HIGGS POTENTIAL 2024
HIGGS POTENTIAL AND BSM OPPORTUNITIES

University of Science and Technology of China

Two loop master integrals for mixed QCD-EW corrections to VWW vertex

$V \in \{H, Z, \gamma\}$

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Outline

1 Motivation and overview

2 Integral family

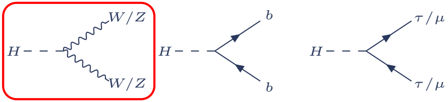
3 Canonical differential equation

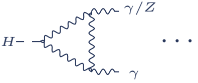
4 Solutions
- Multiple polylogarithms
- Rationalization of square roots
- Elliptic multiple polylogarithms

5 Summary

Motivation

Higgs decay channels:



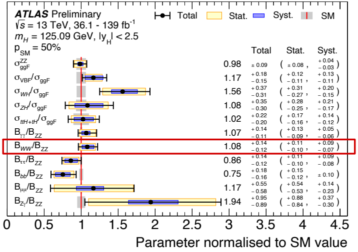


$\cdots$

Also crucial for studying

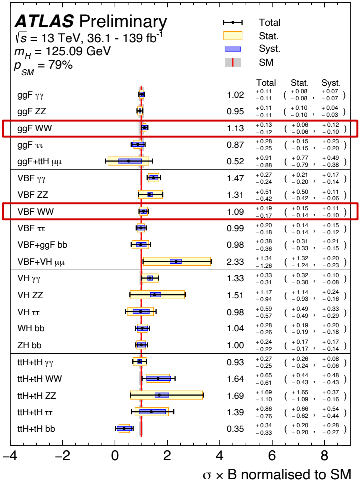
HVV coupling

Higgs mass



[ATLAS,2021]

Ratios of branching fractions to $H \to ZZ^*$



Cross sections times branching fraction

Overview

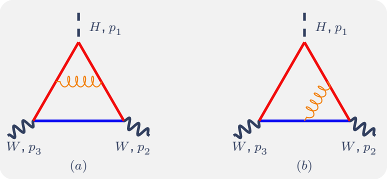
HVV vertex

- **LO:**
  [Thomas G.Rizzo,1980], [G.Pócsik,T.Torma,1980], [W.Keung,W.Marciano,1980] (HWW).
- **NLO:**
  - Higgs-boson decays in the Weinberg-Salam model: [J.Fleischer,F.Jegerlehner,1981].
  - HZ production at e^+e^- colliders: [J.Fleischer,F.Jegerlehner,1983], [Bernd A.Kniehl,1992], [A.Denner,J.Küblbeck,R.Mertig,M.Böhm,1992]
  - Higgs-boson production in association with W or Z bosons: [M.L.Ciccolini,S.Dittmaier, M.Krämer,2003], [Ansgar Denner,Stefan Dittmaier,Stefan Kallweit,Alexander Mück,2012]
- **Beyond the NLO:**
  - [Bernd A. Kniehl,1996], [A. Frink,B.A.Kniehl,D.Kreimer,K.Riesselmann,1996], [A.Djouadi, P.Gambino,B.A.Kniehl,1998], [Bernd A.Kniehl,Oleg L.Veretin,2012]
  - Mixed QCD-EW corrections to HZ production at e^+e^- colliders: [Yinqiang Gong,Zhao Li,Xiaofeng Xu et.al,2017], [Qing-Feng Sun,Feng Feng,Yu Jia,Wen-Long Sang,2017], [Wen Chen,Feng Feng,Yu Jia,Wen-Long Sang,2019]
  - Master integrals for mixed QCD-EW corrections to HVV vertex: [Yuxuan Wang,Xiaofeng Xu,Li Lin Yang,2019], [Ekta Chaubey,Mandeep Kaur,Ambresh Shivaji,2022] (HZV),
    [Stefano Di Vita, Pierpaolo Mastrolia et. al,2017], [Chichuan Ma, Yuxuan Wang, Xiaofeng Xu et. al,2021] (HWW vertex ignoring b quark mass)
    [Zhe Li,Ren-You Zhang,Shu-Xiang Li,Xiao-Feng Wang et. al,2024], [Zhe Li,Ren-You Zhang, Shu-Xiang Li,Xiao-Feng Wang et. al,2024] (VV production at e^+e^- colliders with on-shell V)

VWW vertex with full dependence on massive propagators

This talk

Family and kinematics



Definition of Feynman(scalar) integrals:

$$I_{a_1,\cdots,a_7}(\boldsymbol{x},\epsilon) = \int \mathcal{D}^d k_1 \mathcal{D}^d k_2 \prod_{i=1}^{7} \frac{1}{D_i^{a_i}}$$

with $\mathcal{D}^d k_i = \frac{\mathrm{d}^d k_i\,(m_t^2)^\epsilon}{i\pi^{d/2}\Gamma(\epsilon)}$.

Kinematics: $(p_1^2, p_2^2, p_3^2, m_b^2, m_t^2)$

Denominators identified by:

$$D_1 = (k_1 - k_2)^2, \quad D_2 = k_1^2 - m_b^2, \quad D_3 = k_2^2 - m_b^2$$
$$D_4 = (k_1 + p_2)^2 - m_t^2 \quad D_5 = (k_1 - p_3)^2 - m_t^2$$
$$D_6 = (k_2 + p_2)^2 - m_t^2 \quad D_7 = (k_2 - p_3)^2 - m_t^2$$

A complete set of propagators defines the family.

