Resummation in Higgs Physics: Effective Potential and Higgs Scaling Dimension

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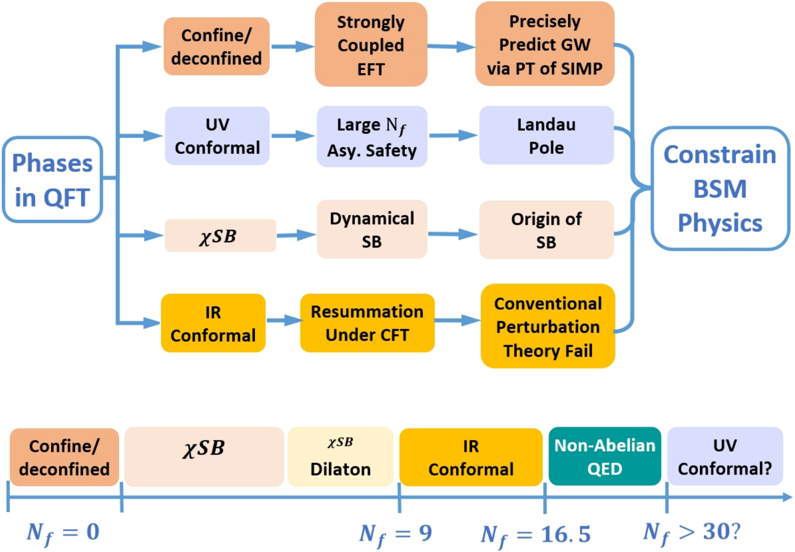
Related Research Works for this Talk

- T. G. Steele and Z. W. Wang, “Is Radiative Electroweak Symmetry Breaking Consistent with a 125 GeV Higgs Mass?,” Phys. Rev. Lett. **110** (2013) 151601 (PRL Editor Suggestion).
- O. Antipin, J. Bersini, P. Panopoulos, F. Sannino and Z. W. Wang, “Infinite order results for charged sectors of the Standard Model,” JHEP **02** (2024) 168.

A Landscape of Phases in QFT and its Relation to BSM Physics



Conventional Symmetry Breaking vs. Radiative Symmetry Breaking

Conventional:

- 1. Input a quadratic term (opposite sign to mass term) into the Higgs sector.
- 2. VEV occurs at tree level effective potential.
- 3. Disadvantage: require to input the abnormal mass term “by hand” to trigger symmetry breaking.
- 4. The mass term of the scalar field will cause the fine-tuning problem.

Radiative:

- 1. Higgs sector without mass (quadratic) term and fulfill scale invariance.
- 2. Dynamically generate a VEV and a mass term through radiative loop corrections.
- 3. Advantage: the mass term can be dynamically generated and is not input by hand.
- 4. Can dynamically generate scale hierarchies via dimensional transmutation

Model Construction: The Effective Potential in the Standard Model

- In Standard Model, the effective potential is written as:

$$V = V_0 + V_{\text{vector}} + V_{\text{scalar}} + V_{\text{fermion}}$$

where V_0 is the classical potential and $V_{\text{vector}}, V_{\text{scalar}}, V_{\text{fermion}}$ are the loop corrections.

- Originally, Coleman and Weinberg assumed a small Higgs coupling λ in the radiative scenario. So, the vector and fermion contributions cannot be easily neglected. The small Higgs coupling leads to a 10 GeV Higgs prediction.

  [Coleman, Weinberg, PRD 7 (1973) 1888.]

- The existence of the large top quark Yukawa coupling will destabilize the original small Higgs coupling solution but a large coupling solution exists. The large Higgs coupling leads to a much larger Higgs prediction. [Elias, Mann, McKeon, Steele, PRL 91 (2003) 25.]

- With large Higgs coupling, the scalar contribution becomes **dominant** and the scalar sector of the Standard Model can be simplified to $O(4)$ scalar model.

  [Chishtie, Hanif, McKeon, Steele, PRD 77 (2008) 065007.]

Model: Massless $O(N)$-Symmetric $\lambda\phi^4$ Theory

- RG improvement of the $O(N)$ model effective potential

$$V(\lambda, \phi, \mu) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \lambda^{n+1} T_{nm} L^m \phi^4, \quad L = \log\left(\phi^2/\mu^2\right)$$

- The general effective potential with loop correction is:

$$V(\lambda, \phi, \mu) = \sum_{n=0}^{\infty} \lambda^{n+1} S_n(\lambda L) \phi^4, \quad S_n(\lambda L) = \sum_{m=0}^{\infty} T_{n+m\, m} (\lambda L)^m$$

The summation includes the leading logarithm (LL), next-to-leading logarithm (NLL), etc. while S_n denotes the $N^n LL$ contribution.

- The summation of the logarithms $S_n(\lambda L)$ can be determined from the renormalization group equation:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + \gamma(\lambda) \phi \frac{\partial}{\partial \phi}\right) V(\lambda, \phi, \mu) = 0,$$

- where renormalization group functions are:

$$\beta(\lambda) = \mu \frac{d\lambda}{d\mu} = \sum_{k=2}^{\infty} b_k \lambda^k, \quad \gamma(\lambda) = \frac{\mu}{\phi} \frac{d\phi}{d\mu} = \sum_{k=1}^{\infty} g_k \lambda^k .$$

Model Construction: Solving RG Equation

- The RG Equation leads to the following coupled differential equations for the functions $S_n(\xi)$

$$0 = \left[(-2 + b_2\xi)\frac{d}{d\xi} + (n+1)b_2 + 4g_1\right] S_n + \sum_{m=0}^{n-1} \left\{ \left(2g_{n-m} + b_{n+2-m}\xi \frac{d}{d\xi}\right) + [(m+1)b_{n+2-m} + 4g_{n+1-m}] \right\} S_m,$$

- From the above recurrence relation, we thus see that the $n+1$-loop renormalization-group functions are needed to determine S_k for $k = 0, 1, 2, \cdots, n$ up to $N^n LL$ order
- To fully solve the above differential equations, it also requires the boundary condition $S_n(0) = T_{n0}$ which will be determined by the RG condition (see next page)

Model Construction: Truncating the Series

- The series is necessary to be truncated with limited information of β and γ (remember $n+1$-loop RG functions determine N^nLL order S_n).
- When truncating the series, counter-terms are needed to compensate for the lost information, and the effective potential with counter-terms is written as:

$$V_m = \sum_{n=0}^{m} \lambda^{n+1} S_n(\lambda L)\phi^4 + \sum_{i=m+1}^{\infty} T_{i,0}\lambda^{i+1}\phi^4$$

- The counter-terms are determined from the renormalization condition (Coleman-Weinberg Scheme):

$$\left.\frac{d^4V}{d\phi^4}\right|_{\phi=\mu} = 24\lambda$$

- The Higgs mass and the corresponding coupling constant are determined respectively with:

$$m^2 = \left.\frac{d^2V}{d\phi^2}\right|_{\phi=\mu} \quad \text{and} \quad \left.\frac{dV}{d\phi}\right|_{\phi=\mu} = 0$$

Model Construction: Renormalization Scheme Transformation

- Originally, $\tilde{\beta}$ and $\tilde{\gamma}$ are determined in the minimal subtraction scheme which have been calculated to five loops. (Kleinert, Neu, Schulte-Frohlinde, Chetyrkin, Larin, PLB 272 (1991) 39.)
- The five loop minimal subtraction scheme result cannot be directly used in our Coleman-Weinberg scheme.
- The Coleman-Weinberg scheme and the minimal subtraction scheme renormalization group functions can be related by: (Ford, Jones, PLB 274 (1992) 409.)

$$\beta(\lambda) = \frac{\tilde{\beta}(\lambda)}{1 - \frac{\tilde{\beta}(\lambda)}{2\lambda}}, \quad \gamma(\lambda) = \frac{\tilde{\gamma}(\lambda)}{1 - \frac{\tilde{\beta}(\lambda)}{2\lambda}}$$

Extrapolating the Theory Further

Problem: Limited β and γ information

- Higgs mass prediction decreases with increasing loop level.
- With limited β and γ information, we can only predict the Higgs mass to five loop **order** and it gives $165\,\mathrm{GeV}$.
- Will extrapolation to higher loop order reduce Higgs mass and eventually converge to $125\,\mathrm{GeV}$ LHC result?

