Causality and stability analysis for the minimal causal spin hydrodynamics

Xin-Qing Xie[1] (谢心晴), Dong-Lin Wang[1], Chen Yang[1], and Shi Pu[1]
1 Department of Modern Physics, USTC

Based on PRD 108 (2023) 9,094031, arXiv: 2306.13880

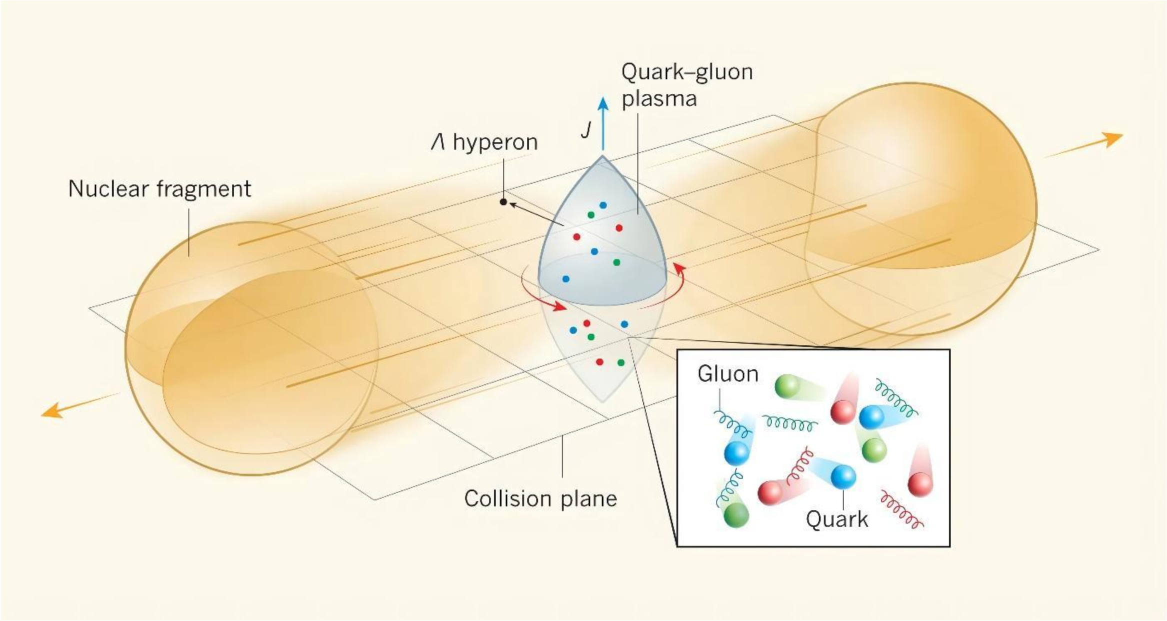
Outline

- Introduction and motivation
- First order spin hydrodynamics
- Minimal causal spin hydrodynamics
- Minimal causal spin hydrodynamics for extended q^{μ} and $\phi^{\mu\nu}$
- Causality and stability in the moving frame
- Summary

