An $\widehat{\mathfrak{su}}(8)_1$ theory of the SM quarks and leptons

an endeavor to the SM flavor puzzle

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Historical reviews

GUTs were proposed in terms of the simple Lie algebras of $\mathfrak{su}(5)$ with $3 \times \left[\overline{\mathbf{5_F}} \oplus \mathbf{10_F}\right]$ by ['74, Georgi-Glashow], and $\mathfrak{so}(10)$ with $3 \times \mathbf{16_F}$ by ['75, Fritzsch-Minkowski]. The main ingredients are

i The $\mathfrak{g}_{\rm SM} \equiv \mathfrak{su}(3)_c \oplus \mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_Y \subset \mathfrak{g}_{\rm GUT}$ can be unified based on the AF of the QCD ['73, Gross, Wilczek, Politzer]. The SUSY extension to the $\mathfrak{su}(5)$ can unify three SM gauge couplings at $\mu \sim 10^{16}\,{\rm GeV}$ ['81, Dimopoulos-Georgi], with sparticle masses $\sim \mathcal{O}(1)$ TeV.

ii The two-generational chiral fermions (with charm quark theorized in '70 by Glashow-Iliopoulos-Maiani, and discovered in late '74),

$$\overline{\mathbf{5_F}} = \underbrace{(\overline{\mathbf{3}}\,, \mathbf{1}\,, +\frac{1}{3})_{\mathbf{F}}}_{d_R{}^c} \oplus \underbrace{(\mathbf{1}\,, \overline{\mathbf{2}}\,, -\frac{1}{2})_{\mathbf{F}}}_{\ell_L} \text{ and}$$

$$\mathbf{10_F} = \underbrace{(\mathbf{3}\,, \mathbf{2}\,, +\frac{1}{6})_{\mathbf{F}}}_{q_L} \oplus \underbrace{(\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}}}_{u_R{}^c} \oplus \underbrace{(\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}}}_{e_R{}^c}.$$

Historical reviews

- The formulation of the QM solved several fundamental puzzles in the late 19^{th} century: (i) blackbody radiation [1900, Max Planck], (ii) photoelectric effects [1905, Albert Einstein], (iii) hydrogen spectrum [1913, Niels Bohr], by hypothesizing the quantized energies/angular momenta of particles.
- Besides of the well-acknowledged challenges within the *minimal* GUTs, there are two longstanding problems within the SM that have never been solved with/without the susy extension, which are: (i) the SM flavor puzzle (quarks+leptons+neutrinos), (ii) the PQ quality problem of the QCD axion.
- My understanding: the flavor sector is the "rough water" sector of the SM where every BSM builder must confront with in your preferred framework, according to S. Weinberg's lesson two.

The flavor puzzle: origin

- The SM flavor puzzle: (i) inter/intra-generational non-universal mass hierarchies, and (ii) the CKM mixing pattern of the quarks and the PMNS mixing pattern of the neutrinos

$$V_{\rm CKM} \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} . \quad (1)$$

- Both the SM and the *minimal* GUTs exhibit the simple repetitions in terms of their chiral irreducible anomaly-free fermion sets (IRAFFSs).

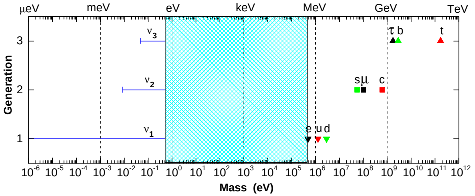


Figure: The SM fermion mass spectrum, [1909.09610, Z.Z. Xing].

The flavor puzzle: Yukawa couplings

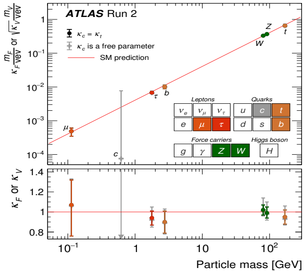


Figure: The LHC measurements of the SM Higgs boson, [2207.00092].

- The hierarchical/non-universal Yukawa couplings of the *single* SM Higgs boson $y_f = \sqrt{2} m_f / v_{\rm EW}$ for all SM quarks/leptons.
- Symmetry dictates interactions [‘80, Chen-Ning Yang]: flavor non-universality in the BSM.

The origin of generations

- The main conjecture: the flavor non-universality in the UV theories can be naturally realized w.o. the simply repetitive structure, à la the generations, such as in the $\mathfrak{su}(N > 5)$ GUTs [‘79, Georgi, ‘80, Nanopolous].
- The anti-symmetric chiral fermions of

$$\{f_L\}_{\mathrm{SU}(N)} = \sum_k n_k \, [N\,, k]_{\mathbf{F}} \;,\; n_k \in \mathbb{Z}\,. \tag{2}$$

No exotic fermions in the spectrum with the $[N\,, k]_{\mathbf{F}}$.

