Spicy Gluon 2024.5.18

Relativistic stochastic hydrodynamics from nonlinear projection operator

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Ref: 2403.15825, Jin Hu

to appear (multiplicative noise), Jin Hu

Outline

- Introduction
- Nonlinear projection operator
- Relativistic stochastic hydrodynamics
- Summary and Outlook

Introduction

1. Hydrodynamic frameworks work well in high-energy physics.

2. There are still unsolved doubts about its applicability.

fast equilibration

Michal P. Heller etc, PRL 115, 072501 (2015) …

What really defines a hydro !

small systems / p-p, p-A collisions

James L. Nagle etc, Ann. Rev. Nucl. Part. Sci., 68:211–235 (2018)…

fluctuations / phase transition

L. D. Landau and E. M. Lifshitz, Fluid Mechanics …

Thermodynamic fluctuations are inherently there within the fluids !

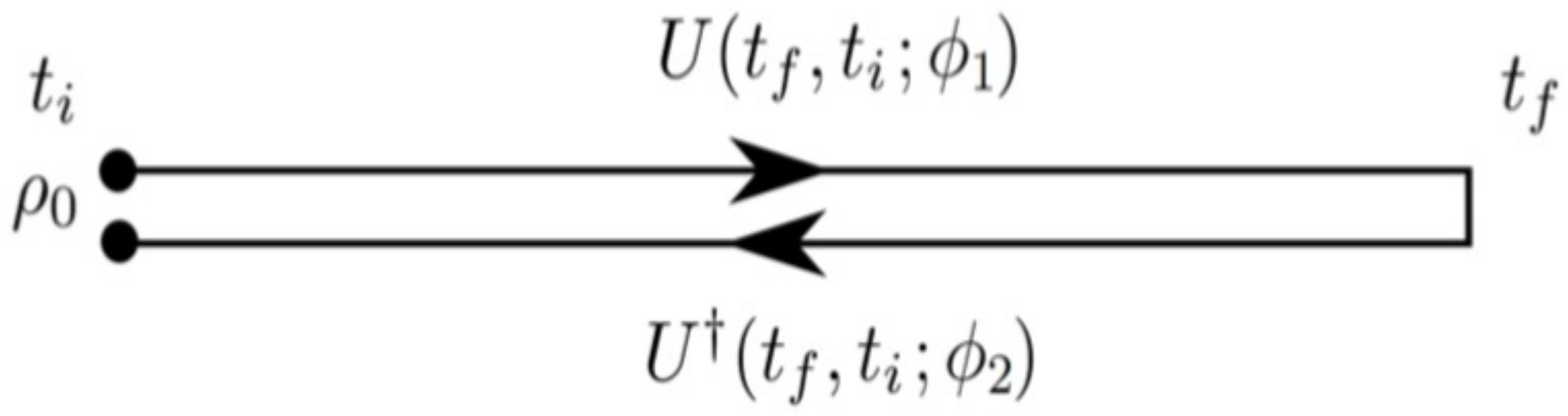
Introduction

Hydrodynamics is a low energy effective theory (often dissipative).

Then the question is how to include dissipation in an effective field theory ?

Schwinger-Keldysh transport

In a Schwinger-Keldysh path, we need to double dofs: one coin, two sides



$$\phi_r = \phi_1 + \phi_2 \qquad \text{physical dof}$$

$$\phi_a = \frac{1}{2}(\phi_1 - \phi_2) \qquad \text{noise}$$

FDT (Fluctuation Dissipation Theorem)

fluctuation ⟷ dissipation

Michael Crossley etc, JHEP 09 (2017) 095…

Introduction

Dissipation + fluctuation

Hydrodynamic fluctuations of the medium are often ignored.

They are crucially enhanced approaching the critical point !

P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49 (1977), 435 …

Introduction to projection operator

What is projection operator suitable for

A **clear separation** of typical scales should exist.