A set of powers $(a_1, \cdots, a_7)$ represents an integral,

The concept of a sector can then be defined as:

$$(s_1, \cdots, s_7), \quad s_i = \begin{cases} 0 & a_i \leqslant 0 \\ 1 & a_i > 0 \end{cases}$$

Note: s^1 is sub-sector of s^2 for $\forall\, s_i^1 \leqslant s_i^2$

Scaleless variables:

$$w = -\frac{p_1^2}{m_t^2}, \quad x = -\frac{p_2^2}{m_t^2}$$
$$y = -\frac{p_3^2}{m_t^2}, \quad z = \frac{m_b^2}{m_t^2}$$
$$\boldsymbol{\rho} = (w, x, y, z)$$

Topology (a): sector $(1,0,1,1,1,1,1)$

Topology (b): sector $(1,1,1,1,1,1,0)$

Integrals in the family are non-independent !

IBP identities and master integrals

The integrals in a given family satisfy the IBP identities [Chetyrkin,Tkachov,1981] and the constraints of Lorentz identities.

Integration-by-parts identities:

No surface term

$$0=\int\prod_{i}^{L}\mathcal{D}^d k_i\frac{\partial}{\partial l^\mu}\left(\frac{v^\mu}{D_1^{a_1}\dots D_N^{a_N}}\right)=\sum_j C_j(\boldsymbol{x},\epsilon)I_{\boldsymbol{a}+\boldsymbol{\delta}_j}(\boldsymbol{x},\epsilon)$$

Rational function

where vector $v\in\{p_1,\cdots,p_N,k_1,\cdots,k_L\}$.

Non-independent

Lorentz identities:

$$\left(p_1^v\frac{\partial}{\partial p_{1\mu}}-p_1^\mu\frac{\partial}{\partial p_{1v}}+\cdots+p_N^v\frac{\partial}{\partial p_{N\mu}}-p_N^\mu\frac{\partial}{\partial p_{Nv}}\right)I\left(p_1,\dots,p_N\right)=0$$

Using the IBP identities and the symmetries of the integrals, the integrals can be reduced to a finite set of master integrals [A.V.Smirnov, A.V.Petukhov,2010] .

Tools: Kira2 [J.Klappert et al.,2021], FIRE6 [A.V.Smirnov,F.S.Chukharev,2020] ...
New approaches: Syzygy equations [Zihao Wu,Y.Zhang et al.,2020] (NeatIBP1.0)
Block-triangular form [Xin Guan,Y.Q.Ma et al.,2024] (Blade)

Topologies $(a\&b)$ ⟶ 47 MIs

Differential system

Establishing the differential system for the basis of the master integrals:

$$\frac{\partial I}{\partial \boldsymbol{\rho}_i} = A_i(\boldsymbol{\rho}, \epsilon) \cdot I$$

$$\mathrm{d}I + A \cdot I = 0, \quad A = \sum_i A_i \, \mathrm{d}\boldsymbol{\rho}_i$$

$$\mathrm{d}A + A \wedge A = 0$$

Integrability condition

ϵ factorization

$$H = B(\boldsymbol{\rho}, \epsilon) I$$

$$\frac{\partial H}{\partial \boldsymbol{\rho}_i} = \epsilon \, A'_i(\boldsymbol{\rho}) \cdot H$$

$$\mathrm{d}H + A' \cdot H = 0$$

Letters

$$A' = \sum_i C_i \, \mathrm{d}\log[\alpha_i(\boldsymbol{\rho})]$$

$$\mathrm{d}A' = 0\,, \quad A' \wedge A' = 0$$

Canonical basis [J.M.Henn,2013]

Properties of integrands

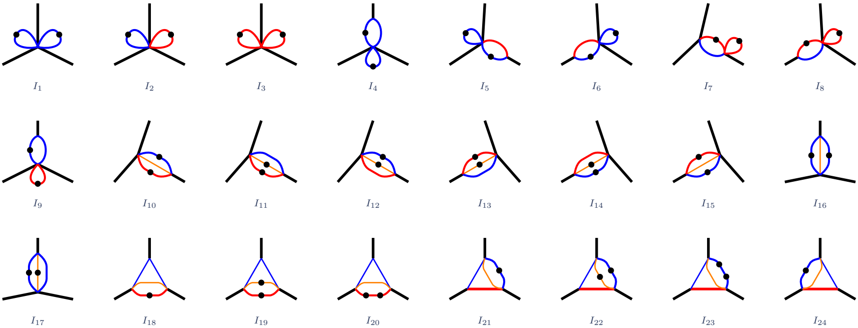
Leading singularities, dlog form...
[J.M.Henn,2015], [C.Dlapa,X.Li,Y.Zhang,2021],
[S.He,Z.Li,Y.Zhang et al.,2022]

Fibre transformations

Magnus series [M.Argeri et al.,2014]
Semi-algorithmic [M.Becchetti,2018]
Moser's algorithm [R.N.Lee et al.,2017]
Leinartas decomposition [C.Meyer,2017]
...

Starting basis

Choose a starting basis based on educated-guess and analysis of the coefficient matrix. (crucial and non-trivial)



The orange lines denote massless propagators and a black dot on a propagator indicates an increase by one in the power of the propagator.

