A Charming Path to Quantum Chromodymamics

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Personal Charming Excursion

- Gao, Huber, Ji, YMW, *Next-to-next-to-leading-order QCD prediction for the photon-pion form factor*, Phys. Rev. Lett. **128** (2022) 062003 [this talk!].
  The first two-loop calculation of the $\gamma\gamma^* \to \pi^0$ form factor at LP.

- Khodjamirian, Melić, YMW, Wei, *The $D^*D\pi$ and $B^*B\pi$ couplings from light-cone sum rules*, JHEP **2103** (2021) 016 [this talk!].
  The complete NLO calculation of the $H^*H\pi$ couplings at twist-3.

- Li, Lü, Wang, YMW, Wei, *QCD calculations of radiative heavy meson decays with subleading power corrections*, JHEP **2004** (2020) 023 [this talk!].
  The first-ever calculation of the hadronic photon corrections to the $H^*H\gamma$ couplings at NLO.

- YMW, Shen, *Subleading power corrections to the pion-photon transition form factor in QCD*, JHEP **1712** (2017) 037 [this talk!].
  Explicit demonstration of the renormalization-scheme dependence.

- Li, Shen, YMW, *Joint resummation for pion wave function and pion transition form factor, JHEP* **1401** *(2014) 004.*
  Resummation improved TMD factorization for the $\pi - \gamma$ form factor.

- Khodjamirian,Klein, Mannel, YMW, *How much charm can PANDA produce?, Eur. Phys. J. A* **48** *(2012) 31.*

- Khodjamirian, Klein, Mannel, YMW, *Form Factors and Strong Couplings of Heavy Baryons from QCD Light-Cone Sum Rules*, JHEP **1109** (2011) 106.
  New strategy to get rid of the background contributions from negative-parity baryons.

Personal Charming Excursion

- Khodjamirian, Mannel, Pivovarov, YMW, *Charm-loop effect in* $B \to K^{(*)}\ell^+\ell^-$ *and* $B \to K^*\gamma$, JHEP **1009** (2010) 089.
  The first QCD calculation of the non-factorizable charm-loop effect.

- YMW, Zou, Wei, Li, Lü, *FCNC-induced semileptonic decays of* J/ψ *in the Standard Model*, J. Phys. G **36** (2009) 105002.

- YMW, Zou, Wei, Li, Lü, *Weak decays of* J/ψ*: The Non-leptonic case*, Eur. Phys. J. C **55** (2008) 607.

- YMW, Lü, *Weak productions of new charmonium in semi-leptonic decays of* B_c, Phys. Rev. D **77** (2008) 054003.

- YMW, Zou, Wei, Li, Lü, *The Transition form-factors for semi-leptonic weak decays of* J/ψ *in QCD sum rules*, Eur. Phys. J. C **54** (2008) 107.

- Chang, Li, Li, YMW, *Lifetime of doubly charmed baryons*, Commun. Theor. Phys. **49** (2008) 993.

- Lü, Zou, YMW, *Twist-3 distribution amplitudes of scalar mesons from QCD sum rules*, Phys. Rev. D **75** (2007) 056001.

- He, Li, Li, YMW, *Calculation of* $\mathrm{BR}(\bar{B}^0 \to \Lambda_c \bar{p})$ *in the PQCD approach*, Phys. Rev. D **75** (2007) 034011.
  The first-ever PQCD calculation of the baryonic bottom-meson decays.

- Li, Liu, YMW, *Calculation of the Branching Ratio of* $B^- \to h_c K^-$ *in PQCD*, Phys. Rev. D **74** (2006) 114029.

Calculational Tools for Charming Physics

- Lattice QCD Technique:
  - First-principles calculations numerically.
  - Very challenging in particular for non-local matrix elements.

- Light-Cone Sum Rules in QCD/SCET [Khodjamirian, Melić, YMW, arXiv: 2311.08700]:
  - QCD factorization for the correlation function in the appropriate kinematical region.
  - Hadronic dispersion relation for the the correlation function.
  - Matching with the aid of the quark-hadron duality ansatz.
  - Constructions of the sum rules with different LCDAs possible.

- QCD/SCET Factorization:
  - Less assumptions theoretically, more challenging conceptuallly/technically.
  - Parametrically power suppressed corrections numerically dominant.
    $\rightarrow$ Rapidity divergences in the subleading-power factorization formulae.
  - Heavy-quark expansion less effectively in comparison with beautiful physics.

- TMD Factorization:
  - Including the Sudakov mechanism, but no definite power counting scheme.
  - Definitions of TMD parton densities more complicated.

Part I: Radiative $M^* \to M\gamma$ decays

- Why radiative heavy-hadron decays?
  - Explore the emerged symmetries of the QCD Lagrangian in the limit of $m_Q \to \infty$ and $m_q \to 0$. Heavy-hadron chiral perturbation theory [Burdman and Donoghue, 1992; Wise, 1992; Yan et al, 1992].
  - Determine the magnetic susceptibility of the quark condensate $\chi(\mu)$ [Ioffe, Smilga, 1984].

$$\langle 0|\bar{q}\,\sigma_{\mu\nu}q|0\rangle_{\rm F} = g_{\rm em}\,\chi(\mu)F_{\mu\nu}\,\langle 0|\bar{q}\,\sigma_{\mu\nu}q|0\rangle\,.$$

    Already discussed in the context of $\Sigma^+ \to p\gamma$ [Balitsky, Braun, Kolesnichenko, 1988].
  - Fundamental ingredient of describing the soft photon correction to any exclusive heavy-hadron decay.
    QED factorization for heavy-hadron reactions works for the hard/collinear photon exchange.
  - Nontrivial applications of the double dispersion sum rules in QCD.
    Originally discussed in the context of $J/\psi \to \eta_c\gamma$ [Khodjamirian, 1979].
  - Delicate interplay of the LP and NLP effects numerically.
    Formally LP contribution can be of minor importance in practice.
- Major task: Hadronic photon correction to radiative $M^* \to M\gamma$ decays.

General aspects of radiative $M^* \to M\gamma$ decays

- The magnetic coupling $M^* M \gamma$ in QCD:

$$\langle \gamma(p,\eta^*) M(q) | M^*(p+q) \rangle = -g_{\rm em}\, g_{M^* M\gamma}\, \varepsilon_{\mu\nu\rho\sigma}\, \eta^{*\mu}\, \varepsilon^{\nu}\, p^{\rho}\, q^{\sigma} \,.$$

- Alternative definition in terms of the vector-to-pseudoscalar transition form factor:

$$\langle M(q) | j_\mu^{\rm em} | M^*(p+q,\varepsilon) \rangle = g_{\rm em} \, \frac{2\, i\, \mathscr{V}(p^2)}{m_V + m_P} \, \varepsilon_{\mu\nu\rho\sigma}\, \varepsilon^{\nu}\, p^{\rho}\, q^{\sigma} \,.$$

The exact relation in QCD:

$$g_{M^* M\gamma} = \frac{2}{m_V + m_P} \, \mathscr{V}(p^2 = 0).$$

- The general strategy of constructing the sum rules [Li, Lü, Wang, YMW, Wei, 2020]:

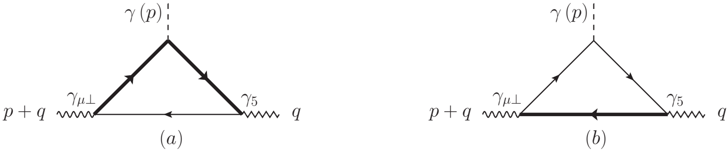
$$\Pi_\mu(p,q) = \int d^4 x\, e^{-i(p+q)\cdot x} \, \langle \gamma(p,\eta^*) | {\rm T} \left\{ \bar{q}(x) \gamma_{\mu\perp} Q(x), \bar{Q}(0) \gamma_5 q(0) \right\} | 0 \rangle \,.$$

Power counting scheme:

$$n\cdot p \sim \mathscr{O}(m_Q), \qquad |(p+q)^2 - m_Q^2| \sim \mathscr{O}(m_Q^2), \qquad |q^2 - m_Q^2| \sim \mathscr{O}(m_Q^2) \,.$$

The $M^* M \gamma$ coupling @ LP

- The “point-like" photon contribution:



- The on-shell photon radiation off the heavy quark computed from the local OPE technique.