- The anomaly-free condition can be expressed in terms of a Diophantine equation

$$\sum_k n_k \, \mathrm{Anom}([N\,, k]_{\mathbf{F}}) = 0\,, \tag{3}$$

$$\mathrm{Anom}([N\,, k]_{\mathbf{F}}) = \frac{(N-2k)\,(N-3)!}{(N-k-1)!\,(k-1)!}\,. \tag{4}$$

Georgi's counting of the SM generations

- To decompose the $\mathfrak{su}(N)$ irreps under the $\mathfrak{su}(5)$, e.g., $\mathbf{N_F} = (N-5) \times \mathbf{1_F} \oplus \mathbf{5_F}$. Decompositions of other irreps can be obtained by tensor products, ['79, Georgi].
- All fermion irreps in Eq. (2) can be decomposed into the $\mathfrak{su}(5)$ irreps of $(\mathbf{1_F}\,, \mathbf{5_F}\,, \mathbf{10_F}\,, \overline{\mathbf{10_F}}\,, \overline{\mathbf{5_F}})$, and we denote their multiplicities as $(\nu_{\mathbf{1_F}}\,, \nu_{\mathbf{5_F}}\,, \nu_{\mathbf{10_F}}\,, \nu_{\overline{\mathbf{10_F}}}\,, \nu_{\overline{\mathbf{5_F}}})$.
- Their multiplicities should satisfy $\nu_{\mathbf{5_F}} + \nu_{\mathbf{10_F}} = \nu_{\overline{\mathbf{5_F}}} + \nu_{\overline{\mathbf{10_F}}}$ from the anomaly-free condition.
- The total SM fermion generations are determined by the net $\overline{\mathbf{5_F}}$'s or net $\mathbf{10_F}$'s

$$n_g = \nu_{\overline{\mathbf{5_F}}} - \nu_{\mathbf{5_F}} = \nu_{\mathbf{10_F}} - \nu_{\overline{\mathbf{10_F}}}\,. \tag{5}$$

- The SM $\overline{\mathbf{5_F}}$'s are from the $\overline{[N\,,1]}_{\mathbf{F}}$, and the SM $\mathbf{10_F}$'s are from the $[N\,,k\geq 2]_{\mathbf{F}}$.

Georgi's counting of the SM generations in GUTs

- Georgi's "third law" of GUT ['79]: no repetition of a particular irrep of $\mathfrak{su}(N)$, i.e., $n_k = 0$ or $n_k = 1$ in Eq. (2). Georgi's minimal solution is

$$\{f_L\}_{\mathrm{SU}(11)} = [11\,,4]_{\mathbf{F}} \oplus [11\,,8]_{\mathbf{F}} \oplus [11\,,9]_{\mathbf{F}} \oplus [11\,,10]_{\mathbf{F}}\ , \tag{6}$$

  with $\dim_{\mathbf{F}} = 561$.
- The $\left[\overline{\mathbf{5}_{\mathbf{F}}} \oplus \mathbf{10}_{\mathbf{F}}\right]$ is one chiral irreducible anomaly-free fermion set (IRAFFS) of the $\mathfrak{su}(5)$, so do one-generational SM fermions.
- A chiral IRAFFS is a set of left-handed anti-symmetric fermions of $\sum_{\mathcal{R}} m_{\mathcal{R}}\, \mathcal{F}_L(\mathcal{R})$, with the anomaly-free condition of $\sum_{\mathcal{R}} m_{\mathcal{R}}\, \mathrm{Anom}(\mathcal{F}_L(\mathcal{R})) = 0$. We also demand that
  1. A chiral IRAFFS can no longer be partitioned into smaller chiral IRAFFS.
  2. No singlet, self-conjugate, or adjoint fermions in a chiral IRAFFS.
- My "third law" [2307.07921]: *only distinctive chiral IRAFFSs without simple repetitions and can lead to $n_g = 3$ at the EW scale are allowed in an $\mathfrak{su}(N)$ theory.*

The $\mathfrak{su}(8)$ theory

- The $\mathfrak{su}(8)$ theory with rank-2 and rank-3 chiral IRAFFSs of

$$
\{f_L\}^{n_g=3}_{\mathfrak{su}(8)} = \left[\overline{\mathbf{8_F}}^{\omega} \oplus \mathbf{28_F} \right] \bigoplus \left[\overline{\mathbf{8_F}}^{\dot\omega} \oplus \mathbf{56_F} \right] , \ \dim_{\mathbf{F}} = 156 ,
$$
$$
\Omega = (\omega , \dot\omega) , \ \omega = (3 , \mathrm{IV} , \mathrm{V} , \mathrm{VI}) , \ \dot\omega = (\dot{1} , \dot{2} , \dot{\mathrm{VII}} , \dot{\mathrm{VIII}} , \dot{\mathrm{IX}}) . \qquad (7)
$$

