Collider Physics

Toni Baroncelli
Haiping Peng
USTC

Year 2024

Overview of the Course

The Course will be shared between
Haiping Peng and me, Toni Baroncelli

And is organized in several parts, lasting in total 16 weeks + 2 weeks for examinations:

(Tentative schedule)

Topic	Weeks	Who	from	→	# lectures
Introduction to basic concepts	2	T.Baroncelli	27/02/24	08/03/24	4
Deep Inelastic Scattering	1	T.Baroncelli	05/03/24	15/03/24	6
Accelerators	1	T.Baroncelli	12/03/24	22/03/24	8
Detectors	1	T.Baroncelli	19/03/24	29/03/24	10
Measurements at Colliders	3	T.Baroncelli	09/04/24	19/04/24	16
Standard Model Theory	2	H.Peng	24/04/24	03/05/24	4
CPV theory and experiment (BELLE,BABAR, LHCb)	2	H.Peng	08/05/24	17/05/24	8
Hadron physics (BESIII, STCF)	2	H.Peng	22/05/24	31/05/24	12
Higher Symmetries (GUT, SUSY, Superstrings....)	2	H.Peng	05/06/24	14/06/24	16

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Practicalities

Find these *slides* and *Lecture Notes* (Check for new versions regularly)

https://indico.pnp.ustc.edu.cn/category/152/

Do not hesitate to contact me:

- After lecture;
- By sending a mail to toni.baroncelli@cern.ch
- By WeChat (I am Toni Baroncelli).

Introductory Part

Collider Physics
Toni Baroncelli
Haiping Peng
USTC

Year 2024

Collider Physics

Haiping Peng
Toni Baroncelli

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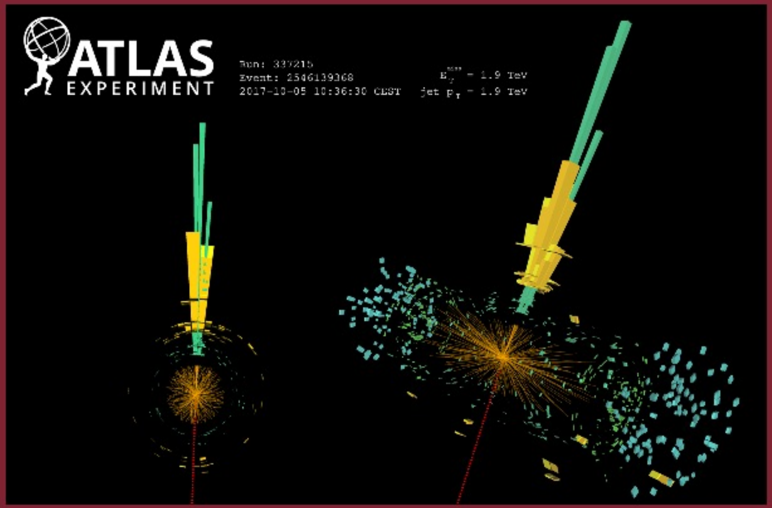


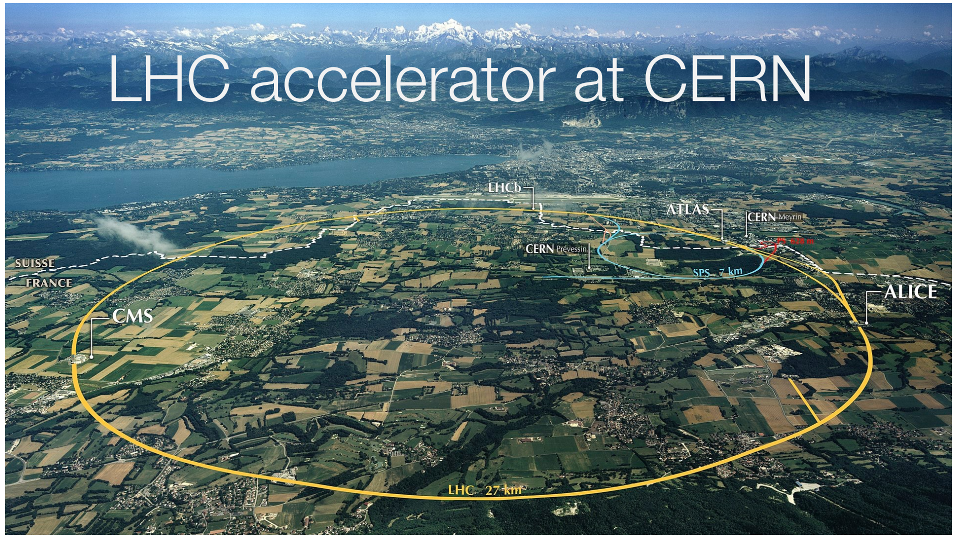
Lecture Notes for the 2024
"Collider Physics" Course

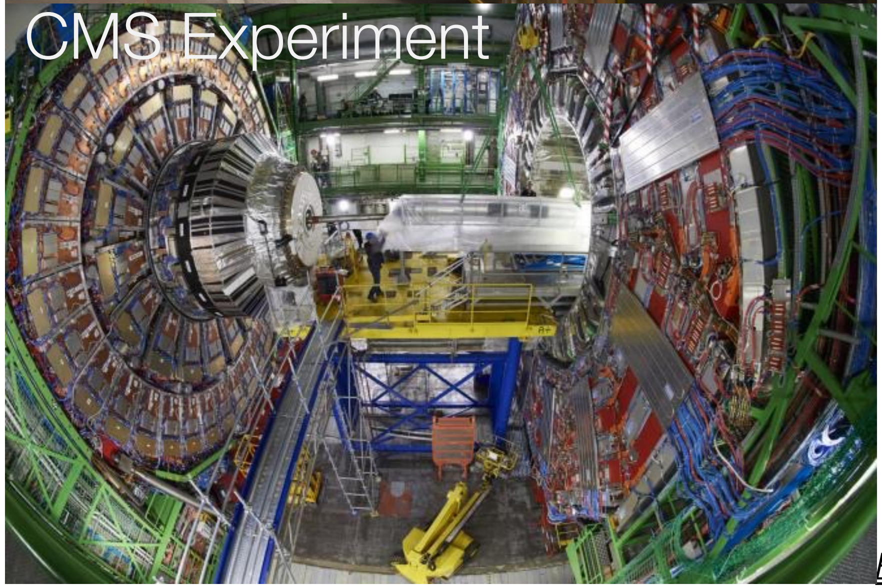
Forward: why 'Collider Physics'?

We will see that to access the intimate structure of matter we have to use probes with wave-lengths as small as possible →

1. Accelerate particles onto targets (used in the past) or
2. Collide two beams against each other

The second option became accessible only with ~modern technologies. Unprecedented high energies are reached → many discoveries





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Topics

1. Probing the Structure of Matter
2. Constituents of Matter & Quantum Numbers
3. The Standard Model, Interactions and Vector Bosons
4. Symmetries and Conservation Laws
5. The Electro-Magnetic Case
6. Feynman diagrams
7. Cross Sections & the Golden Rule
8. Electron – Nucleus Scattering
9. Rutherford Scattering
10. Form Factors

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Prologue: Many Order of Magnitude

The uncertainty principle:
"position x (with uncertainty Δx) and momentum p_x (with uncertainty Δp_x) cannot simultaneously be known to better than

$$\Delta x \Delta p_x \sim \hbar/2.$$

A relation for the energy is obtained by multiplying c,

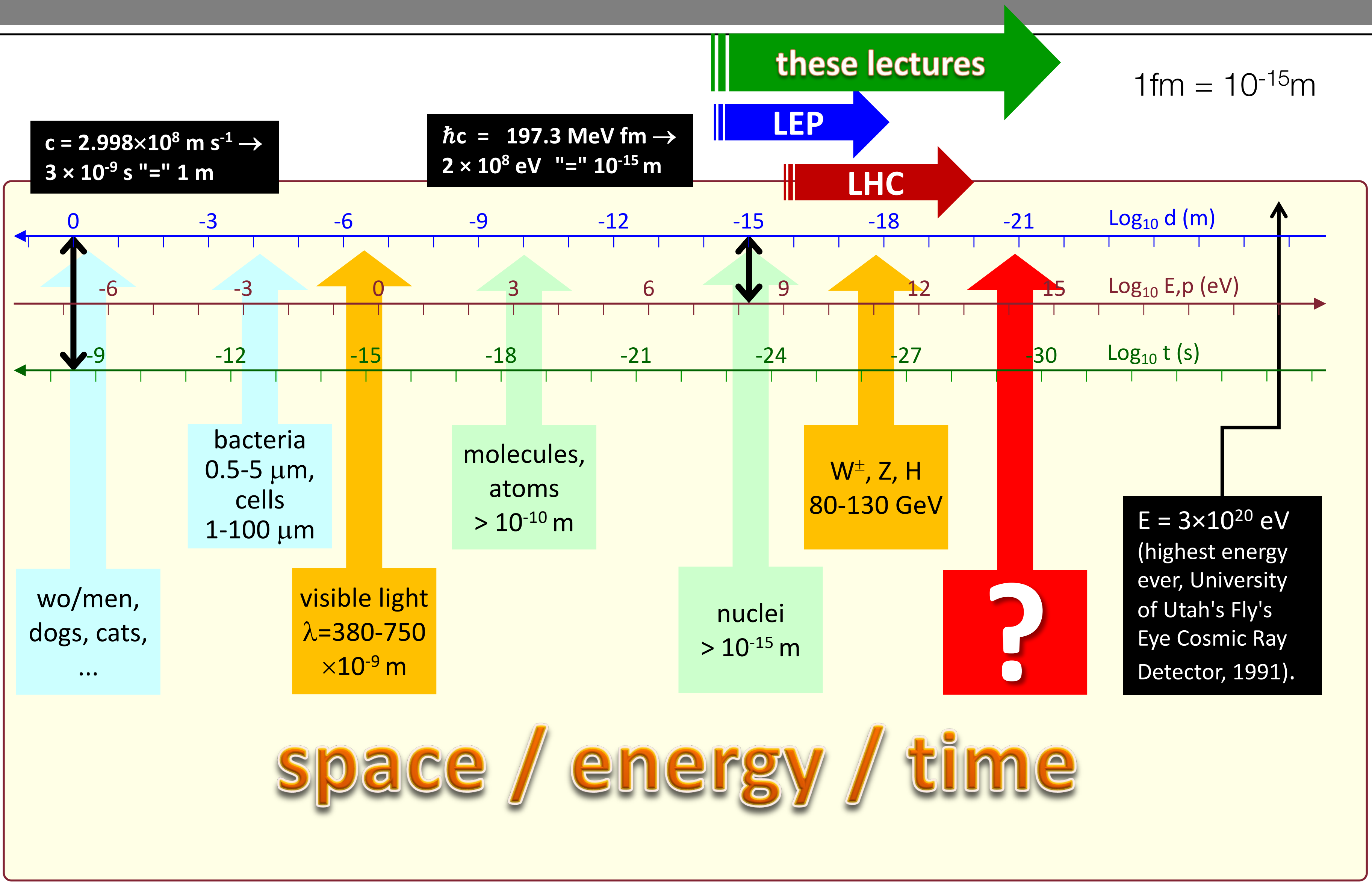
$$\Delta x \Delta E \sim \frac{\hbar c}{2}$$

which gives numerically,

$$\Delta E(MeV) = \frac{1.973^{-11}(MeV\ cm)}{2\Delta x(cm)}$$

Also $\Delta x = c\Delta t \rightarrow \Delta t \Delta E \sim \frac{\hbar}{2}$

1fm = 10^{-15}m



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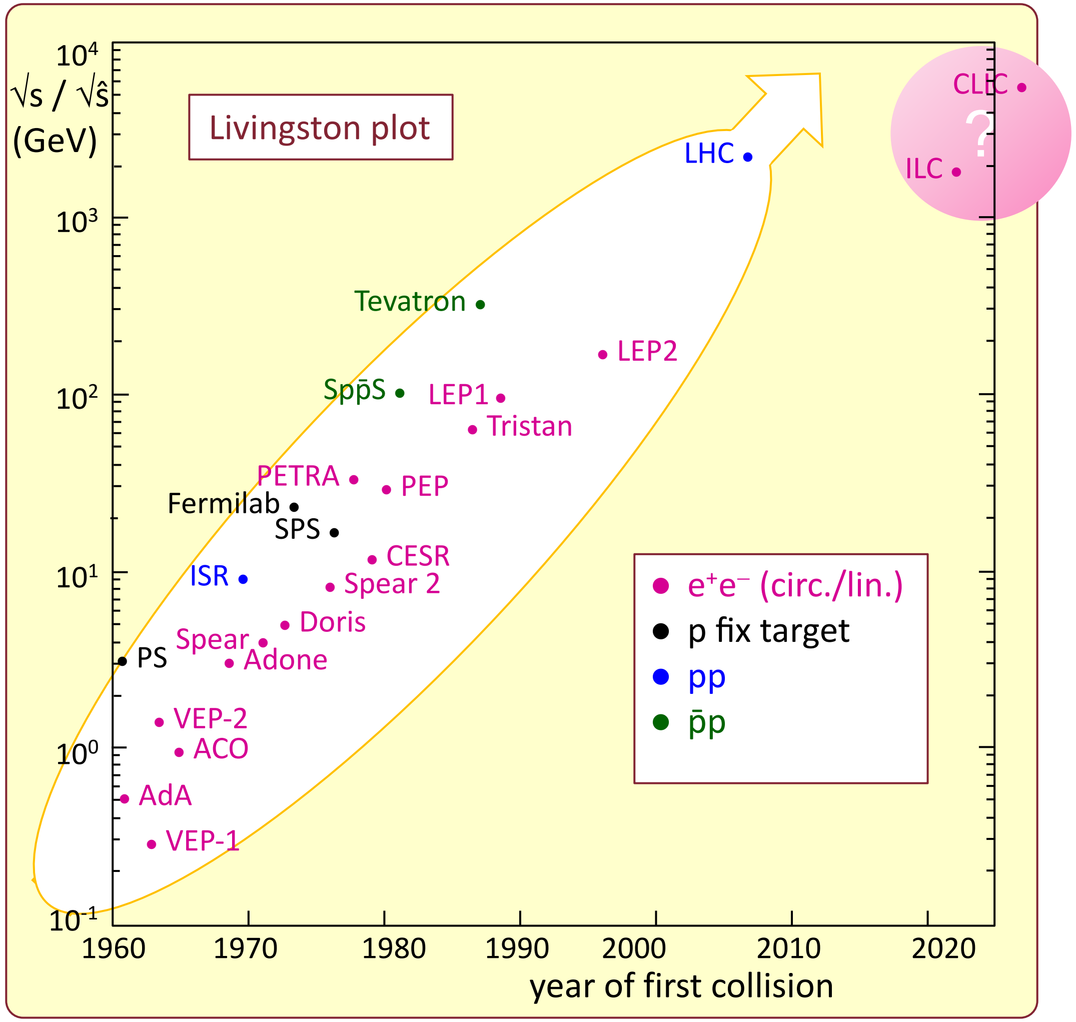
Prologue: the Quest for High Energy

- Discovery range is limited by available data, i.e. by resources (like a microscope).
- The true variable is the resolving power of our microscope.
- Resolving Power $\propto \sqrt{Q^2}$ [i.e. $\propto \sqrt{s}$, the CM energy]
- For non point-like objects, replace $\sqrt{s}$ with the CM energy at component level, called $\sqrt{s}$ ($\sqrt{\hat{s}} < \sqrt{s}$) (quarks in a proton, will see later).
- In the last half a century, the physicists have been able to gain a factor 10 in $\sqrt{s}$ every 10 years (see the "Livingston plot").
- Hope it will continue like that, but needs IDEAS, since not many $$$ (or €€€) will be available.

CEPC in China?



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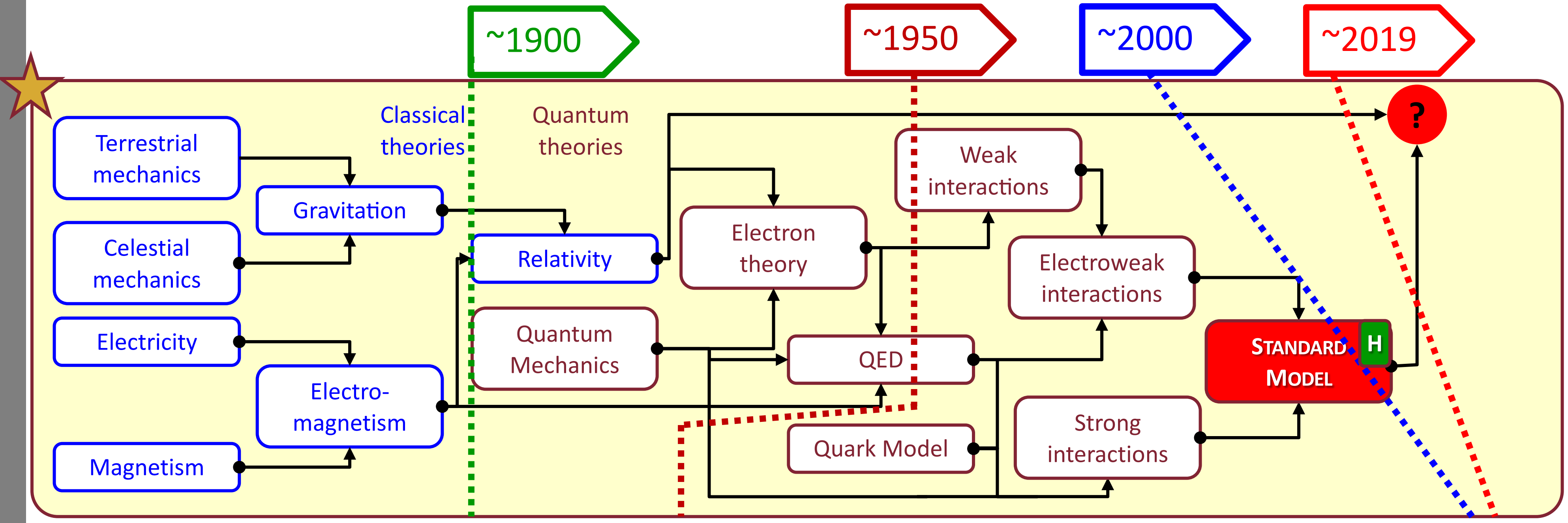
Prologue: The Standard Model

- SM designates the theory of the Electromagnetic, Weak and Strong interactions. The theory has grown in time, the name went together.
- The development of the SM is an interplay between new ideas and measurements.
- Many theoreticians contributed : since the G-S-W (S.Glashow, A.Salam,S.Weinberg) model is at the core of the SM, it is common to quote them as the main authors.



LHC is planned to operate up to year 2040!

SM

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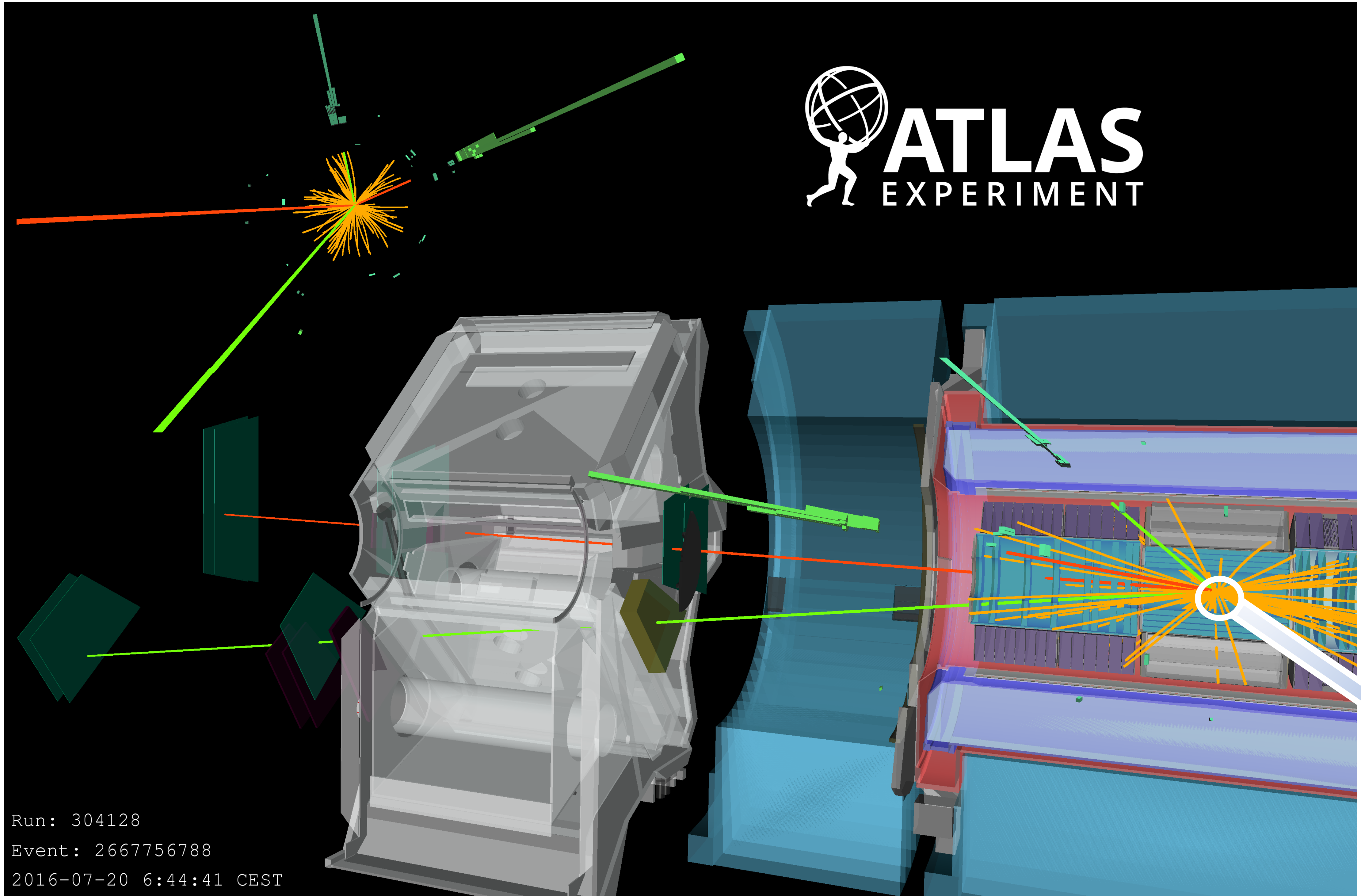
Theory and Experiments



Experiments: instruments and devices that allow you to 'see' the result of an interaction described by Theory.

'see' is a proxy for 'visualise'!

Theory! You do not see what is (happens) inside.

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Quantum Mechanics: a micro-reminder (1) *

(*) More and better in "Lecture Notes"

Free particles = superposition of wave packets (natural units, $c = \hbar = 1$)

$$\psi(\boldsymbol{x}, t) = N \cdot e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - Et)}$$

Observation (a) can be represented as the action of an operator $(\hat{A})$ on the wavefunction (ψ) resulting in an eigenvalue:

$$\hat{A}\psi = a\psi$$

Since an observable must be a real number, → the operator $\hat{A}$ *is* Hermitian.
The application of the momentum operator $\hat{\boldsymbol{p}}$ and of the energy operator $\hat{E}$ *on* ψ return $\boldsymbol{p}$ and E respectively →

$$\hat{p} = -i\nabla \text{ and } \hat{E} = i\frac{\partial}{\partial t}$$

In classical physics (in presence of a potential "V") $E = H = T + V = \frac{\boldsymbol{p}^2}{2m} + V \rightarrow$

Schrödinger equation:

$$i\frac{\partial\psi(\boldsymbol{x}, t)}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi(\boldsymbol{x}, t)}{\partial x^2} + \hat{V}\psi(\boldsymbol{x}, t)$$

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Quantum Mechanics: a micro-reminder (2)

$$i\frac{\partial\psi(\boldsymbol{x},t)}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi(\boldsymbol{x},t)}{\partial x^2} + \hat{V}\psi(\boldsymbol{x},t)$$

First order in time and a second order in space → non-conservation under relativistic transformations → low energy

We must move to a relativistic expression of Energy: from $E = H = T + V = \frac{\boldsymbol{p}^2}{2m}$ to

$$E^2 = \boldsymbol{p}^2 + m^2 \quad \Rightarrow \quad \hat{E}^2\psi(x) = \hat{p}^2\psi(x) + m^2\psi(x)$$

we use $\partial^\mu\partial_\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

$$(\partial^\mu\partial_\mu + m^2)\psi(\boldsymbol{x},t) = 0$$ (Klein-Gordon equation)

Also this equation has a problem: if applied to $\psi(\mathbf{x},t) = N\cdot e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$ it returns $E = \pm\sqrt{p^2 + m^2}$

Unphysical negative energy solutions!