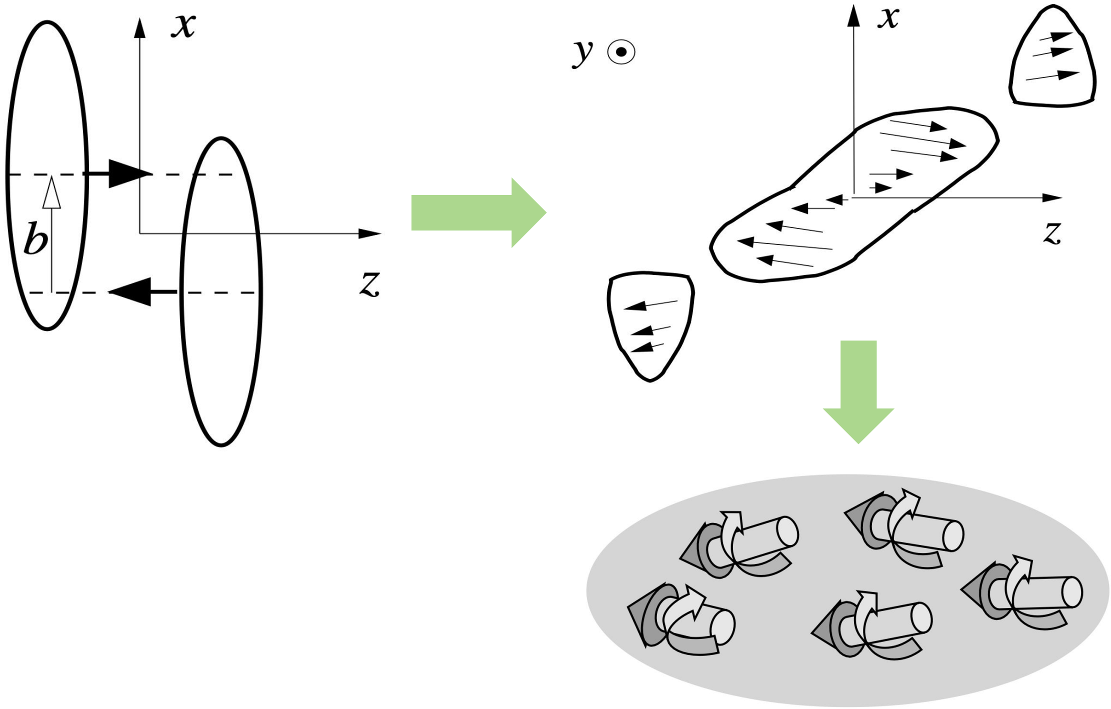
Outline

- Introduction and motivation
- First order spin hydrodynamics
- Minimal causal spin hydrodynamics
- Minimal causal spin hydrodynamics for extended q^{μ} and $\phi^{\mu\nu}$
- Causality and stability in the moving frame
- Summary

Spin-orbit coupling and spin polarization



Petersen, Nature 548, 34–35 (2017)

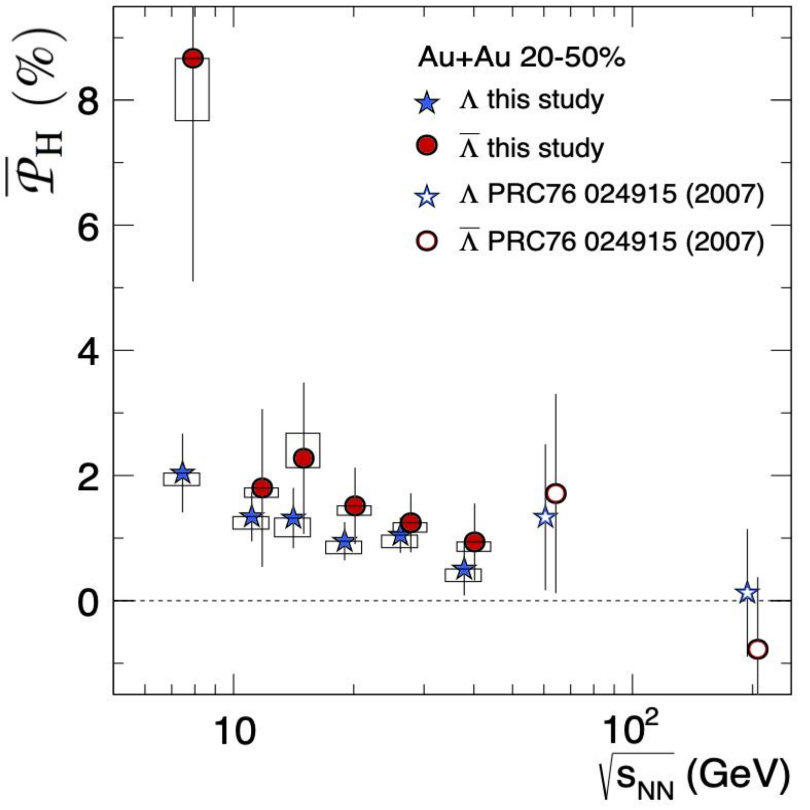


Liang and Wang, PRL 94,102301(2005); PLB 629, 20(2005)
Gao, Chen, Deng, Liang, Wang, Wang, PRC (2008)