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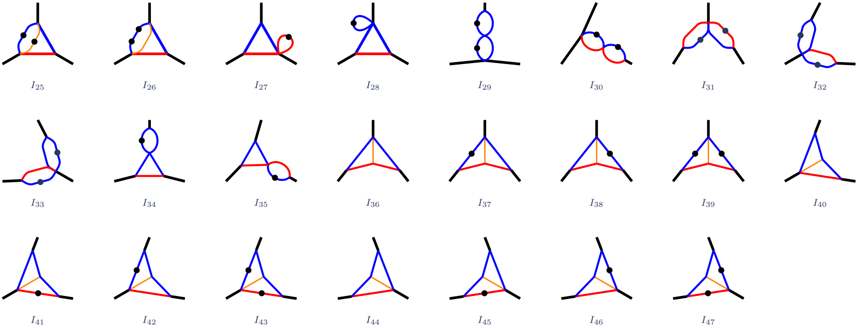

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Canonical basis

Some of integrals I_i are related by $p_2^2 \leftrightarrow p_3^2$ symmetry:

$$I_5 \leftrightarrow I_6\,, \qquad I_{32} \leftrightarrow I_{33}\,, \qquad I_{37} \leftrightarrow I_{38}\,,$$

$$I_{10,11,12} \leftrightarrow I_{15,13,14}\,, \qquad I_{21,22,23} \leftrightarrow I_{24,25,26}\,, \qquad I_{40,41,42,43} \leftrightarrow I_{44,45,46,47}\,.$$

Obtaining the canonical basis $H = B(\boldsymbol{\rho}, \epsilon) I$ (not always doable for algebraic matrix B), we demonstrate several examples of transformations here.

A funtions: polynomials

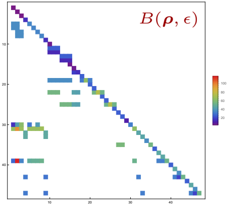
$$H_{36} = \epsilon^4 R_4 I_{36}\,, \qquad H_{37} = \epsilon^3 R_3 R_4 I_{37}\,, \qquad H_{38} = \epsilon^3 R_2 R_4 I_{38}\,,$$

$$\begin{aligned} H_{39} = & \frac{\epsilon^2 A_{38,1}}{A_{5,1}(x) A_{5,1}(y)} I_1 + \frac{\epsilon^2 A_{38,2}}{A_{5,1}(x) A_{5,1}(y) A_{7,2}(x) A_{7,2}(y)} I_2 + \frac{\epsilon^2 A_{38,3}}{A_{7,2}(x) A_{7,2}(y)} I_3 - \frac{\epsilon^2 x A_{22,10}}{A_{5,1}(x) A_{5,1}(y)} I_5 \\ & - \frac{\epsilon^2 y A_{22,10}}{A_{7,2}(x) A_{7,2}(y)} I_8 + \frac{\epsilon^2 A_{22,10}}{A_{5,1}(y) A_{7,2}(x)} (y I_6 + x I_7 - 2xy I_{31}) \\ & - \epsilon^3 A_{22,21}(y,x) I_{37} - \epsilon^3 A_{22,21}(x,y) I_{38} + \epsilon^2 A_{19,19} I_{39}\,. \end{aligned}$$

Four square roots introduced into our transformations:

$$R_1 = \sqrt{w(w+4)}\,, \qquad R_2 = \sqrt{(x+z-1)^2 + 4x}\,,$$

$$R_3 = \sqrt{(y+z-1)^2 + 4y}\,, \qquad R_4 = \sqrt{(w+x-y)^2 - 4wx}\,.$$



Alphabet

The coefficient matrix of the $\mathrm{d}\log$ form, there are 34 independent letters:

$$\alpha_1 = w\,, \quad \alpha_2 = x\,, \quad \alpha_3 = y\,, \quad \alpha_4 = z$$
$$\alpha_5 = R_1^2\,, \quad \alpha_6 = R_2^2\,, \quad \alpha_7 = R_3^2\,, \quad \alpha_8 = R_4^2$$
$$\alpha_9 = w(x+1)(y+1) - (x-y)^2\,, \quad \alpha_{10} = -A_{19,19}\,, \quad \alpha_{11} = \alpha_{12}(x \leftrightarrow y)\,,$$
$$\alpha_{12} = w\big[(x+z)(y+z) - (w+1)\,z + x\big] + (y-x)(x+z) + x\,.$$

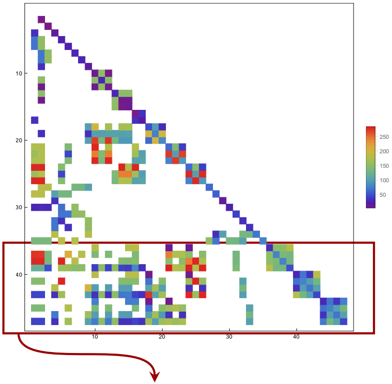
Rational letters

$$\alpha_{13} = \frac{R_1-w}{R_1+w}\,, \quad \alpha_{14} = \frac{R_2-A_{7,2}(x)}{R_2+A_{7,2}(x)}\,, \quad \alpha_{15} = \alpha_{14}(x \leftrightarrow y)\,, \quad \alpha_{16} = \frac{R_1R_4+A_{19,18}}{R_1R_4-A_{19,18}}\,,$$
$$\alpha_{17} = \frac{R_4-(w-x+y)}{R_4+(w-x+y)}\,, \quad \alpha_{18} = \alpha_{17}(x \leftrightarrow y)\,, \quad \alpha_{19} = \frac{R_1R_2-A_{19,10}(x,y)}{R_1R_2+A_{19,10}(x,y)}\,, \quad \alpha_{20} = \alpha_{19}(x \leftrightarrow y)\,,$$
$$\alpha_{21} = \frac{R_2R_4+A_{22,21}(x,y)}{R_2R_4-A_{22,21}(x,y)}\,, \quad \alpha_{22} = \alpha_{21}(x \leftrightarrow y)\,, \quad \alpha_{23} = \frac{R_2R_4+[A_{22,21}(x,y)-2x]}{R_2R_4-[A_{22,21}(x,y)-2x]}\,, \quad \alpha_{24} = \alpha_{23}(x \leftrightarrow y)\,,$$
$$\alpha_{25} = \frac{R_2R_3-A_{22,10}}{R_2R_3+A_{22,10}}\,, \quad \alpha_{26} = \frac{(x+1)R_2+A_{30,2}(-z,-x)}{(x+1)R_2-A_{30,2}(-z,-x)}\,, \quad \alpha_{27} = \alpha_{26}(x \leftrightarrow y)\,,$$
$$\alpha_{28} = \frac{R_1R_4+(A_{19,18}+2wz)}{R_1R_4-(A_{19,18}+2wz)}\,, \quad \alpha_{29} = \frac{(w+1)R_4+(A_{19,18}-w-x+y)}{(w+1)R_4-(A_{19,18}-w-x+y)}\,, \quad \alpha_{30} = \alpha_{28}(x \leftrightarrow y)\,,$$
$$\alpha_{31} = \frac{(y+1)R_1-[w(y+1)+2(y-x)]}{(y+1)R_1+[w(y+1)+2(y-x)]}\,, \quad \alpha_{32} = \frac{(w+1)R_2-[A_{19,10}(x,y)-A_{7,2}(x)]}{(w+1)R_2+[A_{19,10}(x,y)-A_{7,2}(x)]}\,, \quad \alpha_{33} = \alpha_{31}(x \leftrightarrow y)\,,$$
$$\alpha_{34} = \frac{(y+1)R_4-[w(y+1)+(y-1)(x-y)]}{(y+1)R_4+[w(y+1)+(y-1)(x-y)]}\,.$$