- The on-shell photon radiation off the light quark:

$$\Pi_{\mu}^{1(\mathrm{b})}(p,q) = N_c\, e_q\, g_{\mathrm{em}} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{[(\ell-p-q)^2 - m_Q^2 + i0][\ell^2 + i0][(\ell-p)^2 + i0]}$$

$$\Big\{ m_Q \underbrace{\mathrm{Tr}\left[\gamma_{\mu\perp}\, \gamma_5\, \not{p}\, \not{\eta}^*\, \not{\ell}\right]}_{\text{hard region}} - m_q \underbrace{\mathrm{Tr}\left[\gamma_{\mu\perp}\, \gamma_5\, \not{p}\, \not{\eta}^*\, (\not{\ell} - \not{q})\right]}_{\text{hard and collinear regions}} \Big\} .$$

The photon distribution amplitudes needed to parameterize the collinear contribution.

The $M^* M \gamma$ coupling @ LP

- The OPE result in the dispersion form:

$$\Pi_\mu^{(\text{per})}(p,q) \quad \supset \quad -\left(\frac{N_c}{4\pi^2}\right) g_{\text{em}}\, m_Q\, \varepsilon_{\mu p q \eta} \int ds_1 \int ds_2 \, \frac{\rho^{(\text{per})}(s_1,s_2)}{[s_1-(p+q)^2-i0][s_2-q^2-i0]} \,.$$

The QCD spectral density at one loop:

$$\rho^{(\text{per})}(s_1,s_2) \quad = \quad -\delta(s_1-s_2)\left\{e_Q\left[(1-r_q)\left(1-\frac{m_Q^2}{s_2}\right)+\ln\left(\frac{m_Q^2}{s_2}\right)\right]-e_q\left(1-\frac{m_Q^2}{s_2}\right)\right\} \times \theta(s_2-m_Q^2)+\mathcal{O}(\alpha_s)\,.$$

  - The resulting sum rule insensitive to the shape of the duality region.
  - Both the e_Q and e_q terms lead to the LP contributions to $\Pi_\mu(p,q)$.

- Hadronic dispersion relation:

$$\Pi_\mu(p,q) \quad = \quad -\varepsilon_{\mu p q \eta}\left\{\frac{g_{\text{em}} f_P f_V\, m_V\, g_{M^* M \gamma}}{[m_V^2-(p+q)^2-i0][m_P^2-q^2-i0]}\,\frac{m_P^2}{m_Q+m_q} + \int ds_1 \int ds_2 \, \frac{\rho^{(\text{had})}(s_1,s_2)}{[s_1-(p+q)^2-i0][s_2-q^2-i0]}\right\}.$$

Parton-hadron duality ansatz:

$$\iint_{\Sigma} ds_1\, ds_2 \, \frac{\rho^{\text{had}}(s_1,s_2)}{(s_1-(p+q)^2)(s_2-q^2)} = \iint_{\Sigma_0} ds_1\, ds_2 \, \frac{\rho^{(\text{per})}(s_1,s_2)}{(s_1-(p+q)^2)(s_2-q^2)} \,.$$

The $M^* M \gamma$ coupling @ LP

- The yielding LCSR for the “point-like" photon contribution [Li, Lü, Wang, YMW, Wei, 2020]:

$$f_P f_V \mu_P m_V g^{(\mathrm{per})}_{M^* M \gamma} = -\left(\frac{N_c}{4\pi^2}\right) m_Q \int_{m_Q^2}^{s_0} ds \exp\left[-\frac{2s - m_V^2 - m_P^2}{M^2}\right] \times \left\{ e_Q \left[(1 - r_q)\left(1 - \frac{m_Q^2}{s}\right) + \ln \frac{m_Q^2}{s}\right] - e_q \left(1 - \frac{m_Q^2}{s}\right) \right\}.$$

  - Power counting scheme for the sum-rule parameters:

$$m_V \sim m_P \sim m_Q + \Lambda_{\mathrm{QCD}}, \qquad s_0 \sim (m_Q + \omega_0)^2, \qquad f_P \sim f_V \sim \Lambda_{\mathrm{QCD}}^{3/2} / m_Q^{1/2}.$$

  - Decomposition of the LP effect:

$$g^{(\mathrm{per})}_{M^* M \gamma} = e_Q \, g^{(\mathrm{per,I})}_{M^* M \gamma} + e_Q \, r_q \, g^{(\mathrm{per,II})}_{M^* M \gamma} + e_q \, g^{(\mathrm{per,III})}_{M^* M \gamma}.$$

    Asymptotic scaling laws for the separate terms:

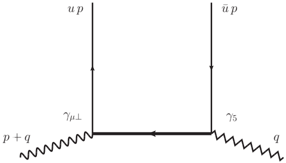
$$g^{(\mathrm{per,I})}_{M^* M \gamma} \sim \frac{1}{m_Q}\left(\frac{\omega_0}{\Lambda_{\mathrm{QCD}}}\right)^3, \qquad g^{(\mathrm{per,II})}_{M^* M \gamma} \sim g^{(\mathrm{per,III})}_{M^* M \gamma} \sim \frac{1}{\Lambda_{\mathrm{QCD}}}\left(\frac{\omega_0}{\Lambda_{\mathrm{QCD}}}\right)^2.$$

  - In agreement with the HHχPT predictions [Manohar, Wise, 2000].
  - The substantial cancellation for the magnetic couplings $D^{*+}_{(s)} D^{+}_{(s)} \gamma$.

The resolved photon correction at twist-2

- The non-perturbative hadronic input.
  - The twist-2 photon LCDA in SCET:

$$\langle \gamma(p,\eta^*)|(\bar{\xi}\, W_c)(\tau n)\, \sigma_{\alpha\beta}\, (W_c^\dagger\, \xi)(0)|0\rangle$$
$$= -i\, g_{\rm em}\, e_q\, \chi(\mu)\, \langle \bar{q}q\rangle(\mu)\, (p_\beta\, \eta_\alpha^* - p_\alpha\, \eta_\beta^*)$$
$$\times \int_0^1 du\, e^{iun\cdot p\,\tau}\, \phi_\gamma(u,\mu)\,.$$



- The resulting LO factorization formula:

$$\Pi_{\mu,\rm LO}^{\rm tw2}(p,q) = -g_{\rm em}\, e_q\, \chi(\mu)\, \langle \bar{q}q\rangle(\mu)\, \varepsilon_{\mu\, p\, q\, \eta^*} \int_0^1 du\, \frac{\phi_\gamma(u,\mu)}{\bar{u}(p+q)^2 + uq^2 - m_Q^2 + i0}\,.$$

- The twist-2 LCSR at tree level:

$$f_P\, f_V\, \mu_P\, m_V\, g_{M^* M\gamma}^{(\rm tw2, LO)} = -e_q\, \chi(\mu)\, \langle \bar{q}q\rangle(\mu)\, \phi_\gamma\left(\frac{1}{2},\mu\right) \int_{m_Q^2}^{s_0} ds\, \exp\left[\frac{m_V^2 + m_P^2 - 2\,s}{M^2}\right].$$

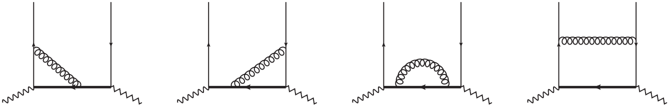
Asymptotic scaling in the heavy quark limit:

$$g_{M^* M\gamma}^{(\rm tw2, LO)} \sim \frac{1}{\Lambda_{\rm QCD}} \left(\frac{\omega_0}{\Lambda_{\rm QCD}}\right).$$

$\hookrightarrow$ Suppressed by a factor of $\Lambda_{\rm QCD}/\omega_0$ compared with the “point-like" photon effect.