Methods:

- 1. Pade Approximation
- 2. Averaging Method
- 3. Combination of Pade Approximation and Averaging Method

Extrapolating the Theory: Pade Approximation

- Pade approximation can be used to predict the next order renormalization group function β and γ to a very high accuracy. [Samuel, Ellis, Karliner PRL 74 (1995) 4380.]
- For Padé approximations to the MS-scheme beta function $\tilde{\beta}$ we write:

$$\tilde{\beta} = \tilde{b}_2 \lambda^2 \left(1 + \frac{\tilde{b}_3}{\tilde{b}_2}\lambda + \frac{\tilde{b}_4}{\tilde{b}_2}\lambda^2 + \frac{\tilde{b}_5}{\tilde{b}_2}\lambda^3 + \frac{\tilde{b}_6}{\tilde{b}_2}\lambda^4 \right)$$

  The Padé prediction of the $R_5 x^5$ term with known coefficients $\{R_1, R_2, R_3, R_4\}$ in the series $P(x) = 1 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4$ is given by

$$R_5 = \frac{R_4^2 \left(R_1 R_3^3 - 2R_3^2 R_4 + R_1 R_2 R_3 R_4 \right)}{R_3 \left(2R_1 R_3^3 - R_2^2 R_3^2 - R_4 R_3^3 \right)}.$$

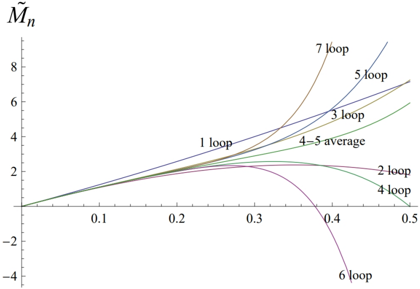
- Pade approximation can be only used in MS Scheme. We need to transform the result from MS Scheme to Coleman-Weinberg Scheme.
- With Pade approximation to seven loop order (two loops more than originally), we get the Higgs mass prediction $m_H = 150$ GeV and the corresponding coupling constant $\lambda = 0.308$. (Compared with five loop result: $m = 165$ GeV, $\lambda = 0.354$).

Extrapolating the Theory: Average Method

- We define the mass without counterterm contributions as:

$$\tilde{M}_n = \frac{1}{v^2}\frac{d^2(v_n - \pi^2 K_n \phi^4)}{d\phi^2}\bigg|_{\phi=v},$$

- $\tilde{M}_n$ predictions to odd loop order are upward while to the even loop order are downward.
- S_n shows alternating series behavior in this figure.
- If the theory is correct, we can expect finally the even and odd order prediction should converge.
- The final curve to a very high loop order should lie somewhere in between the odd order upper bound and even order lower bound.



Extrapolate the Theory Further: Construction of Average Method

- The simplest way to approximately find the higher order curve lies in between the even and odd curves is to do the average.
- This method is based on Euler transformation for accelerating convergence of alternating series and set $s = 0$ as the lowest order approximation:

$$\sum_{n=0}^{\infty}(-1)^n u_n = u_0 - u_1 + u_2 - \cdots - u_{N-1} + \sum_{s=0}^{\infty} \frac{(-1)^s}{2^{s+1}} \left[\Delta^s u_N\right].$$

- For example, if we try to construct the average of four-loop and five-loop summation, the effective potential with counterterm can be written as:

$$V_{\text{average}}(\lambda, \phi, \mu) = \frac{1}{2}\lambda\phi^4 \Big[(S_0(\lambda L) + \lambda S_1(\lambda L) + \lambda^2 S_2(\lambda L) + \lambda^3 S_3(\lambda L)) \\ +(S_0(\lambda L) + \lambda S_1(\lambda L) + \lambda^2 S_2(\lambda L) + \lambda^3 S_3(\lambda L) + \lambda^4 S_4(\lambda L)) \\ +\phi^4(\lambda^6 S_5(0) + \lambda^7 S_6(0) + \lambda^8 S_7(0) + \lambda^9 S_8(0)) \Big]$$

- By four and five loop average method, the mass prediction is 153 GeV compared with five loop prediction 165 GeV.

Extrapolating the Theory: Combine Padé Approximation with Average Method

- The average method works well and in fact it gives two loops more accuracy.
- If we combine the Padé approximation method and the average method, we can predict the Higgs mass with more accuracy.
- We first use Padé approximation to predict the six and seven loop order summation and then use average method of six-seven loop to give mass prediction 141 GeV (approximate 9 loop result).

Loop	λ	M_H	$\lambda_{\rm CSB}$	Average	λ	M_H	$\lambda_{\rm CSB}$
1 loop	0.534	221	0.101				
3 loops	0.417	186	0.072	2,3 loop	0.514	167	0.230
5 loops	0.354	165	0.056	4,5 loop	0.418	153	0.194
7 loops	0.308	150	0.047	6,7 loop	0.352	141	0.041
Extrapolate	0.233	124	0.032				

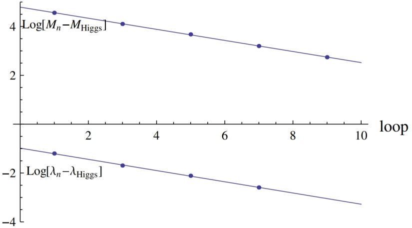
图: Here the $\lambda_{\rm CSB}$ is obtained via $\frac{1}{4}\frac{M_H^2}{2v^2}$

Convergence to a Final Mass?

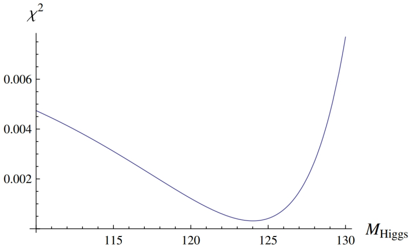
- The mass differences between subsequent loop orders n decrease in a fashion consistent with a geometric series converging to M_{Higgs}:

$$M_n - M_{\text{Higgs}} = \Lambda_1 \sigma_1^n.$$

- The plot of $\log(M_n - M_{\text{Higgs}})$ shows linear behaviour with n, which is consistent with the geometric series when $M_{\text{Higgs}} = 125$ GeV.
- The dependence of the χ^2 deviation from this linear fit is shown as a function of M_{Higgs}, providing an optimized value $M_{\text{Higgs}} = 124$ GeV.
- Thus, the radiatively-generated Higgs mass ultimately converges to a value consistent with the 125 GeV ATLAS/CMS value.



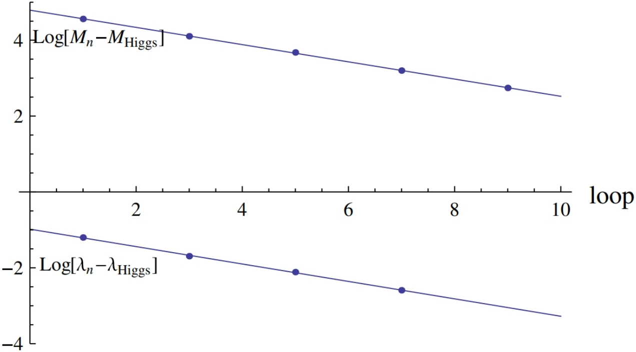
(a) Log plot of $M_n - M_{\text{Higgs}}$



(b) χ^2 vs. M_{Higgs}

Convergence to a Final Coupling?

- Since the mass pattern fulfills a geometric series, does the coupling constant also fulfill a geometric series?
- Assuming $\lambda_n - \lambda_{\text{Higgs}} = \Lambda_2 \sigma_2^n$, we can use chi square method to determine $\lambda_{\text{Higgs}} = 0.233$.
- The plot of $\log(\lambda - \lambda_{\text{Higgs}})$ also shows linear behaviour with n, which means the coupling pattern we predicted is consistent with the geometric series when $\lambda_{\text{Higgs}} = 0.233$.
- The Higgs coupling finally converges to $\lambda_{\text{Higgs}} = 0.233$.



