **exact eigendecompositions**!

1. A magnetic field with a chiral chemical potential:

$$A_\mu = \frac{B}{2}(\delta^1_\mu x^2 - \delta^2_\mu x^1)\,, \quad A_5^\mu = g^{\mu 0}\mu_5$$

2. An electric field with spatial axial gauge:

$$A_\mu = \frac{E}{2}(\delta^0_\mu x^3 - \delta^3_\mu x^0)\,, \quad A_5^\mu = g^{\mu 1}\omega_5 \quad \textbf{vorticity} \text{ interpretation}$$

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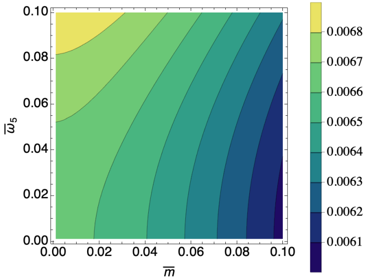
Schwinger Pair Production with Spatial Axial Gauge

$$\lambda_N = \pm\sqrt{p_{\rm n}^{\parallel 2} - p_1^2 + \omega_5^2 + 2s|\omega_5|p_{\rm n}^{\parallel}}\,, \quad p_{\rm n}^{\parallel} = \sqrt{2i|qE|{\rm n} - p_2^2}\,, \quad s = \pm 1.$$

Imaginary part of Γ for small ω_5^4:

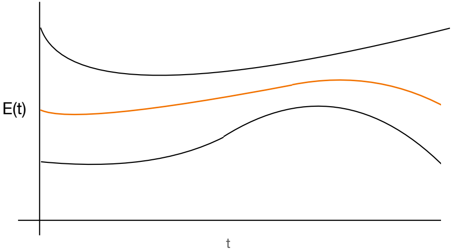
Enhancement from the spatial axial gauge!

$$\bar{\omega}_5 = \frac{\omega_5}{\sqrt{|qE|}} \quad \bar{m} = \frac{m}{\sqrt{|qE|}}$$

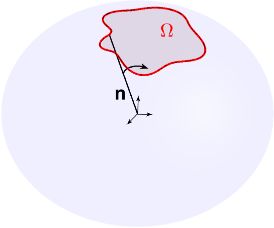


Outline

- Can we see the anomaly at a *classical* level? With the **Berry phase** [M. V. Berry, *Royal Soc. London. A.* 392, 45 (1984).] it is thought possible.
- If the system evolves slowly *(adiabatically)* enough it will stay fixed in the same eigenstate.



$$H(t)|\psi(t)\rangle = E(t)|\psi(t)\rangle$$



Quintessential case of electron in a magnetic field:

$$H(t) = \frac{e}{m}\boldsymbol{B}\cdot\boldsymbol{\sigma}$$

For closed path, electron picks up nonvanishing phase $\sim$ solid angle traversed by magnetic field.

- Magnetic field → **Momentum space**
- Consider for Weyl fermions in quantum mechanics with a positive helicity eigenstate

$$\boldsymbol{p}(t)\cdot\boldsymbol{\sigma}\, u^+(t) = |\boldsymbol{p}(t)|\, u^+(t)$$

In addition to a dynamic factor $\int_{t_i}^{t_f} dt|\boldsymbol{p}|$ if we take a closed path such that $\boldsymbol{p}(t_i) = \boldsymbol{p}(t_f)$ we will acquire an additional phase

$$i\hbar u^+ \boldsymbol{\nabla}_{\boldsymbol{p}} u^+$$

the Berry phase!

- Easiest to see this as a time-dependent gauge transformation:

$$H(t) \to U^\dagger H(t) U + i\hbar U^\dagger(\boldsymbol{\nabla}_{\boldsymbol{p}} U)\cdot\frac{d\boldsymbol{p}}{dt}$$

Then keep the diagonal parts, with $U = (u^-, u^+)$.

- We can also write the worldline formalism in **phase space** representation [A. Migdal, *Nuclear Physics B* 265, 594 (1986).] (from which a Berry phase in momentum space can be found)

$$
\begin{aligned}
G(A,x,y) &= \langle x| \frac{-\hbar}{i\hbar\partial\!\!\!/ - \frac{e}{c}A\!\!\!/ - mc + i\epsilon} |y\rangle \\
&= \int_0^\infty dT\, i\, \langle x|\, e^{-\frac{i}{\hbar}(-i\hbar\partial\!\!\!/ + \frac{e}{c}A\!\!\!/ + mc - i\epsilon)T}\, |y\rangle \\
&= i\int_0^\infty dT \int_{x(0)=y}^{x(T)=x} \mathcal{D}x \int \frac{\mathcal{D}p}{2\pi\hbar}\, e^{\frac{i}{\hbar}S_{\mathrm{A}}}\mathcal{W}_{\mathrm{D}}
\end{aligned}
$$

$$
S_{\mathrm{A}} := \int_0^T d\tau\left[-mc - p_\mu \dot{x}^\mu - \frac{e}{c}A_\mu \dot{x}^\mu\right],
$$

$$
\mathcal{W}_{\mathrm{D}} := \mathcal{P}\exp\left\{\frac{i}{\hbar}\int_0^T d\tau\, p\!\!\!/\right\}.
$$

- **All of the Berry phase contained in $\mathcal{W}_{\mathrm{D}}$!**

- To begin, let's look at the simpler **massless Weyl fermion** case, just the left-handed case, i.e., stemming from the action:

$$i\hbar \int d^4x\, \psi_{\mathrm{L}}^{\dagger} \bar{\sigma}_{\mu} D^{\mu} \psi_{\mathrm{L}} \qquad \bar{\sigma}^{\mu} = (\mathrm{I}_2, -\sigma^i)$$

- Same effective action as before but with

$$\mathcal{W}_{\mathrm{D}} \to \mathcal{W}_{\mathrm{W}} := \mathcal{P} \exp\Big\{ \frac{i}{\hbar} \int_0^T d\tau\, p_{\mu} \bar{\sigma}^{\mu} \Big\}$$

- Insert complete sets of unitary transform, U, into the path ordered element, where

$$U^{\dagger} p_{\mu} \bar{\sigma}^{\mu} U = p^0 \mathrm{I}_2 + |\mathbf{p}| \sigma_3$$

- p^0 is already diagonal and **does not contribute** to Berry's phase $\to$ $U(\boldsymbol{p})$ only depends on spatial momentum.