The resolved photon correction at twist-2

- Hard-collinear factorization at NLO in QCD:



  - Perturbative QCD matching with the evanescenet operator approach.
  - The NLL resummation of the enhanced logarithms with the RG formalism.
  - The yielding QCD factorization formula [Li, Lü, Wang, YMW, Wei, 2020]:

$$\Pi^{\rm tw2}_{\mu,\rm NLL}(p,q) = -g_{\rm em}\, e_q\, \chi(\mu)\, \langle \bar{q} q\rangle(\mu)\, \epsilon_{\mu p q \eta^*} \int_0^\infty ds_1 \int_0^\infty ds_2\, \frac{1}{[s_1-(p+q)^2][s_2-q^2]} \times \left[\rho_0^{\rm tw2}(s_1,s_2) + \frac{\alpha_s(\mu)\, C_F}{4\pi}\, \rho_1^{\rm tw2}(s_1,s_2)\right].$$

The structure of the NLO spectral density:

$$m_Q^2\, \rho_1^{\rm tw2}(s_1,s_2) = \{[\left[\rho_{\rm I}(r,\sigma) + \rho_{\rm II}(r,\sigma)\ln r + \rho_{\rm III}(r,\sigma)\,(\ln^2 r - \pi^2)\right]\delta^{(2)}(r-1) + [\rho_{\rm II}(r,\sigma) + 2\,\rho_{\rm III}(r,\sigma)\ln r]\, \frac{d^3}{dr^3}\ln|1-r|\}\, \theta(s_1 - m_Q^2)\, \theta(s_2 - m_Q^2).$$

- The NLO twist-2 LCSR depends on the shape of the duality region.

The two-particle higher twist corrections

- The two-particle light-cone corrections [Ball, Braun, Kivel, 2002]:

$$\langle\gamma(p,\eta^*)|\bar{q}(x)\,W_c(x,0)\,\sigma_{\alpha\beta}\,q(0)|0\rangle$$
$$= -i\,g_{\rm em}\,Q_q\,\langle\bar{q}q\rangle(\mu)\,(p_\beta\,\eta^*_\alpha - p_\alpha\,\eta^*_\beta)\int_0^1 dz\,e^{izp\cdot x}\left[\chi(\mu)\,\phi_\gamma(z,\mu) + \frac{x^2}{16}\,\mathbb{A}(z,\mu)\right]$$
$$-\frac{i}{2}\,g_{\rm em}\,Q_q\,\frac{\langle\bar{q}q\rangle(\mu)}{q\cdot x}\,(x_\beta\,\eta^*_\alpha - x_\alpha\,\eta^*_\beta)\int_0^1 dz\,e^{izp\cdot x}\,h_\gamma(z,\mu)\,.$$

- The two-particle twist-3 LCDAs:

$$\langle\gamma(p,\eta^*)|\bar{q}(x)\,W_c(x,0)\,\gamma_\alpha\,q(0)|0\rangle = g_{\rm em}\,Q_q f_{3\gamma}(\mu)\,\eta^*_\alpha\int_0^1 dz\,e^{izp\cdot x}\,\psi^{(v)}(z,\mu)\,.$$
$$\langle\gamma(p,\eta^*)|\bar{q}(x)\,W_c(x,0)\,\gamma_\alpha\,\gamma_5\,q(0)|0\rangle = \frac{g_{\rm em}}{4}\,Q_q f_{3\gamma}(\mu)\,\varepsilon_{\alpha\beta\rho\tau}\,p^\rho\,x^\tau\,\eta^{*\beta}\times\int_0^1 dz\,e^{izp\cdot x}\,\psi^{(a)}(z,\mu)\,.$$

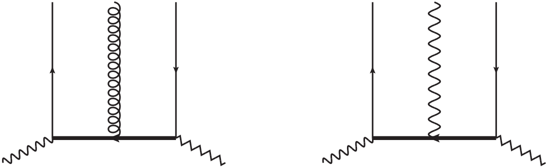
- The yielding tree-level LCSR [Li, Lü, Wang, YMW, Wei, 2020]:

$$f_P f_V\,\mu_P\,m_V\,g^{(2{\rm PHT})}_{M^*M\gamma}\,\exp\left[-\left(\frac{m_V^2+m_P^2}{M^2}\right)\right]$$
$$= \frac{e_q}{2}\left[m_Q f_{3\gamma}(\mu)\,\psi^{(a)}\left(\frac{1}{2},\mu\right) + \left(\frac{m_Q^2}{M^2}+\frac{1}{2}\right)\langle\bar{q}q\rangle(\mu)\,\mathbb{A}\left(\frac{1}{2},\mu\right)\right]\exp\left(-\frac{2\,m_Q^2}{M^2}\right).$$

$\hookrightarrow$ Suppressed by 1 and 2 power(s) of $\Lambda_{\rm QCD}/\omega_0$ compared with the twist-2 effect.

The three-particle higher twist corrections

- The three-particle higher-twist correction at LO in QCD:



- The quark propagator in the background gluon field [Balitsky, Braun, 1989]:

$$\langle 0|\mathrm{T}\{\bar{q}(x), q(0)\}|0\rangle \supset i\, g_s \int_0^\infty \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x} \int_0^1 du \left[\frac{u\, x_\mu\, \gamma_\nu}{k^2 - m_q^2} - \frac{(\not{k} + m_q)\, \sigma_{\mu\nu}}{2\,(k^2 - m_q^2)^2} \right] G^{\mu\nu}(ux)\,.$$

- The three-particle photon distribution amplitudes [Ball, Braun, Kivel, 2002]:

$$\langle \gamma(p,\eta^*)|\bar{q}(x)\, W_c(x,0)\, g_s\, \widetilde{G}_{\alpha\beta}(vx)\, \gamma_\rho\, \gamma_5\, q(0)|0\rangle$$
$$= -g_{\rm em}\, Q_q f_{3\gamma}(\mu)\, p_\rho\, (p_\beta\, \eta^*_\alpha - p_\alpha\, \eta^*_\beta) \int [\mathscr{D}\alpha_i]\, e^{i(\alpha_q + v\,\alpha_g)\, p\cdot x} A(\alpha_i, \mu)\,.$$
$$\langle \gamma(p,\eta^*)|\bar{q}(x)\, W_c(x,0)\, g_s\, G_{\alpha\beta}(vx)\, i\,\gamma_\rho\, q(0)|0\rangle$$
$$= -g_{\rm em}\, Q_q f_{3\gamma}(\mu)\, p_\rho\, (p_\beta\, \eta^*_\alpha - p_\alpha\, \eta^*_\beta) \int [\mathscr{D}\alpha_i]\, e^{i(\alpha_q + v\,\alpha_g)\, p\cdot x} V(\alpha_i, \mu)\,.$$

Conformal expansion at the P-wave accuracy in practice.

The three-particle higher twist corrections

- The resulting tree-level LCSR [Li, Lü, Wang, YMW, Wei, 2020]:

$$
\begin{aligned}
f_P f_V \mu_P m_V g_{M^* M\gamma}^{(3\mathrm{PHT})} = & -\langle \bar{q} q \rangle(\mu) \exp\left[\left(\frac{m_V^2 + m_P^2 - 2 m_Q^2}{M^2}\right)\right] \\
& \times \left\{ e_q \left[\widehat{S}\left(\frac{1}{2}, 0, \mu\right) + \widehat{T}_1\left(\frac{1}{2}, 0, \mu\right) - \widehat{T}_2\left(\frac{1}{2}, 0, \mu\right) - \widehat{\widetilde{S}}\left(\frac{1}{2}, 0, \mu\right) + \widehat{T}_3\left(\frac{1}{2}, 0, \mu\right) \right.\right. \\
& \left. - \widehat{T}_4\left(\frac{1}{2}, 0, \mu\right) + 2\widehat{\widetilde{S}}\left(\frac{1}{2}, 1, \mu\right) - 2\widehat{T}_3\left(\frac{1}{2}, 1, \mu\right) + 2\widehat{T}_4\left(\frac{1}{2}, 1, \mu\right) \right] \\
& \left. + e_Q \left[\widehat{S}_\gamma\left(\frac{1}{2}, 0, \mu\right) - \widehat{T}_4^\gamma\left(\frac{1}{2}, 0, \mu\right) + 2\widehat{T}_4^\gamma\left(\frac{1}{2}, 1, \mu\right) \right] \right\}.
\end{aligned}
$$

  Only depends on the midpoint values of the photon LCDAs.