- There are no simply repetitive "generations" in the UV, and the generations only emerge when the theory flows to the IR.
- Georgi's decompositions:

$$
\begin{aligned}
\overline{\mathbf{8_F}}^{\Omega} &= 3 \times \mathbf{1_F}^{\Omega} \oplus \overline{\mathbf{5_F}}^{\Omega} , \\
\mathbf{28_F} &= 3 \times \mathbf{1_F} \oplus 3 \times \mathbf{5_F} \oplus \mathbf{10_F} , \\
\mathbf{56_F} &= \mathbf{1_F} \oplus 3 \times \mathbf{5_F} \oplus 3 \times \mathbf{10_F} \oplus \overline{\mathbf{10_F}} . \qquad (8)
\end{aligned}
$$

Six $(\mathbf{5_F} , \overline{\mathbf{5_F}})$ massive VL pairs, one $(\mathbf{10_F} , \overline{\mathbf{10_F}})$ massive VL pair from the $\mathbf{56_F}$, and $3 \times \left[\overline{\mathbf{5_F}} \oplus \mathbf{10_F} \right]_{\mathrm{SM}}$.

The $\mathfrak{su}(8)$ theory

- The global symmetries of the $\mathfrak{su}(8)$ theory:

$$\widetilde{\mathcal{G}}_{\rm global}\,[\mathfrak{su}(8)] = \Big[\widetilde{\rm SU}(4)_\omega \otimes \widetilde{\rm U}(1)_{T_2} \otimes \widetilde{\rm U}(1)_{\rm PQ_2}\Big]$$
$$\bigotimes \Big[\widetilde{\rm SU}(5)_{\dot\omega} \otimes \widetilde{\rm U}(1)_{T_3} \otimes \widetilde{\rm U}(1)_{\rm PQ_3}\Big]\,,$$
$$[\mathfrak{su}(8)]^2 \cdot \widetilde{\rm U}(1)_{T_{2\,,3}} = 0\,, \quad [\mathfrak{su}(8)]^2 \cdot \widetilde{\rm U}(1)_{\rm PQ_{2\,,3}} \neq 0\,. \qquad (9)$$

- The Higgs fields and the Yukawa couplings:

$$-\mathcal{L}_Y = Y_{\mathcal{B}} \overline{\mathbf{8_F}}^\omega \mathbf{28_F} \overline{\mathbf{8_H}}_{\,,\omega} + Y_{\mathcal{T}} \mathbf{28_F} \mathbf{28_F} \mathbf{70_H}$$
$$+ Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\dot\omega} \mathbf{56_F} \overline{\mathbf{28_H}}_{\,,\dot\omega} + \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \overline{\mathbf{28_H}}^\dagger_{\,,\dot\omega} \mathbf{63_H} + H.c.\,. \qquad (10)$$

NB: $\mathbf{56_F}\mathbf{56_F}\mathbf{28_H} = 0$ ['08, S. Barr], $d = 5$ operator suppressed by $1/M_{\rm pl}$ is possible, with $M_{\rm pl} = (8\pi G_N)^{1/2} = 2.4 \times 10^{18}\,{\rm GeV}$.

- *Gravity breaks global symmetries.*

Global symmetries in the $\mathfrak{su}(8)$ theory

Fermions	$\overline{\mathbf{8_F}}^{\Omega=\omega,\dot{\omega}}$	$\mathbf{28_F}$	$\mathbf{56_F}$	
$\widetilde{\mathrm{U}}(1)_T$	$-3t$	$+2t$	$+t$	
$\widetilde{\mathrm{U}}(1)_{\mathrm{PQ}}$	p	q_2	q_3	
Higgs	$\overline{\mathbf{8_H}}_{,\omega}$	$\overline{\mathbf{28_H}}_{,\dot{\omega}}$	$\mathbf{70_H}$	$\mathbf{63_H}$
$\widetilde{\mathrm{U}}(1)_T$	$+t$	$+2t$	$-4t$	0
$\widetilde{\mathrm{U}}(1)_{\mathrm{PQ}}$	$-(p+q_2)$	$-(p+q_3)$	$-2q_2$	0

Table: The $\widetilde{\mathrm{U}}(1)_T$ and the $\widetilde{\mathrm{U}}(1)_{\mathrm{PQ}}$ charges, $p : q_2 \neq -3 : +2$ and $p : q_3 \neq -3 : +1$.

- The symmetry breaking pattern [‘74, L.F.Li] of $\mathfrak{su}(8) \xrightarrow{\mathbf{63_H}} \mathfrak{g}_{441} \to \mathfrak{g}_{341} \to \mathfrak{g}_{331} \to \mathfrak{g}_{\mathrm{SM}} \to \mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_{\mathrm{EM}}$. Other patterns are also possible in the $\mathcal{N}=1$ SUSY extension [‘81, Witten].

