A system in thermal equilibrium.

Assumption: x. $\rightarrow$ $\langle x \rangle = \xi = 0$.

a quantity. mean value.

$\langle x^2 \rangle \neq 0$.

squared fluctuation.

Q: what is the probability distribution of x?

$W(x) = \text{const} \times e^{S(x)}$

$S(x) = k_B \ln \Gamma$. $\Gamma = \int d^n p \, d^n x \, \delta(H-E)$

*) complete eq. $\langle x \rangle = 0$.

$S = S(0)$ max entropy.

*) with fluctuation.

$x \Rightarrow \langle x \rangle + \delta x$. $S = S(\delta x) \leq S(0)$

$S(x) = S(0) - \frac{1}{2}\beta x^2 + \ldots$

$W(x) = A e^{-\frac{1}{2}\beta x^2}$, A: normalization factor.

$A = \sqrt{\beta/2\pi}$

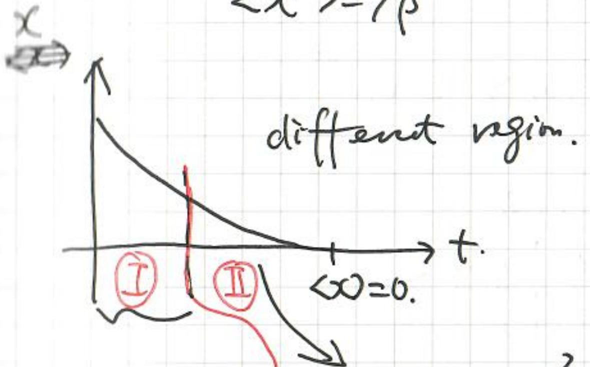
1

$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 W(x) dx = 1/\beta$.

macroscopic p

perturbate the system.

x: $\langle x \rangle = 0$, $\langle x^2 \rangle = 1/\beta$ $\rightarrow$ $x(t)$. $\langle x(t) \rangle = ?$, $\langle x^2(t) \rangle = 1/\beta$.



$\langle x \rangle \gg \langle x^2 \rangle$

$\dot{x} = \dot{x}(x)$

$= -\lambda x + \ldots$

$\langle x \rangle \sim \langle x^2 \rangle$.

$\dot{x} = \dot{x}(x, y)$

$= -\lambda x + y$.

random source of fluctuations.

(Landau-Lifshitz)

$\phi(t'-t) \equiv \langle x(t') x(t) \rangle, \quad t' \geq t$

near equilibrium.

*) $\phi(t) = \phi(-t)$.

*) $\phi(0) = \langle x^2(t) \rangle = \langle x^2(t_0) \rangle = \langle x^2(0) \rangle = 1/\beta$.

(1) in region I. $\dot{x} = -\lambda x + \dots \quad x = x(0) e^{-\lambda t}$

$\phi(t) = e^{-\lambda t} \langle x^2(0) \rangle = e^{-\lambda t} \, 1/\beta . \ (t>0.)$

$t<0. \quad \dot{x} = +\lambda x.$

$\phi(t) = e^{+\lambda t}/\beta . \ (t<0)$

$\phi(t) = \frac{1}{\beta} e^{-\lambda |t|}$.

In hydro.

(2) $\langle x(t') x(t) \rangle = \int d\omega \, d\omega' \langle x(\omega) x(\omega') \rangle e^{-i\omega t} e^{-i\omega' t}$.

$\phi(t'-t) = \int d\omega' e^{+i\omega'(t-t')} \langle x(\omega') x(-\omega') \rangle$

$= \int d\omega \, e^{-i\omega(t'-t)} \langle x(\omega) x(-\omega) \rangle$.

spectrum function.

$$\boxed{\langle x(\omega) x(-\omega) \rangle = \int_{-\infty}^{+\infty} dt \, \langle x(t) x(0) \rangle e^{+i\omega t}.}$$

$\langle x_\omega x_{-\omega} \rangle^{(+)} = \int_0^{+\infty} dt \, \langle \cdots \rangle e \cdots$

$\langle x_\omega x_{-\omega} \rangle^{(-)} = \int_{-\infty}^{0} dt \, \langle \cdots \rangle e \dots$

$\langle x_\omega x_{-\omega} \rangle = \langle \ \rangle^{(+)} + \langle \ \rangle^{(-)}$.