- The scaling behaviour in the heavy quark limit:

$$
g_{M^* M\gamma}^{(3\mathrm{PHT})}\Big|_{\bar{q} q G} \sim g_{M^* M\gamma}^{(3\mathrm{PHT})}\Big|_{\bar{q} q \gamma} \sim \mathcal{O}\left(\frac{1}{m_Q}\right).
$$

  Doubly suppressed by a factor of $\Lambda_{\mathrm{QCD}}^2/(m_Q\,\omega_0)$ compared with the twist-2 effect.

- The final expression of the magnetic $M^* M\gamma$ coupling:

$$
g_{M^* M\gamma} = g_{M^* M\gamma}^{(\mathrm{per})} + g_{M^* M\gamma}^{(\mathrm{tw2,NLL})} + g_{M^* M\gamma}^{(\mathrm{2PHT})} + g_{M^* M\gamma}^{(\mathrm{3PHT})}.
$$

Theoretical predictions of the $M^* M\gamma$ couplings

- Numerical patterns of distinct mechanisms [Li, Lü, Wang, YMW, Wei, 2020]:

	$D^{*+} D^+ \gamma$	$D^{*0} D^0 \gamma$	$D_s^{*+} D_s^+ \gamma$	$B^{*+} B^+ \gamma$	$B^{*0} B^0 \gamma$	$B_s^{*0} B_s^0 \gamma$
$g^{(\rm per)}_{M^* M \gamma}\,(\rm GeV^{-1})$	-0.032	0.73	0.024	0.84	-0.56	-0.47
$g^{(\rm tw2, LL)}_{M^* M \gamma}\,(\rm GeV^{-1})$	-0.50	0.99	-0.41	0.96	-0.48	-0.37
$g^{(\rm tw2, NLL)}_{M^* M \gamma}\,(\rm GeV^{-1})$	-0.45	0.89	-0.36	0.79	-0.40	-0.31
$g^{(\rm 2PHT)}_{M^* M \gamma}\,(\rm GeV^{-1})$	0.18	-0.37	0.14	-0.18	0.089	0.067
$g^{(\rm 3PHT)}_{M^* M \gamma}\,(\rm GeV^{-1})$	0.14	0.23	0.11	-0.017	-0.036	-0.027
$g_{M^* M \gamma}\,(\rm GeV^{-1})$	-0.15	1.48	-0.079	1.44	-0.91	-0.74

  - The LP effects of the magnetic couplings $D^{*+}_{(s)} D^+_{(s)} \gamma$ highly suppressed.
  - The twist-2 hadronic photon effects determined by the electric charge of the light quark.
  - The NLO QCD corrections to the hadronic photon contributions around $(10-20)\%$.
  - The 2- and 3-particle higher-twist corrections of minor importance in the bottom sector.
  - Heavy quark expansion indeed less effectively in the charm sector.

Theoretical predictions of the $M^*M\gamma$ couplings

- Summary of various theory predictions:

	$g_{D^{*+}D^+\gamma}$ (GeV^{-1})	$g_{D^{*0}D^0\gamma}$ (GeV^{-1})	$g_{D_s^{*+}D_s^+\gamma}$ (GeV^{-1})	$g_{B^{*+}B^+\gamma}$ (GeV^{-1})	$g_{B^{*0}B^0\gamma}$ (GeV^{-1})	$g_{B_s^{*0}B_s^0\gamma}$ (GeV^{-1})
this work	$-0.15^{+0.11}_{-0.10}$	$1.48^{+0.29}_{-0.27}$	$-0.079^{+0.086}_{-0.078}$	$1.44^{+0.22}_{-0.20}$	$-0.91^{+0.12}_{-0.13}$	$-0.74^{+0.09}_{-0.10}$
HHχPT	-0.27 ± 0.05	2.19 ± 0.11	0.041 ± 0.056	1.45 ± 0.11	-1.01 ± 0.05	-0.70 ± 0.06
HQET+VMD	$-0.29^{+0.19}_{-0.11}$	$1.60^{+0.35}_{-0.45}$	$-0.19^{+0.19}_{-0.08}$	$0.99^{+0.19}_{-0.23}$	$-0.58^{+0.12}_{-0.10}$	–
HQET+CQM	$-0.38^{+0.05}_{-0.06}$	1.91 ± 0.09	–	$1.45^{+0.11}_{-0.12}$	$-0.82^{+0.06}_{-0.05}$	–
Lattice QCD	-0.2 ± 0.3	2.0 ± 0.6	–	–	–	–
LCSR	-0.50 ± 0.12	1.52 ± 0.25	–	1.68 ± 0.17	-0.85 ± 0.17	–
QCDSR	$-0.19^{+0.03}_{-0.02}$	0.62 ± 0.03	-0.20 ± 0.03	–	–	–
RQM	-0.44 ± 0.06	2.15 ± 0.11	-0.19 ± 0.03	1.66 ± 0.11	-0.93 ± 0.05	0.65 ± 0.03
experiment	-0.47 ± 0.06	1.77 ± 0.03	–	–	–	–

- Our LCSR calculations generally consistent with the HHχPT and Lattice QCD predictions.

$$g_{M^*M\gamma} = \frac{e_Q}{m_Q}\left(1 + \frac{2}{3}\frac{\bar{\Lambda}}{m_Q}\right) + e_q\,\beta + \delta\mu_q^{(\ell)}\,.$$

- The experimental value of $g_{D^{*+}D^+\gamma}$ from the CLEO data. By contrast, $g_{D^{*0}D^0\gamma}$ from the BaBar and BESIII data of $\mathscr{BR}(D^{*0} \to D^0\gamma)$ with the estimated $\Gamma(D^{*0}) = 55.4 \pm 1.4\,\mathrm{keV}$.
- The D_s^{*+} decay width dominated by the QED interaction instead of the strong interaction.
- NLO QCD correction to the “point-like" photon contribution in high demand.

Part II: The $D^* D\pi$ and $B^* B\pi$ couplings

- Why strong couplings of the heavy hadrons with a pion?
  - Among the most important hadronic parameters of heavy flavour physics.

$$\mathscr{L}_{\rm eff} = \left(\frac{2ig_\pi}{f_\pi} P_a^{*\nu\dagger} P_b \partial_\nu M_{ba} + h.c. \right) - \frac{2ig_\pi}{f_\pi} P_a^{*\alpha\dagger} P_b^{*\beta\dagger} \partial^\nu M_{ba} \, \varepsilon_{\alpha\lambda\beta\nu} v^\lambda \, .$$

  - Phenomenologically relevant to the determination of the CKM matrix element $|V_{ub}|$.
  - Longstanding puzzle between the LCSR result and the experimental value of $g_{D^* D\pi}$.
  - Nontrivial constraint on the twist-2 pion distribution amplitude.
    $\hookrightarrow$ Fundamental theory input for the QCD description of the pion physics.

- Current theory status:
  - The first-ever LCSR calculation of the strong couplings $H^* H\pi$ [Belyaev, Braun, Khodjamirian, Rückl, 1995]: $g_{D^* D\pi} = 12.5 \pm 1.0$ $\left[g_{D^* D\pi}|_{\rm exp} = 16.8 \pm 0.2 \right]$.
  - The NLO QCD correction to the twist-2 LCSR [Khodjamirian, Rückl, Weinzierl, Yakovlev, 1999]: $g_{D^* D\pi} = 10.5 \pm 3.0$.
  - The complete NLO LCSR at twist-3 and the rigours LO calculation at twist-4 [Khodjamirian, Melić, YMW, Wei, 2020]: $g_{D^* D\pi} = 14.1^{+1.3}_{-1.2}$.