The Signals of Radiative Symmetry Breaking

- At five loop level, the coupling constant predicted by radiative symmetry breaking is six times larger compared with the conventional symmetry breaking.
- The ratio of the coupling in radiative and conventional sceneries become even larger when the loop order becomes much higher (around 7 times at convergence point).
- Most processes involving Higgs and gauge bosons are identical in both conventional and radiative symmetry breaking. (See Chishtie, Elias, Steele, *Int. J. Mod. Phys. A20* (2005) 6241.)
- The exceptions are two processes which are sensitive to their Higgs coupling difference: a 3-fold enhancement of $W_L^+ W_L^- \to HH$ and a significant enhancement of $HH \to HH$.

Question

Using large charge Q resummation to calculate all order scaling dimension of Higgs field?

Comparing Different Large Parameter Expansion

- Large N_c: Planar limit: t'Hooft coupling $A_c \equiv g^2 N_c$ is fixed
- Large N_f: Bubble diagrams: t'Hooft coupling $A_f \equiv g^2 N_f$ is fixed
- Large Q: t'Hooft coupling $A_Q \equiv \lambda Q$ is fixed
- We have:

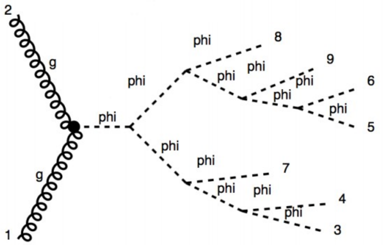
$$\text{Observable} \sim \sum_{l=\text{loops}} g^l P_l(N) = \sum_k \frac{1}{N_i^k} F_k(\mathcal{A}_i)$$

where $N = \{N_c, N_f, Q\}$ and $\mathcal{A} = \{A_c, A_f, A_Q\}$.

Multiparticle Production Problem: Higgs Explosion

(Khoze, “Higgs Explosion", indico.cern.ch/event/677640/contributions/2938636)

- It was proposed that multiparitlce production processes are problematic: higgs explosion and instanton-like processes in baryogenesis.
  (V. A. Rubakov, “Nonperturbative aspects of multiparticle production,” hep-ph/9511236.)
- In the process of $A_{gg\to n\times h} = \sum_{\text{polygons}} A^{\text{polygons}}_{gg\to k\times h^*} \sum_{n_1+\cdots+n_k=n} \Pi^k_{i=1} A_{h_i^*\to n_i\times h}$, perturbation theory fails when around 130 Higgses are produced at $O(100\,\text{TeV})$.



Multiparticle Production Problem: Higgs Explosion

(V. A. Rubakov, "Nonperturbative aspects of multiparticle production," hep-ph/9511236.)

- The "exact" result for the tree amplitude at threshold $(E = nm)$ is

$$A_{1\to n}^{\text{tree}} = n! \left(\frac{\lambda}{8m^2}\right)^{\frac{n-1}{2}}$$

- For multiparticle production of $\lambda\phi^4$ theory (mimic Higgs explosion), the amplitude at one loop level:

$$A_{1\to n}^{\text{tree}} + A_{1\to n}^{\text{one loop}} = A_{1\to n}^{\text{tree}} \left(1 + B\lambda n^2\right) \,, \quad A_{1\to n}^{\text{tree}} = n! \left(\frac{\lambda}{8m^2}\right)^{\frac{n-1}{2}}$$

- Two folds problems:
  - The factorial behavior of the tree amplitudes indicates that the cross section also increase with n and at $n \sim 1/\lambda$, the cross section will exceed the unitarity limit at sufficent large n

$$\sigma_{1\to n}^{\text{tree}} \sim \frac{1}{n!} \left|A_{1\to n}^{\text{tree}}\right|^2 \times (\text{phase space}) \sim n!\lambda^n \epsilon^n \qquad \epsilon = \frac{E - nm}{n}$$

  - Loop corrections fails and conventional perturbation theory fails!

Towards the Holy Grail Summation Function $F\left(\lambda n, \epsilon\right)$

(V. A. Rubakov, “Nonperturbative aspects of multiparticle production,” hep-ph/9511236.)

- Rubakov's insight: $\sigma_{1\to n}\left(E\right) \propto \exp\left[nF\left(\lambda n, \epsilon\right)\right]$
- However, only a few terms in the expansion of $F\left(\lambda n, \epsilon\right)$ at small λn and ϵ are known:

$$F\left(\lambda n, \epsilon\right) = \ln\frac{\lambda n}{16} + \frac{1}{2} + \frac{3}{2}\ln\frac{\epsilon}{3\pi} - \frac{17}{12}\epsilon + B\lambda n + \cdots$$

- Using Large charge method, we can instead calculate LO and NLO scaling dimensions of fixed charge operator $\left[\phi^n\right]$ in $U(1)$ symmetric $\lambda\left(\bar{\phi}\phi\right)^2$ theory. (G. Badel, G. Cuomo, A. Monin and R. Rattazzi, JHEP **1911**, 110 (2019), arXiv:1909.01269.)

$$\begin{aligned} & Z^2_{\phi^n}\lambda_0^n\langle\left[\bar{\phi}^n\right]\left(x_f\right)\left[\phi^n\right]\left(x_i\right)\rangle \\ =&\lambda_0^n n!\exp\left[\frac{1}{\lambda_0}\Gamma_{-1}\left(\lambda_0 n, x_{fi}\right) + \Gamma_0\left(\lambda_0 n, x_{fi}\right) + \Gamma_1\left(\lambda_0 n, x_{fi}\right) + \cdots\right] \end{aligned}$$

Weyl Map and State-Operator Correspondence

- At the (Wilson-Fisher) fixed point, we can exploit the power of conformal invariance.
- Perform a Weyl map from the plane to the cylinder $\mathbb{R}^d \to \mathbb{R} \times S^{d-1}$ where dilations (scaling dimensions $\Delta_{\mathcal{O}}$)on the plane are mapped to time translations (energy spectrum) on the cylinder.
- In coordinates i.e. $(r, \Omega_{d-1}) \to (\tau, \Omega_{d-1})$ using $r = Re^{\tau/R}$. The cylinder metric is related to the flat one by a Weyl re-scaling:

$$ds^2_{\text{cyl}} = d\tau^2 + R^2 d\Omega^2_{d-1} = \frac{R^2}{r^2} ds^2_{\text{flat}}$$

- The two-point function of a scalar primary operator $\mathcal{O}$ on the cylinder is related to the flat space one by:

$$\langle \mathcal{O}^\dagger(x_f)\, \mathcal{O}(x_i) \rangle_{\text{cyl}} = |x_f|^{\Delta_{\mathcal{O}}} |x_i|^{\Delta_{\mathcal{O}}} \langle \mathcal{O}^\dagger(x_f)\, \mathcal{O}(x_i) \rangle_{\text{flat}} \equiv \frac{|x_f|^{\Delta_{\mathcal{O}}} |x_i|^{\Delta_{\mathcal{O}}}}{|x_f - x_i|^{2\Delta_{\mathcal{O}}}}$$

- In the limit $x_i \to 0$ on the plane translates to $\tau_i \to -\infty$ on the cylinder and obtain:

$$\langle \mathcal{O}^\dagger(x_f)\, \mathcal{O}(x_i) \rangle_{\text{cyl}} \overset{\tau_i \to -\infty}{=} e^{-E_{\mathcal{O}}(\tau_f - \tau_i)}, \qquad E_{\mathcal{O}} = \Delta_{\mathcal{O}}/R$$

Leading Order Scaling Dimension Δ_{-1}: $U(1)$ Example

(G. Badel, G. Cuomo, A. Monin and R. Rattazzi, JHEP **1911**, 110 (2019), arXiv:1909.01269.)

