Towards a “bottom-up” construction of spin kinetic theory

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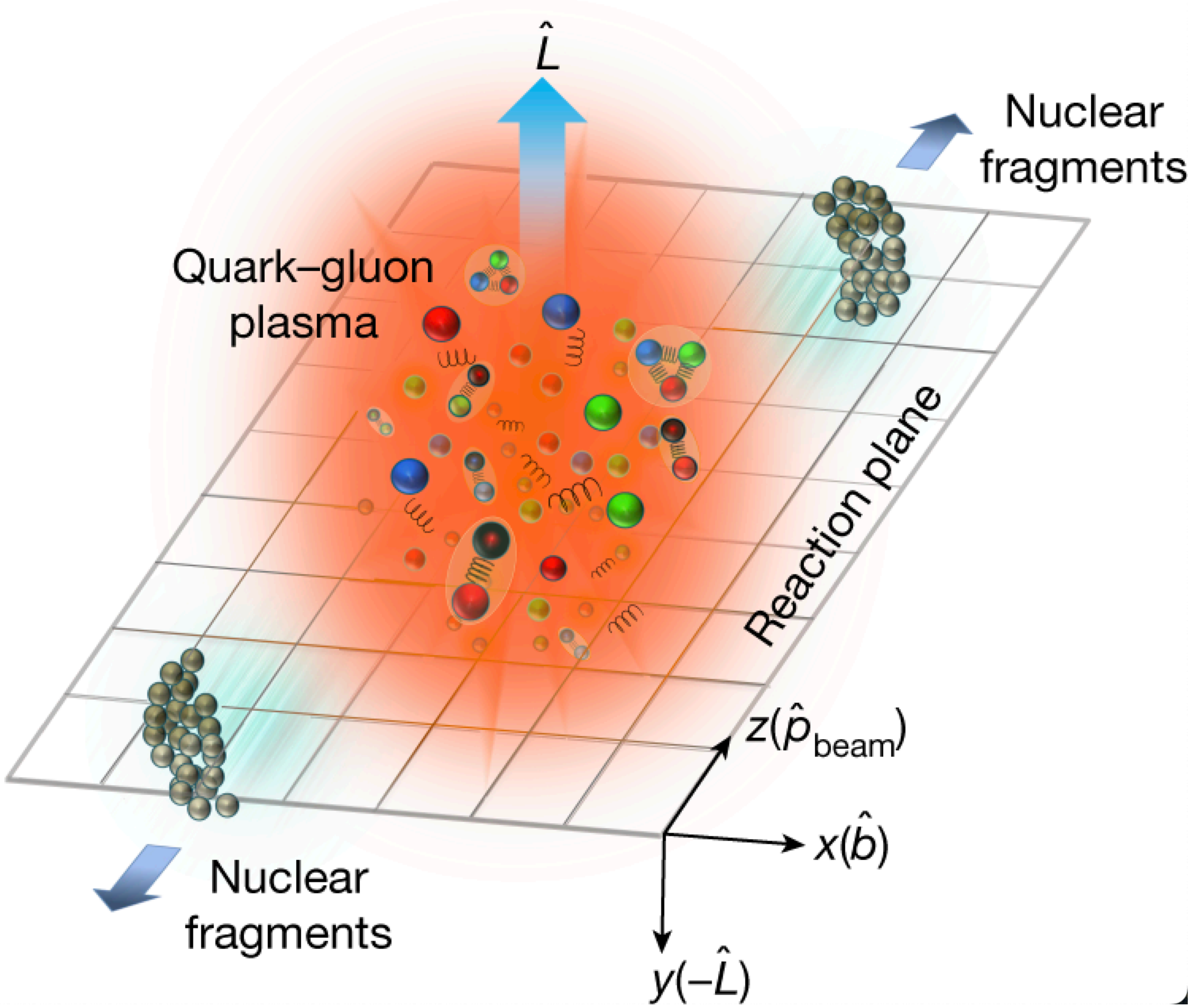
Contents