Aspects of the strong $H^* \to H\pi$ decays

- The strong coupling $H^* H\pi$ in QCD:

$$\langle H^*(q)\pi(p)|H(p+q)\rangle = -g_{H^*H\pi}\, p^\mu \varepsilon_\mu^{(H^*)} .$$

  - The static limit of the strong coupling in the LP approximation:

$$g_{H^*H\pi} = \frac{2m_H}{f_\pi}\, g_\pi + \mathcal{O}(\Lambda/m_Q) .$$

  - Nontrivial relation between $g_{H^*H\pi}$ and $f^+_{H\pi}(q^2)$:

$$f^+_{H\pi}(q^2) \quad = \quad \frac{g_{H^*H\pi} f_{H^*}}{2m_{H^*}(1-q^2/m^2_{H^*})} + \frac{1}{\pi}\int\limits_{(m_H+m_\pi)^2}^{\infty} dt\, \frac{\mathrm{Im} f^+_{H\pi}(t)}{t-q^2} .$$

$$\Downarrow$$

$$g_{H^*H\pi} \quad = \quad \frac{2m_{H^*}}{f_{H^*}} \lim_{q^2 \to m^2_{H^*}} \left[\left(1 - q^2/m^2_{H^*}\right) f^+_{H\pi}(q^2) \right] .$$

- The general strategy of constructing the sum rule:
  - Both interpolating currents are renormalization invariant in QCD.

$$F_\mu(q,p) = i\int d^4x e^{iqx} \langle \pi(p)|T\{\bar{q}_1(x)\gamma_\mu Q(x), (m_Q + m_{q_2})\bar{Q}(0)\, i\gamma_5 q_2(0)\}|0\rangle .$$

  - Power counting scheme for the OPE calculation:

$$m_Q^2 - q^2 \sim m_Q^2 - (p+q)^2 \sim \mathcal{O}(m_Q \tau) , \qquad \tau \gg \Lambda_{QCD} .$$

LCSR of the strong coupling $H^*H\pi$

- Hard-collinear factorization formula in the dispersion form:

$$F^{(\mathrm{OPE})}(q^2,(p+q)^2) = \int_{-\infty}^{\infty} \frac{ds_2}{s_2-(p+q)^2} \int_{-\infty}^{\infty} \frac{ds_1}{s_1-q^2}\, \rho^{(\mathrm{OPE})}(s_1,s_2)\,.$$

  - Double expansion of the correlation function in QCD calculation.
  - A nontrivial task to extract the double spectral density beyond the LO accuracy:

$$\rho^{(\mathrm{OPE})}(s_1,s_2) \equiv \frac{1}{\pi^2} \mathrm{Im}_{s_1} \mathrm{Im}_{s_2} F^{(\mathrm{OPE})}(s_1,s_2)\,.$$

- Hadronic dispersion relation:

$$\begin{aligned} F(q^2,(p+q)^2) &= \frac{m_H^2 m_{H^*} f_H f_{H^*} g_{H^* H\pi}}{(m_H^2-(p+q)^2)(m_{H^*}^2-q^2)} \\ &+ \iint_{\Sigma} ds_2 ds_1 \frac{\rho^h(s_1,s_2)}{(s_2-(p+q)^2)(s_1-q^2)} + \ldots\,. \end{aligned}$$

- Parton-hadron duality ansatz:

$$\iint_{\Sigma} ds_2\, ds_1 \frac{\rho^h(s_1,s_2)}{(s_2-(p+q)^2)(s_1-q^2)} = \iint_{\Sigma_0} ds_2\, ds_1 \frac{\rho^{(\mathrm{OPE})}(s_1,s_2)}{(s_2-(p+q)^2)(s_1-q^2)}\,.$$

Introduce the additional uncertainty due to the dependence on the duality region.

LCSR of the strong coupling $H^* H \pi$

- The resulting LCSR for the coupling $H^* H \pi$ [Khodjamirian, Melić, YMW, Wei, 2020]:

$$
\begin{aligned}
f_H f_{H^*}\, g_{H^* H \pi} &= \frac{1}{m_H^2 m_{H^*}} \exp\left(\frac{m_H^2}{M_2^2} + \frac{m_{H^*}^2}{M_1^2}\right) \\
&\times \iint^{\Sigma_0} ds_2\, ds_1 \, \exp\left(-\frac{s_2}{M_2^2} - \frac{s_1}{M_1^2}\right) \rho^{(\mathrm{OPE})}(s_1, s_2)\,.
\end{aligned}
$$

Can be further improved by including the excited heavy-light states.
But introduce an almost uncontrollable model dependence in the hadronic part.

- Theory summary of the OPE calculation:
  - LO QCD calculations of $B \to P$ form factors [Belyaev, Khodjamirian, Rückl, 1993].
  - NLO QCD calculations of $B \to P$ form factors at twist-2 accuracy [Khodjamirian, Rückl, Weinzierl, Yakovlev, 1997; Bagan, Ball, Braun, 1997].
  - NLO QCD calculations of $B \to P$ form factors at twist-3 accuracy [Ball, Zwicky, 2005; Duplancic, Khodjamirian, Mannel, Melić, Offen, 2008].
  - Improved NLO QCD calculations of $B \to P$ form factors at twist-3 accuracy [Khodjamirian, Mannel, Offen, YMW, 2011].
  - (Partial) NNLO QCD calculations of $B \to P$ form factors at twist-2 accuracy [Bharucha, 2012].
  - Twist-5 and -6 corrections of $B \to P$ form factors in the factorization limit [Rusov, 2017].

LCSR of the strong coupling $H^* H \pi$

- Structure of the QCD spectral density [Khodjamirian, Melić, YMW, Wei, 2020]:

$$\begin{aligned}
\rho^{(\mathrm{OPE})}(s_1,s_2) &= \rho^{(\mathrm{LO})}(s_1,s_2) + \frac{\alpha_s C_F}{4\pi}\rho^{(\mathrm{NLO})}(s_1,s_2)\,, \\
\rho^{(\mathrm{LO})}(s_1,s_2) &= \big[\rho^{(\mathrm{tw2,LO})} + \rho^{(\mathrm{tw3}p,\mathrm{LO})} + \rho^{(\mathrm{tw3}\sigma,\mathrm{LO})} \\
&\quad + \rho^{(\mathrm{tw3},\bar{q}Gq)} + \rho^{(\mathrm{tw4},\psi)} + \rho^{(\mathrm{tw4},\phi)} + \rho^{(\mathrm{tw4},\bar{q}Gq)}\big](s_1,s_2)\,, \\
\rho^{(\mathrm{NLO})}(s_1,s_2) &= \rho^{(\mathrm{tw2,NLO})}(s_1,s_2) + \rho^{(\mathrm{tw3}p,\mathrm{NLO})}(s_1,s_2) + \rho^{(\mathrm{tw3}\sigma,\mathrm{NLO})}(s_1,s_2)\,.
\end{aligned}$$

  - Implement the continuum subtraction for all twist-3 and twist-4 terms at LO.
  - The first derivation of the NLO twist-3 spectral densities:

$$\rho^{(\mathrm{NLO})}(s_1,s_2) = (\sharp)\frac{d^2}{dr_1^2}\,\delta(r_1-r_2) + (\sharp\sharp)\frac{d^3}{dr_1^3}\,\ln|r_1-r_2|\,.$$

    Dependent on the duality region due to the 2nd term.

- Choices of the duality region [Balitsky, Braun, Kolesnichenko, 1988]:

$$\left(\frac{s_1}{s_*}\right)^{\alpha} + \left(\frac{s_2}{s_*}\right)^{\alpha} \le 1\,, \qquad s_1,\, s_2 \ge m_Q^2\,.$$

  Adjust s_* to provide equal diagonal intervals at $\alpha = 1,\, 1/2,\, 2$.

Twist-2 pion distribution amplitude

- The key nonperturbative input $\varphi_\pi(1/2, \mu)$ for the tree-level LCSR:

$$\begin{aligned}\varphi_\pi(1/2, 1\,\mathrm{GeV}) &= 1.5 - 2.25\,a_2 + 2.8125\,a_4 - 3.28125\,a_6 + 3.69141\,a_8 + \ldots\,, \\ \varphi_\pi(1/2, 3\,\mathrm{GeV}) &\simeq 1.5 - 1.471\,a_2 + 1.515\,a_4 - 1.553\,a_6 + 1.585\,a_8 + \ldots\,.\end{aligned}$$

  - The sign-alternating contributions from the different conformal-spin waves.
  - RG evolutions suppress the higher-order terms in the Gegenbauer expansion.
  - Sizeable non-asymptotic corrections at the practical energy scale.
    $\hookrightarrow$ $\varphi_\pi(1/2, \mu)$ provides nontrivial information on the LCDA shape.