There are the following typical slow processes :

Brown motion : slow Brown motion	large mass
hydrodynamics : slow hydro dofs	conservation laws
order parameter near 2nd-order phase transition	critical slowing down

……

Introduction to projection operator

What is projection operator

A splitting way of variables into **relevant** and **irrelevant**.

e.g. Brown motion : **Brown particles** and **pollens**.

$$\hat{B}(x) = P\hat{B}(x) + (1-P)\hat{B}(x)$$

formally, $P\hat{B} \sim = \sum_n \hat{A}_n \langle \hat{A}_n, \hat{B} \rangle$

projection onto **relevant set A**

$$P^2 = 1, \quad P\hat{A} = \hat{A}, \quad (1-P)\hat{A} = 0$$

e.g. hydrodynamics : **slow hydro dofs** and **fast micro dofs**.

$$\hat{A} \equiv \left\{ \delta\hat{n}, \delta\hat{e}, \delta\hat{T}^{0i} \right\}$$

hydrodynamics with noise

Linear projection operator

In Heisenberg picture

$$\partial_t \hat{B}(t) = i[\hat{H}, \hat{B}(t)] \equiv iL\hat{B}(t), \qquad \hat{B}(t) = e^{iLt}\hat{B}(0),$$

with the decomposition

$$\partial_t e^{iLt} = e^{iLt} iL = e^{iLt} PiL + e^{iLt}(1-P)iL \;, \qquad e^{iLt} = e^{(1-P)iLt} + \int_0^t ds e^{iL(t-s)} PiLe^{(1-P)iLs}$$

obtained from Laplace transform

we arrive at the **Generalized Linearized Langevin Equation (GLLE)**

$$\partial_t \hat{B}(t) = e^{iLt} PiL\hat{B} + \int_0^t ds e^{iL(t-s)} PiL\hat{N}(s) + \hat{N}(t)$$

streaming | dissipation | noise

with $\hat{N}(t) \equiv e^{(1-P)iLt}(1-P)iL\hat{B}, \quad P\hat{N}(t) = 0$

Below, we talk about the set A.

Note that $(1-P)iL\hat{A} \neq 0!$

Akira Onuki - Phase transition dynamics

nonlinear projection operator

With linear projection operator

$$P\hat{B} \sim \sum_n \hat{A}_n \langle \hat{A}_n, \hat{B} \rangle$$

how about $\langle \hat{A}_n^2, \hat{B} \rangle, \langle \hat{A}_n^3, \hat{B} \rangle \cdots ?$ $\langle \hat{A}^m, \hat{N} \rangle ! = 0,\ m > 1$

nonlinear projection operator is needed.

Robert Zwanzig, Phys.Rev.124.983 (1961)

$$\hat{f}(a) = \delta(\hat{A} - a) = \frac{1}{(2\pi)^N} \int dx \exp(ix \cdot (\hat{A} - a))$$

Weyl correspondence rule

$$f(a) = \delta(A - a) = \Pi_{n=1}^N \delta(A_n - a_n)$$

H. Weyl, Gruppentheorie und Quantenmechanik

V. G. Morozov, TMF 48, 373(1981)

nonlinear projection operator

$$\hat{f}(a) = \delta(\hat{A} - a) = \frac{1}{(2\pi)^N} \int dx \exp(ix \cdot (\hat{A} - a))$$

linear : $P\hat{B} \sim \hat{A}_i \langle \hat{A}_i, \hat{B} \rangle$

nonlinear: $P\hat{B} \sim \hat{f}(a) \langle \hat{f}(a), \hat{B} \rangle$

why nonlinear? →

$$\hat{A}^n = \int da a^n \delta(\hat{A} - a)$$

$$\hat{f}(a,t) \equiv e^{iLt} \hat{f}(a)$$

ensemble → average

$$f(a,t) \equiv Tr(\hat{f}(a,t)\hat{\rho})$$

averaged over a limited phase space

the distribution of a

$\hat{f}(a,t)$ ↓ Fokker-Planck eq in operator form

Fokker-Planck eq in operator form → ($\int da\, a \cdots$) → Langevin eq (for $\hat{A}_i$)

$f(a,t)$ ↓ Fokker-Planck eq

nonlinear projection operator

V. G. Morozov, TMF 48, 373(1981)

Specify the def of projection $P\hat{B} \sim \hat{f}(a)\langle \hat{f}(a), \hat{B} \rangle$

$$P\hat{B} \equiv \int dada'\hat{f}(a)W_{-1}(a,a')Tr(\hat{B}\hat{f}(a'))$$

with $W(a,a') = Tr(\hat{f}(a)\hat{f}(a'))$ $\quad \int da''W(a,a'')W_{-1}(a'',a') = \delta(a-a')$

new: **non-locality** from the **non-commutativity** of quantum operators

$$W(a,a') = W(a)\big(\delta(a-a') - \underline{R(a,a')}\big), \qquad W_{-1}(a,a') = W^{-1}(a')\big(\delta(a-a') + \underline{r(a,a')}\big)$$

without non-locality, $P\hat{B} = \int da\hat{f}(a)W^{-1}(a)Tr(\hat{B}\hat{f}(a))$ classical version