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The 'Dirac' equation

Dirac tried another approach: what happens if you write an equation with only first derivative terms?

$$\hat{E}\psi(\boldsymbol{x},t) = (\alpha \cdot \hat{\boldsymbol{p}} + \beta \cdot m) \cdot \psi(\boldsymbol{x},t)$$

$$i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi$$

Dirac equation has also to be a solution of the Klein-Gordon equation → consequences (see Lecture's notes). The solution has to be a 4-component wave-function with 4 degrees of freedom

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

Natural to assume that two spinors represent spin up and the other two spin down

and terms α_x and β are not scalar but matrices

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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A bit of math

$$\gamma^0 \equiv \beta, \gamma^1 \equiv \beta\alpha_x, \gamma^2 \equiv \beta\alpha_y, \gamma^3 \equiv \beta\alpha_z$$

and covariant derivatives

$$\partial_0 = \frac{\partial}{\partial t}, \partial_1 = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial y}, \partial_3 = \frac{\partial}{\partial z}$$

The Dirac equation can now be written in a covariant form as

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

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Dirac equation, spin & negative energy values

Let us elaborate a bit more:

$$\psi(\boldsymbol{x}, t) = u(E, \boldsymbol{p})e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-Et)}$$

For a particle at rest

$$\psi(\boldsymbol{x}, t) = u(E, \boldsymbol{0})e^{-iEt}$$

$$E\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} = m\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}$$

With 4 solutions:

Positive energy solutions

$$u_1(E,0) = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, u_2(E,0) = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Negative energy solutions

$$u_3(E,0) = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, u_4(E,0) = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

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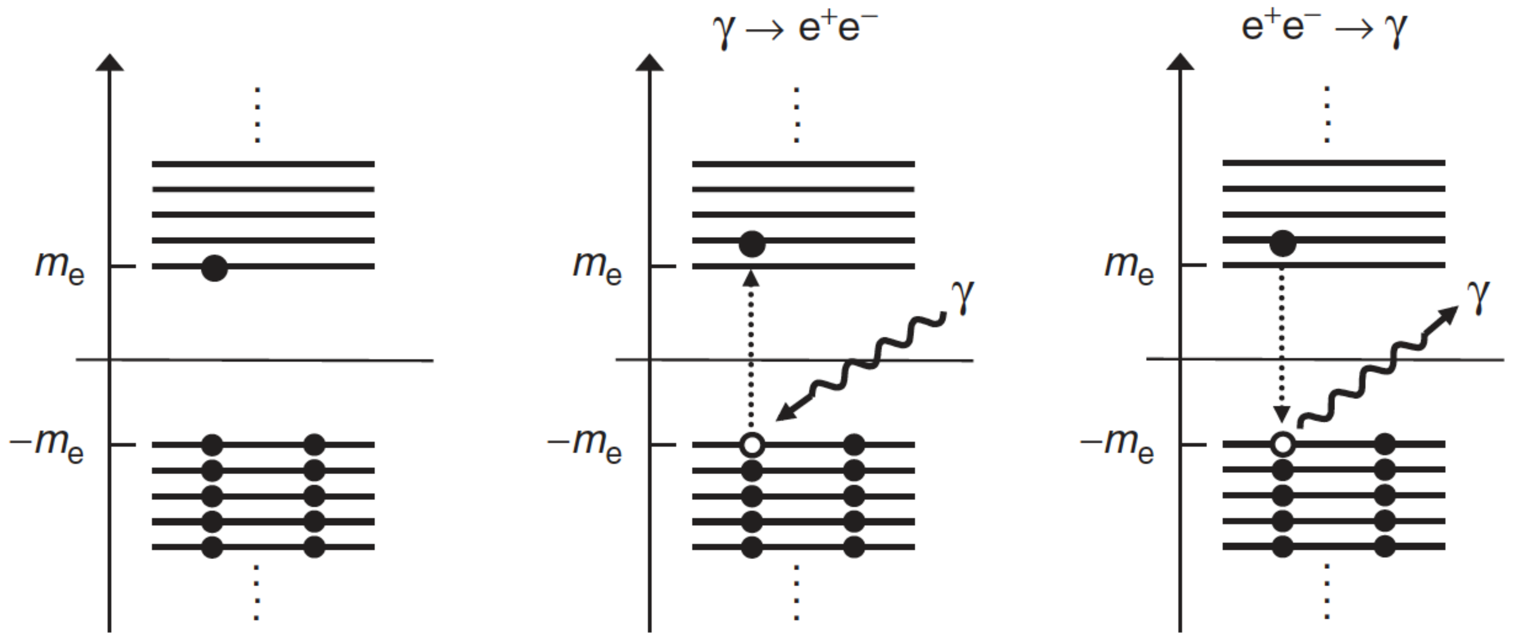
Explicit solutions to the Dirac equation

$$\psi_1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt},\ \psi_2 = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt},\ \psi_3 = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt}\ \text{and}\ \psi_4 = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}.$$



Existence of the physical state of an antiparticle with positive energy, opposite charge to that of the corresponding particle and propagating forward in time

Two explanations:

- Dirac: negative energy states are all occupied, the Pauli principles makes impossible to fill one such state with two identical particles;
- Feynman observed that : the time dependence is contained in the e^{-iEt} term. If you change $t \to -t$ and $E \to -E$ the time behaviour is left unaffected;
- When one 'negative energy' state is excited (by a photon) → leaves a 'hole' state with less negative energy and a positive charge with respect to the fully occupied –E states.

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Constituents of Matter

Table 1 : Fundamental fermions and bosons in the standard model of the microcosm

Fermions				Bosons	
				Fundamental Interactions	Mediators
$\begin{pmatrix} u \\ d \end{pmatrix}$	$\begin{pmatrix} c \\ s \end{pmatrix}$	$\begin{pmatrix} t \\ b \end{pmatrix}$	Quarks	Strong	8 gluons
$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$	$\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}$	$\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}$	Leptons	Electromagnetic	γ
				Weak	W^+, W^-, Z^0
First family	Second family	Third family		Gravitational	Graviton
				Higgs Boson	H^0

the force carriers of the fundamental interactions, the vector bosons, must be added

Leptons and quarks considered to be the ultimate fermionic constituents of matter. Quarks (and antiquarks) appear in three different colours (and anticolours)

scalar (spin 0) Higgs boson. The Higgs boson has been observed; it is thought to be the main ingredient in the mechanism that attributes mass to the particles.

The fact that there are so many particles and that so few constitute the present **stable matter** cannot be currently explained. It is also unclear as to why the ultimate fermionic constituents appear in **three families**, each constituted of two leptons and two quarks, and each being a replica of the same type, see Table 1

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Quantum Numbers of Quarks and Leptons

Table 2 : Leptons and quarks (spin ½ fermions) of the first family

	First family			
	Symbol	Q	L_e	B
Leptons	ν_e	0	1	–
	e^-	−1	1	–
Quarks	u	+2/3	–	+1/3
	d	−1/3	–	+1/3

Q is the electric charge in unit of the proton charge, L_e is the electronic lepton number, B the baryonic number.

For each particle we have an *anti-particle*: it has the opposite quantum numbers of the *particle* and opposite charge

- The first family includes the quarks u, d and the leptons e , ν_e.
- The ordinary matter is constituted of quarks *u, d* and of electrons *e*.
- The second and third families seem to be "replicas" of the first one. Leptons and quarks of generations higher than first can be produced at accelerators

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The Standard Model (2)

Force Carrier	Photon	W & Z Boson	Gluons	Graviton
	EM	Weak	Strong	Gravitational
Quarks	✅	✅	✅	✅
Leptons	✅	✅		✅
Neutrinos		✅		

the neutrinos, being uncharged, are only subject to the weak interaction

The ***hadrons*** are composed of quarks and are known in two topologies:

- those constituted by three quarks (the *baryons*, like the proton and neutron) and
- those constituted by a quark-antiquark pair (the *mesons*).
- As for leptons, antiquarks also exist and particles composed of three antiquarks are called *antibaryons*.

As will be discussed later, the number of baryons and leptons is conserved (one anti-particle counts with a "-" sign). This means that, as described by the relationship $E = mc^2$, the energy can be converted in mass in the form of particles; nevertheless,

the total number of baryons and leptons must be conserved.

→ If an electron is produced, it must be created in association with a positron (its antiparticle, with electric charge and leptonic number of opposite sign) as expected from the Dirac theory.

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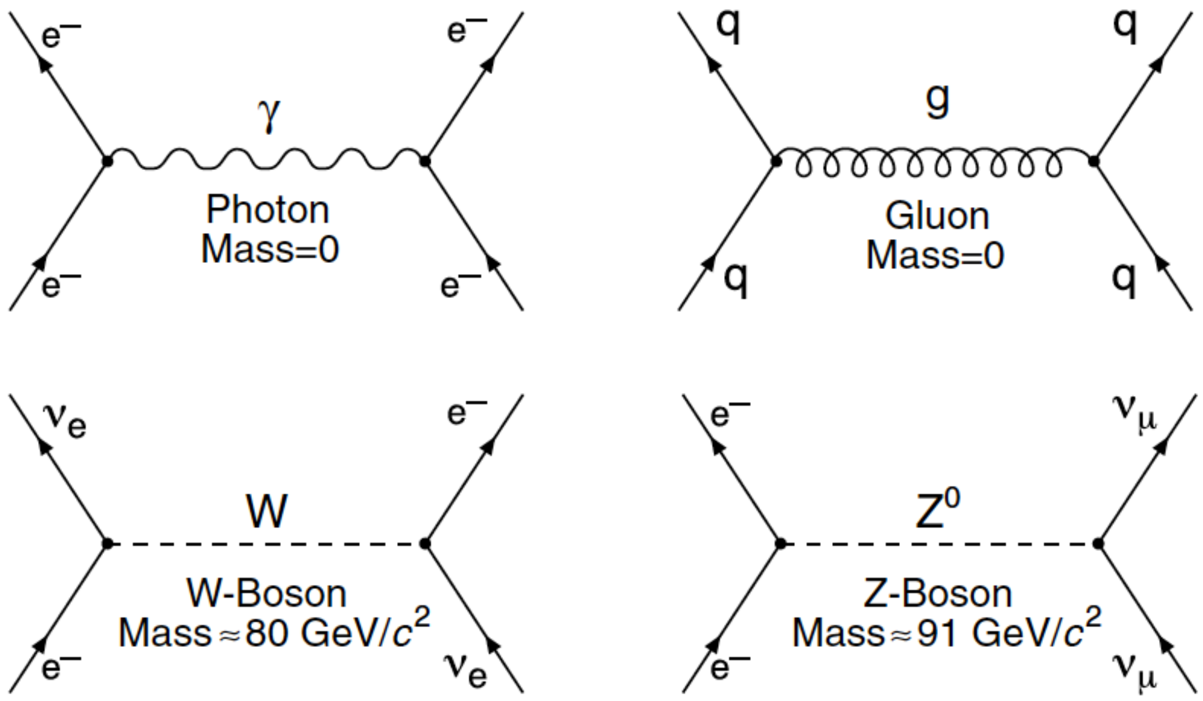
Interactions and Mediators: one comment!

Classical electromagnetism: the electrostatic force is due to a scalar potential. This description is unsatisfactory: in scattering you have transfer of momentum from one particle to the other without any apparent mediating body.

Newton: "It is inconceivable that inanimate brute matter should, without the mediation of something else which is not material, operate upon and affect other matter without mutual contact".

→ the fundamental origin of the electromagnetic interaction (and other interactions) is due to the exchange of particles



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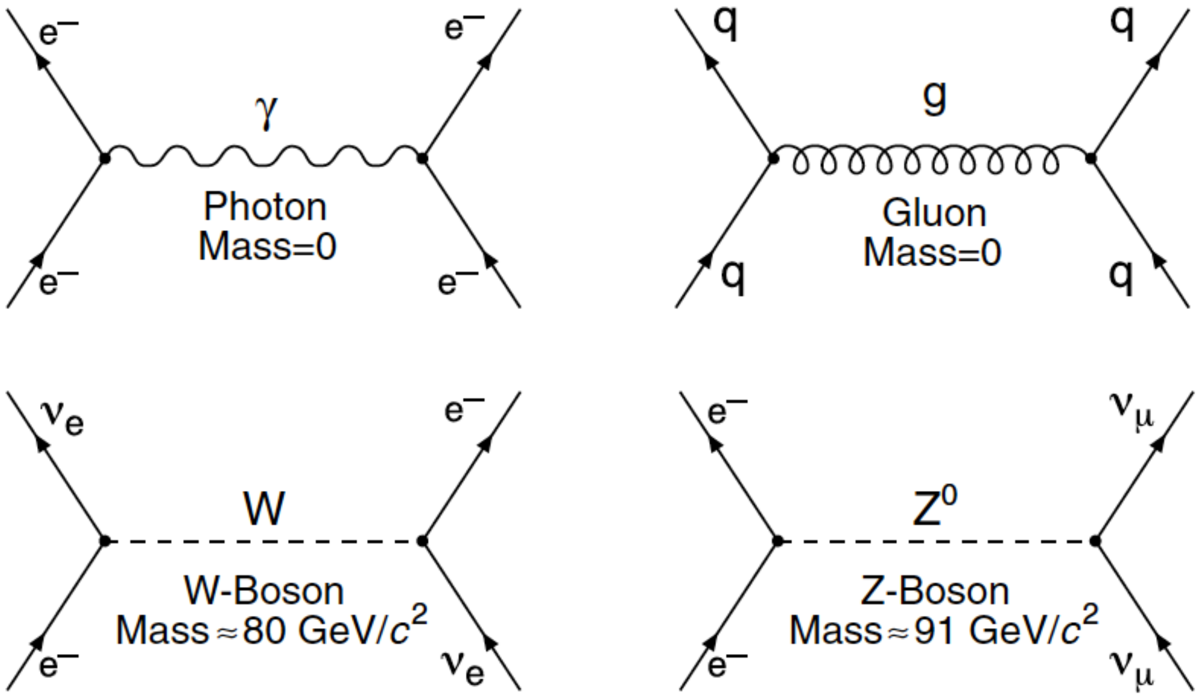
Interactions and Vector Bosons



Interactions are mediated by the exchange of ***vector bosons***, i.e. particles with spin 1: *photons, gluons, W^+ , W^- and Z^0* bosons. Gravity is mediated by a spin 2 boson, the *graviton*

Graphic representation of the different type of interactions between two particles are shown in the diagrams to the left.

- leptons and quarks by straight lines,
- photons by wavy lines,
- gluons by spirals, and
- $W^\pm$ and Z^0 by dashed lines.

Each of these three interactions is associated with a charge: electric charge, weak charge and strong charge. The strong charge is also called colour charge or colour for short.

A particle is subject to an interaction if and only if it carries the corresponding charge:

– Leptons and quarks carry weak charge.
– Quarks and some of the leptons are electrically charged.
– Colour charge is only carried by quarks (not by leptons).

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Interactions

Vector Boson	*Mass*	*Charge*	*Comment*
Photon	0	N	The rest mass of the photon is zero. Therefore, the range of the electromagnetic interaction is infinite. Photons, however, have no electrical charge → do not interact with each other
$W^{\pm}$ Z^0	≈ 80 GeV/c2 ≈ 91 GeV/c2	Y N	Heavy particles can only be produced as virtual, intermediate particles in scattering processes for extremely short times. Therefore, the weak interaction is of very short range
Gluon	0	Y	The gluons, like the photons, have zero rest mass. Gluons, however, carry colour charge. Hence they can interact with each other. As we will see, this causes the strong interaction to be also very short ranged.

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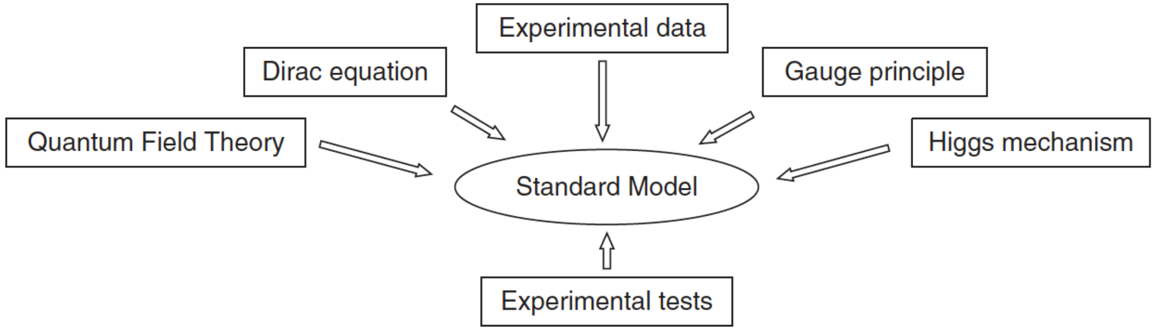
The Standard Model

The Standard Model (SM) is the best description we have today of the microscopic world

- Describes accurately known phenomena
- Important predictive power
- Incorporates known particles, forces and the interaction among them
- Predicted the existence of the Higgs Boson (discovered in 2012).

However the SM is not really a theory, it is rather a ‘Model’:

- Many parameters(*) have to be fixed ‘by-hand’ to describe data
- *SM* created an infrastructure with locations for particles and forces but is not able to explain why it is like that



The *SM* is not the end of the story!
More has to exist and needs to be discovered.

→ *BSM* is called the ‘*Beyond Standard Model*’ Physics. It will incorporate the SM and its capacity to describe / predict microscopic phenomena

(*) SM parameters: masses of twelve fermions, three strengths of gauge interactions, two parameters for the Higgs potential, eight parameters of the mixing matrix CKM → 25 parameters

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Conservation Laws (~Invariance) in Classical Mechanics

In classical mechanics a state with ***n*** degrees of freedom is characterised by ***n*** $\mathbf{q_i}$ coordinates and ***n*** conjugated momenta $\mathbf{p_i}$. The evolution of the system is described, in the Lagrangian formalism, by

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad \frac{dp_i}{d_t} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

If $\mathcal{L}$ does not depend on q_i then $\frac{dp_i}{d_t} = 0$ → the conjugated momentum $\mathbf{p_i}$ is constant

- Translation along x.
  - Let us consider the system Lagrangian Λ Δ Τ ς Δ .1=2/μ Π ξ2. In this case, L does not depend on x and is invariant under translations along the x axis. From the Lagrange Eq. 6.1, one has px D @L=@ P x D mP x D constant, i.e., the linear momentum along x .px D mP x/ is conserved.

- Rotations. The Lagrangian L D T V D .1=2/m'P2r2 where 'Pr D v, does not depend on '; this implies that L is invariant under spatial rotations. From (6.1) follows that p' D @L=@'P D m'Pr2 D mvr= constant, i.e., that the angular momentum is conserved.

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Conservation Laws (~Invariance) in Classical Mechanics

In classical mechanics a state with *n* degrees of freedom is characterised by *n* q_i coordinates and *n* conjugated momenta p_i. The evolution of the system is described, in the Lagrangian formalism, by

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad \frac{dp_i}{d_t} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

If $\mathcal{L}$ does not depend on q_i then $\frac{dp_i}{d_t} = 0 \rightarrow$ the conjugated momentum p_i is constant

- Invariance properties applies to physical systems described by an equation. The system is considered as invariant if the equation describing it is invariant under given transformations (say rotation or translation)
- Invariance properties are closely connected to conservation laws.
- Transformations can be either continuous or discrete.

Symmetries are of great importance in physics. The conservation laws of classical physics (energy, momentum, angular momentum) are a consequence of the fact that the interactions are invariant with respect to their canonically conjugate quantities (time, space, angles). In other words,

physical laws are independent of the time, the location and the orientation in space under which they take place.

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Symmetries in Particle Physics: Parity

- **parity** (P) reflection symmetry: depending on whether the sign of the wave function changes under reflection or not, the system is said to have negative or positive P respectively. For those laws of nature with left-right symmetry, the parity quantum number P of the system is conserved.
- The concept of parity has been generalised in relativistic quantum mechanics. One has to ascribe an intrinsic parity P to particles and antiparticles.
  - Bosons and antibosons have the same intrinsic parity.
  - fermions and antifermions have opposite parities.
  - $W^{\pm}$ and Z do NOT conserve parity in their interactions, so their intrinsic parity is not defined.
  - Convention: P(quarks/leptons) =

    $+1 = P_{e-} = P_{\mu-} = P_{\tau-} = P_u = P_d = P_s = \ldots;$

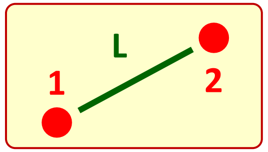
    $-1 = P_{e+} = P_{\mu+} = P_{\tau+} = P_{\bar{u}} = P_{\bar{d}} = P_{\bar{s}} = \ldots$

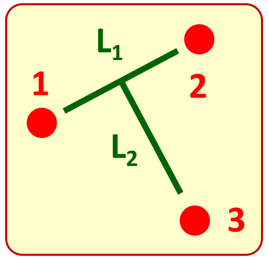
- For a many-body system, P is a multiplicative quantum number :

$$\mathbb{P}\psi(\vec{x}_1,\vec{x}_2\ldots\vec{x}_n,t) = P_1P_2\ldots P_n\psi(\vec{x}_1,\vec{x}_2\ldots\vec{x}_n,t).$$

- Particles in a state of orbital angular momentum are parity eigenstates :
  $Y_{km}(\theta,\phi) = (-1)^k\, Y_{km}(\pi-\theta,\phi+\pi) \rightarrow \mathbb{P}\,|\psi_{km}(\theta,\phi)\rangle = (-1)^k\,|\psi_{km}(\theta,\phi)\rangle$
- Therefore, for a two- or a three-particle system:

$$P_{sys(12)} = P_1P_2\,(-1)^L\,; \qquad P_{sys(123)} = P_1P_2P_3(-1)^{L1+L2}$$





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Symmetries in Particle Physics

- **C, charge conjugation** . Eigenstates of **C** have a quantum number **C-parity** which changes particles into antiparticles and vice versa.
- its eigenvalues are ±1; they are multiplicatively conserved in strong and e.m. interactions.
- Only particles (like π^0, unlike K's) which are their own antiparticles, are eigenstates of $\mathbb{C}$, with values C = (± 1) :

  C = +1 for π^0, η, η';

  C = –1 for ρ^0, ω, ϕ;

  C = –1 for γ. for Z, $\mathbb{C}$ and $\mathbb{P}$ are not defined

- However, few particles are an eigenstate of $\mathbb{C}$; e.g.

  $\mathbb{C}\,|\pi^+\rangle = — |\pi^-\rangle$.

- Why define $\mathbb{C}$? It may be useful when studying reactions where C-conservation plays a role (like in e.-m. decays):

  $\pi^0 \to \gamma\gamma$: +1 $\to$ (–1) (–1) ok;

  $\pi^0 \to \gamma\gamma\gamma$: +1 $\to$ (–1) (–1) (–1) no.