Global symmetries in the $\mathfrak{su}(8)$ theory

- The global $\widetilde{\mathrm{U}}(1)_T$ symmetries at different stages

$$\mathfrak{su}(8) \to \mathfrak{g}_{441_{X_0}} : \mathcal{T}' = \mathcal{T} - 4t\mathcal{X}_0 \,, \quad \mathfrak{g}_{441_{X_0}} \to \mathfrak{g}_{341_{X_1}} : \mathcal{T}'' = \mathcal{T}' + 8t\mathcal{X}_1 \,,$$
$$\mathfrak{g}_{341_{X_1}} \to \mathfrak{g}_{331_{X_2}} : \mathcal{T}''' = \mathcal{T}'' , \quad \mathfrak{g}_{331_{X_2}} \to \mathfrak{g}_{\mathrm{SM}} : \mathcal{B} - \mathcal{L} = \mathcal{T}''' \,. \tag{11}$$

Consistent relations of $(\mathcal{B}-\mathcal{L})(q_L) = \frac{4}{3}t$, $(\mathcal{B}-\mathcal{L})(\ell_L) = -4t$, and etc.

Higgs	$\mathfrak{g}_{441} \to \mathfrak{g}_{341}$	$\mathfrak{g}_{341} \to \mathfrak{g}_{331}$	$\mathfrak{g}_{331} \to \mathfrak{g}_{\mathrm{SM}}$	$\mathfrak{g}_{\mathrm{SM}} \to \mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_{\mathrm{EM}}$
$\overline{\mathbf{8_H}}_{,\omega}$	✓	✓	✓	✓
$\overline{\mathbf{28_H}}_{,\dot\omega}$	✗	✓	✓	✓
$\mathbf{70_H}$	✗	✗	✗	✓

Table: The ✓and ✗represent possible and impossible symmetry breaking directions for a given Higgs field in the $\mathfrak{su}(8)$ theory.

The $\mathfrak{su}(8)$ Higgs fields

- Decompositions of $\overline{\mathbf{8_H}}_{,\omega}/\overline{\mathbf{28_H}}_{,\dot\omega}$

$$\begin{aligned}
\overline{\mathbf{8_H}}_{,\omega} &\supset \underline{(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H},\omega}} \oplus \underline{(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H},\omega}} \supset \underline{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega}}, \\
\overline{\mathbf{28_H}}_{,\dot\omega} &\supset \underline{(\mathbf{1}, \mathbf{6}, -\frac{1}{2})_{\mathbf{H},\dot\omega}} \oplus \underline{(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H},\dot\omega}} \\
&\supset \left[\underline{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})'_{\mathbf{H},\dot\omega}} \oplus (\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H},\dot\omega}\right] \oplus \underline{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\dot\omega}}\,.
\end{aligned} \tag{12}$$

- The global $\widetilde{\mathrm{U}}(1)_{B-L}$ according to Eq. (11)

$$\begin{aligned}
\mathbf{70_H} &\supset \underline{(\mathbf{4}, \overline{\mathbf{4}}, +\frac{1}{2})_{\mathbf{H}}} \oplus (\overline{\mathbf{4}}, \mathbf{4}, -\frac{1}{2})_{\mathbf{H}} \supset \ldots \\
&\supset \underbrace{(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{H}}}_{\mathcal{B}-\mathcal{L}=0} \oplus \underbrace{(\mathbf{1}, \mathbf{2}, -\frac{1}{2})'''_{\mathbf{H}}}_{\mathcal{B}-\mathcal{L}=-8t}\,.
\end{aligned} \tag{13}$$

Conjecture: the $(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{H}} \subset \mathbf{70_H}$ is the unique SM Higgs doublet.

The $\mathfrak{su}(8)$ Higgs fields

- The VEV assignments

$$\mathfrak{g}_{441} \to \mathfrak{g}_{341} \;:\; \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\mathrm{IV}} \rangle \equiv \frac{1}{\sqrt{2}} W_{\overline{\mathbf{4}}\,,\mathrm{IV}}\,,\;\; \zeta_1 \equiv \frac{W_{\overline{\mathbf{4}}}}{M_{\rm pl}} \tag{14a}$$

$$\mathfrak{g}_{341} \to \mathfrak{g}_{331} \;:\; \langle (\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\mathrm{V}\,,\dot{1}\,,\dot{\mathrm{V}}\mathrm{II}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\overline{\mathbf{4}}\,,\mathrm{V}\,,\dot{1}\,,\dot{\mathrm{V}}\mathrm{II}}\,,\;\; \zeta_2 \equiv \frac{w_{\overline{\mathbf{4}}}}{M_{\rm pl}} \tag{14b}$$

$$\mathfrak{g}_{331} \to \mathfrak{g}_{\rm SM} \;:\; \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,3\,,\mathrm{VI}\,,\mathrm{I}\dot{\mathrm{X}}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\overline{\mathbf{3}}\,,3\,,\mathrm{VI}\,,\mathrm{I}\dot{\mathrm{X}}}$$

$$\langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^{\prime}_{\mathbf{H}\,,\dot{2}\,,\dot{\mathrm{V}}\mathrm{III}} \rangle \equiv \frac{1}{\sqrt{2}} V^{\prime}_{\overline{\mathbf{3}}\,,\dot{2}\,,\dot{\mathrm{V}}\mathrm{III}}\,,\;\; \zeta_3 \equiv \frac{V_{\overline{\mathbf{3}}}}{M_{\rm pl}} \tag{14c}$$

$$\mathrm{EWSB} \;:\; \langle (\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})^{\prime\prime\prime}_{\mathbf{H}} \rangle \equiv \frac{1}{\sqrt{2}} v_{\rm EW}\,. \tag{14d}$$

The VEVs in black are the minimal set to integrate out the massive vectorlike fermions. The VEVs in red are necessary for the $(d^i\,,\ell^i)$ masses.