Below, I will talk about slow variables set A (B=A). Note $(1-P)iL\hat{A} \neq 0!$

Plan A (enough time)

Fokker-Planck equation in operator form

$$\partial_t \hat{A}(t) = e^{iLt} P iL \hat{A} + \int_0^t ds e^{iL(t-s)} P iL \hat{N}(s) + \hat{N}(t)$$

replacement of $\hat{A} \to \hat{f}(a)$

$$\frac{\partial \hat{f}(a,t)}{\partial t} = -\frac{\partial}{\partial a_i} \int da' v_i(a,a') \hat{f}(a',t) + \int_0^t du \frac{\partial}{\partial a_i} \int da' K_{ij}(a,a',t-u) \frac{\partial}{\partial a'_j} \int da'' \hat{f}(a'',u) W_{-1}(a'',a') - \frac{\partial}{\partial a_i} \hat{X}_i(a,t),$$

$$iL\hat{f}(a) \equiv -\frac{\partial \hat{J}_i(a)}{\partial a_i}$$

streaming velocity: $v_i(a,a') \equiv \int da'' W_{-1}(a',a'') Tr(\hat{J}_i(a) \hat{f}(a''))$

noise: $\hat{X}_i(a,t) = e^{i(1-P)Lt}(1-P)\hat{J}_i(a),$

diffusion kernel: $K_{ij}(a,a',t) \equiv Tr(\hat{X}_i(a,t)\hat{X}_j(a'))$

Properties:

1. The correlation of slow and fast part is nonlinearly vanishing ! $\quad Tr(G(\hat{A})\hat{X}(a,t)) = 0$

2. Generalized fluctuation-dissipation theorem ! $\quad K_{ij}(a,a',t) = Tr(\hat{X}_i(a,t)\hat{X}_j(a'))$

Generalized langevin equation

$$\frac{\partial \hat{f}(a,t)}{\partial t} = -\frac{\partial}{\partial a_i}\int da' v_i(a,a')\hat{f}(a',t) + \int_0^t du \frac{\partial}{\partial a_i}\int da' K_{ij}(a,a',t-u)\frac{\partial}{\partial a'_j}\int da'' \hat{f}(a'',u)W_{-1}(a'',a') - \frac{\partial}{\partial a_i}\hat{X}_i(a,t),$$

$\int daa\cdots$ the Langevin equation for the slow variables.

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = = \int da \int da' v_\alpha(a,a')\hat{f}(a',t) + \int_0^t du \int da'(\frac{\partial}{\partial a'_j}K_{\alpha j}(a',t-u))\int da'' \hat{f}(a'',u)W_{-1}(a'',a') + \hat{R}_\alpha(t)$$

$$\hat{R}_k(t) \equiv e^{(1-P)iLt}(1-P)iL\hat{A}_k, \quad K_{ij}(a',t) = \int da K_{ij}(a,a',t) = \int da Tr(\hat{X}_i(a,t)\hat{X}_j(a')) = Tr(\hat{R}_i(t)\hat{X}_j(a')).$$

1. The correlation of slow and fast part is nonlinearly vanishing ! $Tr(G(\hat{A})\hat{R}(a,t)) = 0$
2. Generalized fluctuation-dissipation theorem is implicit! $K_{ij}(a',t) = Tr(\hat{R}_i(t)\hat{X}_j(a'))$

Mori, Hazime and Fujisaka, Progress of Theoretical Physics, 49, 764 (1973)

Generalized Langevin equation

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = = \int da \int da' v_\alpha(a, a') \hat{f}(a', t) + \int_0^t du \int da' (\frac{\partial}{\partial a'_j} K_{\alpha j}(a', t - u)) \int da'' \hat{f}(a'', u) W_{-1}(a'', a') + \hat{R}_\alpha(t)$$

local approximation or classical approximation suitable for high temperature

$$v(a, a') = v(a)\delta(a - a'), \quad W_{-1}(a'', a') = W^{-1}(a')\delta(a' - a'')$$

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A}, t) + \int_0^t du \int da' (\frac{\partial}{\partial a'_j} K_{\alpha j}(a', t - u)) W^{-1}(a') \hat{f}(a', u) + \hat{R}_\alpha(t),$$