Square root letters

Boundary

We present here the complexity distribution diagram of canonical coefficient matrix A'



Top sectors generally exhibit higher complexity

Assisted by the method of expansion by regions [Beneke,Smirnov,1998] [Pak,Smirnov,2011]... , $I_{1,\dots,47}$ are regular at the special kinematic point $(w, x, y = 0, z = 1)$. At this specific point, all MIs (I_i) degenerate into vacuum integrals [Zhe Li,Ren-You Zhang et. al,2024].

Then all the canonical integrals (H_i) vanish except for $H_{1,2,3}$, which are simple tadpole diagrams:

$$H_1 = 1\,, \qquad H_2 = (z)^{-\epsilon}\,,$$

$$H_3 = (z)^{-2\epsilon}\,.$$

In this way, all boundary conditions are determined.

Multiple polylogarithms

Expanding our canonical basis in a Taylor series $H_i = \sum_{j=0} \epsilon^j H_i^{(j)}$, we have:

$$\mathrm{d}H_i^{(0)}(\boldsymbol{\rho}) = 0\,, \quad \mathrm{d}H_i^{(n)}(\boldsymbol{\rho}) = -\sum_k C_k \cdot \mathrm{d}H_i^{(n-1)}(\boldsymbol{\rho})\,\mathrm{d}\log[\alpha_j(\boldsymbol{\rho})]\,, \quad n > 1$$

Define a map $\gamma : [0,1] \to M$, M is a n-dimension complex manifold, we can define the r-fold iterated integrals along the path γ as,

Chen's iterated integrals

$$\gamma^* \mathrm{d}\log[\alpha_i(\boldsymbol{\rho})] = \gamma^* \eta_i = f_i(\lambda)\,\mathrm{d}\lambda \qquad \lambda \in [0,1]$$

$$I_\gamma(\eta_1, \cdots, \eta_r; \lambda) = \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int_0^{\lambda_{n-1}} d\lambda_n f_n(\lambda_n)$$

Clearly, the solutions can be expressed as a linear combination of Chen's iterated integrals [K.T.Chen,1977].

For the special integration kernels $f(\lambda) = \frac{1}{\lambda - c_i}$, it introduce the Goncharov multiple polylogarithms (GPLs) [A.B.Goncharov,2001; 2011],

$$G(c_n, ..., c_1; x) = \int_0^x \mathrm{d}\lambda \, \frac{1}{\lambda - c_n} G(c_{n-1}, ..., c_1; \lambda)\,, \qquad c_1 \neq 0$$

n: length/weight

$$G(\underbrace{0, ..., 0}_{n \text{ times}}; x) = \frac{\log^n x}{n!}\,, \qquad G(;x) = 1\,,$$

Regularisation

Multiple polylogarithms

A few properties of GPLs,

Scaling relation:

$$G(c_n, ..., c_1\,; x) = G(\tfrac{c_n}{x}, ..., \tfrac{c_1}{x}\,; 1)\,, \qquad c_1 \neq 0\,, x \in \mathbb{C} \setminus 0\,.$$

Shuffle algebras:

$$G(\vec{a}\,; x) G(\vec{b}\,; x) = \sum_{\vec{c} = \vec{a} \sqcup\!\sqcup \vec{b}} G(\vec{c}\,; x)\,, \qquad \text{with } \vec{a} = (a_1, \cdots, a_n)$$

...

Efficient programs are available for handling GPLs and their numerical evaluation. PolyLogTools [C.Duhr,F.Dulat,2019], handyG [L.Naterop et al.,2019], NumPolyLog...

Back to our letters, it is obviously the rational letter(only simple poles) could be decomposed as,

$$\gamma^* \mathrm{d}\log[\alpha_i(\boldsymbol{\rho})] = \sum_j \frac{1}{\lambda - c_{i,j}} \mathrm{d}\lambda$$

However, a large part of our letters involves square roots, meaning that they cannot be reduced linearly (include branch points). ⟶ Rationalize

Rationalization of square roots

- Can't finding a set of coordinate transformations $(w, x, y, z) \to (w', x', y', z')$ to rationalize all roots simultaneously.
- Upon calculation, we find that four square roots appear simultaneously at weight 2, at which point the integrals can only be expressed in terms of iterated integrals, making further iterations more difficult.