- The transformation matrix is $U = (u^-, u^+)$ with

$$u^- = \begin{pmatrix} e^{-i\omega_p} \cos\frac{\theta_p}{2} \\ \sin\frac{\theta_p}{2} \end{pmatrix} \quad u^+ = \begin{pmatrix} -e^{-i\omega_p} \sin\frac{\theta_p}{2} \\ \cos\frac{\theta_p}{2} \end{pmatrix}$$

Here the momentum is in spherical coordinates $\mathbf{p} = |\mathbf{p}|(\sin\theta_p \cos\omega_p, \sin\theta_p \sin\omega_p, \cos\theta_p)$

- **The adiabatic Berry phase and curvature are**

$$\mathbf{B}^{\pm}_{\mathrm{W}} = -i\hbar u^{\pm}\nabla_{\boldsymbol{p}} u^{\pm} \ ,$$
$$\mathbf{S}^{\pm}_{\mathrm{W}} = \nabla_{\boldsymbol{p}} \times \mathbf{B}^{\pm}_{\mathrm{W}} = \mp\hbar\frac{\mathbf{p}}{2|\mathbf{p}|^3}$$

- And the path ordered element under adiabaticity reads

$$\mathrm{tr}\mathcal{W}_{\mathrm{W}} \approx \sum_{\pm} \exp\left\{\frac{i}{\hbar}\int_0^T d\tau[p^0 \pm |\mathbf{p}| - \mathbf{B}^{\mp}_{W} \cdot \dot{\mathbf{p}}]\right\}$$

- The worldline action is

$$\mathcal{S}_{\mathrm{W}} = \int_0^T d\tau \left[-p_\mu \dot{x}^\mu - \frac{e}{c} A_\mu \dot{x}^\mu + p^0 - |\mathbf{p}| - \mathbf{B}_{\mathrm{W}}^{+} \cdot \dot{\mathbf{p}} \right]$$

- From which we can find the following equations of motion

$$\dot{p}_\mu = \frac{e}{c} F_{\mu\nu} \dot{x}^\nu \,, \quad \dot{\mathbf{x}} = \hat{\mathbf{p}} + \mathbf{S}_{\mathrm{W}}^{+} \times \dot{\mathbf{p}} \,, \quad \dot{x}^0 = 1$$

- Equations of motion become:

$$(1 + \boldsymbol{B} \cdot \boldsymbol{S}_{\mathrm{W}}^{+}) \dot{\boldsymbol{x}} = \hat{\boldsymbol{p}} + \boldsymbol{E} \times \boldsymbol{S}_{\mathrm{W}}^{+} + \boldsymbol{B}(\boldsymbol{S}_{\mathrm{W}}^{+} \cdot \hat{\boldsymbol{p}})$$

$$(1 + \boldsymbol{B} \cdot \boldsymbol{S}_{\mathrm{W}}^{+}) \dot{\boldsymbol{p}} = \boldsymbol{E} + \hat{\boldsymbol{p}} \times \boldsymbol{B} + \boldsymbol{S}_{\mathrm{W}}^{+} (\boldsymbol{E} \cdot \boldsymbol{B})$$

- How does this lead to an anomaly? **Incompressible phase space measure!** [M. Stephanov and Y. Yin *Phys. Rev. Lett.* 109, 162001 (2012).].

- Phase space evolution of a gas of Fermi particles (but with chiral effects) $\rightarrow$ **Chiral Kinetic Theory**
- Liouville equation for distribution f for L/R handed particles:

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial \boldsymbol{x}} \cdot f \dot{\boldsymbol{x}} + \frac{\partial}{\partial \boldsymbol{p}} \cdot f \dot{\boldsymbol{p}} = 0$$

- Incompressible phase space measure: $(1 + \boldsymbol{B} \cdot \boldsymbol{S}^{+}_{\mathrm{W}}) \frac{d^3 x d^3 p}{(2\pi)^3}$ leads to modified distribution function: $f' = (1 + \boldsymbol{B} \cdot \boldsymbol{S}^{+}_{\mathrm{W}}) f$.

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- The phase space current is $j^{\mu} = \int d^3 p (f', f' \dot{\boldsymbol{x}})/(2\pi)^3$, we find

$$\frac{\partial}{\partial t} f' + \frac{\partial}{\partial \boldsymbol{x}} \cdot f' \dot{\boldsymbol{x}} + \frac{\partial}{\partial \boldsymbol{p}} \cdot f' \dot{\boldsymbol{p}} = \boldsymbol{E} \cdot \boldsymbol{B} \, \nabla_{\boldsymbol{p}} \cdot \boldsymbol{S}_{\mathrm{W}}^{+}$$

$$\partial_{\mu} j^{\mu} = \frac{\boldsymbol{E} \cdot \boldsymbol{B}}{4\pi^2} \,, \quad \textbf{the axial anomaly!}$$

- We further explore this phenomenon but with *Dirac* fermions, leading to a richer non-Abelian Berry phase structure. However anomaly is still produced! [PC, S. Pu, Phys. Rev. D, 105 (2022).]

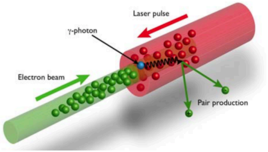
Outline

QED

- Treat perturbatively in α, with high precision.

SFQED

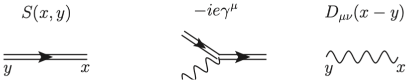
- QED + background field, e.g. ultra-intensity lasers



[M. Marklund, J. Lundin, Eur. Phys. J.D. 55, 319326 (2009).]

- May scale larger that α: $\chi^{2/3}\alpha$? [V. I. Ritus, Sov. Phys. JETP 30, 1181 (1970); N. B. Narozhnyi, Phys. Rev. D 21, 1176 (1980).]
- Must treat non-perturbatively!