$\int_0^\infty dt \langle \dot{x}(t) x(0) \rangle e^{i\omega t} = \int_0^\infty \langle \lambda x(t) x(0) \rangle e^{i\omega t} dt.$

$l.h.s = \int_0^\infty dt \, \partial_t \left(\langle x(t) x(0) \rangle e^{i\omega t} \right)$

$- \int_0^\infty dt \ i\omega \langle x(t) x(0) \rangle e^{i\omega t}.$

$= e^{i\omega t} \langle x(t) x(0) \rangle \Big|_0^\infty - i\omega \langle x_\omega x_{-\omega} \rangle^{(+)}$

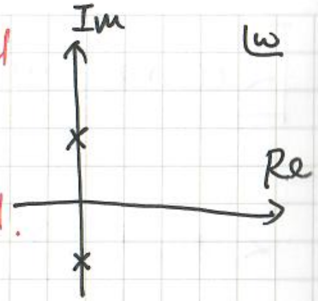
$= -\frac{1}{\beta} - i\omega \langle x_\omega x_{-\omega} \rangle^{(+)}.$

$\langle x(\infty) x(0) \rangle = 0.$ no correlation between ∞. and 0.

$\langle x_\omega x_{-\omega} \rangle^{(+)} = \dfrac{1/\beta}{\lambda - i\omega}.$

$$\langle x_\omega x_{-\omega}\rangle^{(-)} = \frac{1/\beta}{\lambda+i\omega}.$$ Retarded

$$\langle x_\omega x_{-\omega}\rangle = \frac{1}{\beta}\frac{2\lambda}{\lambda^2+\omega^2}$$ Advanced.



MSR formalism. (Martin Siggia Rose).

in region II.

$$\dot x = \lambda x + y. \longrightarrow \quad -i\omega x_\omega = \lambda x_\omega + y_\omega.$$

$$x_\omega = \frac{y_\omega}{\lambda - i\omega}.$$

$$\langle y_\omega y_{-\omega}\rangle = (\lambda - i\omega)(\lambda + i\omega)\langle x_\omega x_{-\omega}\rangle$$

$$= 2\lambda/\beta.$$

$$\langle y(t')y(t)\rangle = \delta(t'-t)\frac{2\lambda}{\beta}.$$

noise is local in time.

↓ has some ~~that~~ probability distribution.

$$\langle y(t')y(t)\rangle = \int Dy\; e^{\frac12\int y(t')\,\underline{W(t',t)}\,y(t)\,dt'}$$

↑ insert ⟨ ⟩

$$= \int Dy.\, e^{-\frac12\int y^2(t)\, 2\lambda/\beta\, dt}.$$

$$\langle x(t')x(t)\rangle = \int Dy\; e^{-\frac12\int y^2(t)\,2\lambda/\beta\, dt}\; x_y(t')x_y(t)$$

x_y: solution of x. in $\dot x = -\lambda x + y$.

$$= \int D(eom)\int Dy\; e^{-\frac12\int y^2(t)W(t)dt} \times \delta(eom)\; x(t')x(t).$$

$$= c\int Dx \int Dy\; e^{-\frac12\int y^2(t)W(t)dt}. \times \int D\theta\; e^{+i\int \theta(eom)dt}.\; x(t')x(t)$$

const.

$$\sim \int Dx\cdot Dy\; D\theta\; \exp\left(-\frac12\int y^2 W + i\theta(\dot x + \lambda x - y)\right) dt$$

...

$$\propto \int Dx D\theta\; \exp\left[\int \left(i\theta(\dot x + \lambda x) - \theta^2/2W\right) dt\right] x(t')x(t).$$

we then can define the EFT

$$S_{EFT} = \int dt \underbrace{\left[\frac{i\theta^2}{2W} + \theta(\dot{x} + \lambda x) \right]}_{\mathcal{L}_{EFT}.}$$

Kovetan 1205.