$\tilde{K} = K/W \qquad Tr(\hat{f}(a, t)) = Tr(\hat{f}(a)) = W(a)$

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A}, t) + \int_0^t du \int da (\frac{\partial}{\partial a_j} \tilde{K}_{\alpha j}(a, t - u)) \hat{f}(a, u) + \int_0^t du \int da \tilde{K}_{\alpha j}(a, t - u) F_j(a) \hat{f}(a, u) + \hat{R}_\alpha(t)$$

$$F_k(a) = \frac{\partial}{\partial a_k} \ln W(a)$$

thermodynamic force

Generalized Langevin equation

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A}, t) + \int_0^t du \int da(\frac{\partial}{\partial a_j}\tilde{K}_{\alpha j}(a, t-u))\hat{f}(a, u) + \int_0^t du \int da\tilde{K}_{\alpha j}(a, t-u)F_j(a)\hat{f}(a, u) + \hat{R}_\alpha(t)$$

with $\tilde{K}_{\alpha j}(a, t-u) \to \int daW(a)\tilde{K}_{\alpha j}(a, t-u) = \int daK_{\alpha j}(a, t-u) = Tr(\hat{R}_\alpha(t-u)\hat{R}_j)$

Gaussian approximation

Mori, Hazime and Fujisaka, Progress of Theoretical Physics, 49, 764 (1973)

→ $$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A}, t) + \int_0^t duTr(\hat{R}_\alpha(t-u)\hat{R}_j)F_j(\hat{A}(u)) + \hat{R}_\alpha(t)$$

Generalized fluctuation-dissipation theorem is obvious !

The diffusion term takes the form of generalized Onsager relation

in a convolution form with memory effects !

Robert Zwanzig, Phys.Rev.124.983 (1961)

Familiar Langevin equation

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A}, t) + \int_0^t du Tr(\hat{R}_\alpha(u)\hat{R}_j)F_j(\hat{A}(t-u)) + \hat{R}_\alpha(t)$$

Markov approximation $\rightarrow$ $t \gg u$

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A}(t)) + \gamma_{\alpha j}F_j(\hat{A}(t)) + \hat{R}_\alpha(t)$$

with bare diffusion coefficients $\gamma_{ij} \equiv \int_0^\infty du Tr(\hat{R}_i(u)\hat{R}_j)$

We have recovered the familiar Langevin equation with white Gaussian noise

$$Tr(\hat{R}_i(t)\hat{R}_j(t')) = 2\gamma_{ij}\delta(t-t')$$

Plan B (all for small system)

Flow Chart

$$\partial_t \hat{A}(t) = e^{iLt} PiL\hat{A} + \int_0^t ds e^{iL(t-s)} PiL\hat{N}(s) + \hat{N}(t)$$

GLLE

replacement of $\hat{A} \to \hat{f}(a)$ ↓

Fokker-Planck equation in operator form

$$\frac{\partial \hat{f}(a,t)}{\partial t} = -\frac{\partial}{\partial a_i}\int da' v_i(a,a')\hat{f}(a',t) + \int_0^t du \frac{\partial}{\partial a_i}\int da' K_{ij}(a,a',t-u)\frac{\partial}{\partial a'_j}\int da'' \hat{f}(a'',u)W_{-1}(a'',a') - \frac{\partial}{\partial a_i}\hat{X}_i(a,t),$$

$$iL\hat{f}(a) \equiv -\frac{\partial \hat{J}_i(a)}{\partial a_i}$$

streaming velocity: $v_i(a,a') \equiv \int da'' W_{-1}(a',a'') Tr(\hat{J}_i(a)\hat{f}(a''))$

noise: $\hat{X}_i(a,t) = e^{i(1-P)Lt}(1-P)\hat{J}_i(a),$

diffusion kernel: $K_{ij}(a,a',t) \equiv Tr(\hat{X}_i(a,t)\hat{X}_j(a'))$

$\int daa\cdots$ ⟶

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = = \int da \int da' v_\alpha(a,a')\hat{f}(a',t) + \int_0^t du \int da' (\frac{\partial}{\partial a'_j} K_{\alpha j}(a',t-u)) \int da'' \hat{f}(a'',u) W_{-1}(a'',a') + \hat{R}_\alpha(t)$$

Generalized quantum Langevin equation

Rethinking the derivation

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = = \int da \int da' v_\alpha(a, a') \hat{f}(a', t) + \int_0^t du \int da' \left(\frac{\partial}{\partial a'_j} K_{\alpha j}(a', t-u) \right) \int da'' \hat{f}(a'', u) W_{-1}(a'', a') + \hat{R}_\alpha(t)$$

local/classical approximation

No quantum non-locality

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A}, t) + \int_0^t du \int da \left(\frac{\partial}{\partial a_j} \tilde{K}_{\alpha j}(a, t-u) \right) \hat{f}(a, u) + \int_0^t du \int da \tilde{K}_{\alpha j}(a, t-u) F_j(a) \hat{f}(a, u) + \hat{R}_\alpha(t)$$