- Particles dressed with background field + perturbative photons [W. H. Furry, Phys. Rev. 81 (1951)]:



[A. Fedotov, et al., Phys. Rept. 1010, (2023)]

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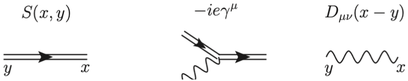

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- Theoretical Shortcomings:
  1. Unrealistic modeling of e.g. high intensity lasers
  2. Higher-order effects need to further address
  3. Resummation required

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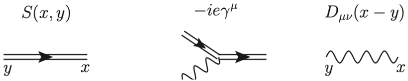

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- Theoretical Shortcomings:
  1. Unrealistic modeling of e.g. high intensity lasers
  2. Higher-order effects need to further address
  3. Resummation required
- Enter the **worldline formalism**
  - All orders in the background field!
  - Sum over all Feynman diagrams at given multiplicity or loop order!

Working in light front coordinates $\mathrm{d}s^2 = \mathrm{d}x^+\mathrm{d}x^- - \mathrm{d}x^\perp\mathrm{d}x^\perp$

Plane Wave

$$eA_\mu(x) = a_\mu(x^+) = \delta_\mu^\perp a_\perp(x^+) \qquad a_\perp(\infty) = a_\perp^\infty$$

Impulsive PP-Waves

$$A_\mu^{(s)}(x) = -n_\mu \delta(n \cdot x)\Phi(x^\perp)$$

1. Shockwaves ,ultra-boosted Coulomb field:
   $\Phi(x^\perp) \propto \log(\mu^2|x^\perp|^2)$
2. Impulsive plane wave:
   $\Phi(x^\perp) = r_\perp x^\perp$

N-Photon Dressed Worldline Formalism

- Expand about $A_{\text{bg}} + A^{\gamma}$ where

$$A^{\gamma}_{\mu}(x) = \sum_{i=1}^{N} \varepsilon_{\mu\, i} \mathrm{e}^{ik_i \cdot x}$$

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- Let's examine the scalar propagator:

$$\mathcal{D}_N^{x'x} = (-ie)^N \int_0^{\infty} \mathrm{d}\, T\, \mathrm{e}^{-im^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau)\, \mathrm{e}^{iS_{\mathrm{B}}[x(\tau), A_{\text{bg}}]} \prod_{i=1}^{N} V^{x'x}[\varepsilon_i, k_i]$$

$$S_{\mathrm{B}}[x(\tau), A_{\text{bg}}] = -\int_0^{T} \mathrm{d}\tau \left[\frac{\dot{x}^2}{4} + eA_{\text{bg}}(x(\tau)) \cdot \dot{x}(\tau)\right]$$

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- Multi-linear **vertex operator** insertions. Place in exponential through lin. operator

$$V^{x'x}[\varepsilon, k] = \int_0^T d\tau\, \varepsilon \cdot \dot{x}(\tau)\, \mathrm{e}^{ik\cdot x(\tau)} = \int_0^T \mathrm{d}\tau\, \mathrm{e}^{ik\cdot x + \varepsilon\cdot \dot{x}}\Big|_{\text{lin.}\, \varepsilon}$$

Outline

- Let's evaluate the propagator in a **plane wave background with N-photons**
- Expand about the straight line: $x^\mu(\tau) = x^\mu + z^\mu \frac{\tau}{T} + q^\mu(\tau)$ with $z^\mu := x'^\mu - x^\mu$.

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- Introduce a Lagrange multiplier $\chi(\tau)$ and auxiliary field $\xi(\tau)$ [C. Schubert and R. Shaisultanov, Phys. Lett. B 843, 137969 (2023)], effectively revealing the Gaussian structure!

$$\mathrm{e}^{-i\int \mathrm{d}\tau\, a(x^+ + z^+\frac{\tau}{T} + q^+)\cdot\dot{q}} = \int \mathcal{D}\xi\mathcal{D}\chi\, \mathrm{e}^{i\int \mathrm{d}\tau\, \left[\chi(\xi - q^+) - a(x^+ + z^+\frac{\tau}{T} + \xi)\cdot\dot{q}\right]}$$

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- The rest of the path integral can be written in terms of the scalar Green function w/

$$\langle q^\mu(\tau) q^\nu(\tau')\rangle = 2i\eta^{\mu\nu}\Delta(\tau,\tau')$$

$$\Delta_{ij} := \Delta(\tau_i,\tau_j) = \frac{1}{2}|\tau_i - \tau_j| - \frac{1}{2}(\tau_i + \tau_j) + \frac{\tau_i\tau_j}{T}$$

Coordinate space propagator:

$$\mathcal{D}_N^{x'x} = (-ie)^N \int_0^\infty \frac{-i\mathrm{d}T}{(4\pi T)^2} \prod_{i=1}^N \int_0^T \mathrm{d}\tau_i \, \mathrm{e}^{-im^2 T + \dots} \int \mathcal{D}\xi \, \mathcal{D}\chi \, e^{iS_N}\Big|_{\text{lin. } \varepsilon}$$

$$-S_N = \int \mathrm{d}\tau_i \mathrm{d}\tau_j \, a_i \cdot a_j \, {}^\bullet\Delta^\bullet_{ij} + 2i \sum_{j=1}^N \int \mathrm{d}\tau_i \left({}^\bullet\Delta^\bullet_{ij} \, a_i \cdot \varepsilon_j + i \, {}^\bullet\Delta_{ij} a_i \cdot k_j \right)$$

$$+ 2i \sum_{j=1}^N \int \mathrm{d}\tau_i \; \chi_i [\Delta^\bullet_{ij} \varepsilon_j^+ + i\Delta_{ij} k_j^+] - \sum_{i,j=1}^N [{}^\bullet\Delta^\bullet_{ij} \varepsilon_i \cdot \varepsilon_j + 2i \, {}^\bullet\Delta_{ij} \varepsilon_i \cdot k_j - \Delta_{ij} k_i \cdot k_j]$$

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1. Kibble mass
2. photon plane wave coupling term
3. ‘hidden’ Gaussianity → implict photon dependence
4. Bern Kosower exponent

LSZ in the Worldline Formalism

Scalars

❶ Start with momentum space propagator w/ $\tilde{p}' = p' + a^\infty$:

$$\mathcal{D}_N^{\tilde{p}'p} = \int \mathrm{d}^4x'\mathrm{d}^4x\, \mathrm{e}^{i\tilde{p}'\cdot x' - ip\cdot x}\, \mathcal{D}_N^{x'x} = \int_0^\infty \mathrm{d}T\, \mathrm{e}^{i(p'^2 - m^2 + i0^+)T} F(T)$$