Introduction

Heavy-ion Collision (HIC)

- Heavy-ion collision (HIC) creates QCD matter in extreme condition



○ Spin observables in HIC

- Informative Λ hyperon polarization
  Xin-Nian Wang, Zuo-Tang Liang, PRL05'; Becattini et al, Annals Phys 13'
- Vector mesons ($K^{\star}$,ϕ, J/ψ)
  e.g. STAR, 2204.02302;ALICE PRL 20', 2204.10171

- an important way to probe the properties of QGP

Quantum Kinetic Theory (QKT)

- Kinetic theory is an effective description of many-body systems

- "Top-down" approach is generally used
  - start from a microscopic theory
  - Derive EoM of the 4×4 Wigner function (its Clifford coefficients)

See Y. Hidaka, S. Pu, Q. Wang, DL, Yang, Prog. Part. Nucl. Phys. 127 (2022) as a review

$$W(x,p) = \int_y e^{-ip\cdot y}\langle \psi(x_+)\bar{\psi}(x_-)\rangle, \quad x_\pm = x \pm \frac{y}{2}$$

- Clifford decomposition

$$W = \frac{1}{4}\left(\mathscr{F}\mathbf{1} + i\mathscr{P}\gamma^5 + \mathscr{V}^\mu\gamma_\mu + \mathscr{A}^\mu\gamma^5\gamma_\mu + \frac{1}{2}\mathscr{S}^{\mu\nu}\sigma_{\mu\nu}\right)$$

"Bottom-up" Methodology

- Effective description of low-energy properties of a many-body system

1. identifying the relevant slow d.o.f., collectively denoted by χ;

2. constructing the equation of motion for χ;

3. expressing other fast observables in term of χ and the resulting expression is referred to as the constitutive relation.

Largely inspired by Jingyuan Chen, Dam T. Son, Annals Phys. 377 (2017)

- Massless particle: distribution $f(t, \boldsymbol{x}; \boldsymbol{p})$ and helicity distribution $f_A(t, \boldsymbol{x}; \boldsymbol{p})$;
- Massive particle: distribution $f(t, \boldsymbol{x}; \boldsymbol{p})$ and spin distribution $\boldsymbol{s}(t, \boldsymbol{x}; \boldsymbol{p})$. [4 d.o.f.]

- In massless limit, $\boldsymbol{s} \propto \hat{\boldsymbol{p}}$, where the helicity distribution $f_A = \hat{\boldsymbol{p}} \cdot \boldsymbol{s}$.

Spin Kinetic Theory (SKT)

Massless EoM as a Heuristic

- Assume $\tau_R^{-1} \ll \omega, q \ll T_{\text{eff}}$ (ignore collision; insure the gradient expansion)

- Kinetic equations up to $\mathcal{O}(\partial)$ and $\mathcal{O}(\phi)$, $\phi = A_\mu, h_{\mu\nu}$

$$\left(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\partial_x} + \boldsymbol{F}(\boldsymbol{x})\right) f(t, \boldsymbol{x}; \boldsymbol{p}) = 0$$

$$(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\partial_x}) f_A(t, \boldsymbol{x}; \boldsymbol{p}) - (\boldsymbol{\partial_x} \Phi_A) \cdot \boldsymbol{\partial_p} f(t, \boldsymbol{x}; \boldsymbol{p}) = 0$$

where $\boldsymbol{v} = \boldsymbol{p}/E_{\boldsymbol{p}}$ with $E_{\boldsymbol{p}}$ the single particle energy.

- $\boldsymbol{F}$: spin independent force

- Φ_s: energy shift of different helicity states generated by external field ϕ, $\Phi_A = \Phi_+ - \Phi_-$.