Gaussian approximation

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A}, t) + \int_0^t du Tr(\hat{R}_\alpha(t-u)\hat{R}_j) F_j(\hat{A}(u)) + \hat{R}_\alpha(t)$$

No multiplicative noise

Markov approximation

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A}(t)) + \gamma_{\alpha j} F_j(\hat{A}(t)) + \hat{R}_\alpha(t)$$

No memory effects

Application to stochastic hydrodynamics

stochastic hydrodynamics $\hat{A} \equiv \left\{ \delta\hat{n}, \delta\hat{e}, \delta\pi_i = \hat{T}^{0i} \right\}$

$$\frac{\partial \delta\hat{n}}{\partial t} = -\nabla\cdot(\hat{n}\delta\hat{v}) + \frac{\kappa n_0^2 T_0}{(e_0+p_0)^2}\nabla^2 \frac{\delta \ln W(\hat{A})}{\delta\hat{n}} + \hat{R}_n(t,x)$$

$$\frac{\partial \delta e}{\partial t} = -\nabla\cdot\pi$$

higher terms in $\delta\hat{n}$ can appear!

$$\frac{\partial\hat{\pi}_i}{\partial t} = -\nabla_i\delta\hat{p} + (\zeta + \frac{1}{3}\eta)\nabla_i(\nabla\cdot\frac{\delta \ln W(\hat{A})}{\delta\hat{\pi}}) + \eta\nabla^2\frac{\delta \ln W(\hat{A})}{\delta\hat{\pi}_i} + \hat{R}_\pi^i(t,x)$$

with $\delta(\frac{\mu}{T}) \sim \frac{\delta \ln W(\hat{A})}{\delta\hat{\pi}}$ $\quad \delta\hat{v} \sim \frac{\delta \ln W(\hat{A})}{\delta\hat{n}}$

$$Tr(\hat{R}_n(t,x)\hat{R}_n(t',x')) = 2\frac{\kappa n_0^2 T_0}{(e_0+p_0)^2}\nabla^2\delta(t-t')\delta(x-x'),$$

$$Tr(\hat{R}_\pi^i(t,x)\hat{R}_\pi^j(t',x')) = 2\Big((\zeta+\frac{1}{3}\eta)\nabla_i\nabla_j + \eta\delta_{ij}\nabla^2\Big)\delta(t-t')\delta(x-x').$$

Minami Yuki, PRD.83.094019 (2011)

Jin Hu, 24903.15825

Application to stochastic hydrodynamics

$$\partial_t A(t,x) = v_\alpha(A) + \gamma_{\alpha j} F_j(A) + R_\alpha(t,x)$$

Fourier transferm

$$-i\omega A(\omega,k) = v_\alpha(A) + \gamma_{\alpha j} F_j(A) + R_\alpha(\omega,k)$$

Generally,

$$A(\omega,k) = N(A) + C(\omega,k) R_\alpha(\omega,k)$$

N(A) = Nonlinearity in A

Then we can follow a perturbation iteration procedure,

$$A = A^{(0)} + A^{(1)} + \cdots \qquad A^{(0)} = C(\omega,k) R_\alpha(\omega,k)$$

the lowest order of A is Gaussian !

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Application to stochastic hydrodynamics

Generally,

$$A(\omega,k) = N(A) + C(\omega,k)R_\alpha(\omega,k)$$

N(A) = Nonlinearity in A

$$A = A^{(0)} + A^{(1)} + \cdots \qquad A^{(0)} = C(\omega,k)R_\alpha(\omega,k)$$

the lowest order of A is Gaussian !

Considering a coarse-grained technique(coarse grain **S**hort wavelength dofs),

$$A = A_S + A_L \qquad N(A) = N_S(A) + N_L(A)$$

Cannot distinguish **S**hort wavelength dofs from noise / as **new source of noise**

Akira Onuki - Phase transition dynamics

Application to stochastic hydrodynamics

Considering a coarse-grained technique(coarse grain **S**hort wavelength dofs),

$$A = A_S + A_L \qquad\qquad N(A) = N_S(A) + N_L(A)$$

Cannot distinguish **S**hort wavelength dofs from noise / as **new source of noise**

e.g: the generalized Kubo formula for bulk viscosity

$$\gamma_R(\omega) = \gamma + \int_0^\infty dt \int dr e^{-i\omega t} \langle N_S(r,t) N_S(0,0) \rangle$$

↓ $A_S^{(0)}$ is Gaussian !