Top quark mass in the $\mathfrak{su}(8)$ theory

- The natural top quark mass from the tree level

$$
\begin{aligned}
& Y_{\mathcal{T}} \mathbf{28_F} \mathbf{28_F} \mathbf{70_H} \supset Y_{\mathcal{T}} (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{4}\,, \mathbf{4}\,, 0)_{\mathbf{F}} \otimes (\mathbf{4}\,, \overline{\mathbf{4}}\,, +\frac{1}{2})_{\mathbf{H}} \\
\supset\ & ... \supset Y_{\mathcal{T}} (\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}\,, \mathbf{2}\,, +\frac{1}{6})_{\mathbf{F}} \otimes (\mathbf{1}\,, \overline{\mathbf{2}}\,, +\frac{1}{2})_{\mathbf{H}}^{\prime\prime\prime} \\
\Rightarrow\ & \frac{1}{\sqrt{2}} Y_{\mathcal{T}} t_L {t_R}^c v_{\rm EW} \,. \qquad (15)
\end{aligned}
$$

- With $(\mathbf{3}\,, \mathbf{2}\,, +\frac{1}{6})_{\mathbf{F}} \equiv (t_L\,, b_L)^T$ and $(\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}} \equiv {t_R}^c$ within the $\mathbf{28_F}$, it is straightforward to infer that $(\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}} \equiv {\tau_R}^c$ also lives in the $\mathbf{28_F}$. This explains why do the third-generational SM $\mathbf{10_F}$ reside in the $\mathbf{28_F}$, while the first- and second-generational SM $\mathbf{10_F}$'s must reside in the $\mathbf{56_F}$.
- Top quark mass conjecture: a rank-2 chiral IRAFFS is necessary so that only the top quark obtains mass with the natural Yukawa coupling at the EW scale. The $\mathrm{SU}(9)$ with $9 \times \overline{\mathbf{9_F}} \oplus \mathbf{84_F}$ is ruled out [2307.07921].

The $\mathfrak{su}(8)$ fermions

$\mathfrak{su}(8)$	$\mathfrak{g}_{441}$	$\mathfrak{g}_{341}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\rm SM}$
$\overline{\mathbf{8}_{\mathbf{F}}}^{\Omega}$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega}\ :\ \mathcal{D}_R^{\Omega\,c}$
		$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega}\ :\ \check{\mathcal{N}}_L^{\Omega}$
	$(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\Omega}\ :\ \mathcal{L}_L^{\Omega}=(\mathcal{E}_L^{\Omega}\,,-\mathcal{N}_L^{\Omega})^T$
				$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega'}\ :\ \check{\mathcal{N}}_L^{\Omega'}$
			$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''}\ :\ \check{\mathcal{N}}_L^{\Omega''}$

$\mathfrak{su}(8)$	$\mathfrak{g}_{441}$	$\mathfrak{g}_{341}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{28}_{\mathbf{F}}$	$(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}\ :\ \mathfrak{D}_L$
		$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}\ :\ {t_R}^c$
	$(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}\ :\ ({\mathfrak{e}_R}^c\,,{\mathfrak{n}_R}^c)^T$
				$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}\ :\ \check{\mathfrak{n}}_R^c$
			$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}'\ :\ ({\mathfrak{n}_R'}^c\,,-{\mathfrak{e}_R'}^c)^T$
				$(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}\ :\ {\tau_R}^c$
	$(\mathbf{4}\,,\mathbf{4}\,,0)_{\mathbf{F}}$	$(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})_{\mathbf{F}}$	$(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}$	$(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}\ :\ (t_L\,,b_L)^T$
				$(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}'\ :\ \mathfrak{D}_L'$
			$(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}''$	$(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}''\ :\ \mathfrak{D}_L''$
		$(\mathbf{1}\,,\mathbf{4}\,,+\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}''$	$(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}''\ :\ ({\mathfrak{e}_R''}^c\,,{\mathfrak{n}_R''}^c)^T$
				$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}'\ :\ \check{\mathfrak{n}}_R'^{\,c}$
			$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}''$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}''\ :\ \check{\mathfrak{n}}_R''^{\,c}$