1. Start with momentum space propagator w/ $\tilde{p}' = p' + a^\infty$:

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2. *How to do LSZ?* [E. Laenen, G. Stavenga, and C. D. White, JHEP 03, 054 (2009); D. Bonocore, JHEP 02, 007 (2021); G. Mogull, J. Plefka, and J. Steinhoff, JHEP 02, 048 (2021).] First, truncate the outgoing state.

$$\begin{aligned} & -i(p'^2 - m^2 + i0^+) \int_0^\infty \mathrm{d}T\, \mathrm{e}^{i(p'^2 - m^2 + i0^+)T} F(T) \\ & = F(0) + \int_0^\infty \mathrm{d}T\, \mathrm{e}^{i(p'^2 - m^2 + i0^+)T} \frac{\mathrm{d}}{\mathrm{d}T} F(T) \end{aligned}$$

and in the on-shell limit

$$\lim_{p'^2 \to m^2} -i(p'^2 - m^2 + i0^+) \int_0^\infty \mathrm{d}T\, \mathrm{e}^{i(p'^2 - m^2 + i0^+)T} F(T) = F(\infty)$$

❸ Do the following variable changes:

$$\tau_0 := \frac{1}{N}\sum_{i=1}^{N}\tau_i \ , \qquad \bar{\tau}_i := \tau_i - \tau_0 \ .$$

$$\prod_{i=1}^{N}\int_0^\infty \mathrm{d}\tau_i = \int_0^\infty \mathrm{d}\tau_0 \prod_{i=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}\bar{\tau}_i\,\delta\Big(\sum_{j=1}^{N}\frac{\bar{\tau}_j}{N}\Big)$$

also shift out τ_0 dependent terms implicit in $a(x_{cl}(\tau))$ with x^+.

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also shift out τ_0 dependent terms implicit in $a(x_{cl}(\tau))$ with x^+.

4. With new 'T' variable, truncate the incoming state (multiply by $p^2 - m^2 + i0^+$ and go on-shell). Same trick!

$$T \to \tau_0 \quad \text{and} \quad p' \to p$$

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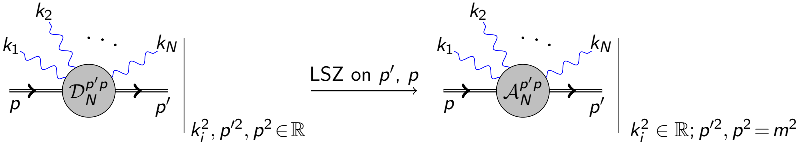

- **Spinor kernel**:

$$\mathcal{K}_N^{p'p} = \int_N \int \mathcal{D}x(\tau)\, \mathrm{e}^{iS_{\mathcal{J}}[x(\tau);a]} \mathrm{Spin}[a](f_{1:N}) \Big|_{\mathrm{lin.}\,\varepsilon}$$

$$\int_N = (-ie)^N \int \mathrm{d}^4x \int_0^\infty \mathrm{d}T \int_0^T \prod_{j=1}^N \mathrm{d}\tau_j$$

- **Spin factor**:

$$\mathrm{Spin}[a](f_{1:N}) = \mathrm{symb}^{-1}\mathfrak{W}_\eta[a](f_{1:N})$$

Use symbolic map to convert η^μ's into γ^μ's.

- Express as average over **plane wave background (f)**:

$$\mathfrak{W}_\eta[a](\tilde{f}_{1:N}) = \left\langle e^{\sum_{i=1}^N \psi_\eta(\tau_i)\cdot\tilde{f}_i\cdot\psi_\eta(\tau_i)} \right\rangle$$

$$\left\langle \dots \right\rangle = 2^{-\frac{D}{2}} \oint_{\mathrm{A/P}} \mathcal{D}\psi(\tau) \dots \mathrm{e}^{i\int_0^T \mathrm{d}\tau \left[\frac{i}{2}\psi\cdot\dot{\psi} + i\psi_\eta(\tau)\cdot f(\tau)\cdot\psi_\eta(\tau)\right]}$$

- Spin factor is Gaussian, to solve just need to find

$$\langle \psi^\mu(\tau)\psi^\nu(\tau')\rangle = \frac{1}{2}\mathfrak{G}^{\mu\nu}(\tau,\tau')$$

which satisfies for A/P BCs:

$$\Big(\frac{1}{2}\eta_{\mu\sigma}\frac{\mathrm{d}}{\mathrm{d}\tau} + f_{\mu\sigma}(\tau)\Big)\mathfrak{G}^{\sigma\nu}(\tau,\tau') = \eta_\mu{}^\nu\,\delta(\tau-\tau')$$

- Can find an exact solution for **any background field**, not just plane or impulsive PP waves:

$$\mathcal{O}(\tau,\tau') = \left[\Theta(\tau-\tau')\mathcal{P} + \Theta(\tau'-\tau)\bar{\mathcal{P}}\right] e^{-2\int_{\tau'}^{\tau} d\sigma f(\sigma)}$$

$$\mathfrak{G}(\tau,\tau') = \mathrm{sgn}(\tau-\tau')\mathcal{O}(\tau,\tau') + \mathcal{O}(\tau,0)\frac{1-\mathcal{O}(T,0)}{1+\mathcal{O}(T,0)}\mathcal{O}(0,\tau')$$

- The fermion Green function takes on a simple form in a **plane wave** background though!

$$[f(\tau), f(\tau')] = 0 \quad \text{and} \quad f(\tau)f(\tau')f(\tau'') = 0$$

since $n^2 = 0$ and $n \cdot a = 0$

- Exact fermion Green function [J. P. Edwards and C. Schubert, Phys. Lett. B 822, 136696 (2021).]:

$$\mathfrak{G}(\tau, \tau') = \mathrm{sgn}(\tau - \tau')\Big[1 - 2\int_{\tau'}^{\tau} \mathrm{d}\sigma\, f(\sigma) + 2\Big(\int_{\tau'}^{\tau} \mathrm{d}\sigma\, f(\sigma)\Big)^2\Big]$$
$$+ T\langle\langle f\rangle\rangle\Big[1 - 2\int_{\tau'}^{\tau} \mathrm{d}\sigma\, f(\sigma)\Big]$$