- We compute the expectation of the evolution operator e^{-HT} in an arbitrary state $|\psi_n\rangle$ with fixed charge n. In the limit $T \to \infty$, the expectation gets saturated by the **lowest energy state**.

$$\langle \psi_n | e^{-HT} | \psi_n \rangle \overset{T\to\infty}{=} \tilde{\mathcal{N}} e^{-E_{\phi^n} T}$$

- It corresponds to the lowest lying operator i.e. operator with minimal classical scaling dimension with the give fixed charge (ϕ^n in $U(1)$).
- Introduce polar coordinates for the field $\phi = \frac{\rho}{\sqrt{2}} e^{i\chi}$, we have:

$$\langle \psi_n | e^{-HT} | \psi_n \rangle = \mathcal{Z}^{-1} \int_{\rho=f}^{\rho=f} \mathcal{D}\rho \mathcal{D}\chi e^{-S_{eff}}$$

$$S_{eff} = \int d\tau \int d\Omega \left[\frac{1}{2} (\partial\rho)^2 + \frac{1}{2} \rho^2 (\partial\chi)^2 + \frac{m^2}{2} \rho^2 + \frac{\lambda_0}{16} \rho^4 + i \frac{n}{R^{d-1}\Omega} \dot{\chi} \right]$$

- Here, $i\frac{n}{R^{d-1}\Omega}\dot{\chi}$ is the boundary term which fixes the value of the total charge and can not be dropped while $m^2 = \left(\frac{d-2}{2R}\right)^2$ arises from the $\mathcal{R}(g)\bar{\phi}\phi$ coupling to the Ricci scalar enforced by conformal invariance.

Leading Order Scaling Dimension Δ_{-1}: $U(1)$ Example

(G. Badel, G. Cuomo, A. Monin and R. Rattazzi, JHEP **1911**, 110 (2019), arXiv:1909.01269.)

- The variation of the S_{eff} provides the E.O.M. and total charge constraint:

$$-\partial^2\rho+\left[(\partial\chi)^2+m^2\right]\rho+\frac{\lambda_0}{4}\rho^3=0,\ \ i\partial_\mu\left(\rho^2 g^{\mu\nu}\partial_\nu\chi\right)=0;\ \ i\rho^2\dot{\chi}=\frac{n}{R^{d-1}\Omega_{d-1}}$$

- The stationary configuration (ground state) is spatially homogeneous given by:

$$\rho=f,\qquad \chi=-i\mu\tau+\text{const}.$$

- Using E.O.M. and the total charge constraint, we obtain the constraints of the VEV f and chemical potential μ as

$$\left(\mu^2-m^2\right)=\frac{\lambda_0}{4}f^2,\qquad \mu f^2 R^{d-1}\Omega_{d-1}=n$$

- The effective action S_{eff} evaluated on the stationary configuration provides the leading order value for the energy and thus Δ_{-1}. We set $\lambda_0=\lambda_*$ at the fixed point and $d=4$ to obtain:

$$\frac{1}{\lambda_*}\Delta_{-1}=\frac{S_{eff}}{T}\bigg|_{\lambda_0=\lambda_*}=\frac{n}{2}\left(\frac{3}{2}\mu+\frac{1}{2}\frac{m^2}{\mu}\right)\bigg|_{\lambda_0=\lambda_*}$$

The Results of Δ_{-1}: $U(1)$ Example

(G. Badel, G. Cuomo, A. Monin and R. Rattazzi, JHEP **1911**, 110 (2019), arXiv:1909.01269.)

- We obtain the leading order result with a closed form

$$\frac{\Delta_{-1}}{\mathcal{A}^*} = \frac{1}{4}F(x)\,, \qquad F(x) \equiv \frac{3^{\frac{2}{3}}x^{\frac{1}{3}}}{3^{\frac{1}{3}}+x^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}\left(3^{\frac{1}{3}}+x^{\frac{2}{3}}\right)}{x^{\frac{1}{3}}},$$

$$x = 9\frac{\mathcal{A}^*}{(4\pi)^2} + \sqrt{-3+81\frac{\mathcal{A}^{*2}}{(4\pi)^4}}\,, \qquad \mathcal{A}^* = \lambda^* n$$

- $F(x)$ is a real and positive function for $x > 0$.
- The closed form results can be expanded in two extreme regimes, $\lambda_* n \ll (4\pi)^2$ and $\lambda_* n \gg (4\pi)^2$

$$\frac{\Delta_{-1}}{\lambda^*} = \begin{cases} n\left[1+\frac{1}{2}\left(\frac{\lambda^* n}{16\pi^2}\right) - \frac{1}{2}\left(\frac{\lambda^* n}{16\pi^2}\right)^2 + \mathcal{O}\left(\frac{(\lambda^* n)^3}{(4\pi)^6}\right)\right], & \lambda^* n \ll (4\pi)^2, \\ \frac{8\pi^2}{\lambda^*}\left[\frac{3}{4}\left(\frac{\lambda^* n}{8\pi^2}\right)^{4/3} + \frac{1}{2}\left(\frac{\lambda^* n}{8\pi^2}\right)^{2/3} + \mathcal{O}(1)\right], & \lambda^* n \gg (4\pi)^2. \end{cases}$$

The Next-to-Leading-Order Δ_0: $U(1)$ Example

(G. Badel, G. Cuomo, A. Monin and R. Rattazzi, JHEP **1911**, 110 (2019), arXiv:1909.01269.)

- The NLO scaling dimension Δ_0 arises from the fluctuation determinant of the Gaussian integrals after we expand the fields around the saddle point configurations $\rho(x) = f + r(x)$, $\chi(x) = -i\mu\tau + \frac{1}{f\sqrt{2}}\pi(x)$.

$$\Delta_0(\lambda n) = \frac{R}{T}\log\frac{\sqrt{\det S^{(2)}}}{\det\left(-\partial_\tau^2 - \Delta_{s^{d-1}} + m^2\right)} = \frac{R}{2}\sum_{\ell=0}^{\infty} n_\ell \left[\sum_i g_i \omega_i(\ell)\right]$$

- Here, n_ℓ is the multiplicity of the Laplacian on the $(d-1)-$dim. sphere where $n_\ell = (1+\ell)^2$ for $d=4$ and ω_i's are the dispersion relations of the fluctuations and g_i is the multiplicity for each ω_i.
- In the small and large $\lambda_* n$ limit, we expand Δ_0 to obtain respectively:

$$\Delta_0 = -\frac{3\lambda_* n}{(4\pi)^2} + \frac{\lambda_*^2 n^2}{2(4\pi)^4} + \mathcal{O}\left(\frac{\lambda_*^3 n^3}{(4\pi)^6}\right), \quad \text{for } (\lambda_* n << 1)$$

$$\Delta_0 = \left[\alpha + \frac{5}{24}\log\left(\frac{\lambda_* n}{8\pi^2}\right)\right]\left(\frac{\lambda_* n}{8\pi^2}\right)^{4/3} + \left[\beta - \frac{5}{36}\log\left(\frac{\lambda_* n}{8\pi^2}\right)\right]\left(\frac{\lambda_* n}{8\pi^2}\right)^{2/3},$$

for $(\lambda_* n >> 1)$ where $\alpha = -0.5753315(3)$, $\beta = -0.93715(9)$.

From Abelian to Non-abelian: the $O(N)$ Model

(Antipin, Bersini, Sannino, Z.W. Wang and Zhang, "Charging the $O(N)$ model," PRD 102 (2020) 045011.)

- In euclidean spacetime, the $O(N)$ theory is defined by the action

$$\mathcal{S} = \int d^d x \left(\frac{1}{2} \partial^\mu \phi_a \partial^\mu \phi_a + \frac{(4\pi)^2 g_0}{4!} (\phi_a \phi_a)^2 \right) \qquad a = 1, \dots, N \, .$$

- The $O(N)$ model with even or odd N has rank $\frac{N}{2}$ or $\frac{N-1}{2}$ corresponding to the number of charges we can fix. Only total charge and VEV matter.

$$\mu^2 - m^2 = \frac{(4\pi)^2}{6} g_0 v_{tot}^2 \quad \text{(EOM)}; \quad \frac{\bar{Q}}{\text{Vol.}} = \mu v_{tot}^2 \quad \text{(Noether Charge)}$$

$$v_{tot}^2 \equiv \sum_i v_i^2 \quad \text{(Sum of VEVs)}; \quad \bar{Q} \equiv \sum_i Q_i \quad \text{(Sum of charges)}$$

- Charge configurations with the same total charge are equivalent
  - Consider a generic charge configuration for $O(N)$

$$\mathcal{Q} = \left(m_1, m_2, \cdots, m_{N/2} \right)$$

  - The corresponding fixed-charge operator is

$$\mathcal{O} = \Pi_{i=1}^{i=N/2} (\varphi_i)^{m_i} \qquad \varphi_k \equiv \frac{1}{\sqrt{2}} (\phi_{2k-1} + i\phi_{2k})$$

The Results of $O(N)$ Model

(Antipin, Bersini, Sannino, Z.W. Wang and Zhang, "Charging the $O(N)$ model," PRD 102 (2020) 045011.)