The $\mathfrak{su}(8)$ fermions

$\mathfrak{su}(8)$	$\mathfrak{g}_{441}$	$\mathfrak{g}_{341}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{56_F}$	$(\mathbf{1},\overline{\mathbf{4}},+\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1},\overline{\mathbf{4}},+\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1},\overline{\mathbf{3}},+\frac{2}{3})'_{\mathbf{F}}$	$(\mathbf{1},\overline{\mathbf{2}},+\frac{1}{2})'''_{\mathbf{F}}\ :\ (\mathfrak{n}_R'''^c,-\mathfrak{e}_R'''^c)^T$
				$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}\ :\ {\mu_R}^c$
			$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}\ :\ {\mathfrak{E}_R}^c$
	$(\overline{\mathbf{4}},\mathbf{1},-\frac{3}{4})_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}\ :\ {u_R}^c$
		$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}\ :\ \mathfrak{E}_L$
	$(\mathbf{4},\mathbf{6},+\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{6},+\frac{1}{6})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{3},0)'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{2},+\frac{1}{6})'_{\mathbf{F}}\ :\ (c_L,s_L)^T$
				$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''_{\mathbf{F}}\ :\ \mathfrak{D}_L'''$
			$(\mathbf{3},\overline{\mathbf{3}},+\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3},\overline{\mathbf{2}},+\frac{1}{6})''_{\mathbf{F}}\ :\ (\mathfrak{d}_L,-\mathfrak{u}_L)^T$
				$(\mathbf{3},\mathbf{1},+\frac{2}{3})_{\mathbf{F}}\ :\ \mathfrak{U}_L$
		$(\mathbf{1},\mathbf{6},+\frac{1}{2})'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{3},+\frac{1}{3})'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})''''_{\mathbf{F}}\ :\ (\mathfrak{e}_R''''^c,\mathfrak{n}_R''''^c)^T$
				$(\mathbf{1},\mathbf{1},0)'''_{\mathbf{F}}\ :\ \check{\mathfrak{n}}_R'''^c$
			$(\mathbf{1},\overline{\mathbf{3}},+\frac{2}{3})''_{\mathbf{F}}$	$(\mathbf{1},\overline{\mathbf{2}},+\frac{1}{2})'''''_{\mathbf{F}}\ :\ (\mathfrak{n}_R'''''^c,-\mathfrak{e}_R'''''^c)^T$
				$(\mathbf{1},\mathbf{1},+1)'''_{\mathbf{F}}\ :\ {e_R}^c$
	$(\mathbf{6},\mathbf{4},-\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{4},-\frac{1}{12})'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{3},0)''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{2},+\frac{1}{6})'''_{\mathbf{F}}\ :\ (u_L,d_L)^T$
				$(\mathbf{3},\mathbf{1},-\frac{1}{3})''''_{\mathbf{F}}\ :\ \mathfrak{D}_L''''$
			$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''''_{\mathbf{F}}\ :\ \mathfrak{D}_L'''''$
		$(\overline{\mathbf{3}},\mathbf{4},-\frac{5}{12})_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{3},-\frac{1}{3})_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{2},-\frac{1}{6})_{\mathbf{F}}\ :\ ({\mathfrak{d}_R}^c,{\mathfrak{u}_R}^c)^T$
				$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})''_{\mathbf{F}}\ :\ {\mathfrak{U}_R}^c$
			$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})'''_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})'''_{\mathbf{F}}\ :\ {c_R}^c$

SM fermion masses in the $\mathfrak{su}(8)$ theory

- To generate lighter SM fermion masses: the gravity breaks the global symmetries in Eq. (9) explicitly.
- The direct Yukawa couplings of $\mathcal{O}_{\mathcal{F}}^{d=5}$:

$$
\begin{aligned}
& c_3 \mathcal{O}_{\mathcal{F}}^{(3\,,2)} \equiv c_3 \overline{\mathbf{8_F}}^{\dot\omega} \mathbf{56_F} \cdot \overline{\mathbf{28_H}}^\dagger_{,\dot\kappa} \cdot \mathbf{70_H}^\dagger \\
\Rightarrow\ & c_3 \left[\dot\zeta_3 (s_L \mathcal{D}_R^{\dot\omega\, c} - \mathcal{E}_L^{\dot\omega} \mu_R{}^c) + \dot\zeta_3' (d_L \mathcal{D}_R^{\dot\omega\, c} - \mathcal{E}_L^{\dot\omega} e_R{}^c) \right] v_{\rm EW} \,, \\
& c_4 \mathcal{O}_{\mathcal{F}}^{(4\,,1)} \equiv c_4 \mathbf{56_F} \mathbf{56_F} \cdot \overline{\mathbf{28_H}}_{\,,\dot\omega} \cdot \mathbf{70_H} \Rightarrow c_4 \dot\zeta_2 (c_L u_R{}^c + \cancel{u_L c_R{}^c}) v_{\rm EW} \,, \\
& c_5 \mathcal{O}_{\mathcal{F}}^{(5\,,1)} \equiv c_5 \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{\,,\omega} \cdot \mathbf{70_H} \\
\Rightarrow\ & c_5 \left[\zeta_1 (u_L t_R{}^c + t_L u_R{}^c) + \zeta_2 (c_L t_R{}^c + t_L c_R{}^c) \right] v_{\rm EW} \,. \qquad (16)
\end{aligned}
$$

- All $(u\,,c\,,t)$ obtain hierarchical masses, while all $(d^i\,,\ell^i)$ are massless.