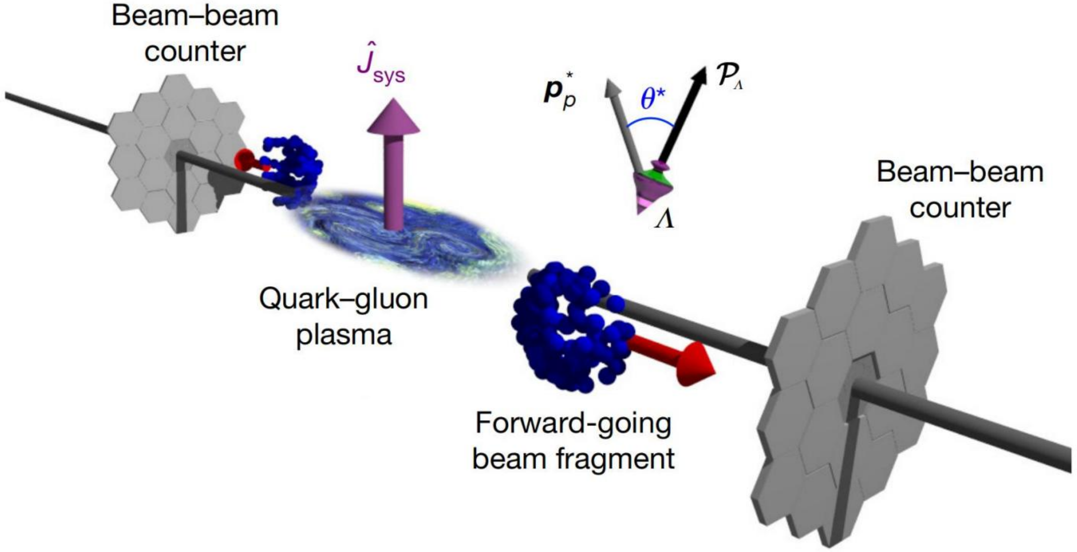
- The averaged spin of final particles produced from QGP are polarized along the direction of the initial orbital angular momentum, as known as the global polarization.

- Hyperons and vector mesons in the final state are pol arized by spin-orbit coupling.

Global polarization of $\Lambda, \bar{\Lambda}$ hyperons



STAR Collabaration, Nature 548,62 (2017)

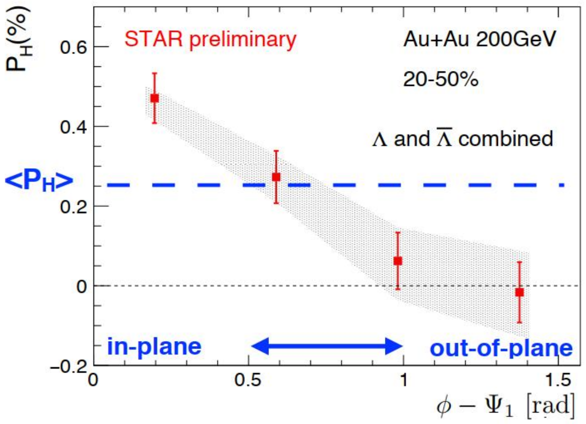


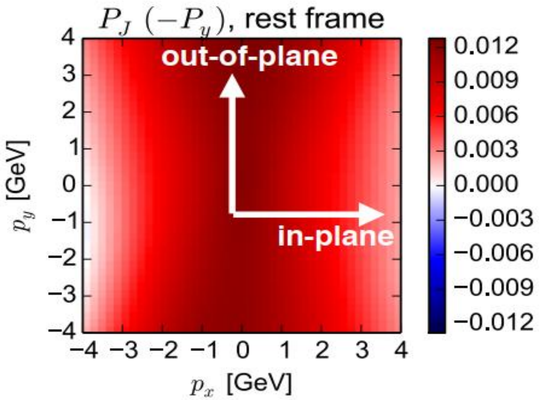
- Weak Decay: $\Lambda \to p + \pi^-$
- Proton distribution:

$$\frac{dN}{d\cos\theta^*} = \frac{1}{2}(1 + \alpha_{\rm H}|\boldsymbol{\mathcal{P}}_{\rm H}|\cos\theta^*), \quad \alpha_\Lambda = -\alpha_{\bar{\Lambda}} = 0.642 \pm 0.013$$

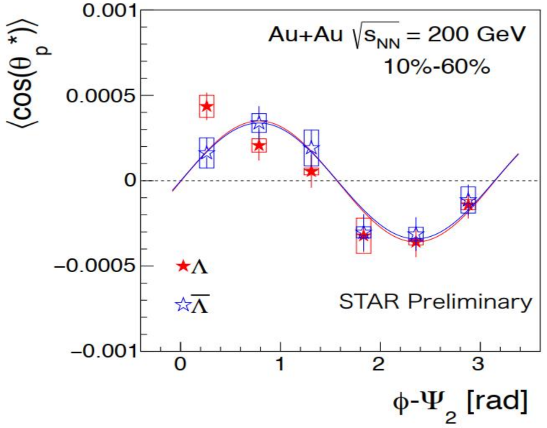
- The most vortical fluid: $\omega \approx (9 \pm 1) \times 10^{21} {\rm s}^{-1}$