The four square roots do not simultaneously appear in a single integrand until weight 4.

$$R_1(t) = \sqrt{wt(wt+4)}\,,$$

$$R_2(t) = \sqrt{(x+z-1)^2\,t^2 + 4xt}\,,$$

$$R_3(t) = \sqrt{(y+z-1)^2\,t^2 + 4yt}\,,$$

$$R_4(t) = R_4\,t\,.$$

Rational

- Parameterizing with respect to (w, x, y, z)
- Handling the parameterized square roots by different parameter transformations for different integrands (without introduce different coordinates).

Define a map: $\gamma_t : [0,1] \to \mathbb{C}^4$

$$\boldsymbol{\rho}_i(t) = \boldsymbol{\rho}_i\,t\,, \quad z(t) = (z-1)t + 1\,. \quad i \in \{1,2,3\}$$

γ_0 represent the boundary point.

Rationalization of square roots

Using N_r to represent the number of square roots involved in the integrand

- If $N_r = 0$, we can integrate directly with respect to t, The pullback of the corresponding letters is given by

$$\gamma_t^* \alpha_i(\boldsymbol{\rho}) = c_0(t - q_1)\dots(t - q_k)\dots(t - q_f)\,.$$

Notably, these functions satisfy $\boxed{\gamma_t^* q_i(\boldsymbol{\rho}) = q_i(\boldsymbol{\rho})/t\,,}$ facilitating iterative calculations. The corresponding integrals H_t are defined as:

$$H_t = \int_{\gamma_t} \gamma_t^* \left[\mathrm{d}\log\alpha_i(\boldsymbol{\rho}), G(\omega_n,\dots,\omega_1;1)\right], \qquad \omega \in \{q_i(\boldsymbol{\rho})\}$$

$$= \int_0^1 \sum_{k=1}^{d} \frac{1}{t - q_k} G(\omega_n,\dots,\omega_1;t)\mathrm{d}t = \sum_{k=1}^{d} G(q_k,\dots,\omega_1;1)$$

- If $N_r = 1$, three distinct square roots appear, applying the following transformations to rationalize $R_1(t)$, $R_2(t)$, and $R_3(t)$, respectively,

$$t = \frac{\sigma_1^2}{(\sigma_1+1)w}\,, \qquad t = \frac{\sigma_2^2}{\sigma_2(x+z-1)+x}\,, \qquad t = \frac{\sigma_3^2}{\sigma_3(y+z-1)+y}\,.$$

Rationalization of square roots

- If $N_r = 2$, three square roots appear in pairs across three different scenarios. we implement the following specific parameter transformations for rationalizing $R_{1,2}(t)$, $R_{1,3}(t)$, and $R_{2,3}(t)$, respectively,

$$t = \frac{16\, wx^2\sigma_4^2(\sigma_4+1)^2}{B_1(\sigma_4,x)\left[B_1(\sigma_4,x)+4wx\sigma_4(\sigma_4+1)\right]}\,, \qquad t = \frac{16\, wy^2\sigma_5^2(\sigma_5+1)^2}{B_1(\sigma_5,y)\left[B_1(\sigma_5,y)+4wy\sigma_5(\sigma_5+1)\right]}\,,$$

$$t = \frac{16\, xy^2(x+z-1)^2\sigma_6^2(\sigma_6+1)^2}{B_2\left[B_2+4y(x+z-1)^2\sigma_6(\sigma_6+1)\right]}\,, \qquad \text{with } B_1(\sigma_4,x) = wx-(x+z-1)^2\sigma_4^2\,,$$

$$B_2 = y(x+z-1)^2 - x(y+z-1)^2\sigma_6^2\,.$$

Defining maps γ_{σ_j}, the letters encountered in two cases above represented as:

$$\gamma_{\sigma_j}^*\gamma_t^*\alpha_i(\boldsymbol{\rho}) = \frac{c_1(\sigma_j-r_1)\ldots(\sigma_j-r_k)\ldots(\sigma_j-r_N)}{c_2(\sigma_j-s_1)\ldots(\sigma_j-s_l)\ldots(\sigma_j-s_D)}\,,$$

$$t(\gamma_t^*\,\boldsymbol{\rho}\,;\sigma_j)=1\,,\quad \sigma_j(\boldsymbol{\rho}\,;t)=\sigma_j(\gamma_t^*\,\boldsymbol{\rho}\,;1)$$
$$\gamma_{\sigma_j}^*\,\gamma_t^*\,h(\boldsymbol{\rho}) = h(\boldsymbol{\rho})\,,\qquad h\in\{r_k,\,s_l\}\,.$$

The corresponding integrals H_{σ_j} take the form

$$H_{\sigma_j} = \int_0^1 \gamma_{\sigma_j}^*\gamma_t^*\left[\mathrm{d}\log\alpha_i(\boldsymbol{\rho})\,G(\omega_n',\ldots,\omega_1'\,;\sigma_j(\boldsymbol{\rho}\,;1))\right] = \int_0^e \sum_{k,l=1}^{N,D}\left[\frac{1}{\sigma_j-r_k}-\frac{1}{\sigma_j-s_l}\right]G(\omega_n',\ldots,\omega_1'\,;\sigma_j)\mathrm{d}\sigma_j$$

$$= \sum_{k,l=1}^{N,D}\left[G(r_k,\ldots,\omega_1'\,;\sigma_j(\boldsymbol{\rho}\,;1)) - G(s_l,\ldots,\omega_1'\,;\sigma_j(\boldsymbol{\rho}\,;1))\right].$$

Endpoint: $\sigma_j(\boldsymbol{\rho};\ 1)$

$\omega' \in \{r_k, s_l\}$

Rationalization of square roots

- If $N_r = 3$, with all roots involved in the integrands, it is impossible to execute any more transformations to rationalize them all. The corresponding terms involve an unrationalizable square root, expressed by

$$\gamma^*_{\sigma_j}\gamma^*_t \alpha_i(\boldsymbol{\rho}) = \frac{A(\sigma_j) - B(\sigma_j)\sqrt{P_{j-3}(\sigma_j)}}{A(\sigma_j) + B(\sigma_j)\sqrt{P_{j-3}(\sigma_j)}} \, .$$