- Two different phenomenological models.
  - Model-I [RQCD Collaboration, Bali et al, 2019]:

$$\varphi_\pi(u) = \frac{\Gamma(2+2\alpha_\pi)}{[\Gamma(1+\alpha_\pi)]^2}\, u^{\alpha_\pi}\,(1-u)^{\alpha_\pi}\,, \qquad \alpha_\pi(2\,\mathrm{GeV}) = 0.585^{+0.061}_{-0.055}\,.$$

    Inspired from the light-front holographic QCD approach [Brodsky, Téramond, Dosch, 2013].

  - Model-II [Cheng, Khodjamirian, Rusov, 2020]:

$$a_2 = 0.270 \pm 0.047, \;\; a_4 = 0.179 \pm 0.060, \;\; a_6 = 0.123 \pm 0.086\,.$$

    Comparing the LCSR calculation of the pion form factor with the experimental data.

Theoretical predictions of the $H^* H\pi$ couplings

- Numerical impacts of the higher-order corrections [Khodjamirian, Melić, YMW, Wei, 2020]:

LCSR result	tw 2 LO	tw 2 NLO	tw 3 LO	tw 3 NLO	tw 4	total
$f_D f_{D^*}\, g_{D^* D\pi}$ [GeV^2]	0.188 (Model I) 0.156 (Model II)	0.049	0.333	0.115	-0.001	0.684 0.652
$f_B f_{B^*}\, g_{B^* B\pi}$ [GeV^2]	0.416 (Model I) 0.367 (Model II)	0.081	0.395	0.148	-0.004	1.037 0.988

  - The non-asymptotic correction to the LO twist-2 term approximately 20% (10%) for the charmed- (bottom-) couplings [comparing the results of Model-I and Model-II].
  - The non-asymptotic corrections to the LO twist-3 term suppressed due to the smallness of $f_{3\pi}(1\text{GeV}) = (4.5 \pm 1.5) \times 10^{-3}\,\text{GeV}^2$.
  - The non-asymptotic corrections to the NLO LCSR indeed of minor importance numerically.
  - The NLO QCD correction to the twist-3 term around 35% for both the charmed- and bottom-couplings.
  - The convergence of the light-cone OPE due to the smallness of the twist-4 terms.
  - The heavy-meson decay constants serve as the fundamental hadronic inputs.

Theoretical predictions of the $H^*H\pi$ couplings

- Systematic uncertainty from the duality shape [Khodjamirian, Melić, YMW, Wei, 2020]:

LCSR result	α	tw 2 NLO	tw 3 NLO
$f_D f_{D^*}\, g_{D^*D\pi}$ [GeV^2]	1/2	0.056	0.128
	1	0.049	0.115
	2	0.038	0.093
$f_B f_{B^*}\, g_{B^*B\pi}$ [GeV^2]	1/2	0.090	0.158
	1	0.081	0.148
	2	0.066	0.131

  - Generally $(10-20)\%$ uncertainties for both the charmed- and bottom-couplings.
  - Earlier discussion on the choices of the duality region with nonrelativistic QM sum rules [Blok, Shifman, 1993; Radyushkin, 2001].

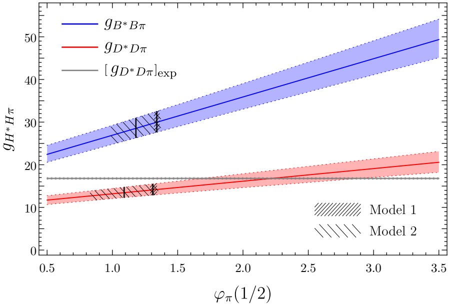
- Final numerical results of the strong couplings:

$\varphi_\pi(1/2)$	decay constants	$g_{D^*D\pi}$	$g_{B^*B\pi}$	$\hat{g}$	δ [GeV]
Model 1	2-point sum rule	$14.5^{+3.5}_{-2.4}$	$24.1^{+4.5}_{-3.8}$	$0.18^{+0.02}_{-0.03}$	$3.28^{+0.62}_{-0.17}$
	Lattice QCD	$14.1^{+1.3}_{-1.2}$	$30.0^{+2.6}_{-2.4}$	$0.30^{+0.02}_{-0.02}$	$1.17^{+0.04}_{-0.04}$
Model 2	2-point sum rule	$13.8^{+3.1}_{-2.3}$	$23.0^{+4.5}_{-3.8}$	$0.17^{+0.03}_{-0.03}$	$3.31^{+0.30}_{-0.01}$
	Lattice QCD	$13.5^{+1.4}_{-1.4}$	$28.6^{+3.0}_{-2.8}$	$0.29^{+0.03}_{-0.03}$	$1.18^{+0.00}_{-0.02}$

Parametrization of the NLP correction in terms of δ/m_H.

Theoretical predictions of the $H^*H\pi$ couplings

- Fitting the hadronic parameter $\varphi_\pi(1/2, \mu_0)$:



- The PDG-2020 avarage:

$$\Gamma(D^{*+} \to D^0\pi^+) = (56.5 \pm 1.3)\,\text{keV}\,.$$

$$\hookrightarrow [g_{D^*D\pi}]_{\rm exp} = 16.8 \pm 0.2.$$

- Our extracted value:

$$\varphi_\pi(1/2,\, 1.5\text{GeV}) \in [1.7, 2.8]\,.$$

Broad interval due to the mild dependence for the $D^*D\pi$ coupling.

- Summary of various theory predictions:

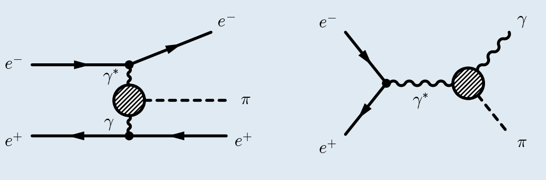
Method	$g_{D^*D\pi}$	$g_{B^*B\pi}$	$\hat{g}$
LQCD, $N_f = 2$	$15.9 \pm 0.7^{+0.2}_{-0.4}$	–	–
LQCD, $N_f = 2+1$	16.23 ± 1.71	–	–
LQCD, $N_f = 2+1$	–	$\frac{2m_B}{f_\pi}(0.56 \pm 0.03 \pm 0.07)$ $= 45.3 \pm 6.0$	–
LCSR (this work)	$14.1^{+1.3}_{-1.2}$	$30.0^{+2.6}_{-2.4}$	$0.30^{+0.02}_{-0.02}$

Part III: The pion-photon transition form factor

- Theoretically simplest hadronic matrix element:

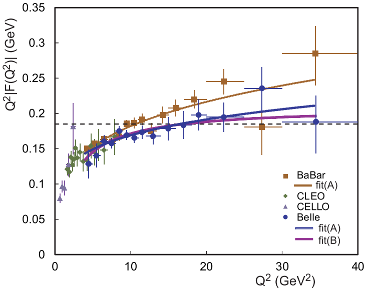
$$\langle \pi(p) | j_\mu^{\rm em} | \gamma(p') \rangle = g_{\rm em}^2 \, \varepsilon_{\mu\nu\alpha\beta} \, q^\alpha \, p^\beta \, \varepsilon^\nu(p') F_{\gamma^* \gamma \to \pi^0}(Q^2) \, .$$

  - Related to the axial anomaly at $Q^2 \equiv -(p-p')^2 = 0$.
  - Golden channel to understand the QCD dynamics.
  - $e^+ e^-$ collisions:



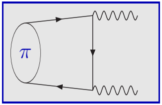
The BaBar puzzle

- Status of experimental measurements:



- Scaling violation?
- Shape of pion wave function?
- The onset of QCD factorization?
- Parton-hadron duality violation?

- Asymptotic limit:



$\longrightarrow$

$$\lim_{Q^2 \to \infty} Q^2 F_{\pi\gamma\gamma^*}(Q^2) = 2 f_\pi$$

Brodsky, Lepage

QCD factorization works perfectly.

Some popular explanations:

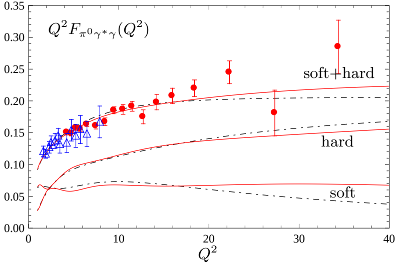
- Non-vanishing pion wave function at the end points [Radyushkin, 2009; Polyakov, 2009].

$$F(Q^2) \quad = \quad \frac{\sqrt{2}}{3} \int_0^1 \frac{\varphi_\pi(x)}{xQ^2} \left[1 - \underbrace{\exp\left(-\frac{xQ^2}{2\bar{x}\sigma}\right)} \right] .$$

$\Uparrow$

from $\mathrm{k_T}$ dependence of pion wave function

- Large soft corrections at moderate Q^2 [Agaev, Braun, Offen, Porkert, 2011].