  $Br(\pi^0 \to \gamma\gamma\gamma)$ measured to be $\sim 10^{-8}$.

- Another symmetry derives from the fact that certain groups ("multiplets") of particles behave almost identically with respect to the strong or the weak interaction. Particles belonging to such a multiplet may be described as different states of the same particle. These states are characterised by a quantum number referred to as **strong** or **weak isospin**. Conservation laws are also applicable to these quantities.

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Isospin, a new Quantum Number

- Proton and neutron have very similar masses. Also their behaviour is very similar (cross section pp ~ pn).
- Nuclei where protons and neutrons are exchanged are very similar: 7Li, made of 3 protons and 4 neutrons ~ &7Be, made of 4 protons and 3 neutrons; 13C(6p,7n)~13N(7p,6n)

On the basis of these observations in years 1930 Heisenberg, Condon and Carren made the hypothesis that the proton and the neutron are *two different states of the same entity, the nucleon*.

In analogy with the spin a new quantum number was introduced, the *Isospin*, indicated with *I*. The nucleon was assumed to have *I=1/2* with two I_3 projections: **the proton (+1/2) and the neutron (-1/2)**

baryons	$m(\mathrm{MeV}/c^2)$	B	Q	S	mesons	$m(\mathrm{MeV}/c^2)$	B	Q	S
p	938.272	+1	+1	0	K^+	493.68	0	+1	+1
n	939.565	+1	0	0	K^0	497.65	0	0	+1
Λ	1115.68	+1	0	-1	η	547.7	0	0	0
Σ^+	1189.4	+1	+1	-1	π^+	139.570	0	+1	0
Σ^0	1192.6	+1	0	-1	π^0	134.977	0	0	0
Σ^-	1197.4	+1	-1	-1	π^-	139.570	0	-1	0
Ξ^0	1314.8	+1	0	-2	$\bar{K}^0$	497.65	0	0	-1
Ξ^-	1321.3	+1	-1	-2	K^-	493.68	0	-1	-1

Isospin combines like the spin: for a value *I* of the Isospin you have (2*I* + 1) possible combinations → I_3 values.
0 → singlet
½ → doublet
1 → triplet
The state np (I_3=+1/2-1/2) may belong to a singlet or triplet

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Isospin and Symmetry of Wave Functions

A rotation in the space of the Isospin does not change the state, Isospin is a conserved quantity in strong interactions (but not in weak and electro-weak). Mass differences are only due to EM interactions

If you have a nucleus with *Z* protons and *N* neutrons the projection of the Isospin will be

$$I_3 = \frac{1}{2}Z - \frac{1}{2}N = \frac{(Z-N)}{2}$$

Considering that the baryon number is B = Z + N and that the charge is Q = Z one can write

$$Q = I_3 + B/2$$

Isospin has to be considered when studying the symmetry of a pair of fermions:

$$\Psi = \psi(Space)\chi(spin)I(Isospin)$$
$$Space \rightarrow -1^L$$
$$Spin \rightarrow -1^{S+1}$$
$$Isospin \rightarrow -1^{I+1}$$

The symmetry of a system with *L, S, I* goes like

$$Symmetry \rightarrow (-1)^L\,(-1)^{S+1}\,(-1)^{I+1}$$

A system with two nucleons has to be anti-symmetric as requested by the Pauli principle

Back

Mesons Isospin

baryons	$m(\text{MeV}/c^2)$	B	Q	S	mesons	$m(\text{MeV}/c^2)$	B	Q	S
p	938.272	+1	+1	0	K^+	493.68	0	+1	+1
n	939.565	+1	0	0	K^0	497.65	0	0	+1
Λ	1115.68	+1	0	-1	η	547.7	0	0	0
Σ^+	1189.4	+1	+1	-1	π^+	139.570	0	+1	0
Σ^0	1192.6	+1	0	-1	π^0	134.977	0	0	0
Σ^-	1197.4	+1	-1	-1	π^-	139.570	0	-1	0
Ξ^0	1314.8	+1	0	-2	$\bar{K}^0$	497.65	0	0	-1
Ξ^-	1321.3	+1	-1	-2	K^-	493.68	0	-1	-1

$$Q = I_3 + B/2$$

Protons and neutrons we saw already, all OK

Pion's masses are close by and may be considered as members of the same triplet with I=1 and I_3=-1,0,1
Also the charges are correctly computed using the standard formula (baryon number=0)

The η has a mass very different from the pion's mass and it is ~isolated → only member of a singlet. Charge is OK, I=0, I_3=0

K^+K^0 are also close in mass, like the pair $K^-\overline{K^0}$, may be assumed to be members of a doublet, I=1/2, I_3=-1/2,1/2. However the Q formula fails.

All is restored if we include S, the *strangeness*, in the charge formula and define a new quantum number, the

Hypercharge Y = B + S

$$Q = I_3 + \frac{B+S}{2} = I_3 + \frac{Y}{2}$$

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Baryon Isospin

baryons	$m(\mathrm{MeV}/c^2)$	B	Q	S	mesons	$m(\mathrm{MeV}/c^2)$	B	Q	S
p	938.272	+1	+1	0	K^+	493.68	0	+1	+1
n	939.565	+1	0	0	K^0	497.65	0	0	+1
Λ	1115.68	+1	0	-1	η	547.7	0	0	0
Σ^+	1189.4	+1	+1	-1	π^+	139.570	0	+1	0
Σ^0	1192.6	+1	0	-1	π^0	134.977	0	0	0
Σ^-	1197.4	+1	-1	-1	π^-	139.570	0	-1	0
Ξ^0	1314.8	+1	0	-2	$\bar{K}^0$	497.65	0	0	-1
Ξ^-	1321.3	+1	-1	-2	K^-	493.68	0	-1	-1

$$Q = I_3 + \frac{B+S}{2} = I_3 + \frac{Y}{2}$$

Protons and neutrons we saw already, all OK

Baryons seem to be organised into multiplets as mesons: 1 singlet, 2 doublets, 1 triplet.
The charge-formula works well for baryons!

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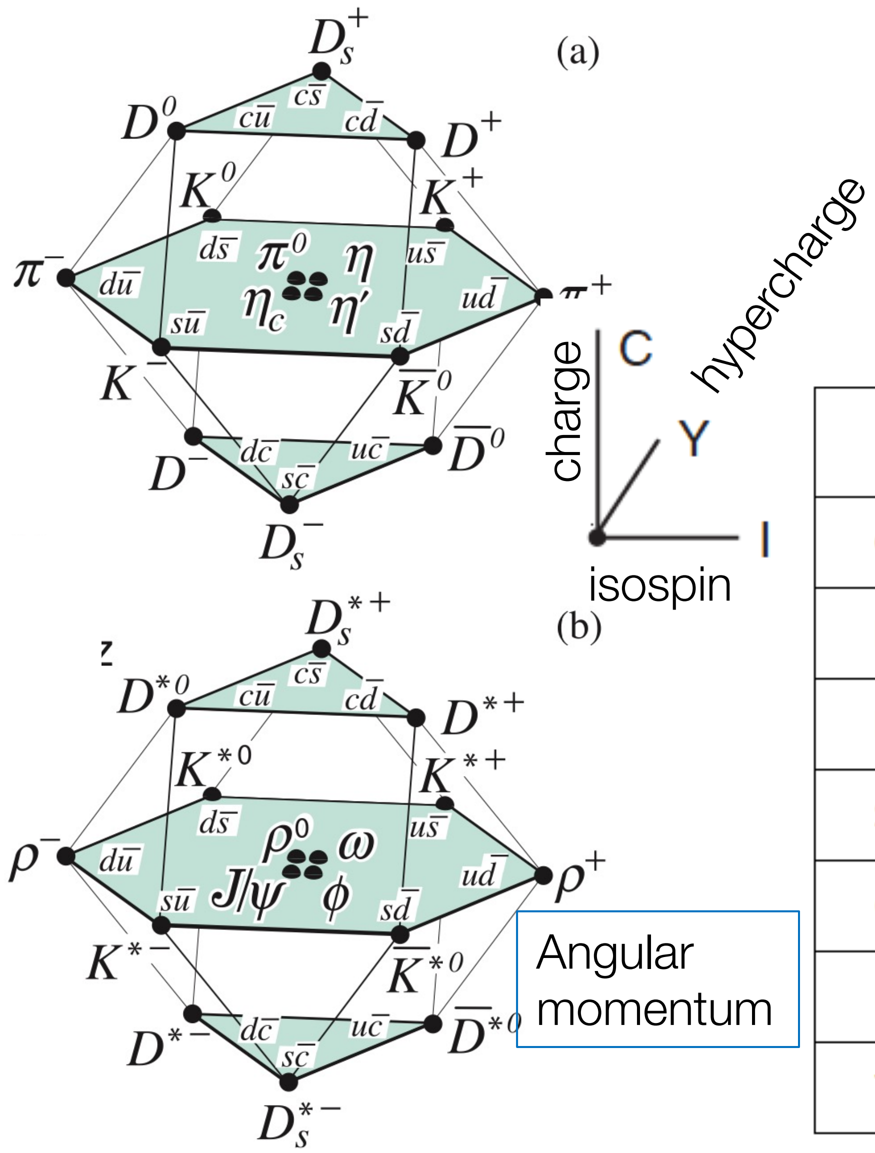
Extension to Other Generations (→c, b, t quarks)



Baryon number

Generalised charge formula $Q = I_z + \frac{\mathcal{B} + S + C + B + T}{2}$,

Bottomness number

Property \ Quark	d	u	s	c	b	t
Q – electric charge	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$
I – isospin	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
I_z – isospin z-component	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	0
S – strangeness	0	0	−1	0	0	0
C – charm	0	0	0	+1	0	0
B – bottomness	0	0	0	0	−1	0
T – topness	0	0	0	0	0	+1

Life-time of Particles

Stability of particles:

- There are stable particles (that are believed to be stable) like the photon γ, the electron e^-, neutrinos ν (and the corresponding antiparticles). Among hadrons only the proton (and the antiproton) is stable.
- In some models 'Beyond Standard Model' BSM also the proton and the neutrinos may unstable.
- Unstable particles classified according to lifetime

Interaction	Life-time (s)	Comment	Example
Weak	10^{-6} to 10^{-12}		$\mu^- \to e^- \overline{\nu}_e \nu_\mu$
Electro-magnetic	10^{-16} to 10^{-20}		$\pi^0 \to \gamma\gamma$
Hadronic	10^{-23}	'Resonances' (more in the following!)	$\Lambda \to p\pi^-$
W^+, W^-, Z^0	$\sim 10^{-25}$	Decay is very fast due to large mass → light particles	$Z^0 \to \mu^+\mu^-$

We will see more in next slides.

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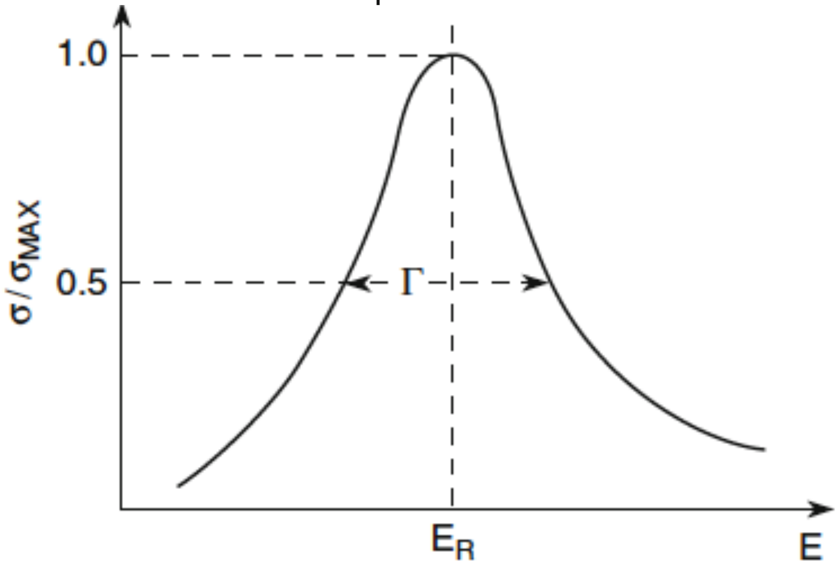
Of the Uncertainty Principle (reminder!)

The *Uncertainty Principle* says that in the microscopic world there are limits to our knowledge of a state and that we determine some observables with limited precision.

Pairs of conjugated physics variables: energy and time, position and momentum

$$\Delta E \Delta t \geq \frac{\hbar}{2} \qquad \Delta p_x \Delta x \geq \frac{\hbar}{2}$$

Resonance is a possible intermediate state produced by the interaction of two particles: $a + b \rightarrow R \rightarrow a' + b'$



In the case of a resonance, we consider the energy (mass) and width Γ (uncertainty on the energy → on the mass), lifetime of the particle at rest τ.

$$\Delta E \sim \Gamma$$

$$\Delta t \sim \tau$$

$$\rightarrow \Delta E \Delta t = \Gamma \tau \geq \frac{\hbar}{2}$$

For a Γ ~ 100 MeV → τ ~ 10^{-23} s

If a resonance is produced with an energy E → lifetime is increased by a factor $\gamma = E/m \rightarrow \tau\gamma$

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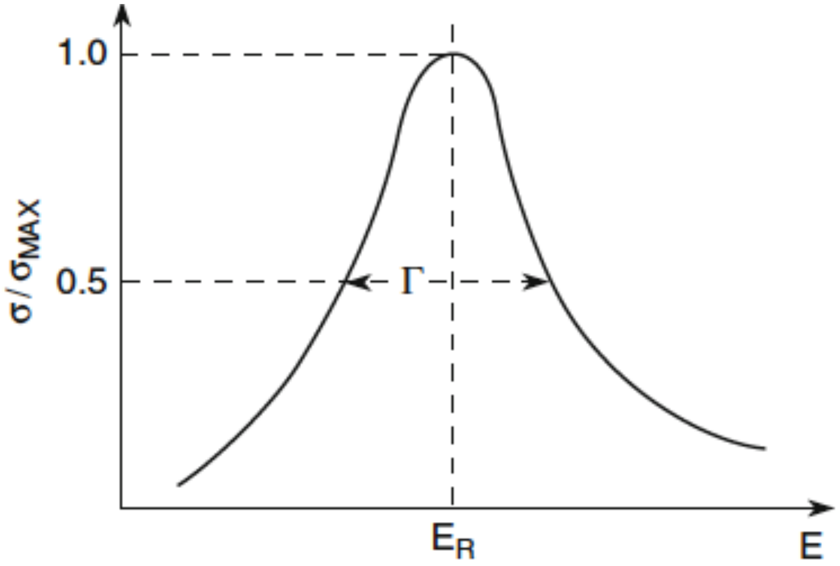
Hadronic Resonances

Some particles are stable. Many others, with a relatively long life-time, longer that 10^{-10} seconds, travel enough to be detected and their decay is due to weak interactions. Hadron resonances undergo strong interactions and their life-time is so short that it cannot be measured.

How to get their life-time? The uncertainty principle says a short life-time implies an uncertainty in the energy of the state → repeated measurements of the mass will give different results that are expected to be distributed like a 'Breit-Wigner' distribution.
The width of the distribution is connected to the life-time: $\Gamma = \Delta Mc^2 = \hbar/\tau$

A width of ~100 MeV corresponds to a life-time of 10^{-23} s → much too short to be measured!



$$|\chi(Mc^2)|^2 \propto \frac{1}{(Mc^2 - M_0c^2)^2 + \frac{\Gamma^2}{4}};$$

- Mc^2: Measured mass value
- M_0c^2: Particle mass value
- Γ: Resonance width Γ

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Getting the Breit Wigner Shape

Elastic scattering case → same particles of the initial state also in the final state
(but different momenta)

Particle *b* at rest

$$a + b \rightarrow R \rightarrow a' + b'.$$

Intermediate excited states may show up in hadronic interactions. These states *R* are called resonances. → described by a wave function with a de Broglie $\lambda = \frac{h}{p} = 2\pi\hbar/pc$ and described by a wave-function like $\psi(t) = \psi(0)e^{-\frac{E_R}{\hbar}t}$. This state is unstable and will decay to a' and b'

Particle *a* in motion. If you increase the energy of this particle and if an intermediate state R is produced this will show up with an increase in the cross section.

dP = probability of decay per unit time
N = number of produced resonances
→ -dN = λN dt where λ is a constant that describes how quickly the resonance decays.

$$N(t) = N_0 e^{-\lambda t} = N_0 e^{-t/\tau}$$

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Getting the Breit Wigner Shape

Let us imagine the elastic formation process of a generic resonance R, which decays with lifetime into the same initial particles. The presence of a interaction process is demonstrated by the different directions and momenta of the particles in the final state, that is,

$a + b \rightarrow a' + b'$ Elastic scattering case

The unstable resonance R is described by the free particle wave function $\psi(0)e^{-i\omega_R t}$ multiplied by a real function describing its decay probability as a function of time, that is,

$$\psi(t) = \psi(0)e^{-i\omega_R t}e^{-\frac{t}{2\tau}} = \psi(0)e^{-\frac{iE_R}{\hbar}t}e^{-\frac{\Gamma}{2\hbar}t},$$

where the relations $\omega_R = \frac{E_R}{\hbar}$ and $\tau = \frac{\hbar}{\Gamma}$ have been inserted in the last equality. The probability of finding the particle at a time t is

$$I(t) = \psi^*\psi = \psi(0)^2 e^{-t/\tau} = I(0)e^{-t/\tau},$$ Exponential life-time

corresponding to the radioactive decay law.

The Fourier transform gives us the energy distribution

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$$

$$\chi(E) = \int \psi(t)e^{iEt}dt = \psi(0)\int e^{-t[(\Gamma/2)+iE_R-iE]}dt =$$

$$= \frac{K}{(E_R - E) - i\Gamma/2}.$$

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Relation between Width and Cross Section

$$\chi(E) = \int \psi(t) e^{iEt} dt = \psi(0) \int e^{-t[(\Gamma/2)+iE_R-iE]} dt = $$
$$= \frac{K}{(E_R - E) - i\Gamma/2}.$$

The constant K has to be related to the cross section: the square of the wave function $\chi(E)$ represents the probability of finding the particle in the energy state E, it must be proportional to the process cross-section, that is

$$\sigma(E) = \sigma_0 \chi^*(E)\chi(E) = \sigma_0 \frac{K^2}{[(E_R - E)^2 + \Gamma^2/4]}.$$

Maximum at $E=E_R$

Since the the distribution has a maximum at E_R we can write $1 = \chi^*(E_R) * \chi(E_R) = \frac{K^2}{\Gamma^2/4}$

$$\left[\frac{\Gamma^2/4}{(E_R - E)^2 + \Gamma^2/4}\right]$$

$$\sigma_0 \sim wave - lenght^2$$

Let us now consider the formation and decay of a resonance with total angular momentum J by the collision of two particles a; b, with spin s_a, s_b . In this case, the cross-section must be averaged over the number of spin states of the incoming particles and multiplied by a factor (2J +1)

$$\sigma_{el}(E; J) = 4\pi\lambda\!\!\!^{-2} \frac{(2J+1)}{(2s_a+1)(2s_b+1)} \left[\frac{\Gamma^2/4}{(E_R - E)^2 + \Gamma^2/4}\right].$$

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Relation between Width and Cross Section

$$\chi(E) = \int \psi(t) e^{iEt} dt = \psi(0) \int e^{-t[(\Gamma/2)+iE_R-iE]} dt = \frac{K}{(E_R - E) - i\Gamma/2}.$$

The constant K has to be related to the cross section: the square of the wave function $\chi(E)$ represents the probability of finding the particle in the energy state E, it must be proportional to the process cross-section, that is

$$\sigma(E) = \sigma_0 \chi^*(E)\chi(E) = \sigma_0 \frac{K^2}{[(E_R - E)^2 + \Gamma^2/4]}.$$

Maximum at E=E_R

Since the the distribution has a maximum at E_R we can write 1 = $\chi^*(E_R) * \chi(E_R) = \frac{K^2}{\Gamma^2/4}$

$$\left[\frac{\Gamma^2/4}{(E_R - E)^2 + \Gamma^2/4}\right]$$

$$\sigma_0 \sim wave - lenght^2$$

The measurement of the cross section also allows to determine the spin of the colliding particles. Spins of the incoming particles are known → Factor (2J+1)

$$\sigma_{el}(E; J) = 4\pi\lambda\!\!\!^{-2} \frac{(2J+1)}{(2s_a+1)(2s_b+1)} \left[\frac{\Gamma^2/4}{(E_R - E)^2 + \Gamma^2/4}\right].$$

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One observation

$$\sigma_{el}(E;J) = 4\pi\lambda\!\!\!^{-2}\frac{(2J+1)}{(2s_a+1)(2s_b+1)}\left[\frac{\Gamma^2/4}{(E_R-E)^2+\Gamma^2/4}\right].$$

Γ^2 is ok for the elastic case because you have the same particles in the initial and final state

Γ tells us how strongly R couples to (ab) once in the initial state and once in the final state → Γ^2

In the inelastic case $a + b \to X$ ('many particles') → $\Gamma_{ab}\,\Gamma_X$

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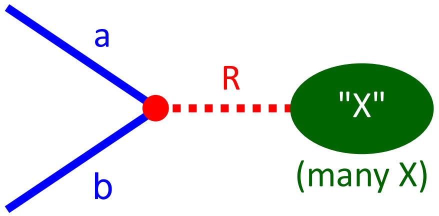
Resonance : σ_R Inelastic Case

$$\sigma_{ab\to R\to X}(E_{CM}=\sqrt{s}) = \frac{\pi}{|\vec{p}_{a,b}|^2}\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\frac{\Gamma_{ab}\Gamma_X}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4} \approx$$



$$\approx \left[\frac{16\pi}{s}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_X}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

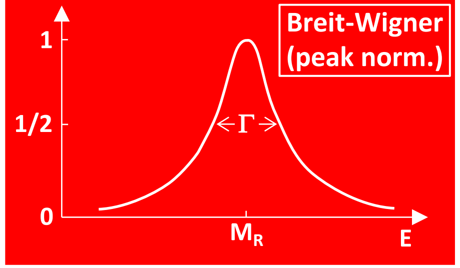
scale factor (1/s)

statistical factor (particle spins)

= BR(R→ab)

= BR(R→X)

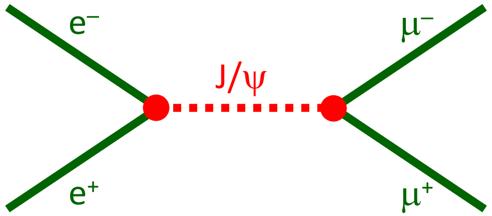
Breit-Wigner (peak norm.)



$(E, \vec{p})$: CM 4-mom.
Γ_R : constant width
$\Gamma_{ab,X}$: couplings
M_R : E_0, mass

$$\sigma(e^+e^-\to J/\psi\to\mu^+\mu^-) = \left[\frac{16\pi}{s}\right]\left[\frac{3}{4}\right]\left[\frac{\Gamma_{ee}}{\Gamma_{tot}}\right]\left[\frac{\Gamma_{\mu\mu}}{\Gamma_{tot}}\right]\left[\frac{(\Gamma_{tot}/2)^2}{\left(\sqrt{s}-M\right)^2+(\Gamma_{tot}/2)^2}\right] =$$

$$= \frac{12\pi}{s}BR_{J/\psi\to e^+e^-}BR_{J/\psi\to\mu^+\mu^-}\left[\frac{(\Gamma_{tot}/2)^2}{\left(\sqrt{s}-M\right)^2+(\Gamma_{tot}/2)^2}\right].$$



$e^+e^- \to J/\psi \to \mu^+\mu^-$
$\sigma_{peak} \propto 1/s$ ($\approx M_R^{-2}$), independent from coupling strength.