- We calculate the LO and NLO in charge expansion (with small 't Hooft coupling) but to all order in couplings scaling dimensions of $\bar{Q}$-index traceless symmetric tensor operator $T_{\bar{Q}} \equiv T^{(\bar{Q})}_{i_1 \ldots i_{\bar{Q}}}$ in the $O(N)$ model

$$\Delta_{T_{\bar{Q}}} = \bar{Q} + \left(-\frac{\bar{Q}}{2} + \frac{\bar{Q}(\bar{Q}-1)}{8+N}\right)\epsilon - \left[\frac{184+N(14-3N)}{4(8+N)^3}\bar{Q} + \frac{(N-22)(N+6)}{2(8+N)^3}\bar{Q}^2 + \frac{2}{(8+N)^2}\bar{Q}^3\right]\epsilon^2 + \left[\frac{8}{(8+N)^3}\bar{Q}^4\right.$$
$$+ \frac{-456-64N+N^2+2(8+N)(14+N)\zeta(3)}{(8+N)^4}\bar{Q}^3 - \frac{-31136-8272N-276N^2+56N^3+N^4+24(N+6)(N+8)(N+26)\zeta(3)}{4(N+8)^5}\bar{Q}^2$$
$$\left. + \frac{-65664-8064N+4912N^2+1116N^3+48N^4-N^5+64(N+8)(178+N(37+N))\zeta(3)}{16(N+8)^5}\bar{Q}\right]\epsilon^3 + \mathcal{O}\left(\epsilon^4\right).$$

- At each ϵ order, the semi-classical computation provides term with leading Q and next leading Q shown in red.

Towards the Standard Model

Antipin, Bersini, Panopoulos, Sannino, Wang, "Infinite order results for charged sectors of the SM," JHEP 2024.

- We calculate the LO and NLO in charge expansion (with small 't Hooft coupling) but to all order in couplings scaling dimensions of the Higgs operator in the Standard Model written as $H^{I_1}...H^{I_Q}$

$$\begin{aligned}\Delta_Q = Q &+ \left\{\frac{1}{3}\lambda Q^2 + \left[N\mathcal{Y}_u + N\mathcal{Y}_d + \mathcal{Y}_l - \frac{3}{4}g'^2 - \frac{\lambda}{3}\right]Q\right\} - \left\{\frac{2}{9}\lambda^2 Q^3 - \left[2N\mathcal{Y}_{uu} + 2N\mathcal{Y}_{dd} + 2\mathcal{Y}_{ll}\right.\right.\\ &\left.\left. - \frac{2}{3}\lambda(N\mathcal{Y}_u + N\mathcal{Y}_d + \mathcal{Y}_l) - \frac{1}{3}\lambda g'^2 + \frac{g'^4}{16} + \frac{\lambda^2}{9}\right]Q^2 + C_{22}Q\right\} + \left\{\frac{8}{27}\lambda^3 Q^4 + \left[\frac{1}{16}g'^6(9\zeta(3)-1)\right.\right.\\ &- \frac{1}{6}g'^4\lambda(1+3\zeta(3)) + \frac{1}{3}g'^2\lambda^2(3-2\zeta(3)) + \frac{4}{27}\lambda^3(9\zeta(3)-8) + \frac{4}{27}\left(3N\left(\lambda^2\mathcal{Y}_u - 3\lambda\mathcal{Y}_{uu}\right.\right.\\ &+ 9\zeta(3)(\lambda\mathcal{Y}_{uu} - 2\mathcal{Y}_{uuu})\right) + 3N\left(\lambda^2\mathcal{Y}_d - 3\lambda\mathcal{Y}_{dd} + 9\zeta(3)(\lambda\mathcal{Y}_{dd} - 2\mathcal{Y}_{ddd})\right) + 3\left(\lambda^2\mathcal{Y}_l - 3\lambda\mathcal{Y}_{ll}\right.\\ &\left.\left.\left.\left. + 9\zeta(3)(\lambda\mathcal{Y}_{ll} - 2\mathcal{Y}_{lll})\right)\right)\right]Q^3 + C_{23}Q^2 + C_{33}Q\right\} + \mathcal{O}\left(\kappa_I^4 Q^5\right) . \qquad (7.1)\end{aligned}$$

where red, blue, and orange colors highlight the terms stemming from the small $\kappa_I Q$ expansion of Δ_{-1}, $\Delta_0^{\rm fm}$ and $\Delta_0^{\rm bos}$.

- $Q = 1$ denotes the Higgs field scaling dimension and can be matched up to three loop order

Acknowledgements

Thank You

Important Concepts

- $U(1)$ Landau Pole Problem
- Asymptotically free vs asymptotically safe
- Large N_f Expansion
- from large N_f to large Q

Fundamental Theory

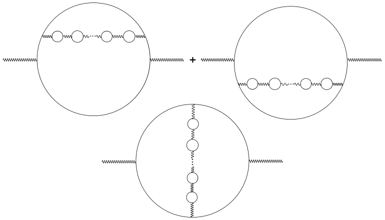
- A fundamental theory has an UV fixed point (K. G. Wilson, Phys. Rev. B **4** (1971) 3174.)
- Couplings stop running with the energy scale at the fixed point
- The Standard Model is not a fundamental theory since it runs into Landau Pole at UV due to the abelian $U(1)$ gauge group
- Asymptotically Free: non-interacting (Gaussian) fixed point (D. J. Gross and F. Wilczek, Phys. Rev. D **8** (1973) 3633; D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343.)
  - non-interacting in the UV
  - coupling runs with logarithmic scale dependence
  - Perturbation theory in UV
- Asymptotically Safe: Interacting fixed point (S. Weinberg(1979). "Ultraviolet divergences in quantum theories of gravitation".)
  - interacting in the UV
  - coupling runs with power law scale dependence
  - Perturbative/Non perturbative theory in UV
  - Smaller critical surface dimension $\Rightarrow$ more IR predictiveness

Large N_f Expansion

- Pascual noticed that it is possible to sum up a subset of the diagrams and the resulting power series is so well behaved to provide a closed-form expression at $1/N_f$ order
- The resummed U(1) beta function reads:

$$\beta_A = \frac{2A^2}{3}\left[1 + \frac{1}{N_f}F_1(A)\right], \qquad A \equiv 4N_f\alpha = 4N_f\frac{g_1^2}{(4\pi)^2}$$

$$F_1(A) = \frac{3}{4}\int_0^A \mathrm{d}x\,\tilde{F}\left(0, \tfrac{2}{3}x\right), \quad \tilde{F}(0,x) = \frac{(1-x)(1-\frac{x}{3})(1+\frac{x}{2})\Gamma(4-x)}{3\Gamma^2(2-\frac{x}{2})\Gamma(3-\frac{x}{2})\Gamma(1+\frac{x}{2})}$$



Game of Asymptotic Safety (Non-Abelian): One Example

- Consider a special gauge-Yukawa system (Gauge: $SU(3)$ and $SU(2)$). It more or less mimic standard model gauge Yukawa sector.
- Positive and Negative (see also Bond, Hiller, Kowalska and Litim, JHEP **1708** (2017) 004)

$$\begin{aligned} SU(3): \quad & \beta_3 = -B_3\alpha_3^2 + \left(C_3\alpha_3 + G_3\alpha_2 - D_3\alpha_y\right)\alpha_3^2 \\ SU(2): \quad & \beta_2 = -B_2\alpha_2^2 + \left(C_2\alpha_2 + G_2\alpha_3 - D_2\alpha_y\right)\alpha_2^2 \\ Yukawa: \quad & \beta_y = \left(E\alpha_y - F_2\alpha_2 - F_3\alpha_3\right)\alpha_y \end{aligned}$$

- At two loop order gauge and one loop order Yukawa coupling, the Yukawa terms are the only negative terms in the gauge RG functions.
- The Yukawa terms occur at next leading order rather than leading order $\Rightarrow$ Non-perturbative Yukawa coupling is required to blance the positive contributions (with lowest dimension of representation) and highly non-perturbative to involve $U(1)$

Non-Perturbative Issue: Veneziaono Limit and Litim-Sannino Model

- In Litim-Sannino Model, the Veneziano limit is implemented to make the leading order terms as small as possible

  (D. F. Litim and F. Sannino, "Asymptotic safety guaranteed," JHEP **1412** (2014) 178.)