- *One needs only to "complete the square!"*

- The exact solution for the spin factor in a plane wave background:

$$\mathfrak{W}_\eta(\tilde{f}_{1:N}) = e^{\sum_{i=1}^N \frac{\delta}{\delta\theta(\tau_i)}\cdot\tilde{f}_i\cdot\frac{\delta}{\delta\theta(\tau_i)}}
\mathrm{e}^{-\int_0^T \mathrm{d}\tau\,[\eta\cdot f(\tau)\cdot\eta+\theta(\tau)\cdot\eta]-\int_0^T \mathrm{d}\tau\mathrm{d}\tau'[\eta\cdot f(\tau)\cdot\mathfrak{G}(\tau,\tau')\cdot\theta(\tau')+\frac{1}{4}\theta(\tau)\cdot\mathfrak{G}(\tau,\tau')\cdot\theta(\tau')]}\Big|_{\theta=0}$$

- Does not spoil the *'hidden' Gaussianity* of the kernel.
- Can find an **exact solution** of the N-photon dressed propagator!
- Spin d.o.f. enter as a multiplicative factor with
  1. Implicit N-photon dependence in $f(\tau)$ (as with $a(\tau)$ for the scalar part)!
  2. Linear operator can act on either spin factor or scalar factor.

- Putting everything together:

$$\mathcal{K}_N^{p'p} = \int_N \mathrm{e}^{i(\tilde{p}'^2-m^2)T+i(\tilde{p}'+K-p)\cdot x+i(2\tilde{p}'+K)\cdot g}$$
$$\mathrm{e}^{-i\sum_{i,j=1}^N[\frac{1}{2}|\tau_i-\tau_j|k_i\cdot k_j-i\ \mathrm{sgn}(\tau_i-\tau_j)\varepsilon_i\cdot k_j+\varepsilon_i\cdot\varepsilon_j\delta(\tau_i-\tau_j)]}$$
$$\mathrm{e}^{-\int_0^T(2\tilde{p}'\cdot a(\tau)-a^2(\tau))\mathrm{d}\tau-2\sum_{i=1}^N[\int_0^{\tau_i}k_i\cdot a(\tau)\mathrm{d}\tau-i\varepsilon_i\cdot a(\tau_i)]}\mathrm{Spin}[a](f_{1:N})\Big|_{\mathrm{lin.}\ \varepsilon}$$

$$a_\mu(\tau) = a_\mu\Big(x^+ + g^+ + (p'+p)^+\tau - \sum_{i=1}^N k_i^+|\tau-\tau_i|\Big)\,, \quad g = \sum_{j=1}^N (k_j\tau_j - i\varepsilon_j)$$

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$$\mathrm{e}^{-i\sum_{i,j=1}^N [\frac{1}{2}|\tau_i - \tau_j| k_i \cdot k_j - i\ \mathrm{sgn}(\tau_i - \tau_j)\varepsilon_i \cdot k_j + \varepsilon_i \cdot \varepsilon_j \delta(\tau_i - \tau_j)]}$$
$$\mathrm{e}^{-\int_0^T (2\tilde{p}' \cdot a(\tau) - a^2(\tau))\mathrm{d}\tau - 2\sum_{i=1}^N [\int_0^{\tau_i} k_i \cdot a(\tau)\mathrm{d}\tau - i\varepsilon_i \cdot a(\tau_i)]} \mathrm{Spin}[a](f_{1:N}) \Big|_{\mathrm{lin.}\ \varepsilon}$$

$$a_\mu(\tau) = a_\mu\left(x^+ + g^+ + (p' + p)^+ \tau - \sum_{i=1}^N k_i^+ |\tau - \tau_i|\right), \quad g = \sum_{j=1}^N (k_j \tau_j - i\varepsilon_j)$$

- And the full propagator in *position space* is

$$\mathcal{S}_N^{x'x} = (-i\partial\!\!\!/_{x'} + a\!\!\!/(x'^+) - m)\mathcal{K}_N^{x'x}(a) + eA\!\!\!/^\gamma(x')\mathcal{K}_{N-1}^{x'x}(a)$$

- N-photon scattering amplitude:

$$\mathcal{M}_{Ns's}^{p'p} = i \lim_{p'^2, p^2 \to m^2} \int \mathrm{d}^4 x' \mathrm{d}^4 x\, \mathrm{e}^{i\tilde{p}' \cdot x' - ip \cdot x}$$
$$\bar{u}_{s'}(p')(i\partial\!\!\!/_{x'} - a\!\!\!/^\infty - m)\mathcal{S}_N^{x'x}(-i\overleftarrow{\partial\!\!\!/}_x - m)u_s(p)$$

- Amplitude w/ subleading terms, **1**, and **2**:

$$\mathcal{M}^{p'p}_{Ns's} = i \lim_{p'^2, p^2 \to m^2} \frac{1}{2m} \int d^4x' d^4x \, e^{i\tilde{p}'\cdot x' - ip\cdot x} \, \bar{u}_{s'}(p')(p'^2 - m^2)$$

$$\left\{ \left[-1 + \frac{1}{2m} \delta \not{a}(x'^+) \right] \mathcal{K}^{x'x}_N + \frac{e}{2m} \sum_{i=1}^{N} \not{\epsilon}_i e^{ik_i \cdot x'} \, \mathcal{K}^{x'x}_{N-1} \right\} (p^2 - m^2) u_s(p)$$

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- From Fourier transform pick up $\delta(x^+ - x'^+ + 2g^+ + 2p'^+ T)$

$\delta a(x'^+) \to \delta a\big(2Tp'^+ + x^+ + 2g^+\big) \to$ **out truncation**$(T \to \infty) \to 0$

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- Poles not in the mass-shell $p'^2 - m^2$, but rather $((p' + k_i)^2 - m^2)$

$$\lim_{p'^2 \to m^2} (p'^2 - m^2)/((p' + k_i)^2 - m^2) = 0$$

Rules:

(i) delete the integral $\int \mathrm{d}T$

(ii) insert a factor of $\delta\Big(\sum_{i=1}^{N}\tau_i/N\Big)$

(iii) change the range of the $\mathrm{d}\tau_i$ integrals to $\mathbb{R}$ + ...