The corresponding part of integrals $H_{\text{one-fold}}$ are written as

$$\begin{aligned} H_{\text{one-fold}} &= \int_0^{\sigma_j(\boldsymbol{\rho};1)} \gamma^*_{\sigma_j}\gamma^*_t \left[\mathrm{d}\log \alpha_i(\boldsymbol{\rho})\, G(\omega'_n, \ldots, \omega'_1\,; \sigma_j(\boldsymbol{\rho}\,; 1))\right] \\ &= \int_0^{\sigma_j(\boldsymbol{\rho};1)} \frac{F(\sigma_j)}{\sqrt{P_{j-3}(\sigma_j)}} G(\omega'_n, \ldots, \omega'_1\,; \sigma_j)\mathrm{d}\sigma_j \, . \end{aligned}$$

$$F(\sigma_j) = \frac{B[A \cdot P'_{j-3} - 2P_{j-3}A'] + 2AB'P_{j-3}}{B^2 P_{j-3} - A^2}$$

Polynomials of degree 4

$A(\sigma_j)$, $B(\sigma_j)$ are polynomials of σ_j

$$H_{\text{one-fold}}: \; H^{(4)}_{37}, H^{(4)}_{38}, H^{(4)}_{41}, H^{(4)}_{42}, H^{(4)}_{45}, H^{(4)}_{46}$$

Associate to elliptic curves

Elliptic multiple polylogarithms

Defining the following class of iterated integrals which is called elliptic multiple polylogarithms (eMPLs) [J,Broedel et al.,2017,2018]

Alternative description [Brown,Levin,2011] [Brown,et al.,2017]

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \psi_{n_1}(c_1, t, \vec{a})\, \mathrm{E}_4\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t, \vec{a}\right),$$

with $n_i \in \mathbb{Z}$ and $c_i \in= \mathbb{C} \cup \{\infty\}$, and the recursion starts with $\mathrm{E}_4(;x,\vec{a}) = 1$. The integration kernels ψ_n are obtained by explicitly constructing a basis of integration kernels with at most simple poles in x on the elliptic curve.

$n > 1$ see back up

$$\psi_0(0,x) = \frac{c_4}{y},$$

$$\psi_1(c,x) = \frac{1}{x-c}, \qquad \psi_{-1}(c,x) = \frac{y_c}{y(x-c)} \qquad c \notin \{a_i\},$$

$$\psi_1(\infty,x) = \frac{c_4}{y} Z_4(x), \qquad \psi_{-1}(\infty,x) = \frac{x}{y}.$$

Transcendental functions

Use 'IBP' to reduce

Length (number of integrations): k, weight: $\sum_i |n_i|$. It contains ordinary MPLs as a special case,

$$\mathrm{E}_4\left(\begin{smallmatrix} 1 & \cdots & 1 \\ c_1 & \cdots & c_k \end{smallmatrix}; x, \vec{a}\right) = G(c_1, \ldots, c_k; x).$$

Elliptic multiple polylogarithms

Elliptic polylogarithms are a natural generalisation of ordinary polylogarithms, and share many of their properties: Shuffle algebra ,Closure under integration ,Rescaling relation ,etc.

Back to our cases, the rational function $F(\sigma_j)$ could decomposed as:

$$F(\sigma_j) = \frac{D(\sigma_j)}{N(\sigma_j)} = d_0 + \sum_i \frac{d_i}{\sigma_j - b_i} \,,$$

where d_i, b_i are complex algebraic functions of $\boldsymbol{\rho}$, b_i are roots of polynomials $N(\sigma_j)$ (only simple pole), the corresponding terms:

$$\begin{aligned} H_{\text{one-fold}} &= \int_0^{\sigma_j(\boldsymbol{\rho};1)} \frac{F(\sigma_j)}{\sqrt{P_{j-3}(\sigma_j)}} G(\omega'_n, \ldots, \omega'_1; \sigma_j) \mathrm{d}\sigma_j \,, \\ &= \int_0^{\sigma_j(\boldsymbol{\rho};1)} \left[\frac{d_0}{c_4}\psi_0(0,\sigma_j) + \sum_i \frac{d_i}{y_{b_i}}\psi_{-1}(b_i,\sigma_j) \right] \mathrm{E}_4\left(\begin{smallmatrix} 1 & \cdots & 1 \\ \omega'_n & \cdots & \omega'_1 \end{smallmatrix}; \sigma_j, \vec{a}_j \right) \,, \\ &= \frac{d_0}{c_4}\mathrm{E}_4\left(\begin{smallmatrix} 0 & 1 & \cdots & 1 \\ 0 & \omega'_n & \cdots & \omega'_1 \end{smallmatrix}; \sigma_j(\boldsymbol{\rho},1), \vec{a}_j \right) + \sum_i \frac{d_i}{y_{b_i}} \mathrm{E}_4\left(\begin{smallmatrix} -1 & 1 & \cdots & 1 \\ b_i & \omega'_n & \cdots & \omega'_1 \end{smallmatrix}; \sigma_j(\boldsymbol{\rho},1), \vec{a}_j \right) \,. \end{aligned}$$

At this stage, the final part is represented in the form of eMPLs that associate to three different elliptic curves.