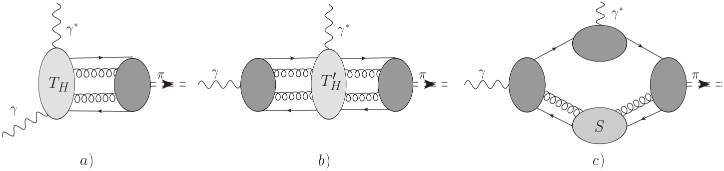
The "hard" and "soft" contributions to the $\pi^0\gamma^*\gamma$ form factor for Model-I (solid curves) and Model-III (dash-dotted curves).

The experimental data are from BaBar (full circles) and CLEO (open triangles).

- Threshold resummation generates power-like $[x(1-x)]^{c(Q^2)}$ distribution [Li and Mishima 2009]. $c(Q^2)$ is around 1 for low Q^2, but small for high Q^2.

The general picture

- Schematic structure of the distinct mechanisms [Agaev, Braun, Offen, Porkert, 2011]:



A: hard subgraph that includes both photon vertices	$\frac{1}{Q^2} + \frac{1}{Q^4} + \ldots$
B: real photon emission at large distances	$\frac{1}{Q^4} + \ldots$
C: Feynman mechanism: soft quark spectator	$\frac{1}{Q^4} + \ldots$

- Operator definitions of different terms needed for an unambiguous classification.

Region A: Leading Twist Contribution

- QCD factorization formula:

$$F^{\rm LP}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}\,(Q_u^2 - Q_d^2) f_\pi}{Q^2} \int_0^1 dx\, T^{\Delta}(x, Q^2, \mu)\, \phi_\pi^{\Delta}(x,\mu)\,.$$

  - Renormalization-scheme dependence due to the IR subtraction.
    $\Rightarrow$ Verify scheme independence of $F_{\gamma^*\gamma\to\pi^0}$ at NLO explicitly.
  - NLO hard function in the NDR scheme [Braaten 1983 + many others]:

$$T^{(1)} = \frac{\alpha_s C_F}{4\pi} \left\{ \frac{1}{\bar{x}} \left[-(2\ln\bar{x} + 3)\ln\frac{\mu^2}{Q^2} + \ln^2\bar{x} + (-1)\,\frac{\bar{x}\ln\bar{x}}{x} - 9 \right] + (x \leftrightarrow \bar{x}) \right\}.$$

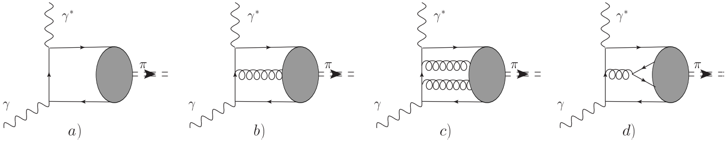
  - Twist-2 pion DA (collinear matrix element):

$$\langle \pi(p) | \bar{\xi}(y)\, W_c(y,0)\, \gamma_\mu\, \gamma_5\, \xi(0) | 0 \rangle = -i f_\pi\, p_\mu \int_0^1 du\, e^{i u p\cdot y}\, \phi_\pi(u,\mu) + \mathcal{O}(y^2)\,.$$

    The ERBL evolution implies Gegenbauer expansion.

Region A: Twist-4 Contribution

- Subleading power correction at $\mathcal{O}(1/Q^4)$:



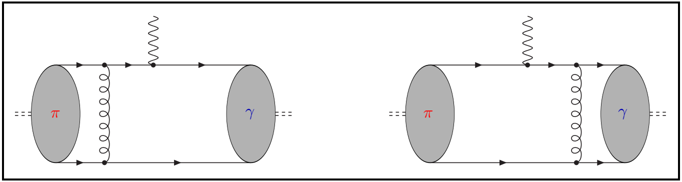
- QCD factorization at tree level [Khodjamirian, 1999]:

$$F^{\mathrm{LP}}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}\,(Q_u^2 - Q_d^2) f_\pi}{Q^2}\left[\int_0^1 dx\,\frac{\phi_\pi(x)}{x} - \frac{80}{9}\,\frac{\delta_\pi^2}{Q^2}\right], \qquad \delta_\pi^2 \simeq 0.2\,\mathrm{GeV}^2\,.$$

  - Asymptotic twist-4 contribution at present.
  - Two-particle and three-particle twist-4 corrections related by the EOM.
  - Four-particle twist-4 correction not included and assumed to be tiny.
  - Asymptotic NLO twist-4 at NLO for future.

Region B: Photon emission at large distances

- Hadronic photon correction:



- QCD factorization at tree level:

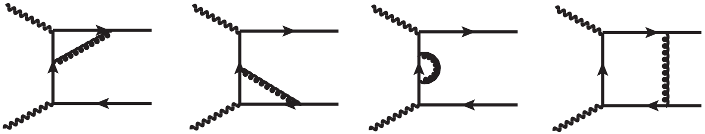
$$F^{\rm LP}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}\,(Q_u^2-Q_d^2)f_\pi}{Q^2}\,\frac{16\,\pi\,\alpha_s\,\chi(\mu)\,\langle\bar{q}q\rangle^2}{9f_\pi^2\,Q^2}\int_0^1 dx\,\frac{\phi^p_{3;\pi}(x)}{x}\int_0^1 dy\,\frac{\phi_\gamma(y)}{\bar{y}^2}\,.$$

Breakdown of QCD factorization due to rapidity singularities.

- QCD calculation of the hadronic photon effect beyond the VMD approximation.
  - ⇒ The NLL LCSRs for the long-distance photon effect [this talk!].
  - ⇒ QCD factorization for the correlation function with a pseudoscalar current.
  - ⇒ γ_5 prescription in dimensional regularization.

Hard-collinear factorization at twist-2

- Feynman diagrams at NLO in QCD:



- The NLO factorization formula [YMW, Shen, 2017]:

$$F^{\rm LP}_{\gamma^* \gamma \to \pi^0}(Q^2) = \frac{\sqrt{2}\,(Q_u^2 - Q_d^2) f_\pi}{Q^2} \int_0^1 dx \left[T_2^{(0)}(x) + T_2^{(1),\Delta}(x,\mu) \right] \phi_\pi^{\Delta}(x,\mu) + \mathcal{O}(\alpha_s^2)\,.$$

- The NLO hard matching coefficient:

$$T_2^{(1)}(x',\mu) = \frac{\alpha_s C_F}{4\pi} \left\{ \frac{1}{\bar{x}'} \left[-\left(2 \ln \bar{x}' + 3\right) \ln \frac{\mu^2}{Q^2} + \ln^2 \bar{x}' + \delta\, \frac{\bar{x}' \ln \bar{x}'}{x'} - 9 \right] + \left(x' \leftrightarrow \bar{x}'\right) \right\}.$$

$\delta = -1$ in NDR scheme and $\delta = +7$ in HV scheme.

- Scheme dependence of the pion DA [Melić, Nižić, Passek, 2002]:

$$\int_0^1 dx\, T_2^{(0)}(x) \left[\phi_\pi^{\rm HV}(x,\mu) - \phi_\pi^{\rm NDR}(x,\mu) \right] = \frac{\alpha_s C_F}{2\pi} (-4) \int_0^1 dy \left(\frac{\ln \bar{y}}{y} + \frac{\ln y}{\bar{y}} \right) \phi_\pi^{\rm NDR}(x,\mu) + \mathcal{O}(\alpha_s^2)\,.$$

$\Rightarrow$ Scheme independence of $F_{\gamma^* \gamma \to \pi^0}$ at NLO.