- Define $\varepsilon = \frac{N_F}{N_c} - \frac{11}{2}$, the general leading order term of the gauge RG function is:

$$\beta_\alpha = -\frac{4}{3}\varepsilon\alpha^2 + O(\alpha^3)$$

- In the limit $N_F \to \infty$ and $N_c \to \infty$, ε could be as small as possible and the perturbative analysis is under control.
- Large N_c will make it difficult to connect to phenomenologies.
- $U(1)$ Landau pole problem is not addressed

Generalize to the Standard Model: $SU(3) \times SU(2) \times U(1)$

- Holdom's system only involves two kinds fields (gauge field+fermions) and one coupling (gauge coupling g)
- Safety can be realized in a more general gauge-Yukawa system: the Standard Model

  Mann, Meffe, Sannino, Steele, Z. W. Wang and Zhang, Phys. Rev. Lett. **119** (2017) 261802
- The standard Model can be safe at UV when including reasonably number of vector-like fermions.
- For $U(1)$, it requires $N_F > 16$ while for $SU(3)$, it requires $N_F > 32$ to suppress $1/N_f^2$ contributions.
- Use vector-like fermions for simplification without involving extra scalars to generate their mass terms

Pati-Salam: Finite Temperature Effective Potential

(Huang, Sannino and Z.W.Wang, PRD **102** (2020) 095025, arXiv:2004.02332.)

- One loop contribution without temperature:
  - $$V_{1\text{loop}} = \sum_i \pm n_i \frac{m_i^4}{64\pi^2}\left(\log\left[\frac{m_i^2}{\mu^2}\right] - C_i\right)$$
  - The total one-loop effective potential is:

$$V_{1\text{loop}} = V_{\text{Higgs}} + V_{\text{Gold}} + V_{\text{lepto}} + V_{W_R^\pm} + V_{Z'} + V_\nu + V_F\ .$$

- Finite temperature effective potential:
  - The one loop finite temperature effective potential:

$$V_T = \sum_i \pm n_i \frac{T^4}{2\pi^2}\int_0^\infty dy y^2 \log\left[1 \mp e^{-\sqrt{y^2+m_i^2/T^2}}\right]\ ,$$

  - The total finite temperature effective potential (without ring contributions) as:

$$V_T^{\text{tot}} = \frac{T^4}{2\pi^2}\left(I_B\left[\frac{M^2_{\text{Higgs1}}}{T^2}\right] + 6I_B\left[\frac{M^2_{\text{Higgs2}}}{T^2}\right] + 9I_B\left[\frac{M^2_{\text{Gold}}}{T^2}\right] + 6I_B\left[\frac{M^2_{W_R^\pm}}{T^2}\right]\right.$$
$$\left. + 3I_B\left[\frac{M^2_{Z'}}{T^2}\right] + 18I_B\left[\frac{M^2_{\text{lepto}}}{T^2}\right] + 4I_F\left[\frac{M^2_\nu}{T^2}\right] + 16I_F\left[\frac{M^2_F}{T^2}\right]\right).$$

Effective Potential: Ring Contributions to the Scalar

(Huang, Sannino and Z.W.Wang, PRD **102** (2020) 095025, arXiv:2004.02332.)

- The general formula for the ring contributions (scalar in the big ring):

$$V^i_{\text{ring}} = -\frac{T}{12\pi}\left(\left[m_i^2(\rho) + \sum_{\text{bosons } j} \pi_i^j(0)\right]^{3/2} - m_i^3(\rho)\right),$$

- The contributions of different species j in the outside ring of the daisy diagram i.e.

$$\pi^j_{\text{scalar}}(0) = \frac{1}{12}\frac{m_j^2(v)}{v^2}T^2 .$$

- The total thermal mass contributions to the Higgs field $\sum_j \pi^j_{\text{scalar}}(0)$:

$$= \pi^{\text{Higgs1}}_{\text{scalar}}(0) + 6\pi^{\text{Higgs2}}_{\text{scalar}}(0) + 9\pi^{\text{Gold}}_{\text{scalar}}(0) + 18\pi^{\text{lepto}}_{\text{scalar}}(0) + 6\pi^{W_R^\pm}_{\text{scalar}}(0) + 3\pi^{Z'}_{\text{scalar}}(0)$$

- The total ring contributions to the scalar fields:

$$V^{\text{scalar,tot}}_{\text{ring}} = V^{\text{Higgs1}}_{\text{ring}} + 6V^{\text{Higgs2}}_{\text{ring}} + 9V^{\text{Gold}}_{\text{ring}} .$$

Effective Potential: Ring Contributions to the Gauge

(Huang, Sannino and Z.W.Wang, PRD **102** (2020) 095025, arXiv:2004.02332.)

- The ring contributions (gauge field in the big ring):

$$V_{\text{ring}}^{\text{gauge,tot}} = -\frac{T}{12\pi}\,\text{Tr}\left(\left[\mathbf{M^2}(\rho) + \mathbf{\Pi}(\mathbf{0})\right]^{3/2} - \mathbf{M^3}(\rho)\right) ,$$

  where both $m_i^2(\rho)$ and $\sum_i \pi_i^j(0)$ are rewritten as matrices $\mathbf{M^2}(\rho)$ and $\mathbf{\Pi}(\mathbf{0})$ respectively since $\mathbf{M^2}(\rho)$ in the gauge field basis is not diagonalized.

- $\mathbf{\Pi}(\mathbf{0})$ is a diagonal matrix and it's eigenvalues are calculated through:

$$SU(N): \quad \pi_{\text{gauge}}^{L,S} = \frac{g^2T^2}{3}\sum_S t_2(R_S)\,,$$

$$\pi_{\text{gauge}}^{L,F} = \frac{g^2T^2}{6}\sum_F t_2(R_F)\,,$$

$$\pi_{\text{gauge}}^{L,V} = \frac{N}{3}g^2T^2\,.$$

Bubble nucleation

(Huang, Sannino and Z.W.Wang, PRD **102** (2020) 095025, arXiv:2004.02332.)

- The tunnelling rate per unit volume $\Gamma(T)$ from the metastable (false) vacuum to the stable one is suppressed by the three dimensional Euclidean action $S_3(T)$:

$$\Gamma(T) = \left(\frac{S_3(T)}{2\pi T}\right)^{3/2} T^4 e^{-S_3(T)/T} ,$$

where the Euclidean action has the form:

$$S_3(\rho, T) = 4\pi \int_0^\infty dr r^2 \left[\frac{1}{2}\left(\frac{d\rho}{dr}\right)^2 + V(\rho, T) - V(0, T)\right] .$$

- The bubble configuration (instanton solution) is give by solving the equation of motion of the action (over shooting under shooting method):

$$\frac{d^2\rho}{dr^2} + \frac{2}{r}\frac{d\rho}{dr} - \frac{\partial F}{\partial \rho}(\rho, T) = 0 ,$$

with the boundary conditions:

$$\frac{d\rho}{dr}(0, T) = 0, \qquad \lim_{r\to\infty} \rho(r, T) = 0 .$$

First Oder Phase Transition and Coleman Weinberg

(Huang, Sannino and Z.W.Wang, PRD **102** (2020) 095025, arXiv:2004.02332.)