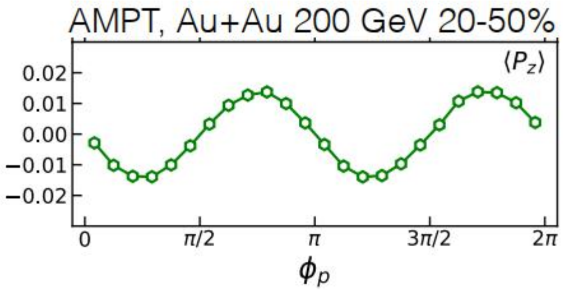
Sign problem in local polarization





Becattini, Karpenko, PRL (2018)





Xia, Li, Tang, Wang, PRC (2018)

- The sign of local polarization in theoretical calculations is opposite to that of experimental data.
- Different approaches: Relativistic spin hydrodynamics, Statistical models, Quantum kinetic theory....
- Several effects: Shear-induced polarization, Spin Hall effects....
- Although there is much important progress, the local polarization has not been fully understood.

Causality and stability in linear modes analysis

Causality and stability are the necessary requirements of relativistic theory!

① Is spin hydrodynamics causal and stable?

② How can causality and stability be ensured in spin hydrodynamics?

- Perturbations: $X = X_{(0)} + \delta X = X_{(0)} + \delta\widetilde{X} e^{i\omega t - ikx}$.
- Irrotational static equilibrium background: $S^{\mu\nu}_{(0)} = 0, \omega^{\mu\nu}_{(0)} = 0$.
- From the constraint equations, one can obtain the linear equations for the independent perturbations. Then one can calculate the dispersion relations $\omega = \omega(k)$.
- Using the causality conditions "$\lim_{k\to\infty} \left|Re\frac{\omega}{k}\right| \leq 1$, $\lim_{k\to\infty} \left|\frac{\omega}{k}\right|$ is bounded" to analyze causality.
- Using the stability condition "$\mathrm{Im}\omega(k) > 0$, for $\forall k \neq 0$" to analyze stability.

Outline

- Introduction and motivation
- First order spin hydrodynamics
- Minimal causal spin hydrodynamics
- Minimal causal spin hydrodynamics for extended q^{μ} and $\phi^{\mu\nu}$
- Causality and stability in the moving frame
- Summary

First order spin hydrodynamics

- Conservation equations for energy, momentum, total angular momentum, and particle number:

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \partial_\lambda J^{\lambda\mu\nu} = 0, \quad \partial_\mu j^\mu = 0.$$

- With the decomposition of the total angular momentum current,

$$J^{\lambda\mu\nu} = x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu} + \boxed{\Sigma^{\lambda\mu\nu}}$$

the conservation equation $\partial_\lambda J^{\lambda\mu\nu} = 0$ can be rewritten as $\partial_\lambda \Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}$

- Decomposition for energy momentum tensor, particle current and rank-3 spin tensor:

$$\Theta^{\mu\nu} = eu^\mu u^\nu - (p+\Pi)\Delta^{\mu\nu} + 2h^{(\mu}u^{\nu)} + \pi^{\mu\nu} + \boxed{2q^{[\mu}u^{\nu]} + \phi^{\mu\nu},}$$

$$j^\mu = nu^\mu + \nu^\mu,$$

$$\boxed{\Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + \Sigma_{(1)}^{\lambda\mu\nu},}$$

- Thermodynamic relations: $e + p = Ts + \mu n + \boxed{\omega_{\mu\nu} S^{\mu\nu},}$

$$de = Tds + \mu dn + \omega_{\mu\nu} dS^{\mu\nu}.$$

$$\partial_\mu \mathcal{S}^\mu \geq 0.$$

First order spin hydrodynamics

- Constitutive relations:

$$
\begin{aligned}
\partial_\mu \mathcal{S}^\mu_{\rm can} = & \left(h^\mu - \frac{e+p}{n}\nu^\mu\right)\left[\partial_\mu \frac{1}{T} + \frac{1}{T}(u\cdot\partial)u_\mu\right] \\
& + \frac{1}{T}\pi^{\mu\nu}\partial_\mu u_\nu - \frac{1}{T}\Pi(\partial\cdot u) + \frac{1}{T}\phi^{\mu\nu}(\partial_\mu u_\nu + 2\omega_{\mu\nu}) \\
& + \frac{q^\mu}{T}\left[T\partial_\mu \frac{1}{T} - (u\cdot\partial)u_\mu + 4\omega_{\mu\nu}u^\nu\right] + \mathcal{O}(\partial^3) \\
\geq 0
\end{aligned}
$$