Kinetic Equation in SKT

- In massive case, the EoM of f is the same as massless limit; the EoM of f_A should be replaced by that of $\boldsymbol{s}$

- We *propose* the EoM for $\boldsymbol{s}$

$$(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\partial_x})\, \boldsymbol{s}(t, \boldsymbol{x}; \boldsymbol{p}) - \partial_i \boldsymbol{\Pi}(\phi, \boldsymbol{p}) \partial_p^j f(t, \boldsymbol{x}; \boldsymbol{p}) = 0$$

where $\boldsymbol{\Pi} = \sum_\tau \lambda_\tau(\boldsymbol{p}) X_\tau(\phi, \boldsymbol{p})$ is a combination of axial vectors constructed by the gradient of the external fields. For example, when ϕ is electromagnetic field we have,

$$X_\tau^i(\phi, \boldsymbol{p}) = \{B^i, \Delta_{\boldsymbol{v}}^{ij} B_j, (\boldsymbol{v} \times \boldsymbol{E})^i\}, \;\; \Delta_{\boldsymbol{v}}^{ij} \equiv \boldsymbol{v}^2 \delta^{ij} - v^i v^j$$

Constitutive Relation

- According to the methodology of the effective theory, the constitutive relation should contains all possible terms (with right parity) constructed by $f, \boldsymbol{s}, \phi, \boldsymbol{p}$.

- The constitutive relation contains two parts

$$J^\mu = \boxed{v^\mu f + \cdots} + \quad \text{direct response to } \phi \text{ in terms of } n_F\,(n_F') \text{ and } \phi$$

$\downarrow$

kinetic equation

$\downarrow$

eg. Hall conductivity

The functional relation $J^\mu[f]$ is general

Largely inspired by Jingyuan Chen, Dam T. Son, Annals Phys. 377 (2017)

Constitutive Relation

- Vector Wigner function: $-\mathscr{V}^\mu = v^\mu f + (\Delta\mathscr{V})^\mu_{\text{ext}}$

  where $(\Delta\mathscr{V})^\mu_{\text{ext}}$ is a combination of vectors constructed by the external fields and should be parity odd.

- Axial Wigner function: $\mathscr{A}^i(f, \boldsymbol{s}, \phi; \boldsymbol{p}) = s^i + \mu_0(\boldsymbol{p})\epsilon^{ijk}v_j\partial_k f + (\Delta\mathscr{A})^i_{\text{ext}}$

  where μ_0 are some momentum-dependent coefficients. Similar to $\boldsymbol{\Pi}$,

$$(\Delta\boldsymbol{\mathscr{A}})_{\text{ext}} = \sum_\tau \kappa_\tau(\boldsymbol{p})\boldsymbol{X}_\tau(\phi, \boldsymbol{p})$$

- In parallel we have $\mathscr{A}^0 = f_A + (\Delta\mathscr{A}^0)_{\text{ext}}, \quad f_A = \boldsymbol{v}\cdot\boldsymbol{s}$.

Response to EM field

- Electromagnetic force $\boldsymbol{F}_{\mathrm{EM}} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$

- Expand f around equilibrium $n_F(\boldsymbol{p})$ (Fermi-Dirac distribution)
  $f = 2n_F + \delta f$, $\delta f \sim \mathcal{O}(\phi)$, $\boldsymbol{s} \sim \mathcal{O}(\phi)$, then the EoM up to $\mathcal{O}(\phi)$ is

$$(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\partial_x})\, \delta f + \boldsymbol{E} \cdot \boldsymbol{v} n_F' = 0$$

$$(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\partial_x})\, \boldsymbol{s} - \boldsymbol{\Pi} \Delta\varepsilon n_F' = 0$$

- The solution in momentum space are

$$\delta f = 2D_F \boldsymbol{v} \cdot \boldsymbol{E},\ \ \boldsymbol{s} = -2iD_F(\boldsymbol{q} \cdot \boldsymbol{v})\boldsymbol{\Pi},\ \ \text{with } D_F = \frac{-in_F}{\omega - \boldsymbol{q} \cdot \boldsymbol{v} + i\epsilon}$$