Products of two point hydrodynamic correlations !
+ some of them are divergent near critical points

↓

The renormalized transport coefficients are divergent !

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Summary and Outlook

- Stochastic hydrodynamics with Gaussian noise
- multiplicative noise hydro (done!)
- dynamic renormalization group applies to multiplicative noise.

  the scaling of transport coefficients

  critical transport in phase transition

  Gas-liquid PT **in progress**!
- totally microscopic construction.
- dissipation-fluctuation gauge ambiguity.

T.Dore etc, *Annals Phys.* 442 (2022) 168902

Back up

Microscopic aspect

$$L \gg \lambda \gg R \qquad\qquad K_n \equiv \lambda / L$$

Standard paradigm: Linearized kinetic equation (Boltzmann or Landau eq)

moments expansion: $\sim \int dP (u \cdot p)^n (\Delta_{\alpha\beta} p^\alpha p^\beta)^q f$

1. First 5 moments — five zero modes or collision invariants

2. First 14 moments — IS : promote the statues of dissipation quantities

Higher moments are beyond the range of hydro description !

Introduction

1. Why hydro can work well ? (why QGP equilibrates so fast ?)

2. How and When hydro behavior emerges from a dynamic QCD system ?

Hydro attractor: an intrinsic structure of hydro

What really defines a hydro

Michal P. Heller etc, PRL 115, 072501 (2015) …

Molecular dynamic simulation: bottom-up picture

What dynamics triggers

R. Baier etc, Phys. Lett. B 502 (2001) …

Generalized Langevin equation

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = = \int da \int da' v_\alpha(a,a')\hat{f}(a',t) + \int_0^t du \int da' (\frac{\partial}{\partial a'_j} K_{\alpha j}(a', t-u)) \int da'' \hat{f}(a'',u) W_{-1}(a'',a') + \hat{R}_\alpha(t)$$

local approximation or classical approximation suitable for high temperature

$$v(a,a') = v(a)\delta(a-a'), \quad W_{-1}(a'',a') = W^{-1}(a')\delta(a'-a'')$$

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A},t) + \int_0^t du \int da' (\frac{\partial}{\partial a'_j} K_{\alpha j}(a', t-u)) W^{-1}(a')\hat{f}(a',u) + \hat{R}_\alpha(t),$$

$\tilde{K} = K/W$

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A},t) + \int_0^t du \int da (\frac{\partial}{\partial a_j} \tilde{K}_{\alpha j}(a, t-u)) \hat{f}(a,u) + \int_0^t du \int da \tilde{K}_{\alpha j}(a, t-u) F_j(a) \hat{f}(a,u) + \hat{R}_\alpha(t)$$

$$F_k(a) = \frac{\partial}{\partial a_k} \ln W(a)$$

thermodynamic force

Introduction

Dissipation + fluctuation

Hydrodynamic fluctuations of the thermal bath are often ignored.

They are crucially enhanced approaching the critical point !

P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49 (1977), 435 …

They may lead to the renormalization of transport coefficients

and give rise to the anomalous long time behavior called long-time tail !

K. Kawasaki, Ann. Phys. 61, 1 (1970), Y. Pomeau and P. Resibois, Phys. Rept. 19 (1975) 63 …

Linear VS nonlinear

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = \Omega \hat{A} + \int_0^\infty du \Phi(u) \hat{A}(t) + \boxed{\hat{F}_\alpha(t)}$$

Equivalent

nonlinearity

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = \boxed{v_\alpha(\hat{A}, t) + \int_0^\infty du Tr(\hat{R}_\alpha(u) \hat{R}_j) F_j(\hat{A}(t)) + \hat{R}_\alpha(t)}$$

$$\sim v_\alpha(\hat{A}, t) + \int_0^\infty du Tr(\hat{R}_\alpha(u) \hat{R}_j) \hat{A}(t) + \hat{R}_\alpha(t)$$

$W(\hat{A})$ is Gaussian

Renormalization of bare kinetic coefficients!

The nonlinearity in noise lies within the range of fast dynamics and thus uncontrollable !