SM fermion masses in the $\mathfrak{su}(8)$ theory

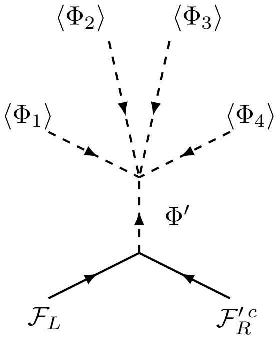


Figure: The indirect Yukawa couplings.

$(d^i\,, \ell^i)$ obtain masses through the EWSB components in renormalizable Yukawa couplings of $\overline{\mathbf{8_F}}^{\omega}\mathbf{28_F}\overline{\mathbf{8_H}}_{\,,\omega}$ and $\overline{\mathbf{8_F}}^{\dot\omega}\mathbf{56_F}\overline{\mathbf{28_H}}_{\,,\dot\omega}$ with the $d=5$ Higgs mixing operators.

SM fermion masses in the $\mathfrak{su}(8)$ theory

- There are two indirect Yukawa couplings with the irreducible Higgs mixing operators of $\mathcal{O}_{\mathcal{H}}^{d=5}$:

$$\begin{aligned} \mathcal{O}_{\mathscr{A}}^{d=5} &\equiv \epsilon_{\omega_1\omega_2\omega_3\omega_4} \overline{\mathbf{8_H}}^{\dagger}_{,\omega_1} \overline{\mathbf{8_H}}^{\dagger}_{,\omega_2} \overline{\mathbf{8_H}}^{\dagger}_{,\omega_3} \overline{\mathbf{8_H}}^{\dagger}_{,\omega_4} \mathbf{70_H}^{\dagger}\,, \\ & \mathcal{PQ} = 2(2p+3q_2) \neq 0\,, \end{aligned} \tag{17a}$$

$$\begin{aligned} \mathcal{O}_{\mathscr{B}}^{d=5} &\supset (\overline{\mathbf{28_H}}^{\dagger}_{,\dot{1}} \overline{\mathbf{28_H}}_{,\dot{\rm VII}}) \cdot \overline{\mathbf{28_H}}^{\dagger}_{,\dot{\rm IIX}} \overline{\mathbf{28_H}}^{\dagger}_{,\dot{1}} \mathbf{70_H}^{\dagger}\,, \\ & (\overline{\mathbf{28_H}}^{\dagger}_{,\dot{1}} \overline{\mathbf{28_H}}_{,\dot{\rm VII}}) \cdot \overline{\mathbf{28_H}}^{\dagger}_{,\dot{\rm IX}} \overline{\mathbf{28_H}}^{\dagger}_{,\dot{2}} \mathbf{70_H}^{\dagger}\,, \\ & \mathcal{PQ} = 2(p+q_2+q_3)\,. \end{aligned} \tag{17b}$$

- Each operator of $\mathcal{O}_{\mathcal{H}}^{d=5}$
  1. *breaks the global symmetries explicitly*;
  2. *can not be further partitioned into subset of renormalizable operators, among which any of them can be allowed by both the gauge and the global symmetries. This relies on the VEV assignments in Eqs.* (14).

SM fermion masses in the $\mathfrak{su}(8)$ theory

- The $(u\,,c\,,t)$ masses

$$m_u \approx c_4 \frac{\zeta_2 \dot{\zeta}_2}{2\zeta_1} v_{\rm EW}\,, \quad m_c \approx c_5^2 \frac{\zeta_1^2}{2\sqrt{2} Y_{\mathcal{T}}} v_{\rm EW}\,, \quad m_t \approx \frac{Y_{\mathcal{T}}}{\sqrt{2}} v_{\rm EW}\,. \qquad (18)$$

- The $(d\,,s\,,b)$ and $(e\,,\mu\,,\tau)$ masses

$$m_d = m_e \approx \frac{c_3 \dot{\zeta}_3}{2} |\tan\lambda| v_{\rm EW}\,, \quad m_s = m_\mu \approx \frac{1}{4}(2c_3 + Y_{\mathcal{D}} d_{\mathscr{B}} \zeta_{23}^{-2}) \dot{\zeta}_3 v_{\rm EW}\,,$$
$$m_b = m_\tau \approx Y_{\mathcal{B}} \frac{d_{\mathscr{A}} \zeta_1 \zeta_2}{4\zeta_3} v_{\rm EW}\,. \qquad (19)$$