Rules:

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$$\mathcal{M}^{p'p}_{Ns's} = (-ie)^N \int \mathrm{d}^4x\, \mathrm{e}^{i(K+p'-p)\cdot x} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \mathrm{d}\tau_i\, \delta\Big(\sum_{j=1}^{N} \frac{\tau_j}{N}\Big)$$

$$\mathrm{e}^{-i\int_{-\infty}^{0}[2\tilde{p}'\cdot a(\tau)-a^2(\tau)]d\tau - i\int_0^{\infty}[2p'\cdot\delta a(\tau)-\delta a^2(\tau)]\mathrm{d}\tau - 2i\sum_{i=1}^{N}[\int_{-\infty}^{\tau_i} k_i\cdot a(\tau)\mathrm{d}\tau - i\varepsilon_i\cdot a(\tau_i)]}$$

$$\mathrm{e}^{i(\tilde{p}'+p)\cdot g - i\sum_{i,j=1}^{N}(\frac{1}{2}|\tau_i-\tau_j|k_i\cdot k_j - i\,\mathrm{sgn}(\tau_i-\tau_j)\varepsilon_i\cdot k_j + \delta(\tau_i-\tau_j)\varepsilon_i\cdot\varepsilon_j)}$$

$$\frac{1}{2m}\bar{u}_{s'}(p')\mathrm{Spin}[a](f_{1:N})u_s(p)\Big|_{\mathrm{lin.}\,\varepsilon}$$

All-multiplicity in compact form by virtue of worldline construction!

Outline

Propagator in a PP-wave Background

Scalars

- Recall gauge is

$$A_\mu^{(s)}(x) = -n_\mu \delta(n \cdot x)\Phi(x^\perp)$$

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$$A_\mu^{(s)}(x) = -n_\mu \delta(n \cdot x)\Phi(x^\perp)$$

- The **scalar propagator w/ Robin BCs** $(\frac{1}{2}\dot{x}_\mu(T) + eA_\mu(x(T)) = p'_\mu)$:

$$\mathcal{D}_N^{p'p} = \int_N \int_{x(0)=x} \mathcal{D}x(\tau)\, \mathrm{e}^{iS_{\mathcal{J}}[x(\tau);A]}\Big|_{\mathrm{lin.}\,\varepsilon}$$

with the abbreviation:

$$\int_N := (-ie)^N \int \mathrm{d}^4x \int_0^\infty \mathrm{d}T \int_0^T \prod_{j=1}^N \mathrm{d}\tau_j$$

- Expand about the classical solution:

$$x^{\pm}(\tau) = x^{\pm}_{\mathsf{cl}}(\tau) + \delta x^{\pm}(\tau)\,, \qquad \ddot{x}^{+}_{\mathsf{cl}} = 2\mathcal{J}^{+}$$

$$\mathcal{J}^{\mu} = i\sum_{i=1}^{N}\big(ik_i^{\mu} - \varepsilon_i^{\mu}\frac{\mathrm{d}}{\mathrm{d}\tau}\big)\delta(\tau - \tau_i)$$

- Remaining path integral:

$$\mathcal{D}_N^{(s)p'p} = \int_N \int_{x^{\perp}(0)=x^{\perp}} \mathcal{D}x^{\perp} \mathrm{e}^{iS_{\mathcal{J}}[(x^{\pm}_{\mathsf{cl}}, x^{\perp});0] + i\int_0^T \mathrm{d}\tau\, e(\dot{x}^{+}_{\mathsf{cl}})\delta(x^{+}_{\mathsf{cl}})\Phi(x^{\perp})}\Big|_{\mathrm{lin.}\,\varepsilon}$$

- Write the Wilson line as product over roots of $x_{\mathrm{cl}}^{+}(\tau) = 0$:

$$\mathrm{e}^{i \int_0^T \mathrm{d}\tau\, \dot{x}_{\mathrm{cl}}^{+} \delta(x_{\mathrm{cl}}^{+})\, e\, \Phi(x^{\perp}(\tau))} = \prod_{j=1}^{\bar{N}} \mathrm{e}^{ie\, \mathrm{sgn}(\dot{x}_{\mathrm{cl}}^{+}(t_j))\, \Phi(x^{\perp}(t_j))}$$

- **Fourier transform $U(1)$ factors** [A. Tarasov and R. Venugopalan, PRD 100 (2019) 054007; T. Adamo, A. Ilderton and A. J. MacLeod, PRD 104 (2021) 116013.]

$$\mathrm{e}^{ie\, \mathrm{sgn}(\dot{x}^{+}) \Phi(x^{\perp})} = \int \hat{\mathrm{d}}^2 r_{\perp}\, W(r_{\perp})\, \mathrm{e}^{i\, \mathrm{sgn}(\dot{x}^{+}) r_{\perp} x^{\perp}}$$

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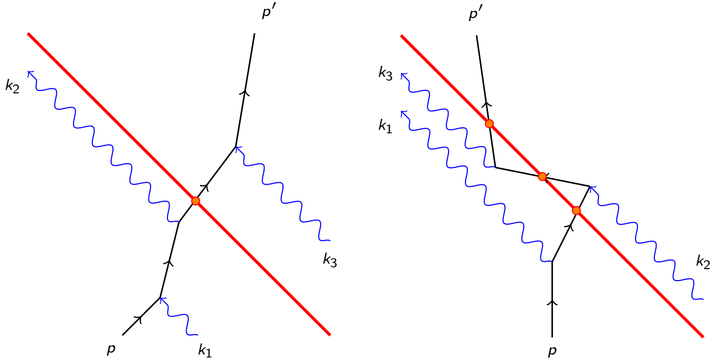
$$\mathrm{e}^{ie \, \mathrm{sgn}(\dot{x}^+) \Phi(x^\perp)} = \int \hat{\mathrm{d}}^2 r_\perp \, W(r_\perp) \, \mathrm{e}^{i \, \mathrm{sgn}(\dot{x}^+) r_\perp x^\perp}$$