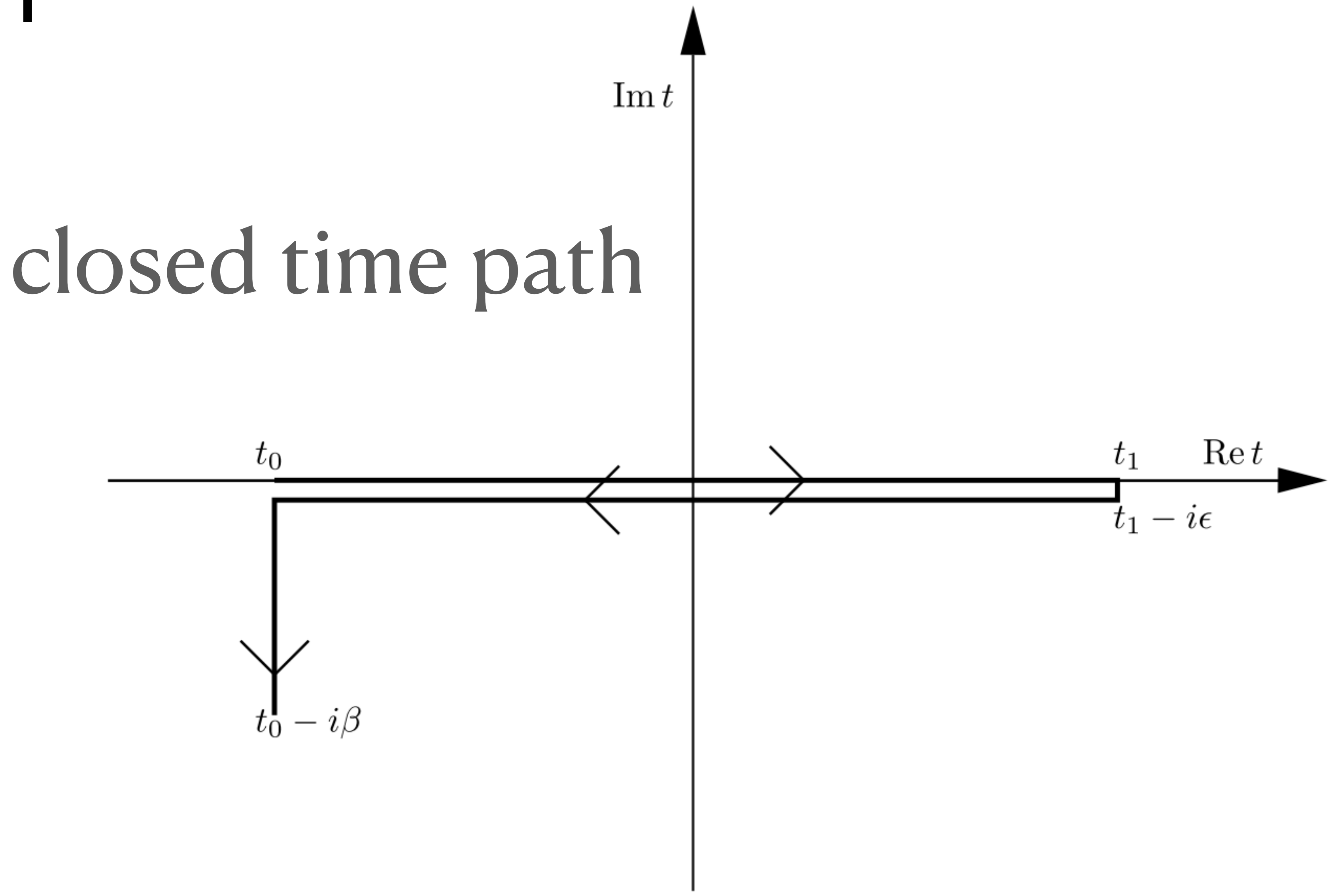
Linear Response matches SKT

Covariant Wigner Function

- Covariant two-point correlation function with gauge link (up to $\mathcal{O}(\partial)$, $\mathcal{O}(\phi)$)