$$Z[J_\theta, J_x] = \int Dx D\theta \, e^{i\int dt \mathcal{L} + i\int (J_x x(t) + J_\theta \theta) dt}.$$

$$i\int d\omega \, (x_\omega, \theta_\omega) \begin{pmatrix} 0 & \frac{-i\omega+\lambda}{2} \\ \frac{i\omega+\lambda}{2} & i/2W \end{pmatrix} \begin{pmatrix} x_{-\omega} \\ \theta_{-\omega} \end{pmatrix}$$

$$+ i\int d\omega \, (J_{x,\omega}, J_{\theta,\omega}) \begin{pmatrix} x_{-\omega} \\ \theta_{-\omega} \end{pmatrix}.$$

$$\langle x_\omega x_{-\omega} \rangle = \frac{2\lambda/\beta}{\lambda^2 + \omega^2}$$

$$\langle x_\omega \theta_{-\omega} \rangle = i/(\lambda - i\omega) \qquad \theta_\omega \sim \phi^A.$$

$$\langle \theta_\omega x_{-\omega} \rangle = i/(\lambda + i\omega)$$

$\langle \theta_\omega \theta_{-\omega} \rangle = 0.$

Part II.

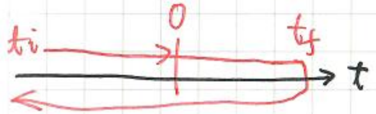
Assuming system is given in an initial state. $t_i = 0$. ρ_0. (density martix).

in QFT. $\rho_0 = |\Omega\rangle\langle\Omega|$

$$\langle \hat{O}(t) \rangle_{\rho_0} = Tr(\rho_0 \hat{O})$$
$$= Tr(\rho_0 U^\dagger O U)$$

U : evolution operator.

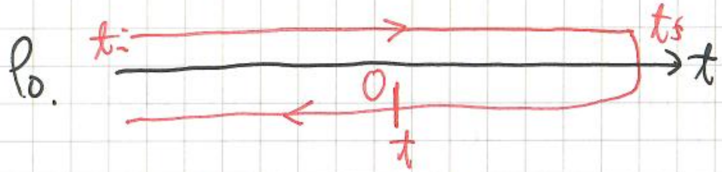
$$\underbrace{U^\dagger(t, t_i)}_{U(t_i, t)} O \, U(t, t_i)$$



$$\langle \hat{O}(t) \rangle_{\rho_0} = Tr(\rho_0 U(t_i, t_f) U(t_f, t) O \, U(t, t_i)).$$

if $\rho = |\Omega\rangle\langle\Omega|$. CTP reduces to ~~that~~ straight line

$$\langle O(t)\rangle = \mathrm{Tr}[\rho_0\, U(t_i, t)\, O\, U(t, t_f)\, U(t_f, t_i)]$$



if $\rho_0 = |\Omega\rangle\langle\Omega|$. Peskin. 4.31

$$\langle O\rangle = \langle\Omega|O|\Omega\rangle = \frac{\langle 0|T\hat{O}e^{-i\int H dt}|0\rangle}{\langle 0|Te^{-i\int H dt}|0\rangle}.$$

~~2pt~~ two-point function.

$$\langle P V(t_1) W(t_2)\rangle_{\rho_0}$$

P : path-ordered. $t_1 \geq t_2$ in CTP.

four-point function.

$$\langle P V(t_1) W(t_2) V(t_3) W(t_4)\rangle.$$

$$= \langle \underset{\substack{\uparrow \\ \text{anti-time -ordered}}}{\tilde{T}}(\quad)\underset{\substack{\uparrow \\ \text{time-ordered}}}{T}(\quad)\rangle_{\rho_0}$$

We can couple the CTP contour to some external sources. and ~~sp~~ study the response.

$$Z[J] = e^{W[J]} = \int D\phi\, e^{i\int(\mathcal{L} + J\phi)d^4x}$$

$$= \int D\phi\, e^{i\int \mathcal{L} dt} \times e^{i\int J\phi\, d^4x}$$

$$= \langle \tilde{T} e^{i\int d^4x J\phi}\rangle$$

$$= \mathrm{Tr}(|\Omega\rangle\langle\Omega|\, T\, e^{i\int J\phi\, d^4x}).$$

Extended to CTP

$$Z[J_1, J_2] = e^{W(J_1, J_2)}$$

$$= \mathrm{Tr}\{\rho_0\, P\, e^{i\int[J_1(x)\phi_1(x) - J_2(x)\phi_2(x)]dt d^3x}\}$$

$$= \mathrm{Tr}\{\rho_0\, \tilde{T}[e^{-i\int J_2\phi_2 d^4x}]\, T[e^{+i\int J_1\phi_1 d^4x}]\}.$$

r-a basis.