- Propagator is now Gaussian!

$$\mathcal{D}_N^{(s)p'p} = \int_N \int_{x(0)=x} \mathcal{D}x^\perp(\tau) \int \prod_{j=1}^{\overline{N}} \hat{\mathrm{d}}^2 r_{j\perp} W(r_{j\perp})$$

$$\mathrm{e}^{iS_{\mathcal{J}}[(x_{\mathrm{cl}}^\pm, x^\perp);0] + \mathrm{sgn}(\dot{x}^+(t_j)) r_{j\perp} x^\perp(t_j)} \Big|_{\mathrm{lin.}\,\varepsilon}$$

$$n \cdot p' > 0 \qquad \text{and} \qquad n \cdot \left(p' + \sum_{i \in \mathcal{U}} k_i\right) > 0 \quad \forall \ \mathcal{U} \subseteq \{1, 2, 3, ..., N\}$$



- Recall the **spinor kernel** is

$$\mathcal{K}_N^{p'p} = \int_N \int_{x(0)=x} \mathcal{D}x(\tau)\, \mathrm{e}^{iS_{\mathcal{J}}[x(\tau);A]} \mathrm{Spin}[A](f_{1:N})\Big|_{\mathrm{lin.}\,\varepsilon}$$

- and the **spin factor** is

$$\begin{aligned}\mathrm{Spin}[A](f_{1:N}) =& \mathrm{symb}^{-1}\mathfrak{W}_\eta[A](f_{1:N})\\ \mathfrak{W}_\eta[A](f_{1:N}) :=& 2^{-\frac{D}{2}} \oint_{\mathrm{A/P}} \mathcal{D}\psi(\tau)\, \mathrm{e}^{i\widetilde{S}_{\mathrm{B}}[\psi(\tau),x(\tau),A]}\\ &\times \mathrm{e}^{\sum_{j=1}^N \psi_\eta(\tau_j)\cdot f_j(x(\tau))\cdot\psi_\eta(\tau_j)}\end{aligned}$$

- with **worldline spinor action** for any background F:

$$\widetilde{S}_{\mathrm{B}}[\psi(\tau), x(\tau), A] = \int_0^T \mathrm{d}\tau \left[\frac{i}{2}\psi\cdot\dot{\psi} + ie\,\psi_\eta(\tau)\cdot F(x(\tau))\cdot\psi_\eta(\tau)\right].$$

- Let's analyze this in the PP-wave background:

- Specialize to the **PP-wave (s) background**

$$\widetilde{S}_{\mathrm{B}}^{(s)} = \int_0^T \mathrm{d}\tau \Big[\frac{i}{2}\psi\cdot\dot{\psi} + 2ie\delta(x^+)\psi_\eta^+(\tau)\psi_\eta^\perp(\tau)\partial_\perp\Phi(x^\perp)\Big]$$

- Expand about the *vacuum* solution $x(\tau) = x_{\mathrm{cl}}(\tau) + q(\tau)$:

$$\widetilde{S}_{\mathrm{B}}^{(s)} = \frac{i}{2}\int_0^T \mathrm{d}\tau\,\psi\cdot\dot{\psi} + 2ie\sum_j^{\bar{N}} \frac{1}{|\dot{x}_{\mathrm{cl}}^+(t_j)|}\psi_\eta^+(t_j)\psi_\eta^\perp(t_j)\partial_\perp\Phi(x^\perp(t_j))$$

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- PP-wave support at *single* t_j. Since Grassmann variables, $\psi_\eta^+(t_j)^2 = 0$:

$$e^{i\widetilde{S}_{\rm B}^{(s)}} \longrightarrow e^{\frac{i}{2}\int_0^T {\rm d}\tau\psi\cdot\dot{\psi}} \prod_j^{\bar{N}} e^{-2e\frac{1}{|\dot{x}_{\rm cl}^+(t_j)|}\psi_\eta^+(t_j)\psi_\eta^\perp(t_j)\partial_\perp\Phi(x^\perp(t_j))}$$

$$= e^{\frac{i}{2}\int_0^T {\rm d}\tau\psi\cdot\dot{\psi}} \prod_j^{\bar{N}} \Big[1 - 2e\frac{1}{|\dot{x}_{\rm cl}^+(t_j)|}\psi_\eta^+(t_j)\psi_\eta^\perp(t_j)\partial_\perp\Phi(x^\perp(t_j))\Big]$$

- Couple $\mathrm{e}^{i\widetilde{S}_{\mathrm{B}}^{(s)}}$ to $e^{ie\int A\cdot dx}$, and write $\partial_\perp \Phi$ as:

$$\mathrm{e}^{i\widetilde{S}_{\mathrm{B}}^{(s)}} \prod_j^{\bar{N}} \mathrm{e}^{ie\,\mathrm{sgn}(\dot{x}_{\mathrm{cl}}^+(t_j))\Phi(x^\perp(t_j))}$$

$$= \mathrm{e}^{\frac{i}{2}\int_0^T \mathrm{d}\tau \psi\cdot\dot{\psi}} \prod_j^{\bar{N}} \left[1 + \frac{2i}{\dot{x}_{\mathrm{cl}}^+(t_j)} \psi_\eta^+(t_j)\psi_\eta^\perp(t_j)\partial_\perp\right] \mathrm{e}^{ie\,\mathrm{sgn}(\dot{x}_{\mathrm{cl}}^+(t_j))\Phi(x^\perp(t_j))}$$

Fouier Transform of U(1) Factors w/ Spinors

Kernel and Spin Factor

- Couple $\mathrm{e}^{i\widetilde{S}_{\mathrm{B}}^{(s)}}$ to $e^{ie\int A\cdot dx}$, and write $\partial_\perp\Phi$ as:

$$\mathrm{e}^{i\widetilde{S}_{\mathrm{B}}^{(s)}}\prod_j^{\bar{N}}\mathrm{e}^{ie\,\mathrm{sgn}(\dot{x}_{\mathrm{cl}}^{+}(t_j))\Phi(x^\perp(t_j))}$$

$$=\mathrm{e}^{\frac{i}{2}\int_0^T \mathrm{d}\tau\psi\cdot\dot{\psi}}\prod_j^{\bar{N}}\left[1+\frac{2i}{\dot{x}_{\mathrm{cl}}^{+}(t_j)}\psi_\eta^{+}(t_j)\psi_\eta^{\perp}(t_j)\partial_\perp\right]\mathrm{e}^{ie\,\mathrm{sgn}(\dot{x}_{\mathrm{cl}}^{+}(t_j))\Phi(x^\perp(t_j))}$$