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Resonance : Different Functions

Many more parameterizations used in literature (semi-empirical or theory inspired), e.g.:

$$\sigma_0 = \left[\frac{16\pi}{(2p)^2}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_{final}}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

original, non-relativistic

$$\sigma_1 = \left[\frac{16\pi}{s}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_{final}}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

m_a, m_b << p

$$\sigma_2 = \left[\frac{16\pi}{M_R^2}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_{final}}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

simpler, neglect s-dependence

$$\sigma_3 = \left[\frac{16\pi}{M_Z^2}\right]\left[\frac{3}{4}\right]\left[\frac{\Gamma_{ee}}{\Gamma_Z}\right]\left[\frac{\Gamma_{f\bar{f}}}{\Gamma_Z}\right]\left[\frac{M_Z^2\Gamma_Z^2}{\left(s-M_Z^2\right)^2+M_Z^2\Gamma_Z^2}\right]$$

relativistic BW for $e^+e^- \to Z \to f\bar{f}$

$$\sigma_4 = \left[\frac{16\pi}{M_Z^2}\right]\left[\frac{3}{4}\right]\left[\frac{\Gamma_{ee}}{\Gamma_Z}\right]\left[\frac{\Gamma_{f\bar{f}}}{\Gamma_Z}\right]\left[\frac{s\Gamma_Z^2}{\left(s-M_Z^2\right)^2+s^2\Gamma_Z^2/M_Z^2}\right]$$

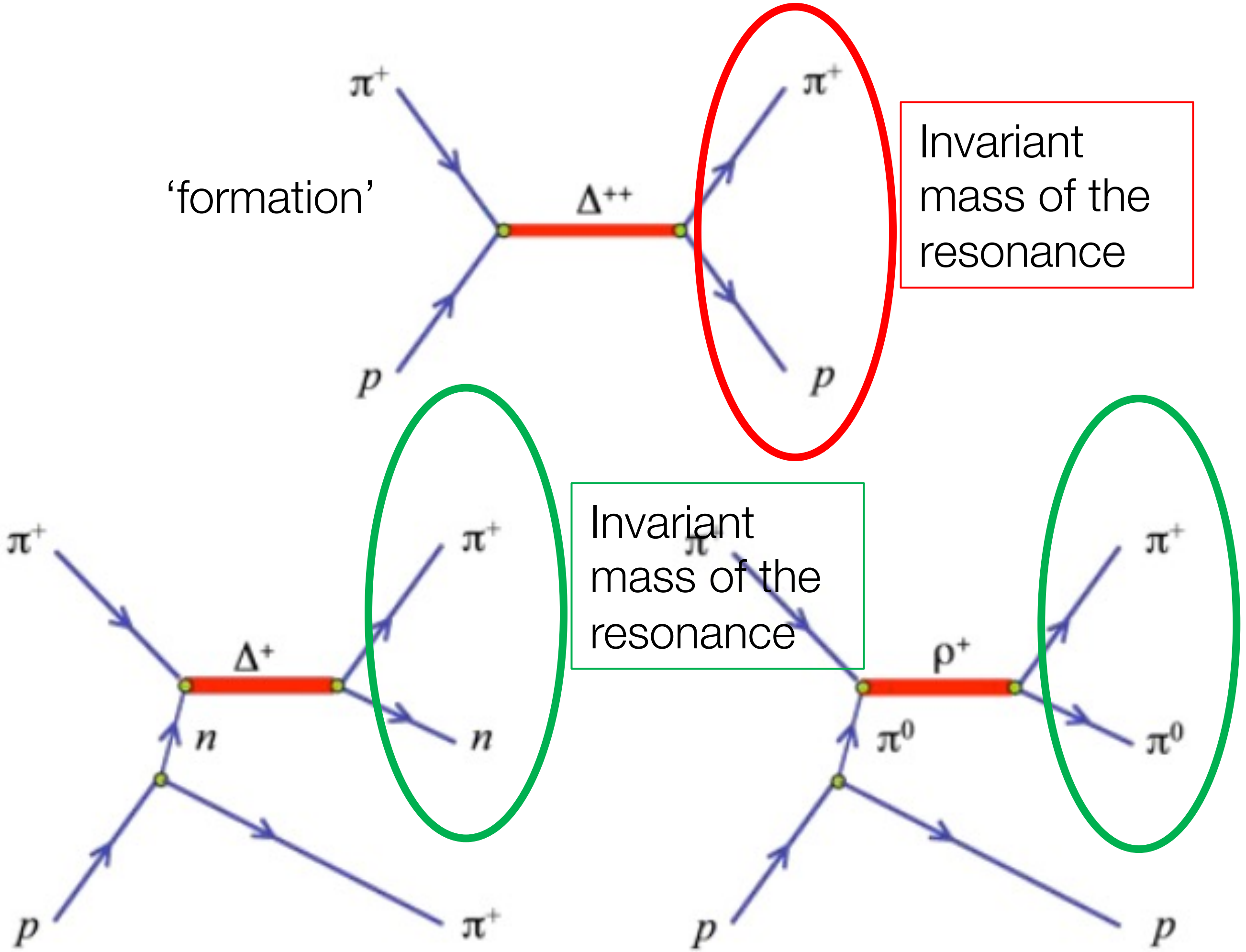
"s-dependent Γ_Z" (used at LEP for the Z lineshape)

Resonances

There are two mechanisms for the observation of resonances:

- Formation: the two interacting particles have both quantum numbers and energy to produce a resonance. It's presence will be determined by an increase of the cross-section at an energy corresponding to the peak of the resonance → mass & width → life-time

- Production: the two interacting particles do NOT have the quantum numbers to create a resonance. In this case an intermediate virtual particle is needed. Determining the presence of this resonance will be more difficult, it will imply the detection of all decay products and the construction of the invariant mass.



Baryonic resonance in 'production'

Mesonic resonance in 'production'

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The Electromagnetic Paradigm

- *Electromagnetic (EM)*: The analytic form of the interaction potential between charged particles is precisely known → Maxwell's original formulation → relativistic representation → quantized field theory.

- *Quantum electrodynamics (QED):* includes the spin of particles, the interaction between charged particles through the exchange of a photon. Many physics quantities (cross-sections, particle lifetimes, magnetic moments, and so on) can be computed very precisely.

- The success of the QED has been extended to the weak and (partially) to the strong interactions. The calculation of the transition probability quantity allows the comparison of theoretical predictions with experimental measurements.

- It has been verified that the electromagnetic and the weak interactions are different manifestations of a single interaction, the electroweak interaction. The unification of the two interactions occurs at an energy larger than the W^+, W^- , and Z^0 boson masses, i.e., for energies above ~90 GeV. At lower energies, the electromagnetic and weak interactions are separate and different.

- At much larger energies, the electroweak and strong interaction unification (the so-called Grand Unification Theory) can be hypothesized.

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The Electro-Magnetic Case

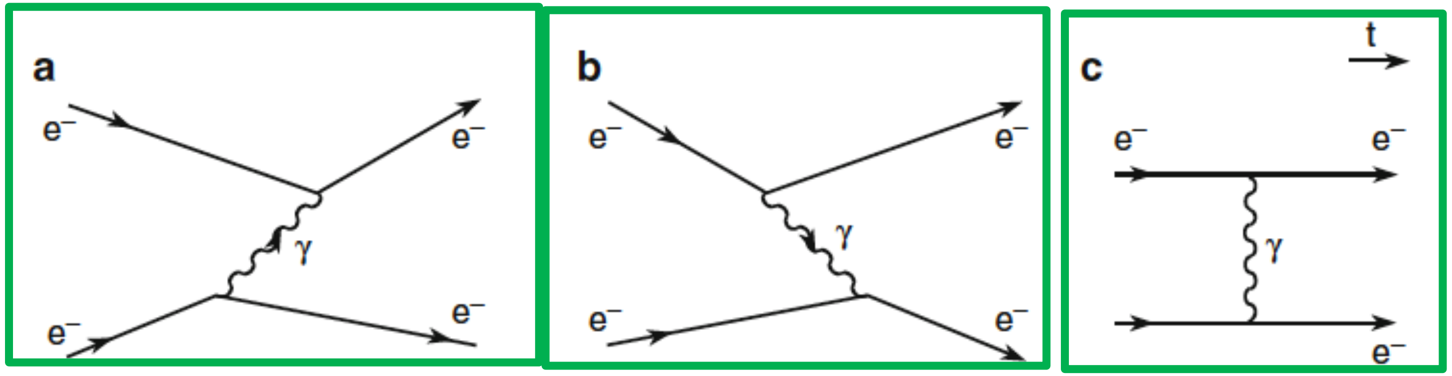
The electrostatic force is ruled by the Coulomb law:

$$F = K\frac{q_1 q_2}{r^2}\vec{r} \qquad \boldsymbol{F}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\boldsymbol{e}_{12} = -\boldsymbol{F}_2.$$

where q_1 e q_2 are the point-like particle electric charges, r is the distance between them, $\vec{r}$ is a unit vector directed from q_1 to q_2 and K is a proportionality constant. The dependence on r is similar to that of Newton's law. The electric charges q_1 and q_2 do not depend on the inertial mass and can assume positive and negative values. The *electrostatic force can attract or repel particles*, depending on the relative sign of the charges.

Today, we know that each magnetic field is generated by electric charges in motion, and that the field is a relativistic effect of such a motion. In the relativistic theory, the electric and magnetic fields are so interrelated that an unique interaction, the electromagnetic interaction, must be considered. The force acting on a moving charge q with velocity ***v*** in an electric field ***E*** and a magnetic field ***B*** is (in the International System):

$$F = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B}$$



Lowest order Feynman diagrams for the elastic electron–electron collision due to the E interaction. The time is in abscissa (from left to right). In (a), the electron in the bottom part emits a "virtual" photon which is then absorbed by the electron at the top; (b) shows the other way around. The diagram (c) represents the interaction without specifying the time direction

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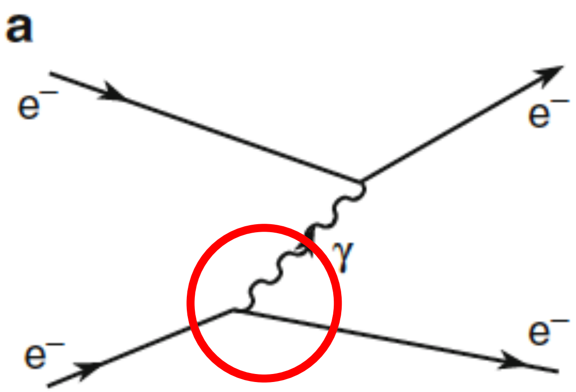
The EM Case & Feynman Diagrams



The representation of Feynman diagrams has been very successful for describing the EM interaction in quantum mechanics. The Feynman diagrams represent a visual method which provides both an intuitive representation of the interaction and a rigorous way to obtain numerical quantities through a perturbative calculation method. Let us first consider the Feynman diagram to the left for the interaction between two electrons. Experimentally, it is observed that the two electrons repel each other. It is conceivable that the interaction occurs through the exchange of a particle, the photon.

An electron at **rest** cannot, however, emit a "real" photon because this would violate the energy conservation law

Process	Initial state energy	Final state energy
$e \to e\gamma$	$m_e c^2$	$\neq m_e c^2 + \frac{p_e^2}{2m_e} + E_\gamma.$

E_γ is the total energy of the emitted photon, p_e is the (nonrelativistic) momentum acquired by the electron, m_e is the electron mass. According to Heisenberg's uncertainty principle, if energy is measured with an uncertainty of ΔE, the uncertainty on the time measurement is

$$\Delta t \geq \hbar/(\Delta E).$$

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The Nature of the Electromagnetic Interaction

Suppose that a photon is emitted from the first electron violating the energy conservation (by a quantity ΔE). Now, suppose that the photon is absorbed by the second electron within a time t, resulting in a second violation of energy conservation by a value of -ΔE. If all this occurs within the time interval defined by the uncertainty principle, *none of the two violations can be observed:* they are "hidden" by the uncertainty principle. Such a process would therefore be considered as possible. *The net effect is an exchange of energy and momentum between the two electrons, and is therefore a way in which two electrons, and more generally two charged particles, can interact*.

$$\Delta t \geq \hbar/(\Delta E).$$

In quantum theory, it is believed that the EM interaction takes place in this way, i.e., by exchanging a virtual, non-observable, photon. The electron quantum numbers, particularly its spin, must remain unchanged. → *As a consequence, the exchanged particle must have integer spin and is therefore a boson* (all the force mediators have spin 1, except the graviton that has spin 2).

Assuming that the boson is moving at the light speed and has no rest mass, it travels in the time interval Δt at a distance of Δr=cΔt. Placing this quantity in the uncertainty relation, one obtains $\Delta E \geq \hbar/(\Delta t) \approx \hbar c/(\Delta r)$. Since the interaction energy V is of the order of E (for a single exchange process), one has

$$\Delta E \simeq V = \alpha_i \hbar c/r.$$

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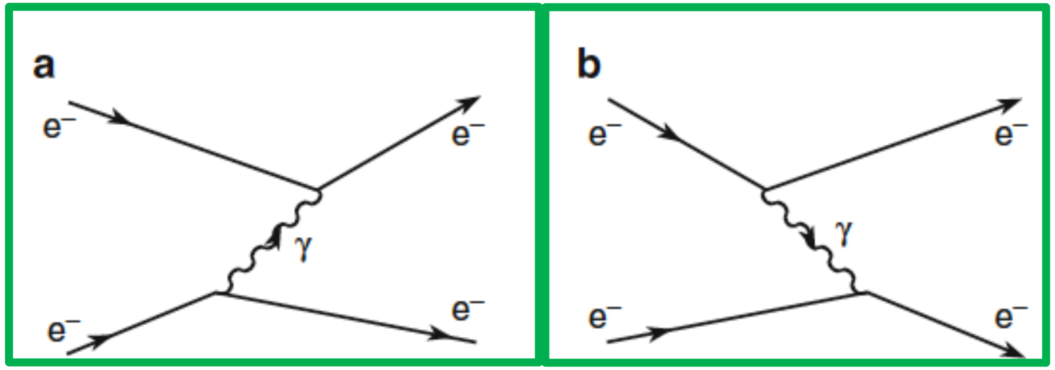
The Photon is the Quantum of EM

The dimensionless constant α_i characterizes the interaction intensity. Equation of page before is obtained, assuming the exchange of a massless boson. As a consequence, the forces due to the exchange of virtual massless particles decrease with the distance r as

$$F \sim dV/dr \sim 1/r^2$$

From the opposite approach, i.e., assuming a force dependence $1/r^2$ one knows that the exchanged virtual particle in the EM interaction is massless. The exchanged virtual particle can therefore be identified with the photon, the real quantum of the EM field. Since the gravitational force has a similar dependence in $1/r^2$, the graviton should also be massless.

In quantum mechanics, the emission of a photon by an electron means that the electron creates a field; when the electron absorbs a photon, it destroys a field. An interaction should therefore be considered as a sequence of two processes: the creation and the destruction of a field represented by the two diagrams below.



Since the virtual field (the photon) is not observable, and since the final states are identical, we do not know if the photon is created (destroyed) by the electron at the top or at the bottom of the diagram: the two processes are indistinguishable.

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Elaborating more

The dimensionless parameter characteristic of the EM interaction is the fine structure constant (also called electromagnetic coupling constant) already known from atomic physics. It can be derived by equating

$$\Delta E \simeq V = \alpha_i \hbar c / r.$$

with the Coulomb energy potential:

$$\frac{\alpha_i \hbar c}{r} = K q^2 / r$$

electromagnetic coupling constant

from which one finds (q = e is the electric charge of the electron):

$$\alpha_i = e^2 / \hbar c$$

$$\boldsymbol{F}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}^2} \boldsymbol{e}_{12} = -\boldsymbol{F}_2.$$

numerically, one has (S.I., cgs, cgs)

$$\alpha_{EM} = \frac{e^2}{4\pi\epsilon_0 \hbar c} = \frac{(1.602 \cdot 10^{-19})^2}{4\pi \cdot 8.85 \cdot 10^{-12} \cdot 1.05 \cdot 10^{-34} \cdot 3 \cdot 10^8} = \mathbf{1/137.1}$$

$$\alpha_{EM} = \frac{e^2}{\hbar c} = \frac{(4.803 \cdot 10^{-10})^2}{1.0546 \cdot 10^{-27} \cdot 3 \cdot 10^{10}} = \mathbf{1/137.1} = 7.294 \cdot 10^{-3}$$

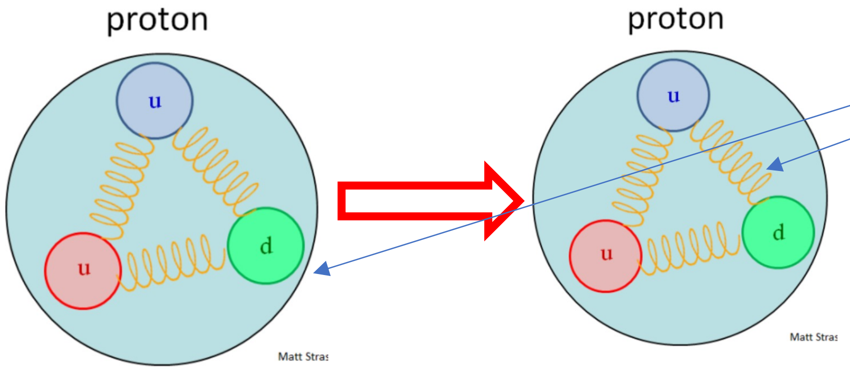
$$\alpha_{EM} = e^2 \qquad (\hbar = c = 1)$$

Back

Nuclear Sizes and Shapes

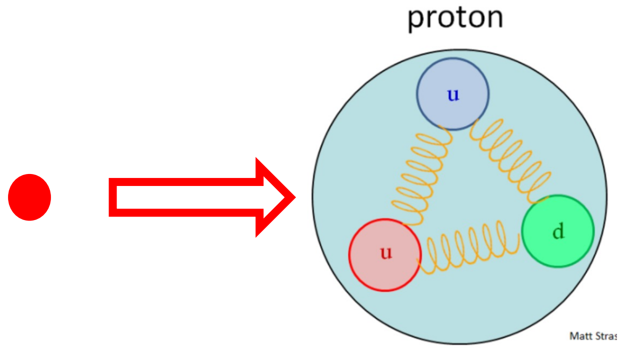
Nuclear sizes and shapes → use scattering technique → use a projectile (accelerated or from radioactivity) that hits a target

projectiles are extended objects



nuclear forces between the projectile and the target are complex and not well understood

Use electrons! Point-like projectiles!



- *The interactions between an electron and a nucleus, nucleon or quark take place via the exchange of a virtual photon — this may be very accurately calculated inside quantum electrodynamics (QED).*
- *These processes are in fact manifestations of the well known electromagnetic interaction, whose coupling constant α ≈ 1/ 137 is much less than one. This last means that higher order corrections play only a tiny role*

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Reminder!

In electron scattering experiments one employs highly relativistic particles. Hence it is advisable to use four-vectors in kinematical calculations. The zero component of space–time four-vectors is time, the zero component of four momentum vectors is energy:

$$x = (x_0, x_1, x_2, x_3) = (ct, \boldsymbol{x}) ,$$
$$p = (p_0, p_1, p_2, p_3) = (E/c, \boldsymbol{p}) .$$

Three-vectors are designated by bold-faced type to distinguish them from four-vectors. The Lorentz-invariant scalar product of two four-vectors a and b is defined by

$$a \cdot b = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 = a_0 b_0 - \boldsymbol{a} \cdot \boldsymbol{b} .$$

In particular, this applies to the four-momentum squared $p^2 = \frac{E^2}{c^2} - \boldsymbol{p}^2 .$

This squared product is equal to the square of the rest mass *m* (multiplied by c²). This is so since a reference frame in which the particle is at rest can always be found and there p = 0, and E = mc². The quantity

$m = \sqrt{p^2}/c$ is called the invariant mass .

From the two relations above we obtain $E^2 - \boldsymbol{p}^2 c^2 = m^2 c^4$

and also (this approximation is valid for electrons already at energies of a few MeV) $E \approx |\boldsymbol{p}| c$ if $E \gg mc^2$.

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(Geometric) Cross Sections – 1 (Povh…)

The inner structure of the matter can only be probed using scattering experiments. The more energetic projectiles are used the smaller is the equivalent de Broglie wave length

$$\lambda = \hbar/p$$

Small wave lengths allow the inspection of the inner structure of the matter.