$$
\begin{aligned}
h^\mu - \frac{e+p}{n}\nu^\mu &= \kappa\Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T - (u\cdot\partial)u_\nu\right], \\
\pi^{\mu\nu} &= 2\eta\partial^{<\mu}u^{\nu>}, \\
\Pi &= -\zeta\partial_\mu u^\mu, \\
q^\mu &= \lambda\Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T + (u\cdot\partial)u_\nu - 4\omega_{\nu\alpha}u^\alpha\right], \\
\phi^{\mu\nu} &= 2\gamma_s\Delta^{\mu\rho}\Delta^{\nu\sigma}(\partial_{[\rho}u_{\sigma]} + 2\omega_{\rho\sigma}),
\end{aligned}
$$

$$\kappa, \eta, \zeta, \lambda, \gamma_s > 0,$$

- The power counting scheme:

$$S^{\mu\nu} \sim O(1),\ \omega_{\mu\nu} \sim O(\partial),\ \Sigma^{\lambda\mu\nu}_{(1)} \sim O(\partial).$$

- We adopt the canonical form in the Landau frame, and neglect the conserved charge current j^μ.
- For simplicity, we consider vanishing $\Sigma^{\lambda\mu\nu}_{(1)}$ and analyze in the rest frame.

Analysis in the first order spin hydrodynamics

- In the $k \to \infty$ limit, $\omega = \omega(k)$ are

$$\omega = -4iD_b\gamma_{\|}^{-1}\lambda'^{-1}k^{-2} + O(k^{-3}),$$
$$\omega = -ic_s^{2/3}\gamma_{\|}^{1/3}k^{4/3} + O(k),$$
$$\omega = (-1)^{1/6}c_s^{2/3}\gamma_{\|}^{1/3}k^{4/3} + O(k),$$
$$\omega = (-1)^{5/6}c_s^{2/3}\gamma_{\|}^{1/3}k^{4/3} + O(k),$$
$$\omega = -2iD_b + O(k^{-1}),$$
$$\omega = 2iD_s\gamma_\perp(\gamma' + \gamma_\perp)^{-1} + O(k^{-1}),$$
$$\omega = \pm ik\sqrt{2\lambda'^{-1}(\gamma' + \gamma_\perp)} + O(k^0),$$

- Different hydrodynamics frames

$$q^\mu = \lambda\left(\frac{2\Delta^{\mu\nu}\partial_\nu p}{e+p} - 4\omega^{\mu\nu}u_\nu\right) + O(\partial^2).$$

$\omega = i(\gamma' + \gamma_\perp)k^2$ as $k \to \infty$,

(B. 2) of Phys. Lett. B 795 (2019) by Hattori, Hongo, Huang, Matsuo, Taya

- The stability conditions need

$$D_s > 0, \quad \lambda' < 0, \quad D_b < -4c_s\lambda\gamma_{\|}^{-1}|\chi_e^{0x}| \le 0.$$

➢ The first order spin hydrodynamics are acausal and unstable!

Outline

- Introduction and motivation
- First order spin hydrodynamics
- Minimal causal spin hydrodynamics
- Minimal causal spin hydrodynamics for extended q^{μ} and $\phi^{\mu\nu}$
- Causality and stability in the moving frame
- Summary

Minimal causal spin hydrodynamics

- Introduce nonzero relaxation time:

$$\begin{aligned}
\tau_q \Delta^{\mu\nu}\frac{d}{d\tau}q_\nu + q^\mu &= \lambda[T^{-1}\Delta^{\mu\alpha}\partial_\alpha T + (u\cdot\partial)u^\mu - 4\omega^{\mu\nu}u_\nu], \\
\tau_\phi \Delta^{\mu\alpha}\Delta^{\nu\beta}\frac{d}{d\tau}\phi_{\alpha\beta} + \phi^{\mu\nu} &= 2\gamma_s \Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_{[\alpha}u_{\beta]} + 2\omega_{\alpha\beta}), \\
\tau_\pi \Delta^{\alpha<\mu}\Delta^{\nu>\beta}\frac{d}{d\tau}\pi_{\alpha\beta} + \pi^{\mu\nu} &= 2\eta\partial^{<\mu}u^{\nu>}, \\
\tau_\Pi \frac{d}{d\tau}\Pi + \Pi &= -\zeta\partial_\mu u^\mu,
\end{aligned}$$