- Question: Is asymptotic safety compatible with the Coleman-Weinberg symmetry breaking? (Remember: Gildener's PhD thesis with Coleman showed the compatibility between Coleman-Weinberg symmetry breaking and asymptotically free.)
- Answer: yes! We find Pati-Salam symmetry can be broken by Coleman-Weinberg mechanism in the **Safe** Pati-Salam scenario.
- We find a strong first order phase transition occurs **only** when spontaneous symmetry breaking happens via the Coleman-Weinberg mechanism (in our model).

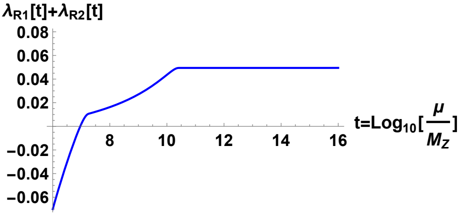


图: We plot the RG running of $\lambda_{R1}(t) + \lambda_{R2}(t)$ from UV to IR. The transition point (the scale $\lambda_{R1}(t) + \lambda_{R2}(t) = 0$) defines the Coleman-Weinberg symmetry breaking scale of the Pati-Salam model

Inverse duration of the Phase Transition β and the ratio of the latent heat α

(Huang, Sannino and Z.W.Wang, PRD **102** (2020) 095025, arXiv:2004.02332.)

- The nucleation temperature which is defined as the temperature at which the rate of bubble nucleation per Hubble volume and time is approximately one:

$$\Gamma(T) \sim H^4 , \quad T \ln \frac{T}{m_{pl}} \simeq -\frac{S_3(T)}{4} \Rightarrow T_n \sim 1260\,\mathrm{TeV}.$$

- The inverse duration of the PT β relative to the Hubble rate H_* at T_n is:

$$\frac{\beta}{H_*} = \left[T \frac{d}{dT} \left(\frac{S_3(T)}{T} \right) \right] \Bigg|_{T=T_n} \Rightarrow \beta / H_* \simeq 183\,.$$

- The ratio of the latent heat released by the phase transition α:

$$\alpha = \frac{\epsilon}{\rho_{\rm rad}} = \frac{1}{\frac{\pi^2}{30} g_* T_n^4} \left(-\Delta V + T_n \Delta s\right), \quad \Delta s = \frac{\partial V}{\partial T}\left(v_{T_n}, T_n\right) - \frac{\partial V}{\partial T}\left(0, T_n\right)$$

where $\Delta V = V\left(v_{T_n}, T_n\right) - V\left(0, T_n\right)$ and we find $\alpha_{T_n} \equiv \alpha(T = T_n) = 0.217$.

Gravitational Wave Spectrum

(Huang, Sannino and Z.W.Wang, PRD **102** (2020) 095025, arXiv:2004.02332.)

- The power spectrum of the acoustic gravitational wave is given by (sound wave only; Collision of scalar field shells and turbulence sub-leading):

$$h^2\Omega_{sw}(f) = 8.5 * 10^{-6}\left(\frac{100}{g_*}\right)^{\frac{1}{3}}\Gamma_{AI}^2\overline{U}_f^4\left(\frac{H_*}{\beta}\right)v_w S_{sw}(f)\ ,$$

  where the adiabatic index $\Gamma_{AI} = \overline{\omega}/\overline{\epsilon} \simeq 4/3$. $\overline{U}_f$ is a measure of the root-mean-square (rms) fluid velocity and is:

$$\overline{U}_f^2 \simeq \frac{3}{4}\kappa_f\alpha_{T_n}\ , \quad \kappa_f \sim \frac{\alpha}{0.73 + 0.083\sqrt{\alpha} + \alpha}$$

  where κ_f is the efficiency parameter and v_ω (wall speed) $\to 1$.
- The spectral shape $S_{sw}(f)$ and peak frequency f_{sw} are:

$$S_{sw}(f) = \left(\frac{f}{f_{sw}}\right)^3\left(\frac{7}{4 + 3\left(f/f_{sw}\right)^2}\right)^{\frac{7}{2}}$$

$$f_{sw} = 8.9\mu\text{Hz}\,\frac{1}{v_\omega}\left(\frac{\beta}{H_*}\right)\left(\frac{z_p}{10}\right)\left(\frac{T_n}{100\,\text{GeV}}\right)\left(\frac{g_*}{100}\right)^{\frac{1}{6}}\ .$$

From Large N_f to Large Q Expansion

- Scaling dimension of the fixed charge operator $\phi^{\bar{Q}}$ in the $U(1)$ model.

$$\frac{\Delta_{\phi^{\bar{Q}}}}{\bar{Q}} = \begin{pmatrix} a_{00} & 0 & 0 & 0 & 0 & \dots \\ a_{11}\hat{\lambda}\bar{Q} & a_{10}\hat{\lambda} & 0 & 0 & 0 & \dots \\ a_{22}\hat{\lambda}^2\bar{Q}^2 & a_{21}\hat{\lambda}^2\bar{Q} & a_{20}\hat{\lambda}^2 & 0 & 0 & \dots \\ a_{33}\hat{\lambda}^3\bar{Q}^3 & a_{32}\hat{\lambda}^3\bar{Q}^2 & a_{31}\hat{\lambda}^3\bar{Q} & a_{30}\hat{\lambda}^3 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{ll}\hat{\lambda}^l\bar{Q}^l & a_{l,l-1}\hat{\lambda}^l\bar{Q}^{l-1} & a_{l,l-2}\hat{\lambda}^l\bar{Q}^{l-2} & a_{l,l-3}\hat{\lambda}^l\bar{Q}^{l-3} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

- Sum the column:

$$\frac{\Delta_{\phi^{\bar{Q}}}}{\bar{Q}} = \Delta_{-1}\left(\mathcal{A}^*\right) + \frac{1}{\bar{Q}}\Delta_0\left(\mathcal{A}^*\right) + + \frac{1}{\bar{Q}^2}\Delta_1\left(\mathcal{A}^*\right) + \cdots \;, \qquad \mathcal{A} \equiv \lambda\bar{Q}$$

Key Obstacles from Abelian to Non-abelian

What is the lowest lying operator?

How to choose the charge configuration?

Charge Config. Model Building Recipe: $U(N) \times U(N)$

(Antipin, Bersini, Sannino, Z.W. Wang and Zhang, Phys. Rev. D **102** (2020) 125033, arXiv:2006.10078 [hep-th]. Antipin, Bersini, Sannino, Z.W. Wang and Zhang, JHEP **12** 204 (2021), arXiv:2102.04390 [hep-th].)

- The charge configuration matters in the $U(N) \times U(N)$ model which in Euclidean spacetime, is written as (H is a $N \times N$ complex matrix)

$$\mathcal{L} = \text{Tr}(\partial_\mu H^\dagger \partial^\mu H) + u_0 \, \text{Tr}(H^\dagger H)^2 + v_0 (\, \text{Tr} H^\dagger H)^2 \, .$$

1. Give a total classical scaling dimension $\bar{Q}$ of the operator $\mathcal{O}$ we are interested
2. Construct all the possible charge configurations satisfying:
   - diagonal elements can only be integer or half-integer;
   - the sum of absolute value of the diagonal element is $\bar{Q}/2$;
   - Tr$\mathcal{Q} = 0$ in $U(N) \times U(N)$ model
3. Determine the “Matrix Form" chemical potential and vacuum expectation value through the E.O.M. and total charge constraints.
4. Following the semi-classical computation recipe to calculate the scaling dimensions.
5. Combining the group theory and semi-classical computation, we can identify the representation of the operator.

An Example to untangle the representation of operator

(Antipin, Bersini, Sannino, Z.W. Wang and Zhang, JHEP **12** 204 (2021), arXiv:2102.04390 [hep-th].)