$$S(x,y;\beta,\phi) = \langle U(x,x_+;\phi)\psi(x_+)\bar{\psi}(x_-)U(x_-,x;\phi)\rangle$$

using field theory in Schwinger-Keldysh formalism



- Invariant under $U(1)$: $A_\mu \to A_\mu + \partial_\mu \theta$

- Covariant under GCT: $x \to x - \xi(x)$

  and LLT: $e_I^\mu \to e_I^\mu + \Omega_I{}^J e_J^\mu$

- The covariant Wigner function: $W(x,p) = \int_y \sqrt{-g}\; e^{-ip\cdot y} S(x,y;\beta,\phi)$

Response to EM field

- $W_\phi = W_0 + G_R\phi + L_\phi\phi$ (1-Loop level), retarded correlation + gauge link
- 1-loop calculation gives the vector Wigner function

$$-\mathscr{V}^\mu = 2v^\mu n_F + v^\mu D_F \boldsymbol{v}\cdot\boldsymbol{E} = v^\mu(2n_F + \delta f) = v^\mu f$$

which matches the solution of EoM. And the axial Wigner function

$$\mathscr{A}^0 = \boldsymbol{v}\cdot\boldsymbol{s} + \kappa_3\boldsymbol{v}\cdot\boldsymbol{B}$$

$$\mathscr{A}^i = s^i + \mu_0\epsilon^{ijk}v_j\partial_k f + \kappa_2\Delta_{\boldsymbol{v}}^{ij}B_j + \kappa_3(\boldsymbol{v}\times\boldsymbol{E})^i$$

with $\lambda_1 = -\lambda_2 = -1/(2E_{\boldsymbol{p}}),\ \lambda_3 = 0$

in $\boldsymbol{s}$, and $\mu_0 = 1/E_{\boldsymbol{p}},\ \kappa_1 = 0,\ \kappa_2 = n_F'/E_{\boldsymbol{p}},\ \kappa_3 = n_F/E_{\boldsymbol{p}}^2$

Response to Weak Gravity

- Gravitational "force": $F_i = dp_i/dt = \Gamma^{\rho}_{i\mu} p^{\mu} p_{\rho}/E_{\boldsymbol{p}}$

- Choose a coordinate s.t. $h_{00} = h_{0j} = 0,\ h_{jk} \neq 0$, and define

$$a_i \equiv h_{ij} p^j/2,\ e_i \equiv \partial_t a_i,\ b_i \equiv \epsilon^{ijk} \partial_j a_k$$

The results are very similar to the response to EM field.

$$-\mathscr{V}^{\mu} = v^{\mu} f - n_F (2h^{\mu\nu} v_{\nu} - \delta^{\mu}_i v^i h_{\nu\rho} v^{\nu} v^{\rho})$$

$$\mathscr{A}^0 = \boldsymbol{v} \cdot \boldsymbol{s}$$

$$\mathscr{A}^i = s^i + \mu_0 \epsilon^{ijk} v_j \partial_k f + \kappa_2 \Delta^{ij}_{\boldsymbol{v}} b_j$$

with $\lambda_1 = -\lambda_2 = -1/(2E_{\boldsymbol{p}}),\ \lambda_{3\sim7} = 0$

in $\boldsymbol{s}$, and $\mu_0 = 1/E_{\boldsymbol{p}},\ \kappa_1 = 0,\ \kappa_2 = n_F'/E_{\boldsymbol{p}},\ \kappa_{3\sim7} = n_F/E^2_{\boldsymbol{p}}$

Covariant Kinetic Equation

- The EoM of f given by horizontal lift derivative $D_\mu = \partial_\mu + \Gamma^\rho_{\mu\nu} p_\rho \partial^\nu_p$