Ideal Simplified Experiment:

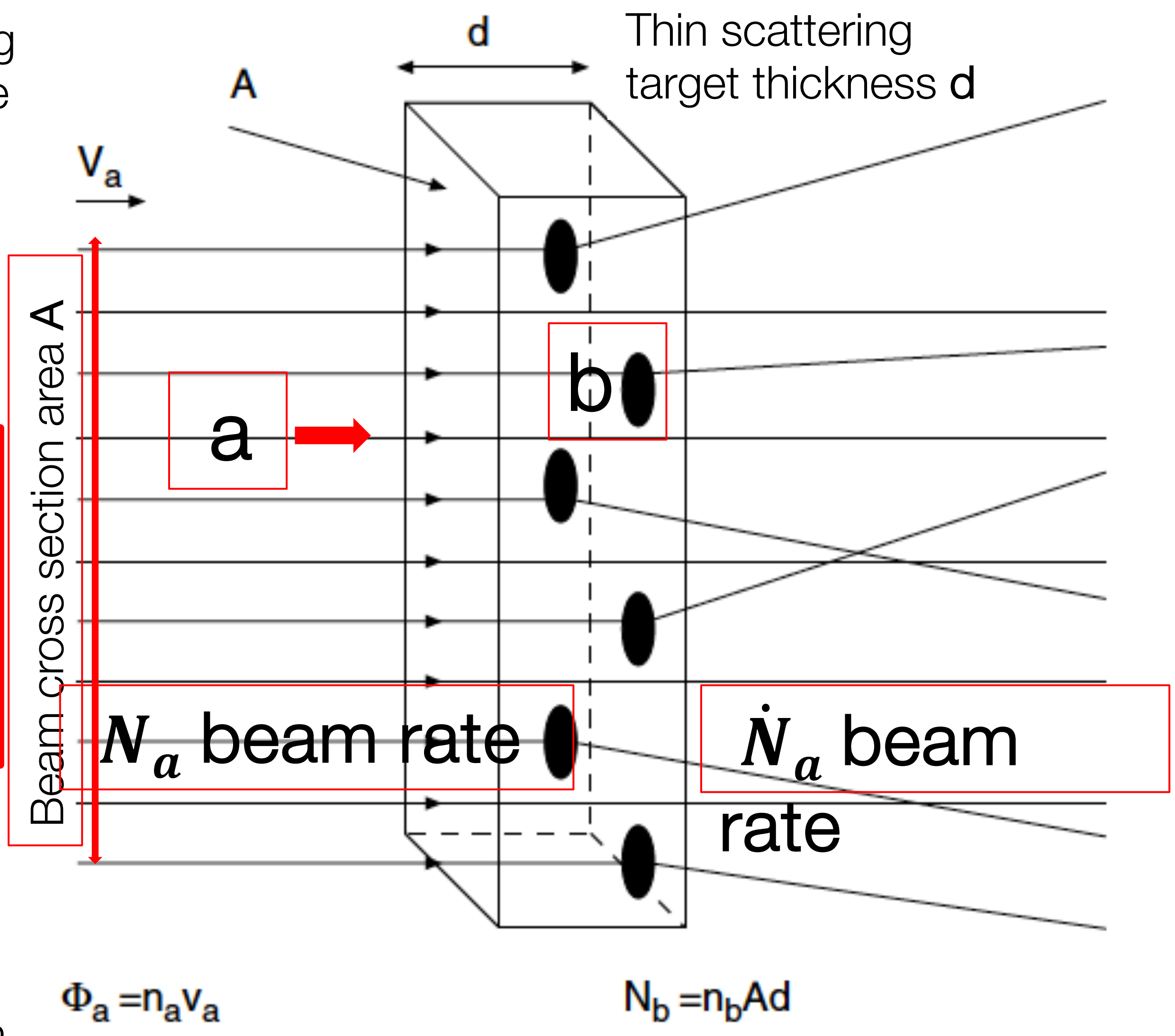
Beam particles **a** bombard scattering centres b. We define that a reaction occurred whenever **a** hits **b**. The beam particle **a** disappears

$$a + b \rightarrow anything$$

Particle beam a coming from left with density n_a and velocity v_a The corresponding flux is

$$\phi_a = n_a \times v_a$$

Target with N_b scattering centres **b** and particle density n_b



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(Geometric) Cross Sections - 2 (Povh…)

Ideal Simplified Experiment:

After the interaction beam particles disappear (we do not distinguish different final topologies, we sum elastic + inelastic cross sections). Reaction rate is

$$\dot{N} = N_a - \dot{N}_a$$

Particle beam **a** coming from left with density n_a and velocity v_a The corresponding flux is

$$\phi_a = n_a \times v_a = \frac{N_a}{A}(area \times time)^{-1}$$

Target with N_b scattering centres **b** and particle density n_b. Target particles within the beam area A are

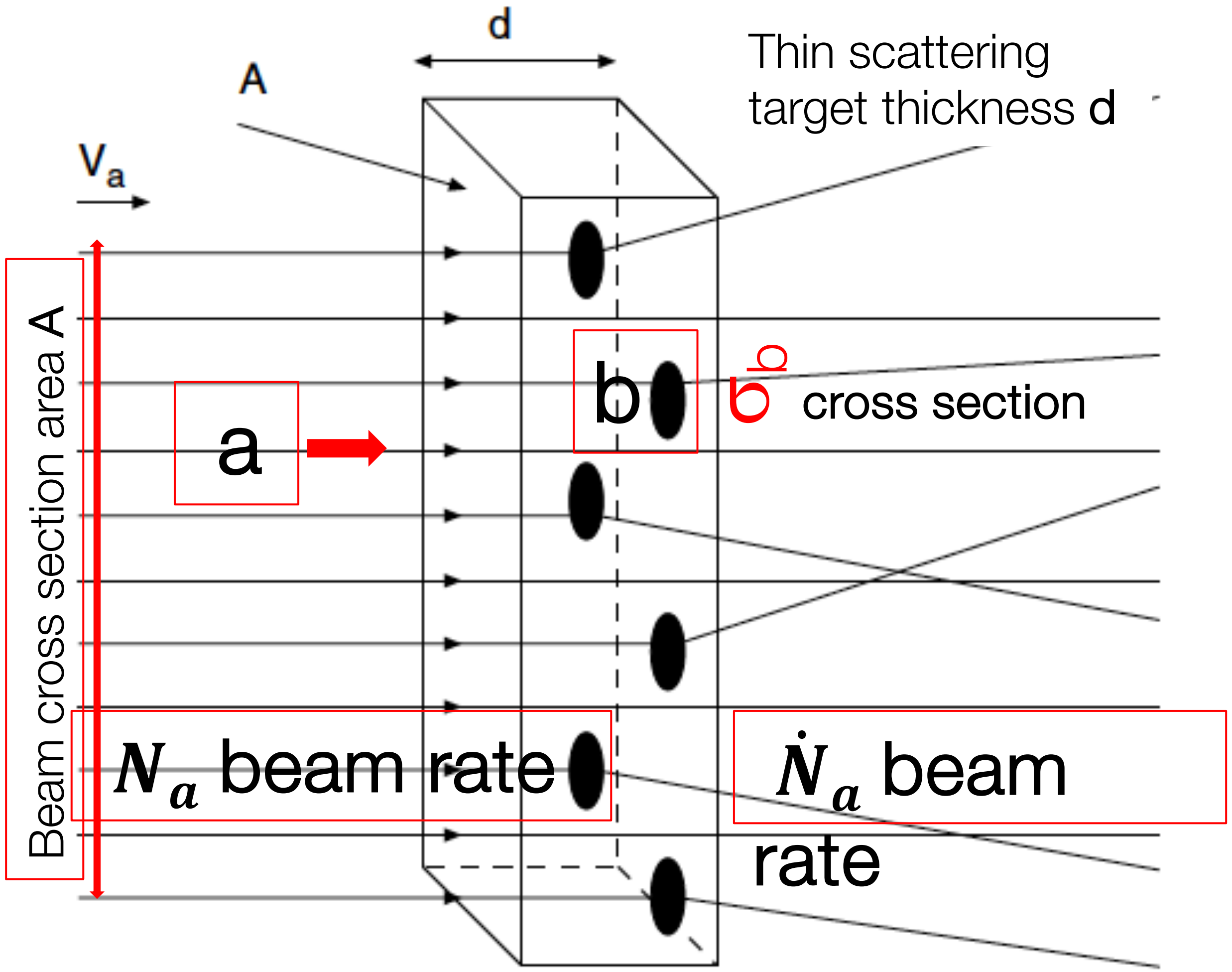
$$N_b = A \times d \times n_b$$

→ the reaction rate $\dot{N}$ is

$$\dot{N} = \phi_a \times N_b \times \sigma_b$$

$$\sigma_b = \frac{\dot{N}}{\phi_a \times N_b}$$

$$= \frac{\textbf{number or reactions per unit time}}{\textbf{beam particles per unit time per unit area} \times \textbf{scattering centres}}$$



Limitations: scattering centres do not overlap + only one scattering

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(Geometric) Cross Sections - 3 (Povh…)

If the density of the beam is not uniform but the target is homogeneous the equivalent expression is used

$$\sigma_b = \frac{\dot{N}}{\phi_a \times N_b} = \frac{\textbf{number or reactions per unit time}}{\textbf{beam particles per unit time} \times \textbf{scattering centres per unit area}}$$

In the expression

$$\sigma_b = \frac{\dot{N} \quad \text{Physics!}}{\phi_a \times N_b \quad \text{Experiment}}$$

the product $\phi_a \times N_b$ is sometimes called Luminosity, $\mathcal{L}$ in this case $\dot{N} = \mathcal{L} \times \sigma_b$ (we will discuss more about Luminosity)

Description is very simplified…

- Energy dependence
- Particle types (proton scattering, electron scattering, neutrino scattering on the same target have cross sections that change by orders of magnitude)

The total cross section is defined as the sum of elastic and inelastic cross section $\sigma_{tot} = \sigma_{el} + \sigma_{inel}$ and has dimensions of area. Given the typical dimensions a common unit to define cross sections is the **barn** (see table on the right):

σ_{pp}(10 GeV) ~ 40 mb, $\sigma_{\nu p}$(10 GeV) ~ 70 fb

Unit	Symbol	m^2	cm^2
megabarn	Mb	10^{-22}	10^{-18}
kilobarn	kb	10^{-25}	10^{-21}
barn	b	10^{-28}	10^{-24}
millibarn	mb	10^{-31}	10^{-27}
microbarn	µb	10^{-34}	10^{-30}
nanobarn	nb	10^{-37}	10^{-33}
picobarn	pb	10^{-40}	10^{-36}
femtobarn	fb	10^{-43}	10^{-39}
attobarn	ab	10^{-46}	10^{-42}
zeptobarn	zb	10^{-49}	10^{-45}
yoctobarn	yb	10^{-52}	10^{-48}

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The Luminosity

Beam on a target

The quantity

$$\mathcal{L} = \phi_a \cdot N_b$$

is called the *luminosity* . Like the flux, it has dimensions of [(area x time)$^{-1}$]. From $\phi_a = n_a \times v_a$ and $N_b = n_b \cdot d \cdot A$ we have

$$\mathcal{L} = \varphi_a \cdot N_b = \dot{N}_a \cdot n_b \cdot d = n_a \cdot v_a \cdot N_b$$

Luminosity → defined as one of two products below

1. number of incoming beam particles per unit time $\dot{N}_a$, the target particle density in the scattering material n_b, and the target's thickness d;
2. beam particle density n_a, their velocity v_a and the number of target particles N_b exposed to the beam.

There is an analogous equation for the case of *two particle beams colliding in a storage ring*. Assume that j particle packets, each of N_a or N_b particles, have been injected into a ring of circumference U. The two particle types circulate with velocity v in opposite directions and cross at an interaction point times per unit time. The luminosity is then:

$$\mathcal{L} = \frac{N_a \cdot N_b \cdot j \cdot v/U}{A}$$

… and have to be well aligned: LHC ~27Km circumference!

where A is the beam cross-section at the collision point. For a Gaussian distribution of the beam particles around the beam centre (with horizontal and vertical standard deviations σ_x and σ_y respectively), A is given by:

$$A = 4\pi\sigma_x\sigma_y \ .$$

To achieve a high luminosity, the beams must be focused at the interaction point into the smallest possible cross-sectional area possible. Typical beam diameters are of the order of tenths of millimetres or less.

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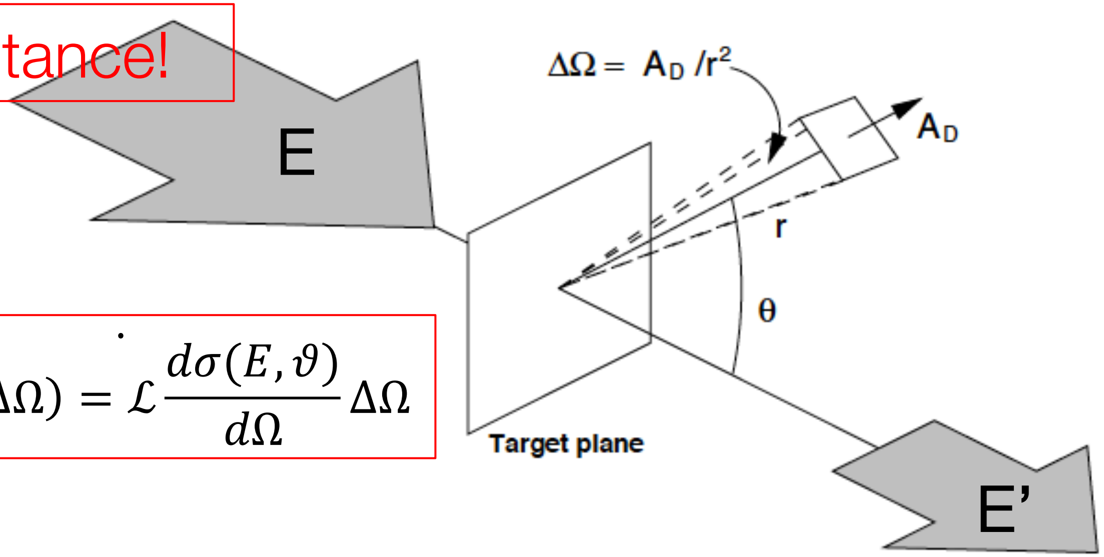
Differential and Doubly-Differential Cross Sections

In all experiments only a fraction of all reactions are measured or accessible because of limited acceptance of the experimental set-up. If a detector of area A_D is positioned at a distance r and at an angle θ, it covers a solid angle equal to $\Delta\Omega = A_D/r^2$.

Acceptance!



The reaction rate (assumed to depend on the energy of the incoming beam and on the angle θ) will be:

$$N(E,\theta,\Delta\Omega) = \dot{\mathcal{L}}\frac{d\sigma(E,\vartheta)}{d\Omega}\Delta\Omega$$

And, if the detector can also determine the energy of the outgoing products then the doubly differential cross section is also measured $d^2\sigma(E,E',\theta)/d\Omega dE'$. The total cross section, in this case, will be the integral over the solid angle and over the scattering energies

$$\sigma_{tot}(E) = \int_{E_{min}}^{E_{max}}\int_{\theta_{min}}^{\theta_{max}}\frac{d^2\sigma(E,E',\theta)}{d\Omega dE}\,d\Omega dE'$$

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Cross Sections from Theory: The Golden Rule

Scattering processes → cross sections. Can we compute it with theory?

The Hamiltonian transforms the initial state wave function into the final one:

$$M_{fi} = \langle \psi_f | \mathcal{H}_{int} | \Psi_i \rangle$$

$$\Delta t \geq \hbar/(\Delta E).$$

M_{fi} is also called the probability amplitude for the transition.

Let's remember that the reaction cross section also depends on the number of final states available to the reaction. According to the Heisenberg uncertainty principle each particle occupies a volume $h^3 = (2\pi\hbar)^3$ in the 6-dim position-momentum space.

Spherical shell radius p', p'+δp'.

Consider a particle scattered into a volume V in a momentum interval **p', p'+δp'.** This volume is $4\pi p'^2 \delta p'$ and the total number of final states is

$$dn(p') = \frac{V 4\pi p'^2 \delta p'}{(2\pi\hbar)^3}$$

Volume occupied by 1 particle

Since (energy E momentum v) $dE' = v' dp'$ we get that the density of final states ρ(E') is

$$\rho(E') = \frac{dn(E')}{dE'} = \frac{V 4\pi p'^2}{v'(2\pi\hbar)^3}$$

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Cross Sections from Theory: The Golden Rule

According to the Fermi second golden rule (not derived here): reaction rate W, transition matrix and density of final states

$$W = \frac{2\pi}{\hbar} |M_{fi}|^2 \rho(E')$$

We have already seen that $\phi_a = n_a \times v_a$ and $\boldsymbol{\sigma_b} = \frac{\dot{N}}{\phi_a \times N_b} \rightarrow$

Theory

$$\sigma = \frac{2\pi}{\hbar v_a} \hbar |M_{fi}|^2 \rho(E')V$$

- If interaction potential is known or calculable → compute the cross section
- if M_{fi} is not known one can measure σ and derive M_{fi} from it.

The Golden Rule applies both to scattering and decay processes. In the second case the lifetime of the process will be

$$\tau = \frac{1}{W}$$

- if the lifetime is (can be) measured then M_{fi} can be derived.
- If τ cannot be measured then the uncertainty principle can be used and we can take $\Delta E = \hbar/\tau$

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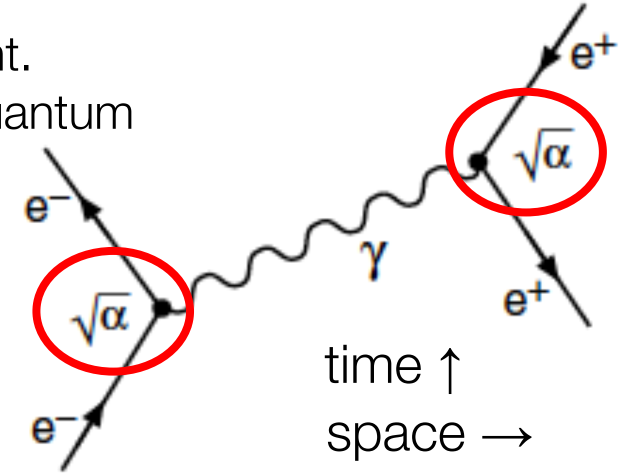
Feynman Diagrams (Povh…)

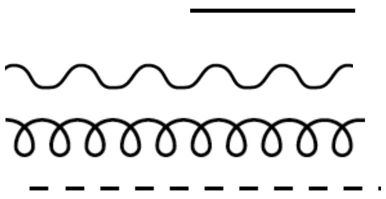
- Scattering processes, say two particles colliding against each other or the decay of a single particle, are normally depicted by Feynman diagrams.
- There are rules to construct these diagrams that map symbols into the matrix element.
- This is true not only for QED and weak interactions but also for strong interactions quantum chromodynamics (QCD).
- No exact calculation but qualitative arguments.

Figure → shows a typical diagram. Drawing rules:



(a)

- The time axis runs ↑ and the space axis →.
- The straight lines correspond to the wave functions of the initial and final fermions.
- Antiparticles → arrows pointing backwards in time; photons by wavy lines; heavy vector bosons by dashed lines; and gluons by corkscrew-like lines.



- The electromagnetic interaction between charged particles → photon exchange.
- Particles appearing neither in the initial nor in the final state, such as this exchanged photon, are called virtual particles.
- Uncertainty principle → virtual particles do not satisfy the energy-momentum relation $E^2 = p^2c^2 + m^2c^4$. This may be interpreted as: *1]* the exchanged particle has a mass different from that of a free (real) particle, or *2]* that energy conservation is violated for a brief period of time.

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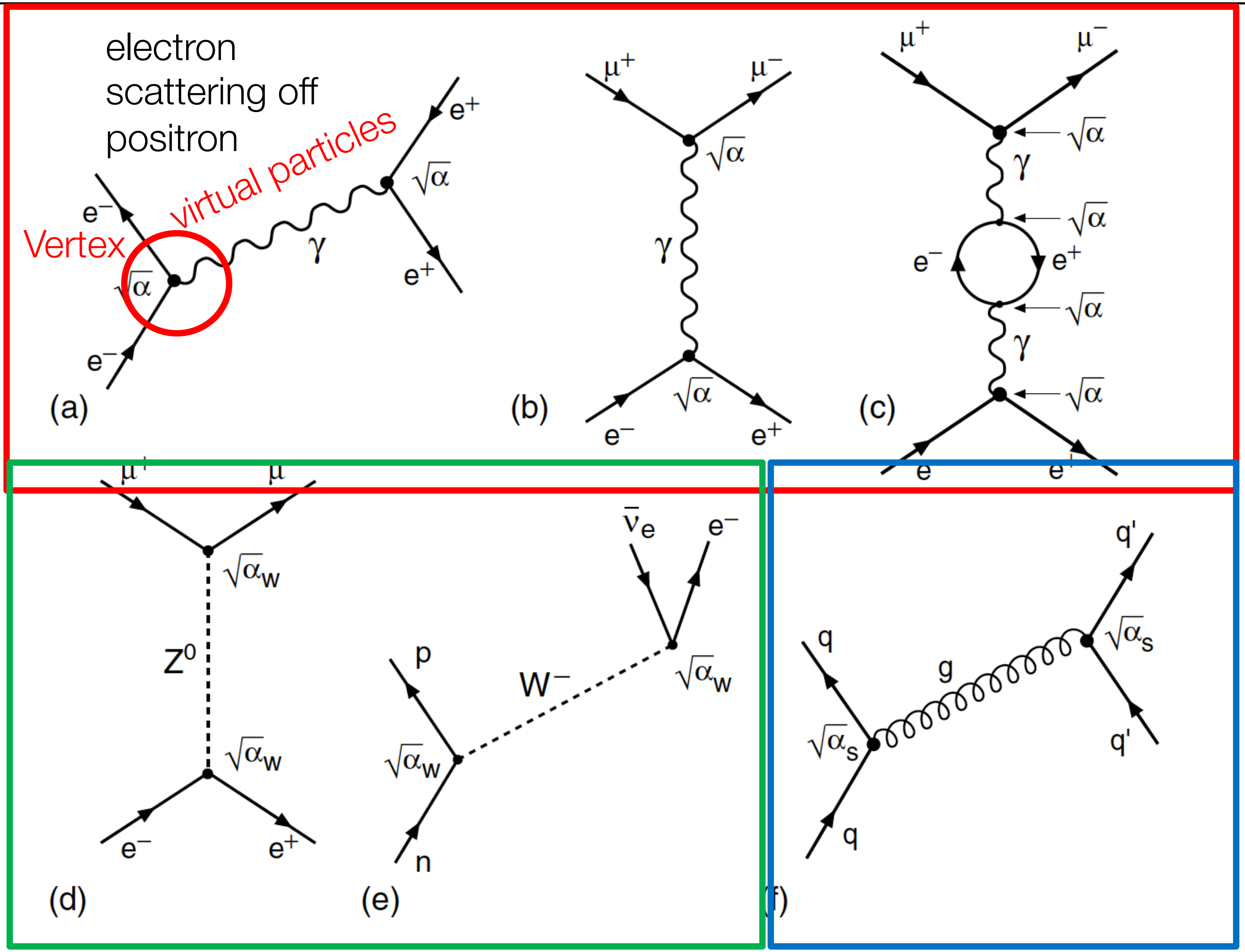
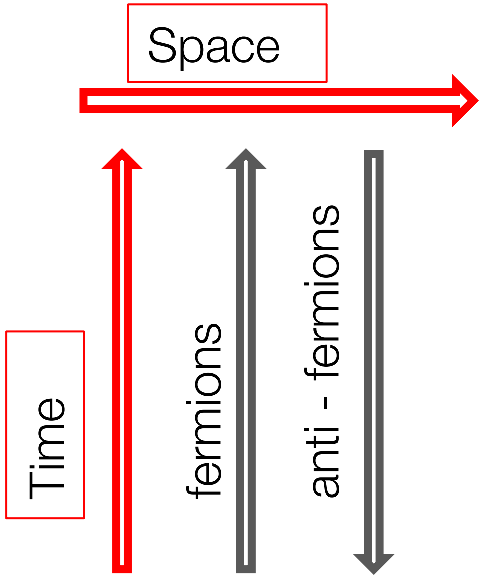
Feynman Diagrams for Em, Weak, Strong Interactions

Points at which three or more particles meet are called vertices.



Feynman diagrams for the electromagnetic (a, b, c), weak (d, e) and strong interactions (f).

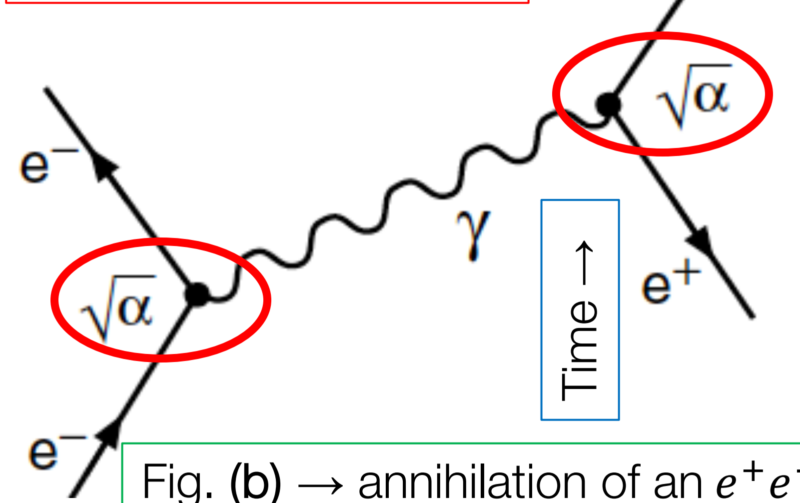


In QED, as in other quantum field theories, we can use the little pictures invented by my colleague Richard Feynman, which are supposed to give the illusion of understanding what is going on in quantum field theory.