Hadronic photon correction

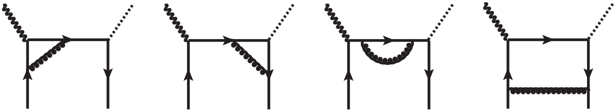
- Breakdown of the direct QCD factorization for the long-distance photon effect
  $\Rightarrow$ Construct the sum rules for the hadronic photon correction.
- Correlation function [YMW, Shen, 2017]:

$$G_\mu(p',q) = \int d^4z\, e^{-iq\cdot z} \langle 0|\mathrm{T}\left\{ j_{\mu,\perp}^{\rm em}(z), j_\pi(0)\right\}|\gamma(p')\rangle = -g_{\rm em}^2\, \varepsilon_{\mu\nu\alpha\beta}\, q^\alpha\, p'^\beta\, \varepsilon_\nu(p')\, G(p^2,Q^2)\,.$$

$$j_{\mu,\perp}^{\rm em} = \sum_q g_{\rm em}\, Q_q\, \bar{q}\,\gamma_{\mu\perp}\, q\,, \qquad j_\pi = \frac{1}{\sqrt{2}}\left(\bar{u}\,\gamma_5\, u - \bar{d}\,\gamma_5\, d\right)\,, \qquad p^2 = (p'+q)^2\,.$$

Explicit dependence on $\gamma_5 \Rightarrow$ scheme dependence of the QCD amplitude.

- The OPE calculation at NLO in QCD:



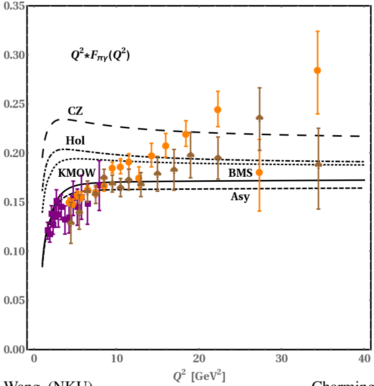
- The NLL LCSRs for the hadronic photon effect:

$$F_{\gamma^*\gamma\to\pi^0}^{\rm NLP}(Q^2) = -\frac{\sqrt{2}\,(Q_u^2 - Q_d^2)}{f_\pi\,\mu_\pi(\mu)\,Q^2}\,\chi(\mu)\,\langle\bar{q}q\rangle(\mu)\int_0^{s_0} ds \exp\left[-\frac{s-m_\pi^2}{M^2}\right] \times\left[\rho^{(0)}(s,Q^2) + \frac{\alpha_s\,C_F}{4\pi}\,\rho^{(1)}(s,Q^2)\right] + \mathcal{O}(\alpha_s^2)\,.$$

Model dependence of the pion-photon form factor

- Models of the pion DA [YMW, Shen, 2017]:

$$\begin{aligned}
a_2(1.0\,\mathrm{GeV}) &= 0.21^{+0.07}_{-0.06}, \qquad a_4(1.0\,\mathrm{GeV}) = -\left(0.15^{+0.10}_{-0.09}\right), \qquad [\mathrm{BMS}, 2004];\\
a_2(1.0\,\mathrm{GeV}) &= 0.17 \pm 0.08, \qquad a_4(1.0\,\mathrm{GeV}) = 0.06 \pm 0.10, \qquad [\mathrm{KMOW}, 2011];\\
a_n(1.0\,\mathrm{GeV}) &= \frac{2n+3}{3\pi}\left(\frac{\Gamma[(n+1)/2]}{\Gamma[(n+4)/2]}\right)^2, \qquad [\mathrm{Holographic}],\\
a_2(1.0\,\mathrm{GeV}) &= 0.5, \qquad a_{n>2}(1.0\,\mathrm{GeV}) = 0, \qquad [\mathrm{CZ}].
\end{aligned}$$



Purple squares from CLEO.
Orange circles from BaBar.
Brown bins from Belle.

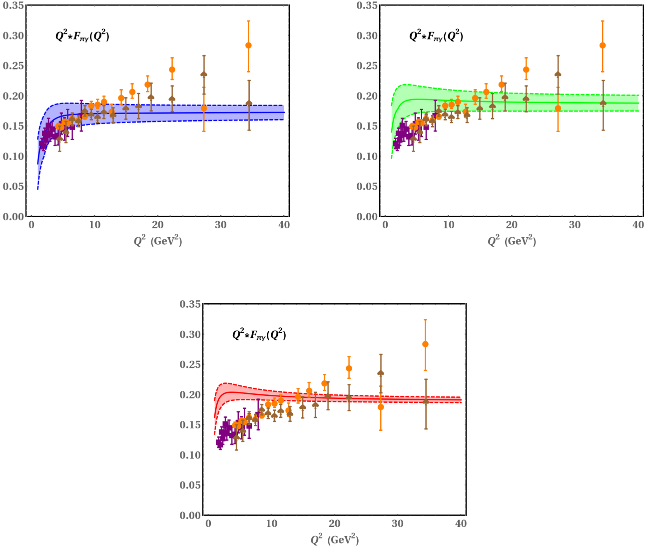
KMOW and Holographic models appear to describe the data at $Q^2 \geq 10\,\mathrm{GeV}^2$.

"Soft" physics more important at low Q^2 .

The nonperturbative modification of the spectral function works better.

Model dependence of the pion-photon form factor

Theory predictions with BMS, KMOW and Holographic models [YMW, Shen, 2017]:



NNLO prediction of the pion-photon form factor

- Models of the pion DA [Gao, Huber, Ji, YMW, 2022]:

$$\text{Model I}: \quad \phi_\pi(x,\mu_0) = \frac{\Gamma(2+2\,\alpha_\pi)}{\Gamma^2(1+\alpha_\pi)}\,(x\bar{x})^{\alpha_\pi}\,, \qquad \alpha_\pi(\mu_0) = 0.422^{+0.076}_{-0.067}\,,$$

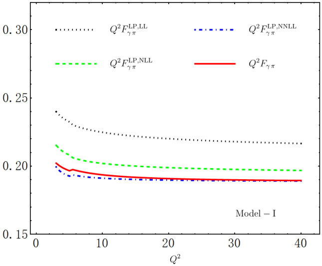
$$[\text{Holographic} \oplus \text{Lattice QCD 2020}]\,;$$

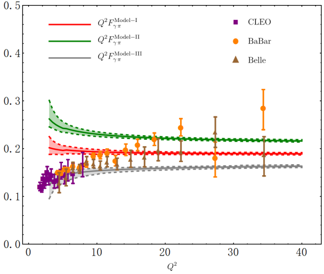
$$\text{Model II}: \quad \{a_2, a_4\}(\mu_0) = \{0.269(47), 0.185(62)\}\,,$$

$$\{a_6, a_8\}(\mu_0) = \{0.141(96), 0.049(116)\}\,, \qquad [\text{CKR}, 2020]\,;$$

$$\text{Model III}: \quad \{a_2, a_4\}(\mu_0) = \{0.203^{+0.069}_{-0.057}\,, -0.143^{+0.094}_{-0.087}\}\,, \qquad [\text{BMS}, 2020]\,.$$

- NNLO QCD predictions [Gao, Huber, Ji, YMW, 2022]:





Very sizeable $\mathcal{O}(\alpha_s^2)$ correction and the golden process to distinguish the various models!

Theoretical wishlist

- Charming physics apparently fascinating but also highly challenging.
  - QCD calculations realistic for certain charmed-hadron decay observables.
  - Formally LP contributions can be strongly suppressed in flavour physics.
  - Hadronic photon contributions suppressed by $\Lambda_{\rm QCD}/\omega_0$ instead of $\Lambda_{\rm QCD}/m_Q$.
  - Convergence of the light-cone OPE in the absence of the hard-collinear scale.

- Further applications of QCD techniques in charming physics:
  - $\rm N^3LO$ QCD corrections to the decay constants f_D and f_D^*.
  - NNLO QCD corrections to the semileptonic $D \to (\pi, K)\ell\nu$ decay form factors.
  - NLO QCD corrections to the radiative charmed-hadron (charmonium) decays at LP.
  - Hadronic decays of charmed hadrons cannot be rigorously treated in QCD.

- The pion-photon form factor as the cornerstone of establishing factorization theorems.
  - NLO corrections to the 2-particle and 3-particle twist-four contributions.
  - Systematic improvement of the dispersion approach to access the soft physics.
  - Rigorous framework of evaluating the NLP contributions in QCD.

- Very promising future for QCD aspects of charming and BESIII physics!