def: $\phi_r = (\phi_1 + \phi_2)/2$ $\quad J_r = (J_1 + J_2)/2$

$\phi_a = (\phi_1 - \phi_2)$. $\quad J_a = (J_1 - J_2)$

$$= \mathrm{Tr}[\rho_0 \, P \, e^{i\int (J_r \phi_a + J_a \phi_r) d^4x}]$$

$$G_{\alpha_1, \alpha_2, \cdots, \alpha_n} = \frac{1}{i^{n_r}} \frac{\delta^n W}{\delta J_{\bar{\alpha}_1} \delta J_{\bar{\alpha}_2} \cdots \delta J_{\bar{\alpha}_n}}$$

$\alpha = \{r, a\}$.

$\bar{\alpha} = \alpha(r \to a, a \to r)$ $\qquad$ n_r: number of "r"

$$G_{rr}^{(x_1, x_2)} = \frac{1}{i^2} \frac{\delta^2 W}{\delta J_a(x_1) \delta J_a(x_2)}$$

We can expand $W[J_1, J_2]$ in terms of J_1, J_2

$$W[J_1, J_2] = \int d^4x \, d^4y \, (J_1(x) \; J_2(x)) \begin{pmatrix} G_F(x-y) & -iG^- \\ -iG^+ & \tilde{G}_F(x-y) \end{pmatrix} \begin{pmatrix} J_1(y) \\ J_2(y) \end{pmatrix}$$

$$G_F^{(x-y)} = \langle T \phi(x) \phi(y) \rangle.$$

$$\tilde{G}_F(x-y) = \langle \tilde{T} \phi(x) \phi(y) \rangle.$$

$$G^+(x-y) = \langle \phi(x) \phi(y) \rangle.$$

$$G^-(x-y) = \langle \phi(y) \phi(x) \rangle$$

translate them into r-a basis

$$W[J_r, J_a] = \frac{i}{2} \int d^4x \, d^4y.$$

$$(J_r(x) \; J_a(y)) \begin{pmatrix} 0 & G^R \\ G^A & iG^S \end{pmatrix} \begin{pmatrix} J_r(y) \\ J_a(y) \end{pmatrix}.$$

$$G^R(x-y) = i\Theta(x^0 - y^0) \langle [\phi(x), \phi(y)] \rangle$$

$$G^A(x-y) = (-i)\Theta(y^0 - x^0) \langle [\phi(x), \phi(y)] \rangle$$

$$G^S(x-y) = \langle \phi(x)\phi(y) + \phi(y)\phi(x) \rangle_{\rho_0}.$$

Ref. 1805.09331.

symmetry properties of $W[J_1, J_2]$.

$W[J_r, J_a]$.

$\sim \frac{i}{2}\left(J_r G^R J_a + J_a G^A J_r + J_a i G^S J_r\right)$

a) $W[J_r, J_a=0] = 0$

b) $W[J_r, -J_a] = W^*[J_r, J_a]$

c) $\mathrm{Re}\, W[J_r, J_a]$ must be ~~positive~~ negative. to make sure. e^W reduce to zero.

$$W[J_1, J_2] = \mathrm{Tr}\left[\rho_0 \tilde{T}\left(e^{-i\int J_2 \phi_2}\right) T\left(e^{+i\int J_1 \phi_1 dt}\right)\right]$$

if $\rho_0 = \dfrac{e^{-\beta_0 H}}{\mathrm{Tr}(e^{-\beta_0 H})}$.

$$= \frac{1}{Z_0} \mathrm{Tr}\left[e^{-\beta_0 H} \tilde{T}\left(e^{-i\int J_2 \phi_2}\right) T\left(e^{+}\right)\right]$$

$$= \frac{1}{Z_0} \mathrm{Tr}\left[e^{-\beta_0 H} \tilde{T}(e^-) e^{+\beta_0 H} e^{-\beta_0 H} T(e^+)\right].$$

$$= \frac{1}{Z_0} \mathrm{Tr}\left[e^{-\beta H_0} T(e^+) \underbrace{e^{-\beta H_0} \tilde{T}(e^-) e^{+\beta H}}\right]$$

$$e^{-\beta H} O(t)\, e^{+\beta H}$$

$$= O(t + i\beta)$$

$$= \frac{1}{Z_0} \mathrm{Tr}\left[\rho_0\, T\left(e^{+i\int \phi_1 J_1}\right) \tilde{T}\, e^{-i\int dt\, \phi_2 J_2(t - i\beta)}\right]$$

$$\equiv W^T[J_1(t), J_2(t - i\beta)]$$ KMS.