Simplification with shuffle algebra

Here, we demonstrate the application of shuffle algebra in the case of weight 2.

$$H_{31}^{(2)} = \big[2G(\omega_{16},\omega_{37},\beta_3) - G(\omega_{16},\omega_{38},\beta_3) + G(\omega_{16},\omega_{39},\beta_3) + G(\omega_{16},\omega_{40},\beta_3) - 2G(\omega_{16},\omega_{84},\beta_3)$$
$$-2G(\omega_{16},\omega_{86},\beta_3) + 2G(\omega_{16},\omega_{89},\beta_3) + 2G(\omega_{16},\omega_{90},\beta_3) - G(\omega_{16},\omega_{92},\beta_3) - G(\omega_{16},\omega_{94},\beta_3)$$
$$\ldots.$$
$$+G(\omega_{98},\omega_{31},\beta_3) - G(\omega_{98},\omega_{36},\beta_3) + G(\omega_{98},\omega_{41},\beta_3) - G(\omega_{98},\omega_{42},\beta_3) + G(\omega_{98},\omega_{43},\beta_3)$$
$$+G(\omega_{98},\omega_{51},\beta_3) - G(\omega_{98},\omega_{52},\beta_3)\big]/8$$

192 terms

↓

$$H_{31}^{(2)} = \big[G(\omega_{16},\beta_3) - G(\omega_{31},\beta_3) + G(\omega_{36},\beta_3) - G(\omega_{41},\beta_3) + G(\omega_{42},\beta_3) - G(\omega_{43},\beta_3) - G(\omega_{51},\beta_3)$$
$$+G(\omega_{52},\beta_3)\big] \cdot \big[G(\omega_{37},\beta_3) + G(\omega_{38},\beta_3) + G(\omega_{39},\beta_3) + G(\omega_{40},\beta_3) - 2G(\omega_{84},\beta_3)$$
$$-2G(\omega_{86},\beta_3) + 2G(\omega_{89},\beta_3) + 2G(\omega_{90},\beta_3) - G(\omega_{92},\beta_3) - G(\omega_{94},\beta_3) - G(\omega_{95},\beta_3)$$
$$-G(\omega_{98},\beta_3)\big]/8$$

More complicated for weight 3 and weight 4

Summary

- Adopt different parameter transformations to address the problem of multiple square roots. (Apply minimal transformations to handle the corresponding integrands.)
- For the unrationalizable square root parts, express them in the form of elliptic multiple polylogarithms.
- Using shuffle relations to simplify the analytic results.

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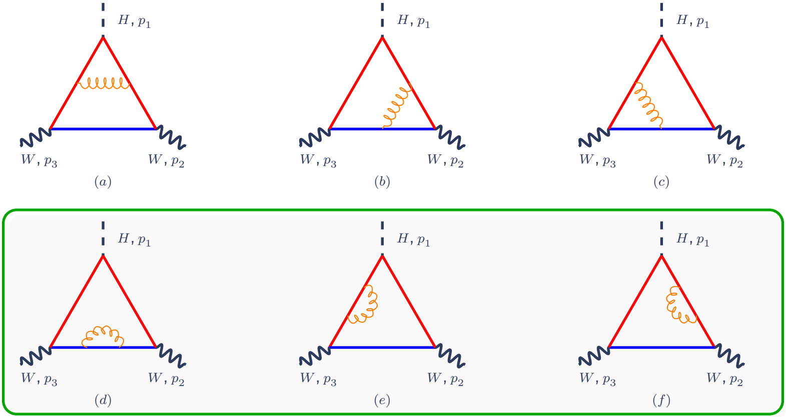
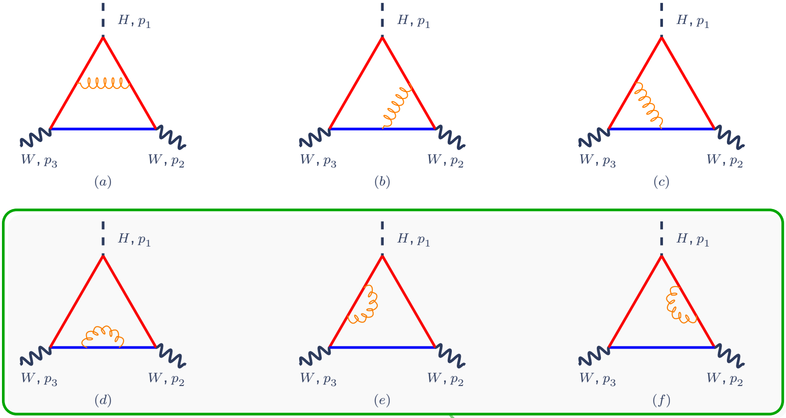
Thank you for your attention!

Representative diagrams



Notion: Same topologies for WWZ/γ vertex

Representative diagrams



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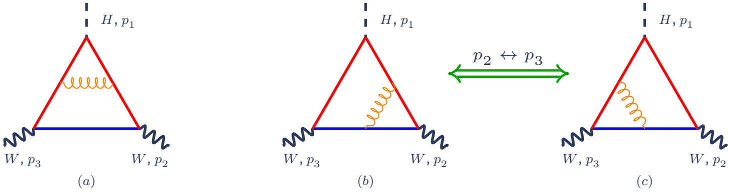
Sub-topologies