M. Gell-Mann [Ge80]

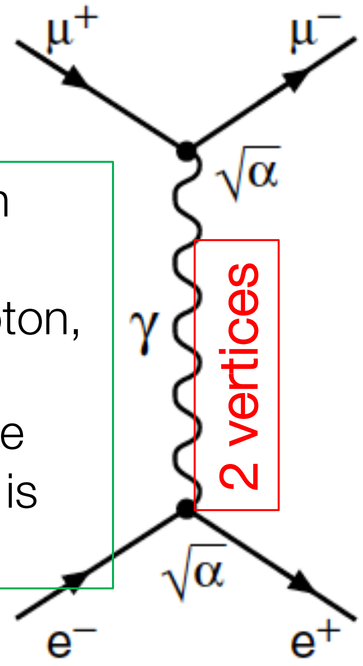
Feynman Diagrams – EM

$$\sigma = \frac{2\pi}{\hbar v_a}\hbar\,|M_{fi}|^2\,\rho(E')V$$

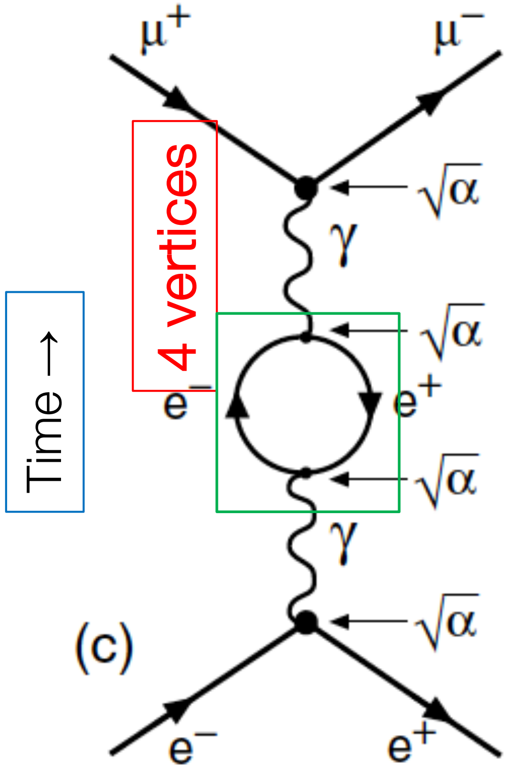


Each vertex corresponds to a term in the transition matrix element which includes the structure and strength of the interaction. In the graph ← the exchanged photon couples to the charge of the electron at the left vertex and to that of the positron at the right vertex. For each vertex the transition amplitude contains a factor which is proportional to e , i.e., √α .

Fig. (b) → annihilation of an e^+e^- pair. A photon is created as an intermediate state which then decays into a $\mu^+\mu^-$ pair. Fig. (c) shows a more complicated graph of the same process: the photon, is briefly transformed into an e^+e^- intermediate state. This and additional, more complicated, diagrams contributing to the same process are called *higher-order diagrams* . The transition matrix is the sum of of all diagrams → same final state.



(b)



(c)

More vertices implies higher powers of α . Diagram (b) is $\propto \alpha$ diagram (c)'s is $\propto \alpha^2$. The conversion $e^+e^- \rightarrow \mu^+\mu^-$ is given by graph (b), higher diagrams produce only small corrections to (b).

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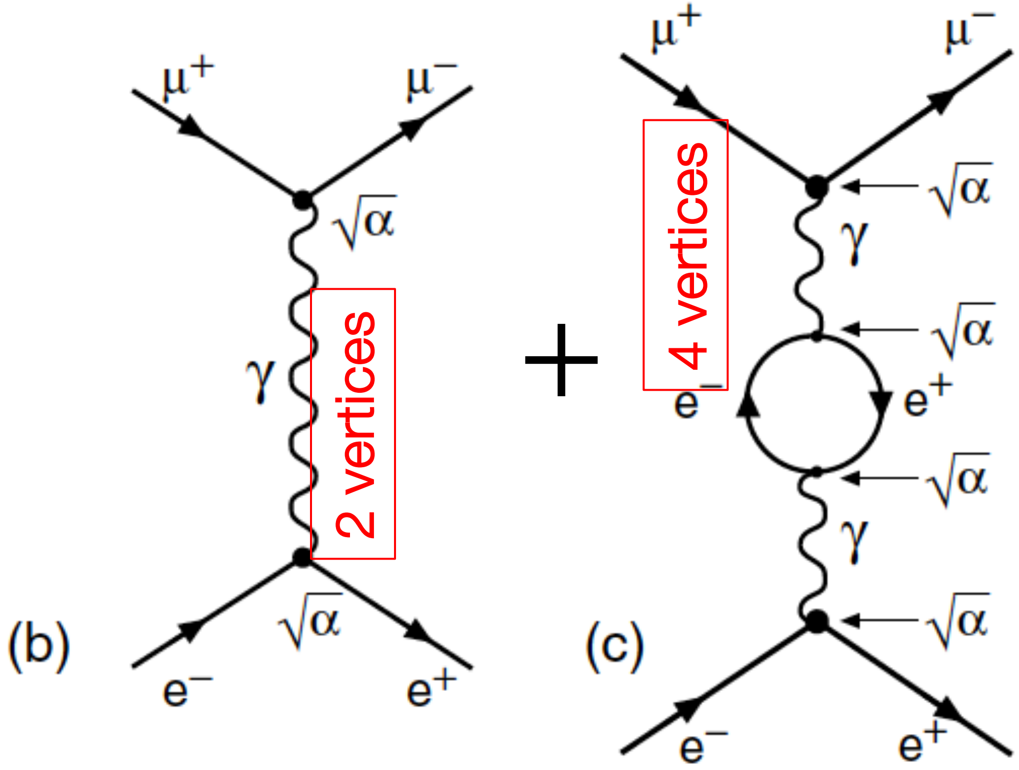
Feynman Diagrams – EM

Annihilation of an e^+e^- pair into a $\mu^+\mu^-$ pair.

$=$



$$= \alpha + \alpha^2 + \ldots.$$

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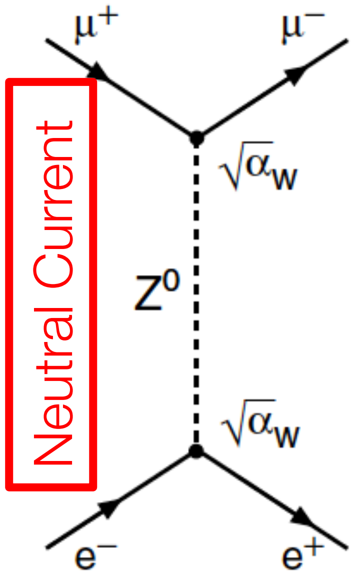
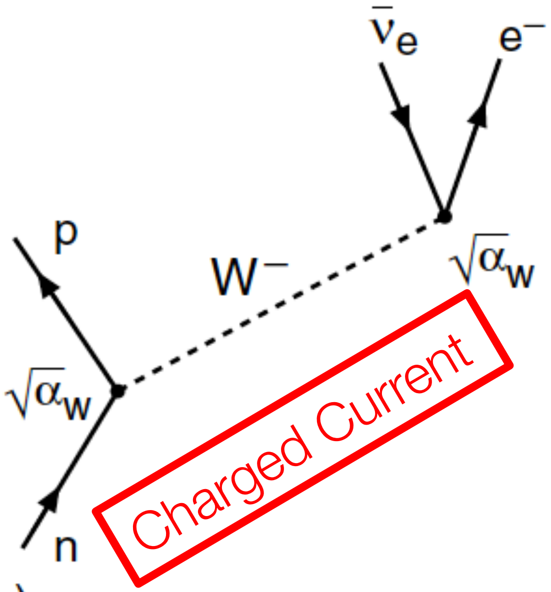
Feynman Diagrams – Weak & Strong



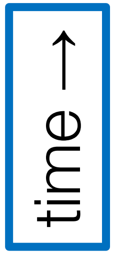
(d)

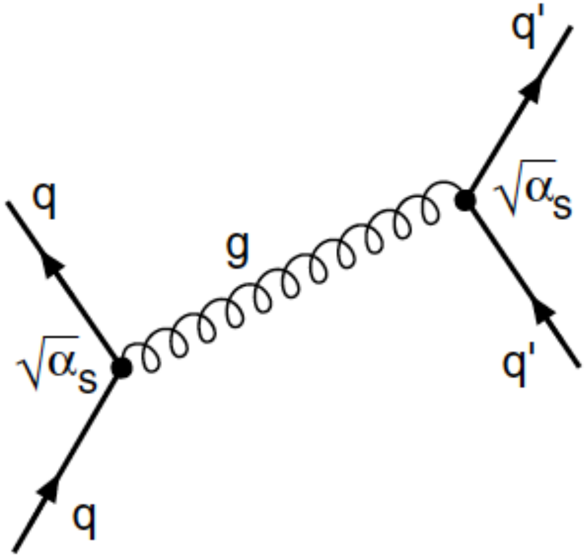
Figure (d) shows electron-positron annihilation followed by muon pair production in a weak interaction proceeding through exchange of the neutral, heavy vector boson Z^0



(e)

In Figure (e), we see a neutron that transform into a proton via β^--decay in which it emits a negatively charged heavy vector boson W^- which subsequently decays into an electron and antineutrino ν_e .





(f)

Figure (f) depicts a strong interaction process between two quarks q and q' which exchange a gluon, the field quantum of the strong interaction

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Feynman Diagrams

- In weak interactions, a heavy vector boson is exchanged which couples to the "weak charge" g and not to the electric charge e . Accordingly, $M_{fi} \propto g^2 \propto \alpha_w$.
- In strong interactions the gluons which are exchanged between the quarks couple to the "colour charge" of the quarks, $M_{fi} \propto \sqrt{\alpha_s} \cdot \sqrt{\alpha_s} = \alpha_s$.

The exchange particles contribute a propagator term to the transition matrix element. This contribution has the general form

$$\frac{1}{Q^2 + M^2c^2}$$

Here Q^2 is the square of the four-momentum which is transferred in the interaction and M is the mass of the exchanged particle. In the case of a virtual photon, this results in a factor $1/Q^2$ in the amplitude and $1/Q^4$ in the cross-section.

In the weak interaction, the large mass of the exchanged vector boson causes the cross-section to be much smaller than that of the electromagnetic interaction — although at very high momentum transfers, of the order of the masses of the vector bosons, the two cross-sections become comparable in size.

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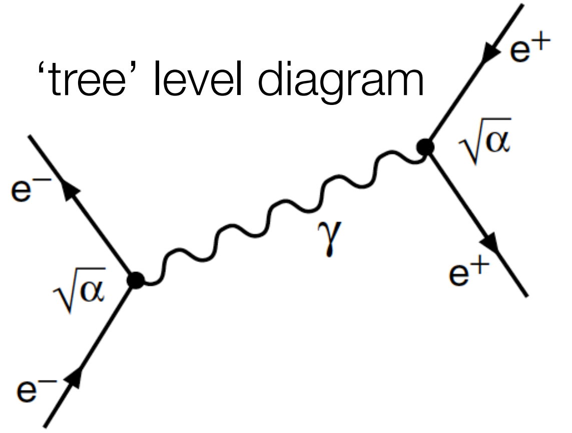
Of the Scattering Processes (real life)

We saw in the previous slides that scattering experiments allow to determine the transition matrix element. However in the real life this is less easy than it seems. Scattering experiment means you shoot projectiles on a target (like a beam on a target or two beams clashing against each other).

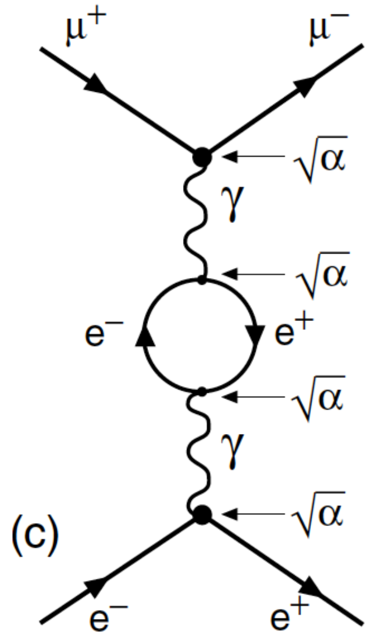
- Projectiles are or may be extended objects (low energy particles have a large de Broglie ; The more energetic projectiles are used the smaller is the equivalent de Broglie wave length $\lambda = \hbar/p$, small wave lengths allow the inspection of the inner structure of the matter).
- protons have an internal structure, we know they are composite objects (alfa particles even more);

- electrons, as far as we know, are point-like objects, the interaction between electrons and a nucleus or a quark proceeds via the exchange of a photon;
- Processes mediated by photons have two advantages:
  1. The first one is that they are well known since long.
  2. The second positive fact is that these interactions are characterised by a strength (i.e. a coupling constant) α=1/137 which is rather small and allows the perturbation theory to be applied.

'tree' level diagram



higher level diagram → small contribution to cross section



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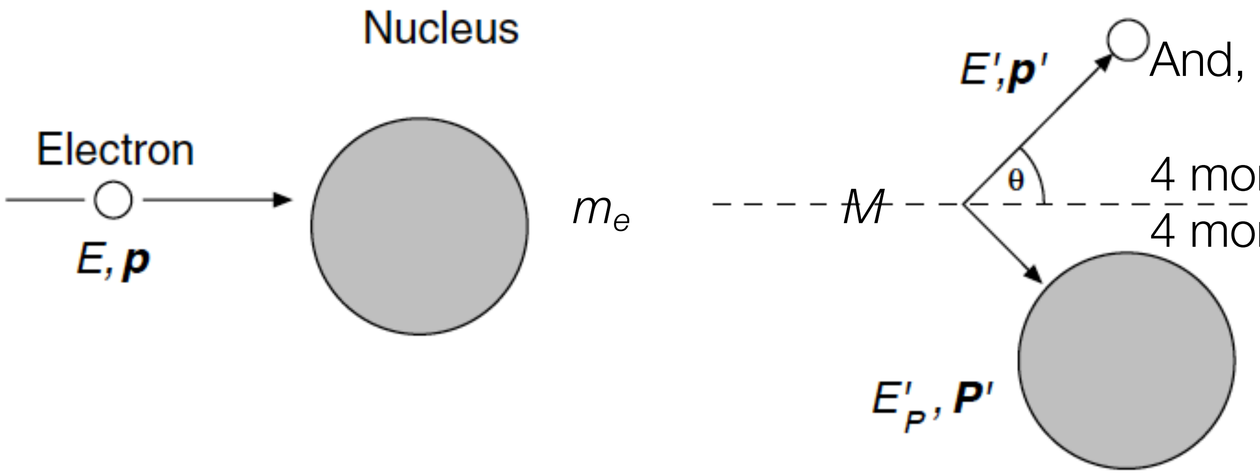
Electron Nucleus Kinematics (Povh…)

Elastic Scattering of an Electron on a Particle at Rest with Mass *M*
(assumed to be a proton)
→ *electron and particle with mass M remain unchanged in the final state*

4 – Momenta

!



Kinematics of elastic electron – nucleus scattering

The energy conservation implies that $p + P = p' + P'$
and also… $P' = p + P - p'$

And, once squared, $p^2 + 2pP + P^2 = p'^2 + 2p'P' + P'^2$

4 momenta p^2 and p'^2 are invariant and equal to m_e
4 momenta P^2 and P'^2 are also invariant and equal to M

$$p^2 = p'^2 = m_e^2 c^2 \quad \text{and} \quad P^2 = P'^2 = M^2 c^2$$

$$p \cdot P = p' \cdot (p + P - p') = p'p + p'P - m_e^2 c^2$$

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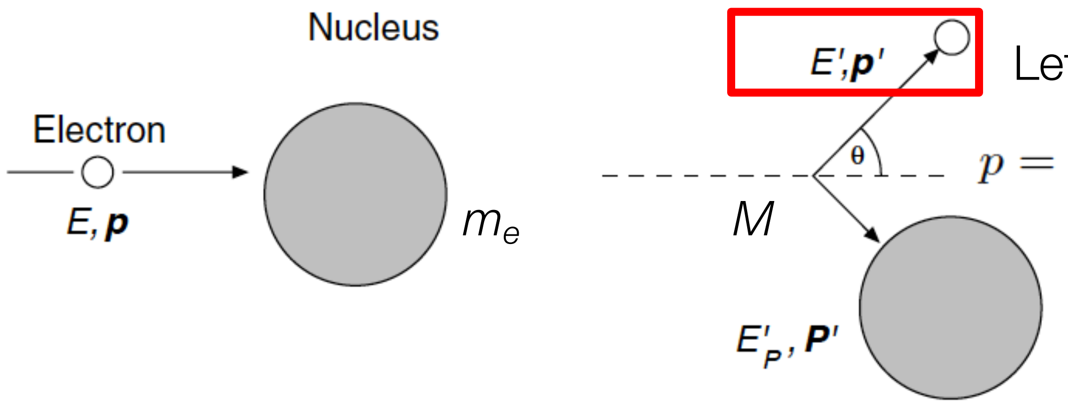
Electron Nucleus Kinematics

$$a \cdot b \;=\; a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 \;=\; a_0 b_0 - \boldsymbol{a} \cdot \boldsymbol{b}$$

In general only the electron is detected

$$p \cdot P \;=\; p' \cdot (p + P - p') \;=\; p'p + p'P - m_e^2 c^2$$



Let's choose the laboratory frame where particle *P* is at rest →

$$p = (E/c, \boldsymbol{p}) \quad p' = (E'/c, \boldsymbol{p}') \quad P = (Mc, \boldsymbol{0}) \quad P' = (E'_P/c, \boldsymbol{P}')$$

$$E \cdot Mc^2 = E'E - \boldsymbol{p}\boldsymbol{p}'c^2 + E'Mc^2 - m_e^2 c^4 \,.$$

Fig. 5.1. Kinematics of elastic electron-nucleus scattering.

At high energy we may neglect the electron mass and take $E \sim |\boldsymbol{p}| \cdot c$ → $E \cdot Mc^2 = E'E \cdot (1 - \cos\theta) + E' \cdot Mc^2$

θ is the scattering angle between p and p'

$$E' = \frac{E}{1 + E/Mc^2 \cdot (1 - \cos\theta)}$$

In elastic scattering (and in elastic scattering only!) there is a one-to-one correlation between the scattering angle θ end the energy of the electron. The recoil energy transferred to the target proton is given by E'-E
→ if the term E/M increases E' decreases → small recoil energy

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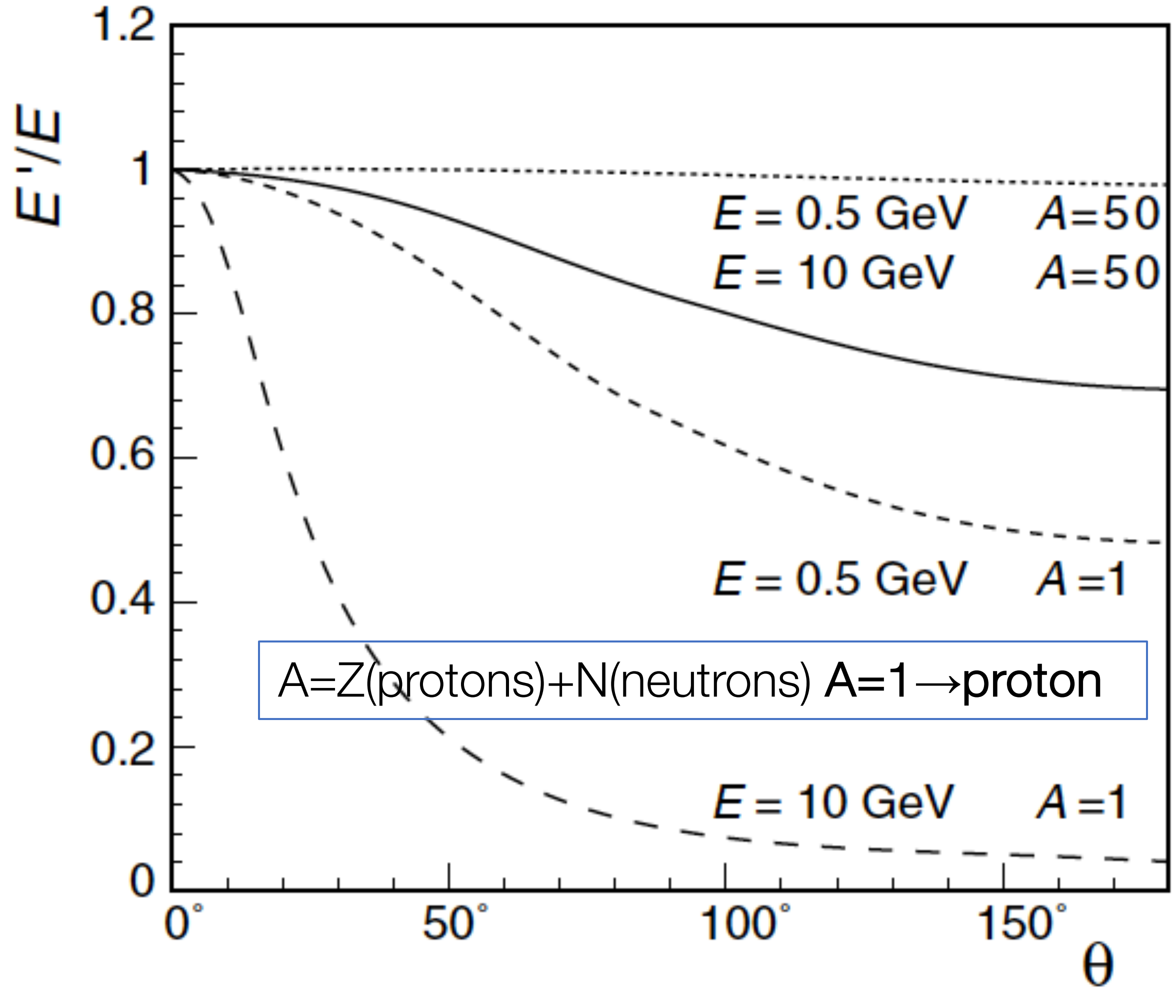
Kinematics of Electron Scattering off Nucleus



A=50 → Sn (tin)

$$E' = \frac{E}{1 + E/Mc^2 \cdot (1 - \cos\theta)}$$

The angular dependence of the scattering energy E is described by the term (1−cos θ) multiplied by E/Mc^2. Hence the recoil energy of the target increases with the ratio E/c^2 to the target mass M. In electron scattering at the relatively low energy of 0.5 GeV off a nucleus with mass number A=50 the scattering energy varies by only 2% between forward and backward scattering. The situation is very different for 10 GeV electrons scattering off protons. The scattering energy E then varies between 10 GeV ($\theta \approx 0^\circ$) and 445 MeV ($\theta = 180^\circ$).