- Next, introduce **Fourier transform of the U(1)** factors of Φ:

$$\int\hat{\mathrm{d}}^2 r_{j\perp}\left[1-\frac{2}{|\dot{x}_{\mathrm{cl}}^{+}(t_j)|}\psi_\eta^{+}(t_j)\psi_\eta^{\perp}(t_j)r_{j\perp}\right]W(r_{j\perp})\mathrm{e}^{i\,\mathrm{sgn}(x^+)r_{j\perp}x^\perp}$$

- Now similar to the scalar case we have a Gaussian in $x_\perp$ worldine (scalar+spinor parts) action!

- Like for the scalars, use the **positivity constraint**: $\overline{N} = 0, 1$; however only $\overline{N} = 1$ will remain after LSZ.

$$(-ie)\widetilde{\mathcal{K}}_N^{(s)p'p} = \int_{N+1} \int \hat{\mathrm{d}}^3 k_{(N+1)} \mathrm{e}^{iS_{\tilde{\mathcal{J}}}[\tilde{x}_{\mathrm{cl}}(\tau);0] + ik_{(N+1)+} x^+_{\mathrm{cl}}(\tau_{(N+1)})} \left[W(k_{(N+1)\perp})\dot{x}^+_{\mathrm{cl}}(\tau_{(N+1)})\right] \mathrm{symb}^{-1}\mathfrak{W}_\eta[A](f_{1:N})\Big|_{\mathrm{lin.}\,\varepsilon}$$

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$$\left[W(k_{(N+1)\perp})\dot{x}_{\mathrm{cl}}^+(\tau_{(N+1)})\right] \mathrm{symb}^{-1} \mathfrak{W}_\eta[A](f_{1:N})\Big|_{\mathrm{lin.}\,\varepsilon}$$

- Look at the term above in brackets + background field part of spin factor:

$$W(k_{(N+1)\perp})\dot{x}_{\mathrm{cl}}^+(\tau_{(N+1)}) - 2W(k_{(N+1)\perp})\psi_\eta^+(\tau_{(N+1)})\psi_\eta(\tau_{(N+1)}) \cdot k_{(N+1)}$$

- Label $r_\perp = k_{(N+1)\perp}$ and $\varepsilon_{(N+1)\mu} = n_\mu W(k_{(N+1)\perp})$:

Looks like a photon vertex operator!

Off-shell Current - Vacuum Analogy

- Kernel looks like one with N+1 photons

$$(-ie)\widetilde{\mathcal{K}}_N^{(s)p'p} = \int \hat{\mathrm{d}}^4 k_{(N+1)}\,\hat{\delta}(k_{(N+1)} \cdot n) K_{N+1}^{p'p}\Big|_{\varepsilon_{\mu(N+1)} \to n_\mu W(k_{(N+1)\perp})}$$

Off-shell Current - Vacuum Analogy

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- We have subleading terms, **1**, and **2**:

$$\mathcal{M}_{Ns's}^{p'p} = i \lim_{p'^2, p^2 \to m^2} \frac{1}{2m} \int d^4x' d^4x\, e^{i\tilde{p}'\cdot x' - ip\cdot x}\, \bar{u}_{s'}(p')(p'^2 - m^2)$$

$$\left\{\left[-1 + \frac{1}{2m}\not{a}(x'^+)\right] \mathcal{K}_N^{x'x} + \frac{e}{2m} \sum_{i=1}^{N} \not{\varepsilon}_i e^{ik_i\cdot x'}\, \mathcal{K}_{N-1}^{x'x}\right\}(p^2 - m^2) u_s(p)$$

but analogously to the plane wave case they disappear.

- Off-shell current is likewise:

$$(-ie)\mathcal{M}_N^{(s)p'p} = \int \hat{\mathrm{d}}^4 k_{(N+1)}\, \hat{\delta}(k_{(N+1)} \cdot n) \mathcal{M}_{N+1}^{p'p}\Big|_{\varepsilon_{\mu(N+1)} \to n_\mu W(k_{(N+1)\perp})}$$

Off-shell Current - Plane Wave Analogy

- May also view the kernel, $\widetilde{\mathcal{K}}_N^{(s)p'p}$, under the *positivity constraint* as one in a **plane wave** since

$$\frac{2}{|\dot{x}^+_{\rm cl}(t_1)|}\psi^+_\eta(t_1)\psi^\perp_\eta(t_1)r_{1\perp} = e\int_0^T \mathrm{d}\tau\,\psi_\eta(\tau)\cdot F^{(\mathrm{pw})}\cdot\psi_\eta(\tau)$$

$$eA^{(\mathrm{pw})}_\mu = -n_\mu\delta(n\cdot x)\,r_\perp x^\perp$$

- Kernel and off-shell current (plane wave analogy):

$$\widetilde{\mathcal{K}}_N^{(s)p'p} = \int \hat{\mathrm{d}}^2 r_\perp\, W(r_\perp)\mathcal{K}_N^{(\mathrm{pw})p'p}$$

$$\mathcal{M}_N^{(s)p'p} = \int \hat{\mathrm{d}}^2 r\, W(r_\perp)\mathcal{M}_N^{(\mathrm{pw})p'p}$$

Conclusions

The Worldline Formalism

1. First-quantized QFT description (use techniques from QM) ✓
2. All orders in gauge field coupling ✓
3. Both perturbative and **non-perturbative** understanding ✓

- *Non-equilibrium in-in construction*:
  Schwinger effect + axial anomaly
- *Semi-classical evaluation*:
  Schwinger effect in non-Abelian and complex fields
- *Plane Wave and Impulsive PP-wave Backgrounds*
  1. Exact master formulae constructed for scalars and spinors
  2. **All multiplicity** N-photon scattering amplitude / off-shell current in compact form on the worldline

Thank you for your time and attention!