Discussion

- The def of thermodynamic average

$$P_{\text{eq}}(a) = \left\langle \prod_j \delta(A_j - a_j) \right\rangle = \text{const.} \exp[-\beta\mathcal{H}(a)], \quad (5.2.3)$$

which is the probability of finding A at a in equilibrium (equilibrium distribution). Hereafter $\langle\cdots\rangle$ denotes the equilibrium average, and the conditional average in which A is fixed at a may be defined by

$$\langle\cdots; a\rangle = \left\langle \cdots \prod_j \delta(A_j - a_j) \right\rangle \Big/ P_{\text{eq}}(a). \quad (5.2.4)$$

$$P_{\text{gra}}(\Gamma) = \frac{1}{\Xi} \exp[-\beta\mathcal{H} + \beta\mu\mathcal{N}]. \quad (1.2$$

The equilibrium average is written as $\langle\cdots\rangle = \int d\Gamma(\cdots)P_{\text{gra}}(\Gamma)$, where

$$\int d\Gamma = \sum_{\mathcal{N}} \frac{1}{\mathcal{N}!(2\pi\hbar)^{d\mathcal{N}}} \int d\boldsymbol{p}_1 \cdots \int d\boldsymbol{p}_{\mathcal{N}} \int d\boldsymbol{r}_1 \cdots \int d\boldsymbol{r}_{\mathcal{N}} \quad (1.2$$

Akira Onuki - Phase transition dynamics.pdf

Application to stochastic hydrodynamics

- Kubo formula $\lambda = T^{-2} \int d\mathbf{r} \int_0^\infty dt \langle q(\mathbf{r},t) q(0,0) \rangle,$

- Divergence of transport coefficients

$$q_{\rm macro} \sim \delta s \delta v, \quad (3)$$

where δs and δv respectively denote the fluctuations of the entropy density and the fluid velocity. The macroscopic current Eq. (3) is of the second order in fluctuations and hence negligible far from the CP. However, it becomes the dominant part near the CP, since the fluctuations are enhanced there. We see that Eq. (1) now has the following form

$$\lambda = \lambda_{\rm micro} + \int d\mathbf{r} \int_0^\infty dt \langle \delta s(\mathbf{r},t) \delta v(\mathbf{r},t) \delta s(0,0) \delta v(0,0) \rangle,$$

Minami Yuki, PRD.83.094019 (2011)

nonlinear projection operator

Below, I will talk about slow variables set A

The complication: $\hat{f}(a) = \delta(\hat{A} - a) = \frac{1}{(2\pi)^N} \int dx \exp(ix \cdot (\hat{A} - a))$

$$iL\hat{f}(a) \equiv -\frac{\partial \hat{J}_i(a)}{\partial a_i} \qquad \hat{J}(a) \equiv \frac{1}{(2\pi)^N} \int dx e^{ix(\hat{A}-a)} \int_0^1 d\tau e^{-i\tau x \hat{A}} iL\hat{A} e^{i\tau x \hat{A}}$$

with Kubo identity

simplification : classical, one component

$$[e^{\hat{B}}, \hat{H}] = e^{\hat{B}} \int_0^1 d\tau e^{-\tau \hat{B}} [\hat{B}, \hat{H}] e^{\tau \hat{B}}$$

Ryogo Kubo, Lectures in theoretical physics, Volumn I (1959)

Generalized Fokker-Planck equation

$$\frac{\partial \hat{f}(a,t)}{\partial t} = -\frac{\partial}{\partial a_i}\int da' v_i(a,a')\hat{f}(a',t) + \int_0^t du \frac{\partial}{\partial a_i}\int da' K_{ij}(a,a',t-u)\frac{\partial}{\partial a'_j}\int da'' \hat{f}(a'',u)W_{-1}(a'',a') - \frac{\partial}{\partial a_i}\hat{X}_i(a,t),$$

$Tr(\cdots\hat{\rho})$

$$\rightarrow \quad \frac{\partial f(a,t)}{\partial t} = -\frac{\partial}{\partial a_i}\int da' v_i(a,a')f(a',t) + \int_0^t du \frac{\partial}{\partial a_i}\int da' K_{ij}(a,a',t-u)\frac{\partial}{\partial a'_j}\int da'' f(a'',u)W_{-1}(a'',a')$$

The motion equation for the distribution of slow variables. No noise term.