New contour  ρ_0.

KMS → Fluctuation Dissipation Theorem.

Enkelberg, U. Heinz.

$= \coth \beta\omega \,\mathrm{Im}\, G^R$ Einstein (first)

$$G_S \approx \frac{2T}{\omega} \mathrm{Im}\, G^R.$$

$W[J_1(t), J_2(t)] = W^T[J_1(t), J_2(t - i\beta)]$.

Now, assuming that ρ_0 has symmetry properties,

e.g. ρ_0 is time reversal invariant, (T)

and parity invariant. (P).

$W[J_1(t), J_2(t)] = W^T[J_1^{PT}, J_2^{PT}]$

$J_{1,2}^{PT}(t,x) = J_{1,2}(-t-x)\, \eta_{\phi PT}$

$\tilde{J}_2(t) = \Theta J_2(t - i\beta, \vec{x})$ $\quad J_2(t, x) = \tilde{J}_2(t + i\beta, \vec{x})$

[illegible]

$$\boxed{W[J_1(t,\vec{x}), J_2(t,\vec{x})] = W[\tilde{J}_1(t,\vec{x}), \tilde{J}_2(t+i\beta,\vec{x})]}$$

$\tilde{J}_1(t,\vec{x}) = \Theta J_1(t,\vec{x})$.

$\Theta = PT$. operator.

$\tilde{J}_2(t,\vec{x}) = \Theta J_2(t,\vec{x})$.

EFT: Generating functional for EFT.

the system describes by. $e^{W[J_1, J_2]}$

contains: $iI_0[\phi_1, J_1] - iI_0[\phi_2, J_2]$

. $\phi_1(t_f) = \phi_2(t_f)$. $J(t_f) \to 0$.

EFT. $E \le E_0$. integrating over. d.o.f. $E > E_0$.

$\phi = \lambda + \psi$.
IR. UV.

$e^{W[J_1, J_2]} \to$ integrate over $D\psi$.

$\to \int D\chi_1 D\chi_2\, e^{iI_{EFT}[\chi_1, J_1, \chi_2, J_2, \rho_0]}$.

a) $W[J_r, J_a = 0] = 0. \to I_{EFT}[J_r, J_a = 0] = 0$

b). $W[J_r, -J_a] = W^*[J_r, J_a]$

$\to I_{EFT}[J_r, -J_a] = I_{EFT}^*[J_r, J_a]$
$\quad \chi_r, -\chi_a \qquad \chi_r, \chi_a.$

c) $\mathrm{Re}\,W < 0$. $\rightarrow$ $\mathrm{Im}\, I_{EFT} > 0$.

Q: KMS gives ~~what~~ ?

$I_{EFT}[\chi_1, J_1, \chi_2, J_2]$

$= I_{EFT}[\tilde{\chi}_1, \tilde{J}_1, \tilde{\chi}_2, \tilde{J}_2]$.

$\partial_t n + \vec{\nabla}\cdot\vec{J} = 0$. $\vec{J} = -D\vec{\nabla} n$. Fick's law.

$\rightarrow$ $\partial_t n - D\nabla^2 \vec{n} = 0$. $\rightarrow$ $\partial_\mu J^\mu = 0$.

What is the χ field here?

$e^{W[A_{1\mu}, A_{2\mu}]} = \cancel{\int D\chi}\ \cancel{e^{i I_0[J_{1\mu}, A_{1\mu}] - i I_0[J_{2\mu}, A_{2\mu}]}}$ $\mathrm{Tr}[\rho_0 P$ $]$

Because of the conservation law

$A_\mu \rightarrow A_\mu + \partial_\mu \lambda$.

$W[A_1^\mu, A_2^\mu] = W[A_1^\mu + \partial^\mu\lambda_1, A_2^\mu + \partial^\mu\lambda_2]$

requires. Eom for $\chi_{1,2}$ should give $\partial_\mu J^\mu = 0$.

~~I_{EFT}~~. requires. $B_\mu = A_\mu + \partial_\mu \chi$.