O.A. Fonarev, J. Math. Phys. 35 (1994)   Y.-C. Liu, L.-L. Gao, K. Mameda and X.-G. Huang, Phys. Rev. D 99 (2019)

$$v^\mu D_\mu f = v^\mu(\partial_\mu + \Gamma^\rho_{\mu\nu} p_\rho \partial^\nu_p)(n_F + \delta f) = 0$$

- In the EoM of $\boldsymbol{s}$, Π^i is not covariant since the first order derivative of $h_{\mu\nu}$ is not covariant, so we replace $\partial_j \Pi^i$ with tensor $\Pi^i{}_j = \sum_\tau \lambda'_\tau(\boldsymbol{p})(X_\tau)^i{}_j(\phi, \boldsymbol{p})$, a combination of tensors constructed by the (second order) gradient of the external fields, which should be parity odd since $\boldsymbol{s}$ is parity even. The curvature is the second order derivative of $h_{\mu\nu}$ and covariant.

$$(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\partial_x})\, s^i(t, \boldsymbol{x}; \boldsymbol{p}) - \Pi^i{}_j(\phi, \boldsymbol{p})\partial^j_p f(t, \boldsymbol{x}; \boldsymbol{p}) = 0$$

Response to Weak Gravity

- $\delta f = iE_{\boldsymbol{p}} D_F \omega h_{\mu\nu} v^\mu v^\nu$, $\quad -\mathscr{V}^\mu = v^\mu f - n_F(2h^{\mu\nu} v_\nu - \delta^\mu_i v^i h_{\nu\rho} v^\nu v^\rho)$

- $s^i = -D_F \Pi^i{}_j v^j$, with

$$(X_\tau)^i{}_j = \{\epsilon^{ijk} R_{jk0l},\ \Delta^{im}_{\boldsymbol{v}} \epsilon^{mjk} R_{jk0l},\ \epsilon^{ijk} v_j R_{0k0l},\ \epsilon^{ijk} v_j R_{0k} v_l,\ \epsilon^{ijk} v_j R_{kl}\}$$

- Axial Wigner function $\quad \mathscr{A}^0 = \boldsymbol{v} \cdot \boldsymbol{s}$

$$\mathscr{A}^i = s^i + \mu_0' \epsilon^{ijk} v_j D_k f$$

with $\quad \lambda_1 = -\lambda_2 = i/2,\ \lambda_{3\sim5} = 0,\ \mu_0' = 1/(2E_{\boldsymbol{p}})$.

$(\mathscr{A})^\mu_{\text{ext}}$ is forbidden in $\mathcal{O}(\partial)$ by covariance. (Again, $R \sim \mathcal{O}(\partial^2)$)

Summary and Outlook

Summary

- We construct an effective "bottom-up" SKT and confirm this approach in finite temperature under EM field and weak gravity, using field theory up to 1-loop level.

- SKT has less d.o.f. (f & s, 4 in total) than traditional QKT.
- The EoM in SKT is simple and easy to solve.
- SKT covers all the possible structure (the spirit of effective theory).

- Outlook: 1. A unified covariant EoM of s;

  2. More constrain to the EoM and the constitutive relation.

  3. SKT is hopeful to become a complement of QKT.

THANK YOU!