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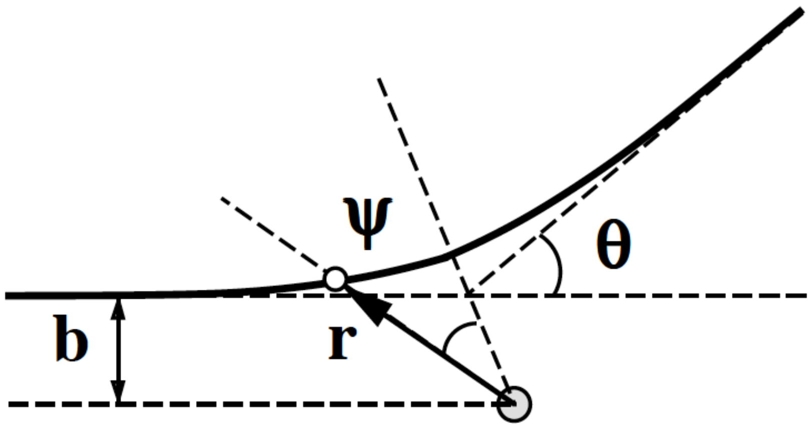
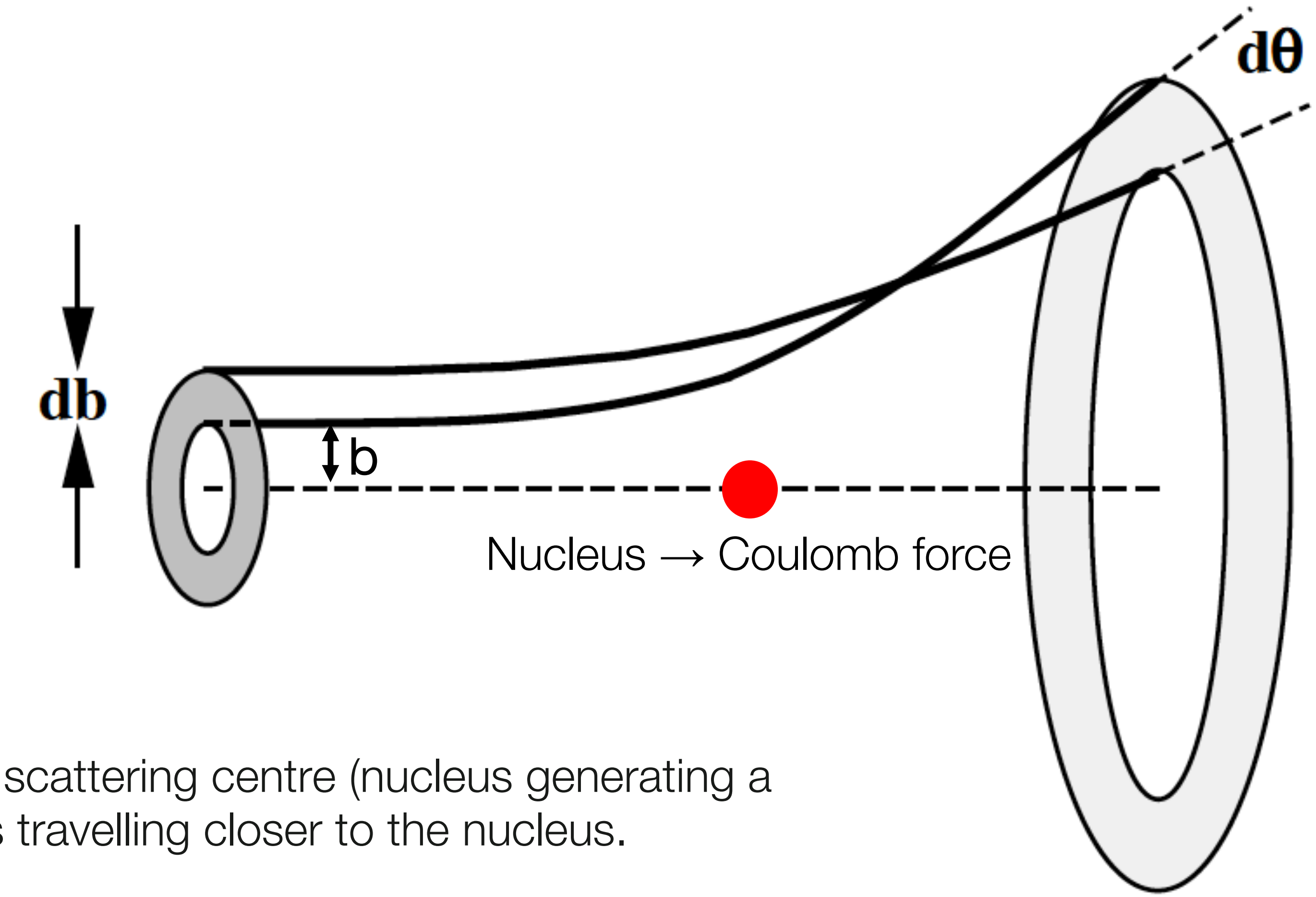
Rutherford scattering – Classical Calculation





Particles travelling at a large distance from the scattering centre (nucleus generating a Coulomb field) are deflected less than particles travelling closer to the nucleus.

→ For a larger impact parameter b, the diffusion occurs at a smaller angle . For this reason, a negative dθ corresponds to a positive db

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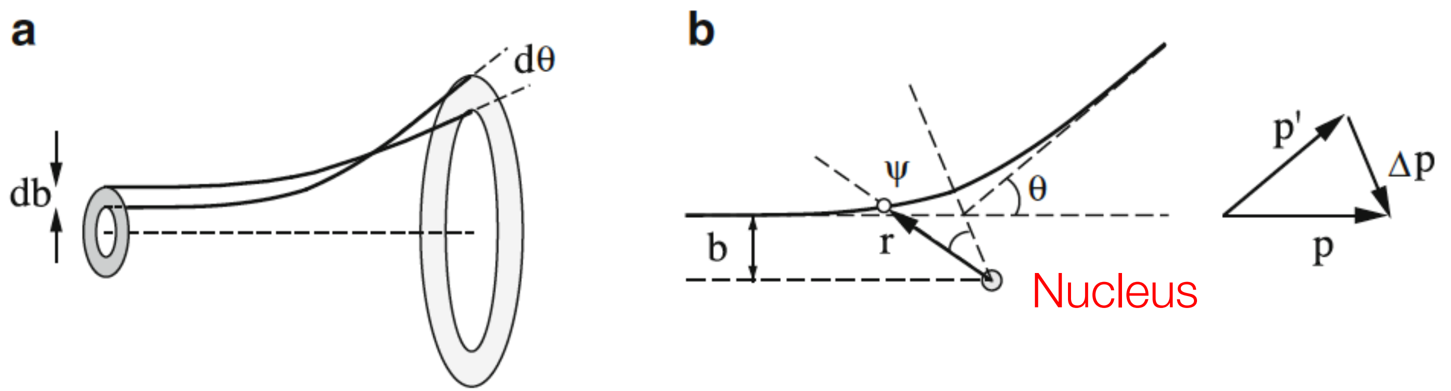
Rutherford scattering – Classical Calculation



Charged particles reaching the region of surface (2πb db) around a fixed center of diffusion (generally, a heavy nucleus) which produces a Coulomb type potential are elastic scattered in the angular range [θ, θ – δθ] (b)
→ Derive the relation between the impact parameter b and the deflection angle in the Coulomb elastic scattering.

You measure this!

The number of incident particles elastically scattered per time unit in the interval (θ , θ – δθ) is

$$dN = 2\pi N_0 b\, db = N_0\, d\sigma \rightarrow d\sigma = 2\pi\, b\, db$$

- N_0 is the number of incident particles per area and time units
- dσ = 2πbdb is the surface of the annular ring hit by the incident particles in the angular range (θ , θ – δθ).
- Note that if one considers dN as the difference between the number of initial and final particles, the variation is < 0

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Rutherford Scattering - continued

The elastic differential cross-section $d\sigma/d\Omega$ is defined as

$$d\sigma(\theta) = \frac{d\sigma}{d\Omega}\, d\Omega = \frac{d\sigma}{d\Omega} 2\pi \sin\theta\, d\theta = -2\pi b\, db,$$

Integrated over 2π in ϕ

$$d\sigma(\theta,\varphi) = b\, db\, d\varphi = -\frac{d\sigma}{d\Omega}(\theta,\varphi)\, d\Omega = -\frac{d\sigma}{d\Omega}(\theta,\varphi)\, \sin\theta\, d\theta\, d\varphi.$$

$$\sigma_{tot}^{el} = \int \frac{d\sigma}{d\Omega}(\theta) d\Omega = -2\pi \int_0^{\pi} \sin\theta \frac{d\sigma}{d\Omega}(\theta) d\theta$$

The total cross-section represents the actual size of the “transverse” area of the diffusion centre seen by the incident particles..

The calculation can be applied to the elastic Coulomb scattering. For example, Rutherford used α particles (that is, ^{4}He nuclei, 2p) with a few MeV of energy, colliding against heavy gold (Z = 79) nuclei. The Coulomb potential of the nucleus with charge Z and an incoming particle with charge ze (Z=2 for α particles) is $U(r) = \frac{Ze}{r}$

Then the Coulomb energy potential V(r) is zeU(r)

$$V(r) = zZe^2/r$$

($z \to 2$, $Z \to 79$)

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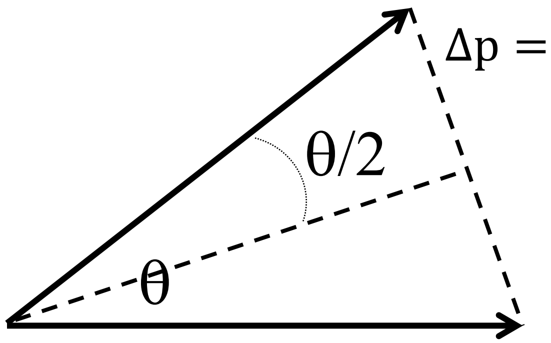
Rutherford scattering - continued



$\tan(\theta/2) = \frac{\sin(\vartheta/2)}{\cos(\theta/2)} = \frac{Potential\ energy\ at\ a\ distance\ 2b}{Kinetic\ energy} = \frac{zZe^2}{2bE_c} \rightarrow b = \frac{zZe^2\cot(\theta/2)}{2E_c}$
Now the expression for db/dθ is (dcot(x)=sin(x)$^{-2}$dx)
$\frac{db}{d\theta} = \frac{zZe^2}{2bE_c\sin(\theta/2)^2}$
And using
We get

α particle, E_C

The α particle with kinetic energy E_C hits the gold nucleus at a distance b and sees a potential energy zZe^2/b. The scattering angle θ is then

$(|\vec{p'}| = |\vec{p}|;\ \Delta p = |\vec{p'} - \vec{p}| = 2\,p\,sin(\frac{\theta}{2})$

Classical Relation

$$\tan(\theta/2) = \frac{\sin(\vartheta/2)}{\cos(\theta/2)} = \frac{Potential\ energy\ at\ a\ distance\ 2b}{Kinetic\ energy} = \frac{zZe^2}{2bE_c} \rightarrow b = \frac{zZe^2\cot(\theta/2)}{2E_c}$$

$$\frac{d\sigma}{d\Omega} 2\pi \sin\theta\, d\theta = -2\pi b\, db,$$

2 slides before

Now the expression for db/dθ is (dcot(x)=sin(x)$^{-2}$dx)

$$\frac{db}{d\theta} = \frac{zZe^2}{2bE_c\sin(\theta/2)^2}$$

And using

$$\frac{d\sigma}{d\Omega}(\theta) = -\frac{b}{\sin\theta}\frac{db}{d\theta}.$$

We get

$$\frac{d\sigma}{d\Omega}(\theta) = -\frac{b}{\sin\theta}\frac{db}{d\theta} = \frac{(Zze^2)^2}{(4E_c)^2}\frac{1}{\sin^4(\theta/2)}.$$

Valid for a spinless particle

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Targets with Extended Charge Distribution

Simplified case with point-like charges.

Let's consider the case of

- an electron beam with a density n_a particles per unit volume,
- scattering on a very heavy target (nucleus), so heavy that the recoil is so small that it can be approximated to zero.
- We can use three-momenta.
- If the total charge of the target *Ze* is small then the interaction potential of the electro-magnetic interaction will be small $Z\alpha<<1$.
- In this case the wave functions Ψ_i and Ψ_f of the initial and final state (i.e. electron) will be described by plane waves:

$$\Psi_i = \frac{1}{\sqrt{V}} e^{i\boldsymbol{p}x/\hbar} \qquad \Psi_f = \frac{1}{\sqrt{V}} e^{i\boldsymbol{p'}x/\hbar}$$

We assume that the process takes place in a volume V (large with respect to the scattering centre) and that wave functions of the incoming and outgoing electrons are normalised in this volume. We have a total number of N_a electrons in the beam

$$\int_V |\psi_i|^2 \, \mathrm{d}V = n_a \cdot V \qquad \text{where} \quad V = \frac{N_a}{n_a}$$

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In previous slides we introduced the Golden Rule:

$$\frac{\sigma v_a}{V} = W = \frac{2\pi}{\hbar}|M_{fi}|^2 \rho(E')$$

which in our case reads

$$W = \frac{2\pi}{\hbar}|M_{fi}|^2 \rho(E') = \frac{2\pi}{\hbar}|\langle\psi_f|\mathcal{H}_{int}|\psi_i\rangle|^2 \frac{dn}{dE'}$$

Since $dE' = v'dp'$ we get

$$dn(p') = \frac{V 4\pi p'^2 dp'}{(2\pi\hbar)^3}$$

$$\mathrm{d}\sigma \cdot v_a \cdot \frac{1}{V} = \frac{2\pi}{\hbar}|\langle\psi_f|\mathcal{H}_{\text{int}}|\psi_i\rangle|^2 \frac{V|\boldsymbol{p}'|^2 \mathrm{d}|\boldsymbol{p}'|}{(2\pi\hbar)^3 \mathrm{d}E_f} \mathrm{d}\Omega$$

The velocity $\mathbf{v_a}$ can be approximated by the velocity of light **c** and |p'|~E'/c →

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{V^2 E'^2}{(2\pi)^2(\hbar c)^4}|\langle\psi_f|\mathcal{H}_{\text{int}}|\psi_i\rangle|^2$$

Kinematics

Physics

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Some Calculation…

The interaction operator that transforms the initial state into a final one for a charge **e** in an electric potential **ϕ** is:

$$\mathcal{H}_{int} = e\phi$$

And the matrix element $\langle\psi_f|\mathcal{H}_{int}|\psi_i\rangle$ becomes

$$\langle\psi_f|\mathcal{H}_{int}|\psi_i\rangle = \frac{e}{V}\int e^{\frac{-i\boldsymbol{p}'x}{\hbar}}\phi(x)e^{\frac{-i\boldsymbol{p}x}{\hbar}}\,d^3x$$

The momentum transfer between **p** and **p'** is defined as **q=p-p'** giving us the possibility of re-writing the formula as

$$\langle\psi_f|\mathcal{H}_{int}|\psi_i\rangle = \frac{e}{V}\int \phi(x)e^{\frac{i\boldsymbol{q}x}{\hbar}}\,d^3x$$

And using Poisson's equation

$$\langle\psi_f|\mathcal{H}_{int}|\psi_i\rangle = \frac{e\hbar^2}{\varepsilon_0 V|\boldsymbol{q}^2|}\int f(x)e^{\frac{i\boldsymbol{q}x}{\hbar}}\,d^3x = \frac{e\hbar^2}{\varepsilon_0 V|\boldsymbol{q}^2|}F(\boldsymbol{q})$$

Where we have defined $F(\boldsymbol{q}) = \int f(x)e^{\frac{i\boldsymbol{q}x}{\hbar}}\,d^3x$, *called form factor of the charge distribution*. The form factor contains all the information of the charge distribution of the target we are probing in our scattering experiment.

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Getting the point-like Rutherford Cross Section

Let's recall the expression we derived few slides before

$$\frac{d\sigma}{d\Omega} = \frac{V^2 E'^2}{(2\pi)^2(\hbar c)^4} \left|\langle\psi_f|\mathcal{H}_{\text{int}}|\psi_i\rangle\right|^2$$

If we neglect the fact that our target has an extended charge distribution then $F(\boldsymbol{q})$ becomes a δ function and becomes equal to 1. If we do this approximation then we get Rutherford cross section in the case of a point-like charge distribution and expressed as a function of the momentum transfer **q**.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{4Z^2\alpha^2(\hbar c)^2 E'^2}{|\boldsymbol{q}c|^4}$$

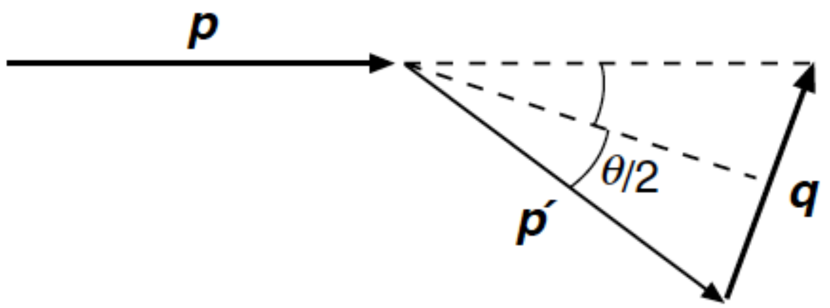
$$\alpha_{EM} = \frac{e^2}{4\pi\epsilon_0\hbar c}$$

*The 1/q⁴ dependence indicates that the cross section drops very quickly for large values of **q** and that the largest part of the cross section is limited at small values of **q**.*

Let's remember that

- If we neglect recoil E=E'
- |p| = |p'|
- E=|p| c is a good approximation

We get the classical expression



$$|\boldsymbol{q}| = 2\cdot|\boldsymbol{p}|\sin\frac{\theta}{2}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{Z^2\alpha^2(\hbar c)^2}{4E^2\sin^4\frac{\theta}{2}}\,.$$

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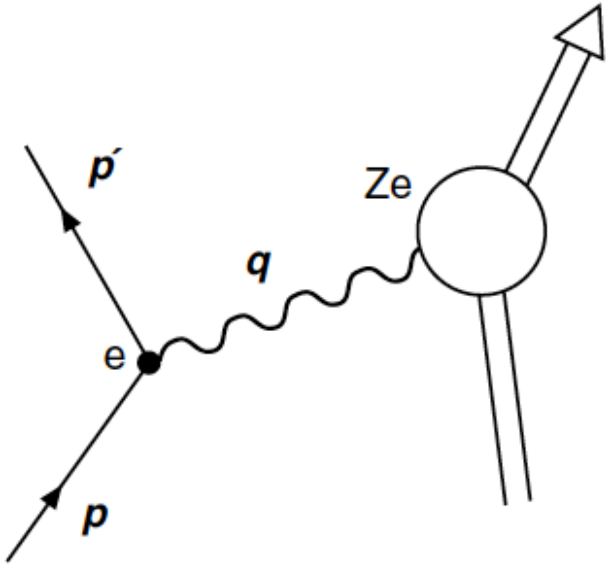
Changing Point of View: Field Theory

In the field theory language the scattering of an electron with charge *e* on a target with charge *Ze* is understood as being mediated by the exchange of a virtual photon emitted by the electron and absorbed by the target (see picture here →).



The transition matrix will contain a term $e \cdot Ze$ and the cross section the square of it $(e \cdot Ze)^2$. In this way of seeing the interaction the momentum transfer q introduced before is the momentum carried by the photon and transferred from the electron to the target. The equivalent de-Broglie wave-length of the photon is

$$\lambda = \frac{\hbar}{\boldsymbol{q}} = \frac{\hbar}{|\boldsymbol{p}|} \frac{1}{2sin\frac{\vartheta}{2}}$$

In this representation the physical interpretation of the scattering is that:

if the de-Broglie wave-length of the photon is NOT small enough with respect to the extension of the target it cannot probe the internal structure of the scattering centres and the target appears to be point-like. The Rutherford cross section was obtained with low energy electrons corresponds to this situation.

In this way of representing the scattering process (exchange of a boson), the propagator in the matrix element $\frac{1}{Q^2+M^2c^2}$ becomes (the mass of the photon is zero) simply $\frac{1}{Q^2}$

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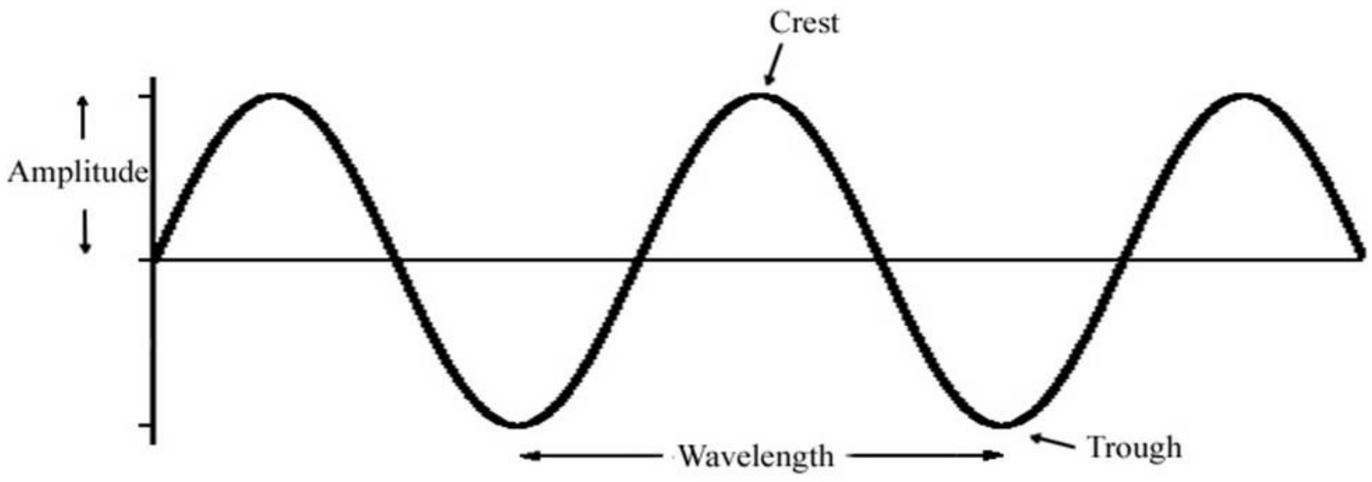
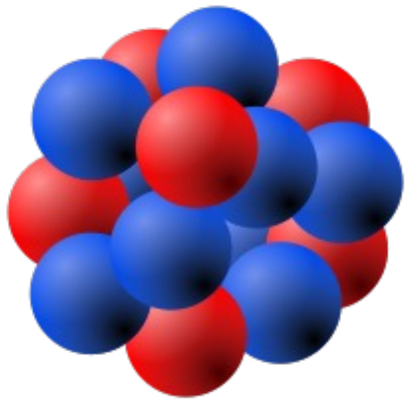
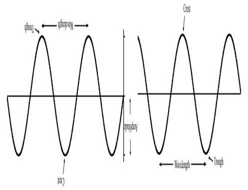
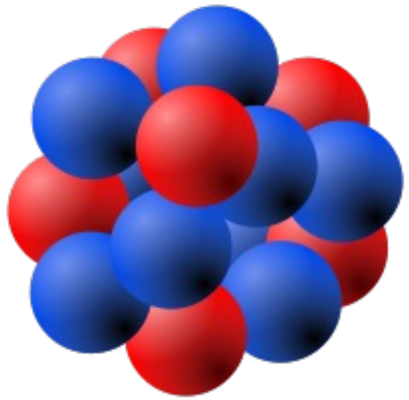
Rutherford Scattering









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Electron Spin →The Mott Cross Section

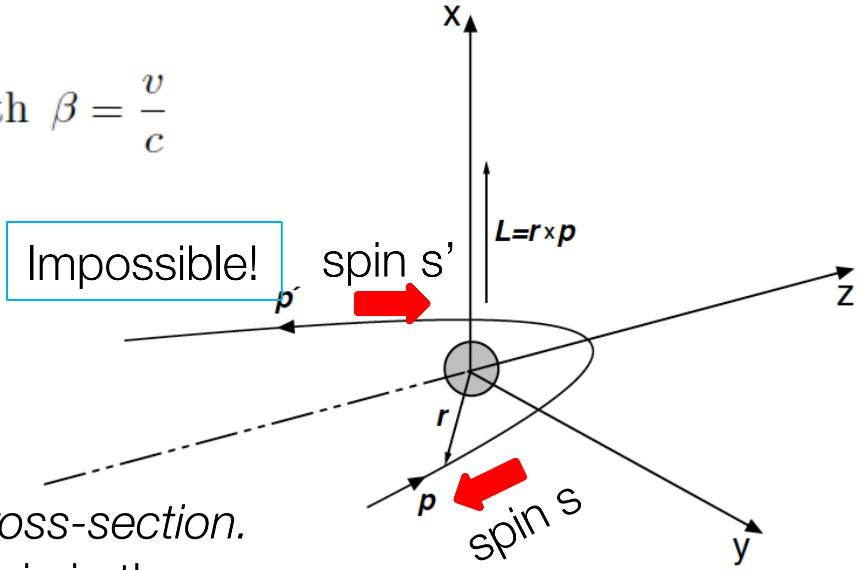
The Rutherford cross section has been studied so far neglecting the effects due to the spin of the electron (later we will also consider possible effects due to the spin of the target).