KMS:

$I_{EFT}[B_{1\mu}, B_{2\mu}] = I_{EFT}[\tilde{B}_{1\mu}, \tilde{B}_{2\mu}]$.

KMS in classical limit.

$\tilde{B}_1^\mu(t,x) = \Theta\, B_1^\mu(t, \vec{x})$

$\tilde{B}_2^\mu(t,x) = \Theta\, B_2^\mu(t+i\beta, x)$.

χ_r, $\chi_a \rightarrow \hbar\chi_a$. χ_a is quantum.

$$\Rightarrow \boxed{\begin{aligned} \tilde{B}_r^\mu(t,x) &= \Theta\, B_r^\mu(t,\vec{x}) \\ \tilde{B}_a^\mu(t,x) &= \Theta\, B_a(t,\vec{x}) + i\,\Theta\,\beta\,\partial_t B_r(t,x) \end{aligned}}$$

EFT. of Diffusion

$I_{EFT} = \int d^4x \, \mathcal{L}_{EFT}$. (Keep Quadrat terms in free theory only)

$\mathcal{L}_{EFT}[B_r, B_a]$

$= \# B_{a,0}^2 + \cancel{\# B_r^2} + \# B_{a,i}^2$

$+ \# B_{r,0} B_{a,0} + \# B_{r,i} \cdot B_{a,i}$

$+ B_{r,i} \partial_i B_{a,0} + \dots$

$+ \dots$

① turn off source term first.

$\mathcal{L}_{EFT} \sim \chi_a \chi_r + \chi_a \chi_a$

$$\frac{\delta \mathcal{L}_{EFT}}{\delta \chi_a} = \bigcirc \chi_r + \bigcirc \chi_a \xrightarrow{\chi_a = 0} \bigcirc \chi_r$$

it is EoM

$\partial_\mu J^\mu = 0$.

$\partial_t H + \partial_i G^i = 0$.

$$\mathcal{L}_{EFT}^{A_\mu = 0} = -(\partial_t H + \partial_i G_i)\chi_a + \bigcirc \chi_a \bigcirc \chi_a$$

↑ for convenience

G_i. operator.

$= \underline{H \partial_t \chi_a + G_i \partial_i \chi_a}$ dissipation

fluctuations

$+ \underline{\# \partial_t \chi_a \partial_t \chi_a + \# (\partial_i \chi_a)(\partial_i \chi_a)}$

replacing $\partial_\mu \chi \rightarrow B_\mu = A_\mu + \partial_\mu \chi$.

if. H. G_i are fixed, the coefficients for other terms can be determined by KMS.

$$J_{ri} = \frac{\delta W}{\delta A_{ai}} \bigg|_{A_{a,r}=0} = G_i. = D \nabla_i n$$

$$= -DC \partial_t B_{r,i}$$

$$J_{r0} = \frac{\delta W}{\delta A_{a,0}} \bigg|_{A_{a,r}} = H. = n = C\mu = C\partial_t \chi_r$$

↑ susceptibility

$$= C B_{r,0}$$

$$D \nabla_i n = DC \nabla_i \mu = DC \nabla_i \partial_t \chi_r$$

(11)

$$\mathcal{L}_{EFT} = \cancel{\mathscr{A}}\, C\, B_{r,0} B_{a,0} - DC\, B_{a,i}\, \partial_t B_{r,i}$$

$$+ B_{a,0} M_1 B_{a,0} + B_{a,i} M_2 B_{a,i}$$

Using KMS.

$$B_{r,\mu}(t,x) = \Theta\, B_{r,\mu}(t,x)$$

$$B_{a,\mu}(t,x) = \Theta\, B_{a,\mu}(t,x) + i\beta\Theta\,\partial_t B_{r,\mu}(t,x)$$

$$\mathcal{L}_{EFT} = \cdots + \chi\, B_{r,0}\,(i\beta\,\partial_t B_{r,0})$$

$\hookrightarrow \partial_t B_{r,0}^2$

$\to 0$

$+$

we need to drop "total" derive terms

and $\partial_t^2 B_{r,0} \to 0$.

$$M_2 = i\sigma T.$$

$$\mathcal{L}_{EFT} = C\, B_{r0} B_{a,0} - \sigma B_{ia} \dot{B}_{ia} + i\sigma T\, B_{ai} B_{ai}.$$

When. $A_r = 0$. $\mathcal{L}_{EFT}$ is MSR