- We are interested in $SU(3) \times SU(3)$ and considering operators $\mathcal{O}$ with classical scaling dimension $\bar{Q} = 4$
- Since $H \sim (\mathbf{F}_L, \bar{\mathbf{F}}_R)$, $H^\dagger \sim (\bar{\mathbf{F}}_L, \mathbf{F}_R)$, thus the operators belongs to (Γ_L, Γ_R). Γ_L and Γ_R are respectively in $(\mathbf{Adj}_L)^{\bar{Q}/2}$ and $(\mathbf{Adj}_R)^{\bar{Q}/2}$
- Operators live in the decomposition of the tensor product $\mathbf{8} \otimes \mathbf{8}$

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus 2(\mathbf{8}) \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}\,.$$

- We can only construct two different charge configuration matrices:

$$\mathcal{Q}_{3A}^{(4)} = \mathrm{diag}\,(1, -1, 0)\,, \qquad \mathcal{Q}_{3\mathrm{B}}^{(4)} = \mathrm{diag}\,(1, -1/2, -1/2)$$

- The weight which corresponds to $\mathcal{Q}_{3A}^{(4)}$ is $(4, -2)$ only appearing in $\mathbf{27}$ while the weight corresponds to $\mathcal{Q}_{3B}^{(4)}$ reads $(3, 0)$ appearing in both $\mathbf{27}$ and $\mathbf{10}$. Thus, we have the following correspondence

$$\mathcal{Q}_{3A}^{(4)} : (\mathbf{27}, \mathbf{27})\,, \quad \mathcal{Q}_{3B}^{(4)} : \begin{cases} (\mathbf{27}, \mathbf{27}) \\ (\mathbf{10}, \overline{\mathbf{10}}) \end{cases} .$$

An Example

(Antipin, Bersini, Sannino, Z.W. Wang and Zhang, JHEP **12** 204 (2021), arXiv:2102.04390 [hep-th].)

- For $N > \sqrt{3}$, the scalar couplings at the WF fixed points are complex and thus the scaling dimensions at the fixed points are also complex.
- We choose the real part of the scaling dimensions.

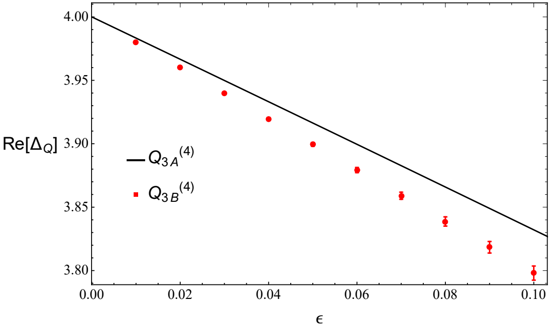


图: The real part of the scaling dimension at the fixed point for the $U(3) \times U(3)$ operators with CSD $\bar{Q} = 4$, carrying the charges $\mathcal{Q}_{3A}^{(4)}$ (black line) and $\mathcal{Q}_{3B}^{(4)}$ (red dots) as a function of ϵ. The error bars encode the numerical error in evaluating Δ_0 for $\mathcal{Q}_{3B}^{(4)}$.

Dire. 3: Large Charge Method and Higgs Explosion

- Can we address the problematic multi-Higgs boson production processes at future colliders?
  Answer: we can calculate the LO and NLO (but to all order in couplings) scaling dimensions of $O(N)$ and $U(N) \times U(M)$ models around 4D for a family of fixed-charge operators by using the semi-classical method and state-operator correspondence.

$$\Delta_{T_{\bar{Q}}} = \bar{Q} + \left(-\frac{\bar{Q}}{2} + \frac{\bar{Q}(\bar{Q}-1)}{8+N}\right)\epsilon - \left[\frac{184+N(14-3N)}{4(8+N)^3}\bar{Q} + \frac{(N-22)(N+6)}{2(8+N)^3}\bar{Q}^2 + \frac{2}{(8+N)^2}\bar{Q}^3\right]\epsilon^2 + \left[\frac{8}{(8+N)^3}\bar{Q}^4\right.$$
$$+ \frac{-456-64N+N^2+2(8+N)(14+N)\zeta(3)}{(8+N)^4}\bar{Q}^3 - \frac{-31136-8272N-276N^2+56N^3+N^4+24(N+6)(N+8)(N+26)\zeta(3)}{4(N+8)^5}\bar{Q}^2$$
$$\left.+ \frac{-65664-8064N+4912N^2+1116N^3+48N^4-N^5+64(N+8)(178+N(37+N))\zeta(3)}{16(N+8)^5}\bar{Q}\right]\epsilon^3 + O\left(\epsilon^4\right) .$$

The scaling dimension of $\bar{Q}$-index traceless symmetric tensor $T_{\bar{Q}} \equiv T^{(\bar{Q})}_{i_1 \ldots i_{\bar{Q}}}$.

- **Outlook** Theory: add fermions, gauge bosons; explore complex CFT
  Phenomenology: found various interesting calculable EFT operators; address full SM Higgs Explosion

What we have achieved so far

(Antipin, Bersini, Sannino, Wang and Zhang, “Charging the $O(N)$ model,” PRD 102 (2020) 045011.)

- Rubakov’s insight: $\sigma_{1\to n}(E) \propto \exp\left[nF(\lambda n, \epsilon)\right]$
- Rattazzi calculate (leading classical, leading quantum) to the scaling dimension of operator $[\phi^n]$ in $U(1)$ symmetric $\lambda\left(\bar{\phi}\phi\right)^2$ theory.(G. Badel, G. Cuomo, A. Monin and R. Rattazzi, JHEP **1911**, 110 (2019), arXiv:1909.01269.)

$$
\begin{aligned}
&Z_{\phi^n}^2 \lambda_0^n \langle \left[\bar{\phi}^n\right](x_f)\left[\phi^n\right](x_i)\rangle \\
=&\lambda_0^n n! \exp\left[\frac{1}{\lambda_0}\Gamma_{-1}(\lambda_0 n, x_{fi}) + \Gamma_0(\lambda_0 n, x_{fi}) + \Gamma_1(\lambda_0 n, x_{fi}) + \cdots\right]
\end{aligned}
$$

- Around the fixed point, the ground state energy of the system corresponds to the scaling dimension of the operator with the lowest classical scaling dimension.
- We did a non-trivial generalization of Rattazi’s work to $O(N)$, $U(N)\times U(N)$, $U(N)\times U(M)$ theory.
- The perturbative expansion of our all order expression can match the existing perturbative loop calculation up to three loop order.

The Effect of PDF

(C. Degrande, V. V. Khoze and O. Mattelaer, Phys. Rev. D **94** (2016), 085031.)

- Question: whether the factorial growth will be completely washed away by PDF suppression?
- For the collider above $50\,\mathrm{TeV}$, the growth of the cross section can not be killed while for lwer energy collider, the PDF are killing the cross-section before reaching the fast growth regime.

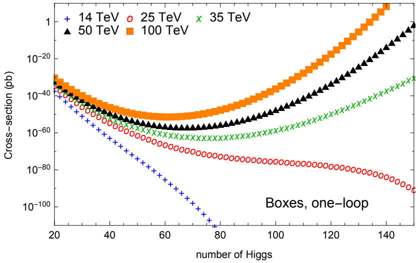


图: Cross-sections for multi-Higgs production (3.6) at proton colliders including the PDFs for different energies of the proton-proton collisions plotted as the function of the Higgs multiplicity. Only the contributions from the boxes are included.

Conclusion

- Asymptotically Safe Standard Model is feasible through GUT embedding with large N_f and $U(1)$ Landau pole problem is addressed.
- Both Safe Pati-Salam model and the Safe Trinification Model can roughly match the SM at IR.
- Strong first order phase transition due to Coleman Weinberg symmetry breaking generates interesting gravitational wave signals within the detection region of near future LIGO Voyager.
- Combining semi-classical computation with CFT, we can calculate the scaling dimensions of a class of fixed charge operator up to NLO in charge expansion but to all order in the couplings.
- We create a recipe to untangle the representations of the fixed charge operator associated with a specific charge configuration.
- Multi-Higgs production is important for the future $100\,\mathrm{TeV}$ collider. Our work is one step towards the “Holy Grail" summation function beyond tree level and has the potential to address the Higgs explosion as a long goal.

The Power of Asymptotic Safety