Schrödinger picture — **V. G. Morozov, TMF 48, 373(1981)**

classical version — **Robert Zwanzig, Phys.Rev.124.983 (1961)**

Introduction to projection operator

What is projection operator

A splitting way of variables into **relevant** and **irrelevant**.

e.g. Brown motion : **Brown particles** and **pollens**.

$$\hat{B}(x) = P\hat{B}(x) + (1 - P)\hat{B}(x)$$

formally, $P\hat{B} \sim \sum_n |a_n\rangle\langle a_n|B\rangle = \sum_n \hat{A}_n\langle\hat{A}_n, \hat{B}\rangle$ projection onto **relevant set A**

$$P^2 = 1, \quad P\hat{A} = \hat{A}, \quad (1 - P)\hat{A} = 0$$

e.g. hydrodynamics : **slow hydro dofs** and **fast micro dofs**.

$$\hat{A} \equiv \left\{\delta\hat{n}, \delta\hat{e}, \delta\hat{T}^{0i}\right\}$$ hydrodynamics with noise

Fokker-Planck equation in operator form

$$\partial_t \hat{A}(t) = e^{iLt} PiL\hat{A} + \int_0^t ds e^{iL(t-s)} PiL\hat{N}(s) + \hat{N}(t)$$

replacement of $\hat{A} \rightarrow \hat{f}(a)$

$$\frac{\partial \hat{f}(a,t)}{\partial t} = -\frac{\partial}{\partial a_i} \int da' v_i(a,a') \hat{f}(a',t) + \int_0^t du \frac{\partial}{\partial a_i} \int da' K_{ij}(a,a',t-u) \frac{\partial}{\partial a'_j} \int da'' \hat{f}(a'',u) W_{-1}(a'',a') - \frac{\partial}{\partial a_i} \hat{X}_i(a,t),$$

$$iL\hat{f}(a) \equiv -\frac{\partial \hat{J}_i(a)}{\partial a_i}$$

streaming velocity: $v_i(a,a') \equiv \int da'' W_{-1}(a',a'') Tr(\hat{J}_i(a)\hat{f}(a''))$

noise: $\hat{X}_i(a,t) = e^{i(1-P)Lt}(1-P)\hat{J}_i(a),$

diffusion kernel: $K_{ij}(a,a',t) \equiv Tr(\hat{X}_i(a,t)\hat{X}_j(a'))$

Properties:

1. The correlation of slow and fast part is nonlinearly vanishing ! $\quad Tr(G(\hat{A})\hat{X}(a,t)) = 0$

2. Generalized fluctuation-dissipation theorem ! $\quad K_{ij}(a,a',t) = Tr(\hat{X}_i(a,t)\hat{X}_j(a'))$

3. $\frac{\partial W(a)}{\partial t} = 0, \quad Tr(\hat{f}(a,t)) = Tr(\hat{f}(a)) = W(a)$

$\delta(\hat{A}-a)$

like micro-canonical partition function

Application to stochastic hydrodynamics

stochastic hydrodynamics $\hat{A} \equiv \left\{ \delta\hat{n}, \delta\hat{e}, \delta\pi_i = \hat{T}^{0i} \right\}$

stochastic hydro → Langevin equation

drift terms → continuity equation

$\partial_t \delta e = -\nabla \cdot \pi$ $\quad\quad$ $\partial_t T^{00} = -\partial_i T^{0i}$

dissipation terms —hydro limit→ hydro dissipation terms

$$\frac{\kappa n_0^2 T_0}{(e_0+p_0)^2} \nabla^2 \frac{\delta \ln W(\hat{A})}{\delta \hat{n}} \quad\quad \frac{\kappa n_0^2 T_0}{(e_0+p_0)^2} \nabla^2 \frac{\mu}{T}$$

noise terms —FDT→ dissipation terms

$$Tr(\hat{R}_n(t,x)\hat{R}_n(t',x')) = 2\frac{\kappa n_0^2 T_0}{(e_0+p_0)^2} \nabla^2 \delta(t-t')\delta(x-x'),$$

Minami Yuki, PRD.83.094019 (2011)

Jin Hu, 24903.15825

Application to stochastic hydrodynamics

短波分量,

the large-wavenumber components of the slow variables $A_j(t)$ in the infinitesimal wavenumber shell,

本来需要精细的k，结果用一个shell替代，粗粒化

$$\Lambda - \delta\Lambda < k < \Lambda, \tag{10}$$

Akira Onuki - Phase transition dynamics.pdf, p300 !!!!!

for Eq. (7). Here, Λ starts from the initial value Λ_0 and is lowered up to $\Lambda \ll \xi^{-1}$. In this way, we infinitesimally make coarse graining of the Langevin equation. Because the coarse-graining procedure is infinitesimal, we need not the rescaling . Inspecting the form of the coarse-

Minami Yuki, PRD.83.094019 (2011)