Back Up

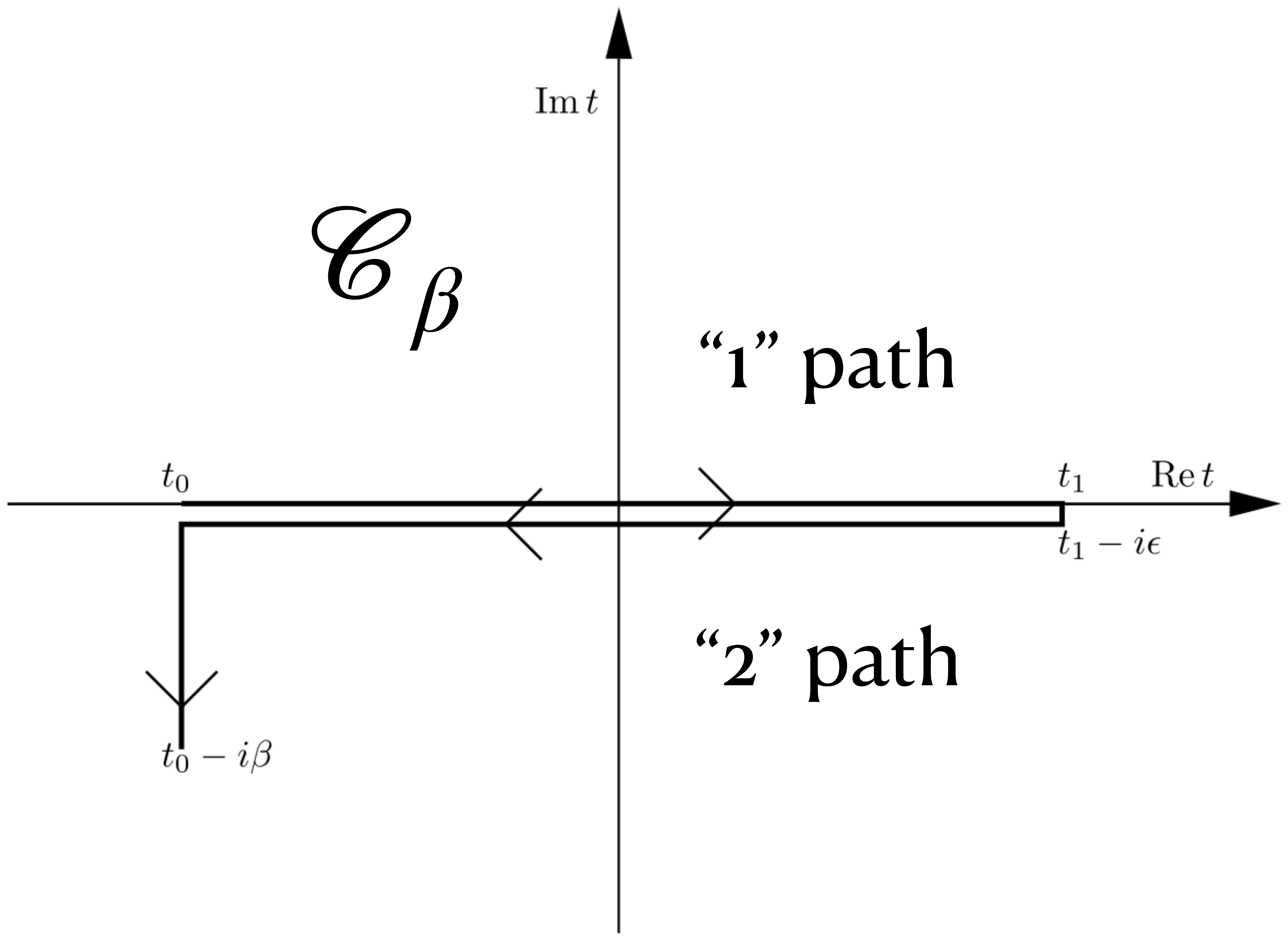
Schwinger-Keldysh Formalism

- Closed time path (CTP)
- r/a basis: for arbitrary operator $\hat{O}$

$$\hat{O}_r = \frac{1}{2}(\hat{O}_1 + \hat{O}_2),\, \hat{O}_a = \hat{O}_1 - \hat{O}_2.$$

  - $r - field$: averaged behavior (classical)

    $a - field$: stochastic effects (quantum)



- Expectation value

$$\langle \hat{O} \rangle = \frac{1}{Z} \int_{\mathscr{C}_\beta} \mathscr{D}\psi\, \hat{O}\, e^{iI[\psi,\phi]}$$

ϕ is the external field ($A_\mu, g_{\mu\nu}/e_I^\mu$).