- The CKM mixing:

$$\hat{V}_{\rm CKM}\Big|_{\mathfrak{su}(8)} \approx \begin{pmatrix} 1-\lambda^2/2 & \lambda & \frac{c_5}{Y_{\mathcal{T}}}\zeta_2 \\ -\lambda & 1-\lambda^2/2 & -\frac{c_5}{Y_{\mathcal{T}}}\zeta_1 \\ -\frac{c_5}{Y_{\mathcal{T}}}(\lambda\zeta_1+\zeta_2) & -\frac{c_5}{Y_{\mathcal{T}}}\zeta_1 & 1 \end{pmatrix}\,. \qquad (20)$$

SM fermion masses in the $\mathfrak{su}(8)$ theory: benchmark

ζ_1	ζ_2	ζ_3	$Y_{\mathcal{D}}$	$Y_{\mathcal{B}}$	$Y_{\mathcal{T}}$
6.0×10^{-2}	2.0×10^{-3}	2.0×10^{-5}	0.5	0.5	0.8
c_3	c_4	c_5	$d_{\mathscr{A}}$	$d_{\mathscr{B}}$	λ
1.0	0.2	1.0	0.01	0.01	0.22
m_u	m_c	m_t	$m_d = m_e$	$m_s = m_\mu$	$m_b = m_\tau$
1.6×10^{-3}	0.6	139.2	0.5×10^{-3}	6.4×10^{-2}	1.5
$\lvert V_{ud}\rvert$	$\lvert V_{us}\rvert$	$\lvert V_{ub}\rvert$			
0.98	0.22	3.0×10^{-3}			
$\lvert V_{cd}\rvert$	$\lvert V_{cs}\rvert$	$\lvert V_{cb}\rvert$			
0.22	0.98	7.5×10^{-2}			
$\lvert V_{td}\rvert$	$\lvert V_{ts}\rvert$	$\lvert V_{tb}\rvert$			
0.019	7.5×10^{-2}	1			

Table: The parameters of the $\mathfrak{su}(8)$ benchmark point and the predicted SM quark/lepton masses (in unit of GeV) as well as the CKM mixings. [2402.10471]

The $\mathfrak{su}(8)$ RGEs

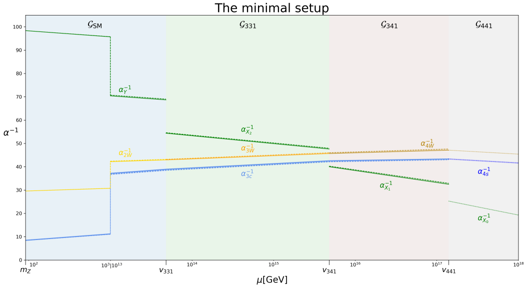
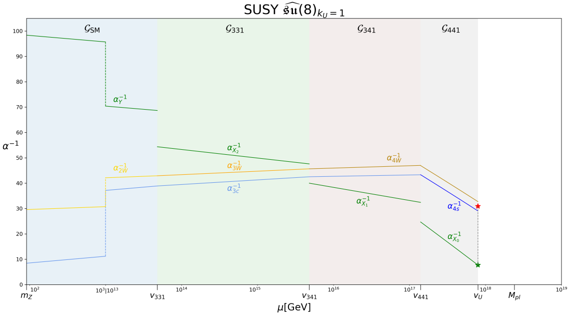


Figure: The minimal non-SUSY $\mathfrak{su}(8)$ setup cannot achieve the unification. [2406.09970]

The SUSY $\widehat{\mathfrak{su}}(8)_1$ RGEs



The SUSY $\widehat{\mathfrak{su}}(8)_1 \supset \widehat{\mathfrak{su}}(4)_{s,1} \oplus \widehat{\mathfrak{su}}(4)_{W,1} \oplus \widehat{\mathfrak{u}}(1)_{X_0,1/4}$ achieves the unification in the affine Lie algebra [2411.12979]:

$$\{H\} = \mathbf{8_H}^{\omega} \oplus \mathbf{28_H}^{\dot\omega}, \quad \alpha_{4s} = \alpha_{4W} = \frac{1}{4}\alpha_{X_0}. \tag{21}$$

Summary

- We propose an $\mathfrak{su}(8)$ theory with the non-trivial embedding to address the SM quark/lepton flavor puzzles.
- The global symmetries based on the chiral IRAFFSs are vital to: (i) determine one single SM Higgs doublet through the non-anomalous $\widetilde{\mathrm{U}}(1)_{B-L}$ symmetry, (ii) count the massless left-handed sterile neutrinos precisely through the 't Hooft anomaly matching, (iii) organize the gravity-induced $d = 5$ operators for the SM fermion mass (mixing) terms.
- All light SM quark/lepton masses (other than the top) and the CKM mixing pattern are due to YM$\oplus$gravity. Observed mass hierarchies are reproduced with the precise flavor identifications.
- To unify: non-SUSY $\mathfrak{su}(8)$ is promoted to the $\mathcal{N} = 1$ SUSY $\widehat{\mathfrak{su}}(8)_1$.