- When we consider the spin hydrodynamics with zero viscous effects, i.e., $\delta\Pi = 0$, $\delta\pi^{ij} = 0$, the causality conditions require,

$$0 \le \frac{c_s^2(3\lambda' + 2\tau_q)}{2\tau_q - \lambda'} \le 1, \; 0 \le \frac{2\gamma'\tau_q}{(2\tau_q - \lambda')\tau_\phi} \le 1.$$

The stability condition in small and large k limits requires,

$$\tau_q > \lambda'/2, \; D_s > 0, \; D_b < 0, \; \chi_e^{0x} = 0.$$

We can implement the Routh-Hurwitz criterion to prove the stability constraints above are sufficient and necessary for stability.
Interesingly, there exist zero modes, i.e., $\omega = 0$. It indicates that there exist nonlinear modes.

Analysis for the minimal causal spin hydrodynamics

- Causality conditions: $0 \leq \frac{b_1^{1/2} \pm (b_1 - b_2)^{1/2}}{6(2\tau_q - \lambda')\tau_\pi \tau_\Pi} \leq 1$ and $0 \leq \frac{2\tau_q(\gamma'\tau_\pi + \gamma_\perp \tau_\phi)}{(2\tau_q - \lambda')\tau_\pi \tau_\phi} \leq 1,$

> - The causality conditions imply that the relaxation time cannot be arbitrarily small.
> - The minimal causal spin hydrodynamics can be causal in the rest frame.

- Stability condition in small and large k limits:

$$
\begin{aligned}
\tau_q &> \lambda'/2, \\
D_s &> 0, \quad D_b < -4c_s\lambda\gamma_{\parallel}^{-1}\left|\chi_e^{0x}\right| \leq 0, \\
b_1 &> b_2 > 0, \quad \frac{c_2}{c_3} > 0.
\end{aligned}
$$

- The satisfaction of stability conditions above relies on the equation of state for $S^{\mu\nu}$ and $\omega^{\mu\nu}$.

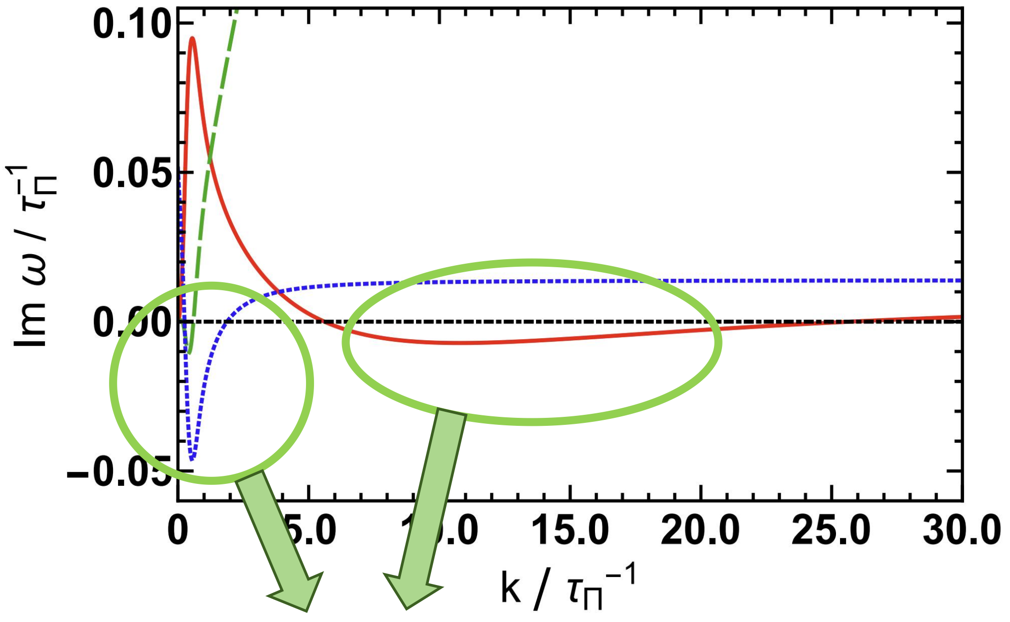
$$\delta\omega^{\mu\nu} = \chi_1 \delta S^{\mu\nu} + \chi_2 \Delta^{\mu\alpha}\Delta^{\nu\beta}\delta S_{\alpha\beta}.$$

$$D_s = 4\gamma_s(\chi_1 + \chi_2), \;\; D_b = 4\lambda\chi_1.$$

$$D_s > 0, D_b < 0 \Longleftrightarrow \chi_2 > -\chi_1 > 0.$$

Non-trivial stability conditions

- Stability at finite k:



$\mathrm{Im}\,\omega(k) < 0$ Violate stability!

$$c_s = \frac{1}{\sqrt{3}}, \quad \lambda\chi_e^{0x} = \frac{1}{8}, \ \tau_\pi = 4\tau_\Pi, \ \tau_\phi = 2\tau_\Pi, \ \tau_q = 10\tau_\Pi, \ \lambda' = \frac{1}{2}\tau_\Pi,$$
$$\gamma_\parallel = \frac{7}{10}\tau_\Pi, \quad \gamma_\perp = \frac{1}{2}\tau_\Pi, \ \gamma' = \tau_\Pi, \ D_s = \frac{1}{2\tau_\Pi}, \ D_b = -\frac{1}{2\tau_\Pi}.$$

- The stability constraints in small and large k limits are necessary but may not be sufficient for spin hydrodynamics.
- It is still unclear whether the unstable modes at finite k indicates the fluid becomes unstable or not.

Outline

- Introduction and motivation
- First order spin hydrodynamics
- Minimal causal spin hydrodynamics
- Minimal causal spin hydrodynamics for extended q^{μ} and $\phi^{\mu\nu}$
- Causality and stability in the moving frame
- Summary

Analysis for extended q^μ and $\phi^{\mu\nu}$

- We introduce the simplest coupling between q^μ and $\phi^{\mu\nu}$,

$$\tau_q \Delta^{\mu\nu} \frac{d}{d\tau} q_\nu + q^\mu = \lambda \left(T^{-1} \Delta^{\mu\nu} \partial_\nu T + u^\nu \partial_\nu u^\mu - 4 u_\nu \omega^{\mu\nu} \right) + g_1 \Delta^{\mu\nu} \partial^\rho \phi_{\nu\rho},$$

$$\tau_\phi \Delta^{\mu\alpha} \Delta^{\nu\beta} \frac{d}{d\tau} \phi_{\alpha\beta} + \phi^{\mu\nu} = 2\gamma_s \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_{[\alpha} u_{\beta]} + 2\omega_{\alpha\beta} \right) + g_2 \Delta^{\mu\alpha} \Delta^{\nu\beta} \partial_{[\alpha} q_{\beta]},$$

- When we consider the cases without viscous effects, the causality conditions require,

$$0 \le \frac{c_s^2 (3\lambda' + 2\tau_q)}{2\tau_q - \lambda'} \le 1, \; 0 \le \frac{m}{4(2\tau_q - \lambda')\tau_\phi} \le 1,$$

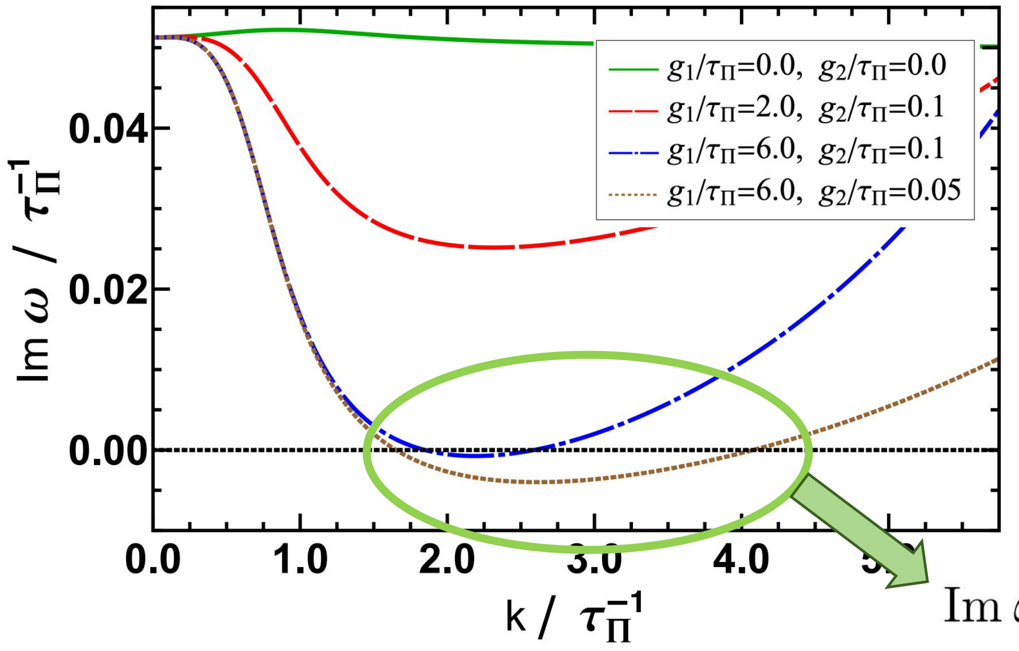
And the stability condition in small and large k limits requires,