Relevant Transformations

- EM field A_μ
  - $U(1)$ gauge transformation: $\psi' = \psi e^{-i\theta(x)},\ A'_\mu = A_\mu + \partial_\mu \theta$
- Weak gravity $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, h_{\mu\nu} \ll 1$
  - general coordinate transformation (GCT)

$$\tilde{x}(x) = x - \xi(x),\ \tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(x) + 2\partial_{(\mu}\xi_{\nu)},\ \psi(x) = \tilde{\psi}(\tilde{x})$$

  - local Lorentz transformation (LLT)

$$\acute{e}^\mu_I = e^\mu_I + \Omega_I{}^J e^\mu_J,\ \acute{\psi}(x) = e^{-\frac{i}{2}\Omega_{IJ}S^{IJ}}\psi(x),\ S^{IJ} \equiv \frac{i}{4}\{\gamma^I, \gamma^J\}$$

Gauge Link

- Covariant two-point correlation function (up to $\mathcal{O}(\partial)$, $\mathcal{O}(\phi)$)

$$S(x, y; \beta, \phi) = \langle U(x, x_+; \phi)\psi(x_+)\psi(x_-)U(x_-, x; \phi)\rangle$$

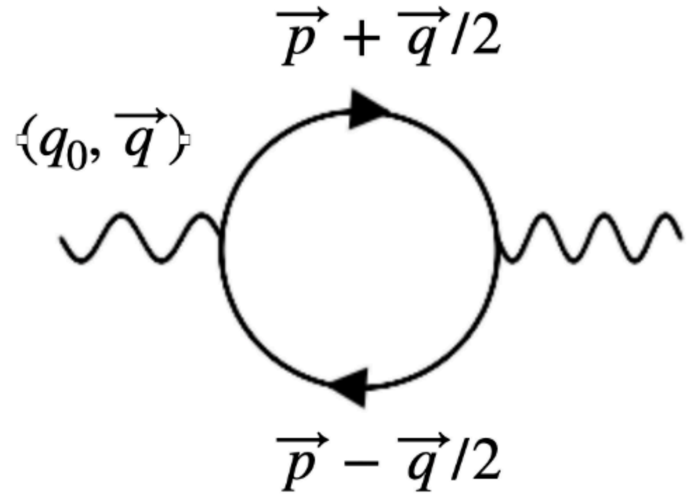
The gauge link is defined as

$$U(x_1, x_2; \phi) = \exp\left[\int_{x_1}^{x_2} dz^\mu \, C_\mu^\phi(z)\right]$$

C_μ^ϕ is the connection. $\phi = A_\mu, C_\mu^\phi = iA_\mu$; $\phi = e_I^\mu, C_\mu^\phi = \Gamma_\mu$ (spin connection).

- The covariant Wigner function: $W(x, p) = \int_y \sqrt{-g}\ e^{-ip\cdot y} S(x, y; \beta, \phi)$

1-Loop and Gauge Link

- $W_\phi = W_0 + G_R\phi + L_\phi\phi, \quad \phi = A_\mu, h_{\mu\nu}$
  - G_R is the retarded correlation. $G_R \sim \langle \psi_r(x_+)\bar{\psi}_r(x_-)j_a \rangle =$ 

    with j the current coupling to the external field ϕ.
  - L_ϕ is the contribution from the gauge link.
- Ward identity

$$U(1)\text{:}\ \ (G_R^\mu + L_A^\mu)\partial_\mu\theta = 0 \ \ \Rightarrow \ \ q_\mu(G_R^\mu + L_A^\mu) = 0.$$

$$\text{GCT:}\ \ (G_R^{\mu\nu} + L_h^{\mu\nu} - p^\nu\partial_p^\mu W_0 - \bar{G}^{\mu\nu})\partial_\mu\xi_\nu = 0 \ \ \Rightarrow \ \ \dots$$