The inclusion of the electron spin modifies the Rutherford cross section introducing a term $\left(1-\beta^2\sin^2\frac{\theta}{2}\right)$ which disfavours too large scattering angles much more than the standard formula.

$$\left(\frac{d\sigma}{d\Omega}\right)^*_{\rm Mott} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm Rutherford}\cdot\left(1-\beta^2\sin^2\frac{\theta}{2}\right), \quad \text{with } \beta=\frac{v}{c}$$

In the limit case β→1 and using *sin²x+cos²x=1* we get

$$\left(\frac{d\sigma}{d\Omega}\right)^*_{\rm Mott} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm Rutherford}\cdot\cos^2\frac{\theta}{2} = \frac{4Z^2\alpha^2(\hbar c)^2E'^2}{|\boldsymbol{q}c|^4}\cos^2\frac{\theta}{2}$$



In the limit case β→1 and for a scattering angle of 180° we have a zero cross-section. This is understood with the conservation of helicity, the projection of the spin in the direction of the motion : $h=\boldsymbol{s}\cdot\boldsymbol{p}/(|\boldsymbol{s}|\cdot|\boldsymbol{p}|)$

Neither the orbital angular momentum $\mathbf{L}$ (pointing up with respect to the plane of motion) nor the spin-less target can compensate the flip of the helicity. The situation changes in case of a target with spin.

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Nuclear Form Factors

- The Rutherford Cross Section is found to represent well data only for small values of the momentum transfer **q**.
- For higher values of q data are lower that predicted by formulas.
- This is understood with the fact that *assuming that the charge distribution F(q) of a nucleus is point-like* is acceptable only when **q** is small → the reduced wave length of the photon is too large to probe the charge distribution of the nucleus.
- In this picture (**q** small) the photon sees the nucleus (or the nucleon) as a unique object.
- When **q** increases the photon starts to see the inner structure of the proton → the photon starts to see only a part of the charge and not all of it
- → the cross section decreases with **q** faster than expected.

- The form factor, F(**q**^{2}), carrying the information on how the charge is distributed inside the nucleus ~ modulates the Mott cross section

$$F(q^2) = \int e^{\frac{iqx}{\hbar}} f(x) d^3x \qquad \left(\frac{d\sigma}{d\Omega}\right)_{exp.} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \cdot |F(q^2)|^2$$

- The ratio between measurements and the Mott cross section for a point-like charge distribution allows the measurement of the charge distribution inside the nucleus.
- You measure the angle of the scattered electron, you compute **q**, you do the ratio.

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More on Form Factors (Povh)

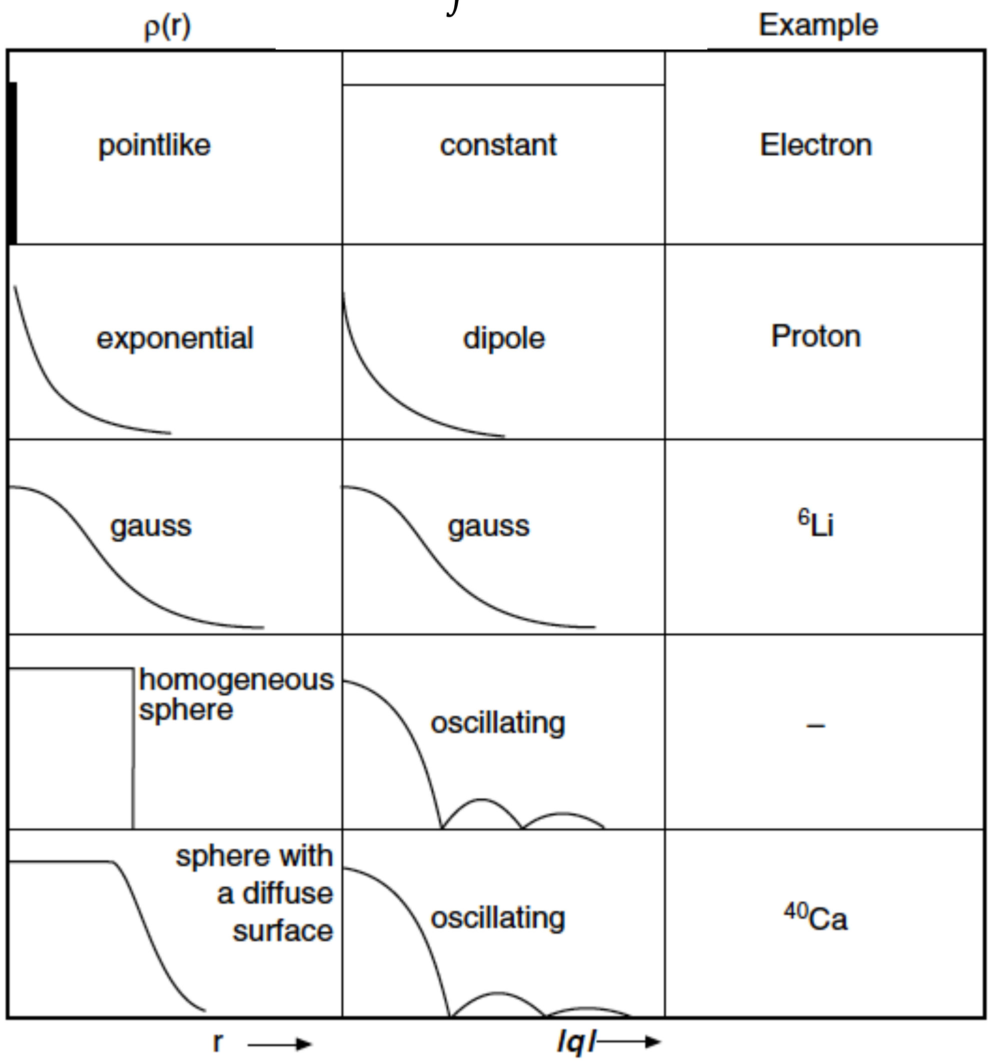
One could measure the ratio

$$|F(q^2)|^2 = (\frac{d\sigma}{d\Omega})_{exp.}/(\frac{d\sigma}{d\Omega})_{Mott}$$

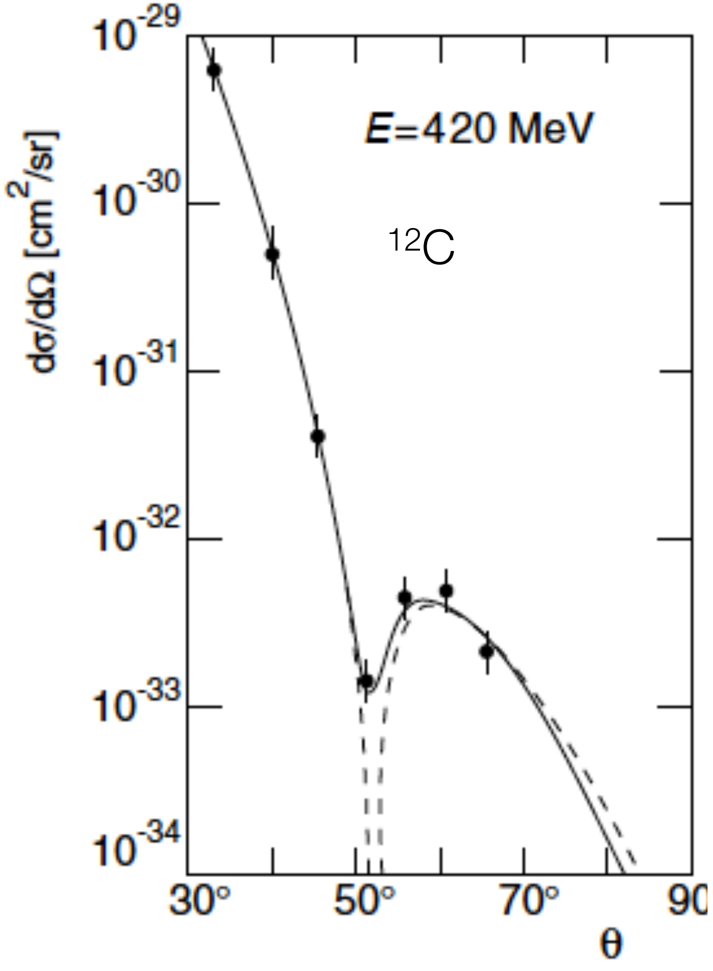
But in practice one assumes different analytical shapes and compares data with predictions

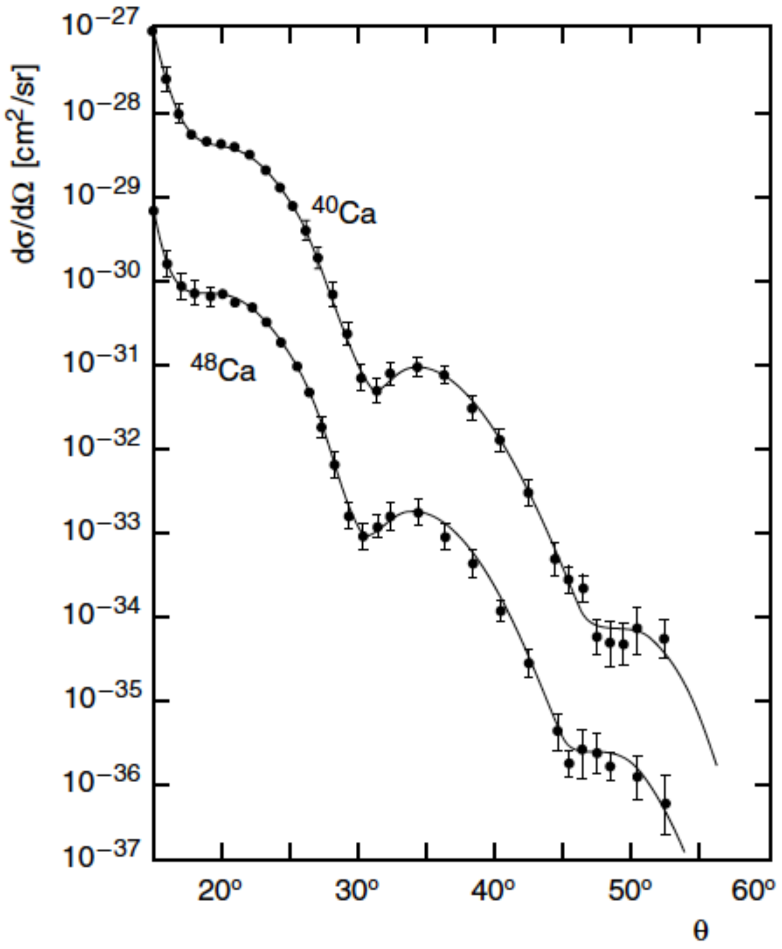
Charge distribution $f(r)$		Form Factor $F(q^2)$	
point	$\delta(r)/4\pi$	1	constant
exponential	$(a^3/8\pi)\cdot\exp(-ar)$	$(1+q^2/a^2\hbar^2)^{-2}$	dipole
Gaussian	$(a^2/2\pi)^{3/2}\cdot\exp(-a^2r^2/2)$	$\exp(-q^2/2a^2\hbar^2)$	Gaussian
homogeneous sphere	$\begin{cases} 3/4\pi R^3 & \text{for } r \le R \\ 0 & \text{for } r > R \end{cases}$	$3\,\alpha^{-3}(\sin\alpha - \alpha\cos\alpha)$ with $\alpha = \lvert q\rvert R/\hbar$	oscillating

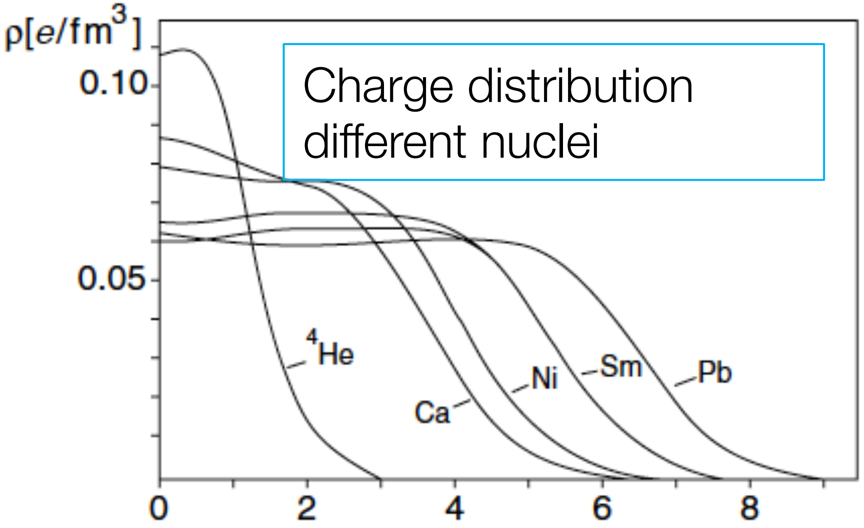
$$F(q^2) = \int e^{\frac{iqx}{\hbar}} f(x)d^3x$$



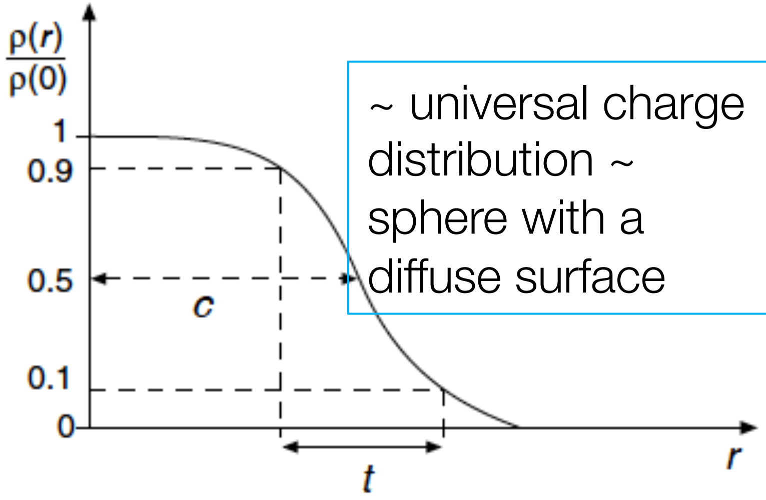
Measuring the Charge Distribution of Nuclei







Charge distribution different nuclei



~ universal charge distribution ~ sphere with a diffuse surface

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Magnetic Moments of Charged Spin ½ Particles

We must now not only take the interaction of the electron with the nuclear charge into account, but also we have to consider the interaction between the current of the electron and the nucleon's magnetic moment.

The scattering (→ deflection of the electron) produces a magnetic field that interacts with the magnetic moment of the nucleon

- The magnetic moment of a charged, spin 1/2 particle which does not possess any internal structure (a Dirac particle) (not a nucleon, known to have an internal structure) is given by

$$\mu = g \cdot \frac{e}{2M} \cdot \frac{\hbar}{2}$$

- *M* is the mass of the particle and the g =2 factor is a result of relativistic quantum mechanics;
- *The magnetic interaction is associated with a flip of the spin of the nucleon*.
- Scattering through 0° is not consistent with conservation of both angular momentum and helicity and scattering through 180° is preferred. The magnetic interaction thus introduces into the interaction an additional factor containing a factor of $sin^2 \theta/_2$.

Interaction with charge of target

Interaction with magnetic moment of target

(Here below we use $tan^2 \theta/_2 = sin^2 \theta/_2 \cdot cos^2 \theta/_2$)

$$\left(\frac{d\sigma}{d\Omega}\right)_{point\ spin\frac{1}{2}} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \left[1 + 2\tau \tan\frac{\theta}{2}^2\right]$$ where $\tau = \frac{Q^2}{M^2c^2}$ $\propto (constant \cdot \mu^2)$

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Magnetic Moments of Charged Spin ½ Particles

$$\left(\frac{d\sigma}{d\Omega}\right)_{point\ spin\frac{1}{2}} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left[1 + 2\tau\tan\frac{\theta}{2}^2\right] \qquad \text{where } \tau = \frac{Q^2}{M^2c^2}$$

- The matrix element of the interaction is proportional to $1/(M^2)$
- It is proportional to the deflection of the electron (i.e., to the momentum transfer Q^2).
- The magnetic term in the expression is large at high four-momentum transfers Q^2 and
- if the scattering angle θ is large.
- This additional term causes the cross section to fall off less strongly at larger scattering angles → al large values of Q^2 and a more isotropic distribution is found then what predicted by the electric interaction alone.

The g-factor of a spin ½ charged particle is exactly 2 (but for small understood very small deviations) while the g-factor of a spin ½ neutral particle is exactly 0.

HOWEVER Nucleons are made up of quarks and this changes things
Their g-factors for protons and neutrons are determined by the internal structure

$$\mu_p = \frac{g_p}{2}\mu_N = +2.79 \cdot \mu_N \ ,$$
$$\mu_n = \frac{g_n}{2}\mu_N = -1.91 \cdot \mu_N \ ,$$

Where μ_N is

$$\mu_N = \frac{e\hbar}{2M_p} = 3.1525 \cdot 10^{-14}\,\text{MeV}\,\text{T}^{-1}$$

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Magnetic Moments of the Proton and Neutron

Similarly to what we did for nuclei, where we introduced form factors, now we have to introduce ~ form factors to describe electric and magnetic interactions (nucleus has a structure → nucleon has a structure)

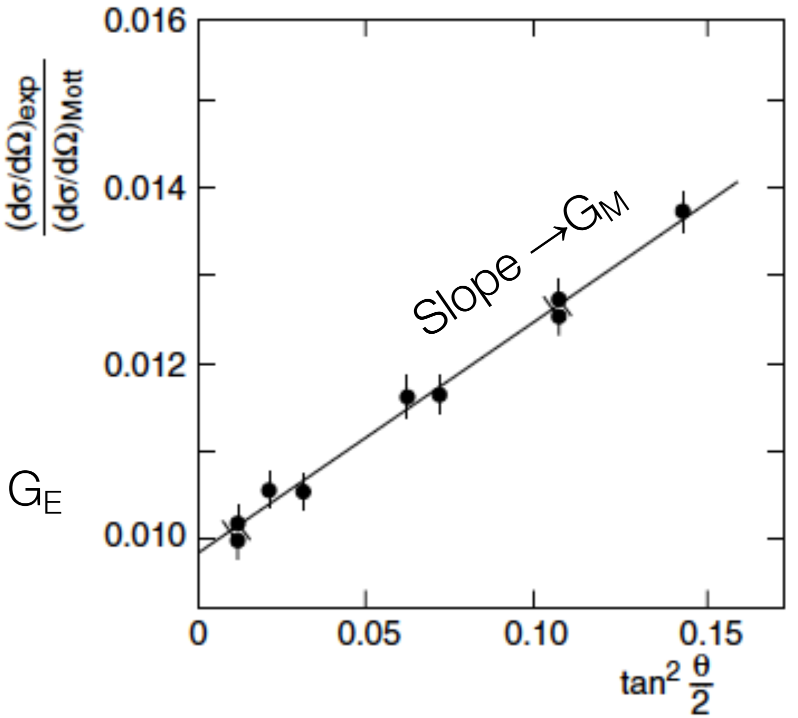
$$\left(\frac{d\sigma}{d\Omega}\right)=\left(\frac{d\sigma}{d\Omega}\right)_{Mott} \cdot \left[\frac{G_E^2(Q^2)+\tau\cdot G_M^2(Q^2)}{1+\tau} + 2\tau G_M^2(Q^2)\, tan\,\frac{\theta}{2}^2\right] \text{where } \tau = \frac{Q^2}{M^2c^2}$$

Here $G_E(Q^2)$ and $G_M(Q^2)$ electric and magnetic form factors both of which depend upon Q^2

The measured Q^2 dependence of the form factors gives us information about the radial charge distributions and the magnetic moments. The limiting case $Q^2 \rightarrow 0$ is particularly important. In this case G_E coincides with the electric charge of the target, normalised to the elementary charge e ; and G_M is equal to the magnetic moment μ of the target, normalised to the nuclear magneton. The limiting values are:

$$G_E^p(Q^2=0)=1 \qquad G_E^n(Q^2=0)=0$$
$$G_M^p(Q^2=0)=2.79 \qquad G_M^n(Q^2=0)=-1.91$$

Intercept → G_E



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Scattering of electrons on nucleus/proton

	electron		Target, charge Ze (Z=1 proton)					
Calculation	electron	Electron with spin	Point-like target, infinite Mass	Point-like target with mass M	Point-like proton	Point-like proton with spin	Finite size proton with spin	Expression
Rutherford	✅		✅					$(\frac{d\sigma}{d\Omega})_R = \frac{Z^2 e^4}{4E_0^2\,(\sin\theta/2)^4}$
Mott		✅		✅				$(\frac{d\sigma}{d\Omega})_M = (\frac{d\sigma}{d\Omega})_R \cdot (\cos\frac{\theta}{2})^2$
σ_{NS}		✅			✅			$(\frac{d\sigma}{d\Omega})_{NS} = (\frac{d\sigma}{d\Omega})_M \cdot 1/(1 - \frac{2E_0}{M}\sin\theta/2^2)$
σ		✅				✅		$(\frac{d\sigma}{d\Omega}) = (\frac{d\sigma}{d\Omega})_M \cdot (1 + \frac{q^2}{2M^2} tan\,\theta/2^2)$
Rosenbluth		✅					✅	$(\frac{d\sigma}{d\Omega}) = (\frac{d\sigma}{d\Omega})_M \cdot \left[\frac{G_E^2(Q^2)+\tau \cdot G_M^2(Q^2)}{1+\tau} + 2\tau G_M^2(Q^2)\,tan\,\theta/2^2\right]$

e nucleus ⟷ (Point-like target, infinite Mass; Point-like target with mass M)

e proton ⟷ (Point-like proton; Point-like proton with spin; Finite size proton with spin)

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Particle Physics
Toni Baroncelli
Haiping Peng
USTC

End of Introductory Part

The Transition Matrix Element

The cross-section can be experimentally determined from the reaction rate $\dot{N}$, as we saw above. We now outline how it may be found from theory. First, the reaction rate is dependent upon the properties of the interaction potential described by the Hamilton operator Hint . In a reaction, this potential transforms the initial-state wave function ψi into the final-state wave function ψf . The transition matrix element is given by:

$$\mathcal{M}_{fi} = \langle \psi_f | \mathcal{H}_{\text{int}} | \psi_i \rangle = \int \psi_f^* \, \mathcal{H}_{\text{int}} \, \psi_i \, \mathrm{d}V \, .$$

Furthermore, the reaction rate will depend upon the number of final states available to the reaction. According to the uncertainty principle, each particle occupies a volume h3 = (2π)3 in phase space, the six-dimensional space of momentum and position. Consider a particle scattered into a volume V and into a momentum interval between p and p +dp. In momentum space, the interval corresponds to a spherical shell with inner radius p and thickness dp which has a volume 4πp2dp. Excluding processes where the spin changes, the number of final states available is:

$$\mathrm{d}n(p') = \frac{V \cdot 4\pi p'^2}{(2\pi\hbar)^3} \mathrm{d}p' \, .$$

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The Transition Matrix Element

The energy and momentum of a particle are connected by:

$$\mathrm{d}E' = v'\mathrm{d}p'.$$

Hence the density of final states in the energy interval dE is given by:

$$\varrho(E') = \frac{\mathrm{d}n(E')}{\mathrm{d}E'} = \frac{V \cdot 4\pi p'^2}{v' \cdot (2\pi\hbar)^3} \,.$$

The connection between the reaction rate, the transition matrix elementand the density of final states is expressed by Fermi's second golden rule (not discussed here). It expresses the reaction rate W per target particle and per beam particle in the form:

$$\boxed{W = \frac{2\pi}{\hbar} \left|\mathcal{M}_{fi}\right|^2 \cdot \varrho(E') \,.}$$

with cross section

$$\sigma = \frac{2\pi}{\hbar \cdot v_{\mathrm{a}}} \left|\mathcal{M}_{fi}\right|^2 \cdot \varrho\left(E'\right) \cdot V \,.$$

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The Transition Matrix Element

The golden rule applies to both scattering, to the decay of unstable particles, to the excitation of particle resonances and to the transitions between different atomic or nuclear energy states. In these cases we have

$$W = \frac{1}{\tau}$$

and the transition probability per unit time can be either directly determined by measuring the lifetime τ or indirectly read off from the energy width of the state

$$\Delta E = \hbar/\tau$$

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