Representative diagrams



Notion: Same topologies for WWZ/γ vertex

backup

The coefficients $A_{i,j}$ in the canonical basis:

$$
\begin{aligned}
&A_{5,1}(x) = x - z + 1\,,\\
&A_{7,2}(x) = x + z - 1\,,\\
&A_{19,10}(x,y) = w(x + z + 1) + 2(x - y)\,,\\
&A_{19,18} = w(w - x - y - 2z + 2)\,,\\
&A_{19,19} = w\left[w\,z - xy - (x+y)(z+1) - (z-1)^2\right] + (x-y)^2\,,\\
&A_{22,10} = xy - 2wz + (x+y)(z+1) + (z-1)^2\,,\\
&A_{22,21}(x,y) = w(x - z + 1) - (x-y)(x+z-1)\,,\\
&A_{30,2}(x,z) = x(z+1) + (z-1)^2\,,\\
&A_{30,3} = x^2(2z+1) + x(z-1)(z-2) - (z-1)^3\,,\\
&A_{31,2} = xy + (z-1)^2\,,\\
&A_{38,1} = z(w - x - y)\,,\\
&A_{38,2} = (x+y)\left[xy(z+1) + (x+y+z+1)(z-1)^2\right] - 2wz\left[xy + (z-1)^2\right],\\
&A_{38,3} = wz - x - y\,.
\end{aligned}
$$

backup

Polynomials of $P_{j-3}(\sigma_j)$:

$$\begin{aligned}
P_1(\sigma_4) = &\left[y(x+z-1)^4 - 4wx(x-y)(xy-(z-1)^2)\right]\sigma_4^4 \\
&+ 4wx\left[2xy^2 - y(x-z+1)^2 + 2x(z-1)^2\right]\sigma_4^3 \\
&+ 2wx\left[2wxy + 2xy^2 - y(x-z+1)^2 + 2x(z-1)^2\right]\sigma_4^2 \\
&+ 4yw^2x^2\sigma_4 + yw^2x^2\,, \\
P_2(\sigma_5) = &\left[x(y+z-1)^4 - 4wy(y-x)(xy-(z-1)^2)\right]\sigma_5^4 \\
&+ 4wy\left[2yx^2 - x(y-z+1)^2 + 2y(z-1)^2\right]\sigma_5^3 \\
&+ 2wy\left[2wxy + 2yx^2 - x(y-z+1)^2 + 2y(z-1)^2\right]\sigma_5^2 \\
&+ 4xw^2y^2\sigma_5 + xw^2y^2\,, \\
P_3(\sigma_6) = &\left[x(y+z-1)^2((x-4y)(z-1)^2 - 6xy(z-1) + xy(y-4x))\right. \\
&\left.+ 4wxy^2(x+z-1)^2\right]\sigma_6^4 - 4xy(x+z-1)^2\left[(y+z-1)^2 - 2wy\right]\sigma_6^3 \\
&+ 2y(x+z-1)^2\left[2wxy + 2yx^2 - x(y-z+1)^2 + 2y(z-1)^2\right]\sigma_6^2 \\
&+ 4y^2(x+z-1)^4\sigma_6 + y^2(x+z-1)^4\,.
\end{aligned}$$

back up

Starting from the polynomial defining the elliptic curve $y^2 = P(x)$ is given in the form,

$$P(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$$

Assuming in the following that the branch points are real, distinct and ordered according to $a_1 < a_2 < a_3 < a_4$, and choose the branches as follows,

$$\sqrt{P(x)} \equiv \sqrt{|P(c)|} \times \begin{cases} -1, & x \leqslant a_1 \text{ or } x > a_4 \,, \\ -i, & a_1 < x < a_2 \,, \\ 1, & a_2 < x < a_3 \,, \\ i, & a_3 < x \leqslant a_4 \,. \end{cases}$$

Two periods of the elliptic curve are defined by

$$\omega_1 = 2c_4 \int_{a_2}^{a_3} \frac{\mathrm{d}x}{y} = 2\mathrm{K}(\lambda) \,, \qquad \omega_2 = 2c_4 \int_{a_1}^{a_2} \frac{\mathrm{d}x}{y} = 2i\mathrm{K}(1-\lambda) \,,$$

with

$$\lambda = \frac{a_{14}a_{23}}{a_{13}a_{24}} \,, \qquad c_4 = \frac{1}{2}\sqrt{a_{13}a_{24}} \,, \qquad a_{ij} = a_i - a_j \,.$$

backup

For $n > 1$ cases,

$$\psi_{-n}(c,x) = \frac{y_c}{y(x-c)} Z_4^{(n-1)}(x)\,, \qquad \psi_{-n}(\infty,x) = \frac{x}{y} Z_4^{(n-1)}(x) - \frac{\delta_{n2}}{c_4}\,,$$

$$\psi_n(c,x) = \frac{1}{x-c} Z_4^{(n-1)}(x) - \delta_{n2}\,\Phi_4(x)\,, \qquad \psi_n(\infty,x) = \frac{c_4}{y} Z_4^{(n)}(x)\,.$$

where we have defined $Z_4^{(0)}(x) \equiv 1$,

$$i\Phi_4(x) \equiv \widetilde{\Phi}_4(x) + 4c_4\,\frac{\eta_1}{\omega_1}\,\frac{1}{y}\,, \qquad Z_4(x) \equiv \int_{a_1}^{x} dx'\,\Phi_4(x')\,.$$

$$\eta_1 = -\frac{1}{2}\int_{a_2}^{a_3} dx\,\widetilde{\Phi}_4(x,\vec{a}) = \mathrm{E}(\lambda) - \frac{2-\lambda}{3}\,\mathrm{K}(\lambda)\,,$$

$$\eta_2 = -\frac{1}{2}\int_{a_1}^{a_2} dx\,\widetilde{\Phi}_4(x,\vec{a}) = -i\,\mathrm{E}(1-\lambda) + i\,\frac{1+\lambda}{3}\,\mathrm{K}(1-\lambda)\,,$$

where E denotes the complete elliptic integral of the second kind, and $\widetilde{\Phi}_4(x,\vec{a})$ is defined by

$$\widetilde{\Phi}_4(x,\vec{a}) \equiv \frac{1}{c_4\,y}\left(x^2 - \frac{s_1}{2}\,x + \frac{s_2}{6}\right)\,.$$

Detials about $Z_4^{(n)}$: [Broedel et al.,2017]