$$\tau_q > \lambda'/2, \; D_s > 0, \; D_b < 0, \; \chi_e^{0x} = 0, \quad m > 8\gamma' \left(\frac{2}{2\tau_q - \lambda'} + \frac{1}{\tau_\phi} \right)^{-1}.$$

We can implement the Routh-Hurwitz criterion again to prove that the constraints are sufficient and necessary for stability. However, there still find the zero modes.

Analysis for extended q^{μ} and $\phi^{\mu\nu}$

- We need to consider the non-vanishing viscous effects. Similarly, we obtain the causality conditions and the stability constraints in small and large k limits, which are modified by the coupling between q^{μ} and $\phi^{\mu\nu}$. Then we choose the same parameters before with $(g_1/\tau_\Pi, g_2/\tau_\Pi) = (0.0,0.0), (2.0,0.1), (6.0,0.1), (6.0,0.05)$ to analyze the stability at finite k,



$\mathrm{Im}\,\omega(k) < 0$

Violate stability!

- The extended q^{μ} and $\phi^{\mu\nu}$ can modify the causality and stability conditions, but cannot remove the zero modes when we turn off other dissipative effects.
- The unstable modes at finite k cannot be cured by the extended q^{μ} and $\phi^{\mu\nu}$.

Outline

- Introduction and motivation
- First order spin hydrodynamics
- Minimal causal spin hydrodynamics
- Minimal causal spin hydrodynamics for extended q^{μ} and $\phi^{\mu\nu}$
- Causality and stability in the moving frame
- Summary

Causality and stability in the moving frame

- **Causality:** some studies show that the system is causal in the moving frame if it is causal in the rest frame. Thus, the minimal causal spin hydrodynamics are causal in the moving frame when the causality conditions in the rest frame are satisfied.

P. Kovtun, JHEP 10, 034 (2019)
D.-L. Wang and S. Pu, PRD 109 (2024) 3, L031504
R. E. Hoult and P. Kovtun, PRD 109 (2024) 4, 046018

- **Stability:** If a causal theory is unstable in the rest frame, it is also unstable in the moving frame. Thus, if the equation of state violates the stability, the minimal causal spin hydrodynamics will be unstable in the moving frame since it has unstable modes in the rest frame.

L. Gavassino, PRX 12 (2022) 4, 041001

Outline

- Introduction and motivation
- First order spin hydrodynamics
- Minimal causal spin hydrodynamics
- Minimal causal spin hydrodynamics for extended q^{μ} and $\phi^{\mu\nu}$
- Causality and stability in the moving frame
- Summary

Summary

- The first order spin hydrodynamics are acausal and unstable.
- The relaxation time cannot be arbitrarily small. The minimal causal spin hydrodynamics can be causal in any reference frame.
- The stability constraints in small and large k limits are necessary but may not be sufficient for spin hydrodynamics, which is different with the conventional hydrodynamics.
- The satisfaction of stability condition relies on the equation of state for $S^{\mu\nu}$ and $\omega^{\mu\nu}$.
- The extended q^{μ} and $\phi^{\mu\nu}$ modifies the causality and stability conditions. However, the zero modes and unstable modes at finite k still exist.

TABLE I. Causality and stablity of different hydrodynamic frames in the rest frame

	first order	minimal causal		minimal causal for extended q^{μ} and $\phi^{\mu\nu}$	
		w/o viscous	w/ viscous	w/o viscous	w/ viscous
Causality conditions	$\times$	✓ (under constrains)	✓ (under constrains)	✓ (under constrains)	✓ (under constrains)
Stability in $k \to 0$ and $k \to \infty$	$\times$	✓ (under constrains)	✓ (under constrains)	✓ (under constrains)	✓ (under constrains)
Stability at finite k		✓ (R-H criterion)	$\times$ (numerical figure)	✓ (R-H criterion)	$\times$ (numerical figure)
Other comments		zero modes!☹	Stability $\leftrightarrow$ EoS	zero modes!☹	Stability $\leftrightarrow$ EoS new modes $\times$ stability

Thank you!