

# Microscopic Encoding of Macroscopic Universality: Scaling properties of Dirac Eigenspectra near QCD Chiral Phase Transition

*How do universal behaviors at macroscale arise from quarks and gluons?*

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based on Phys. Rev. Lett. 131 (2023) 16, 161903,

in collaboration with

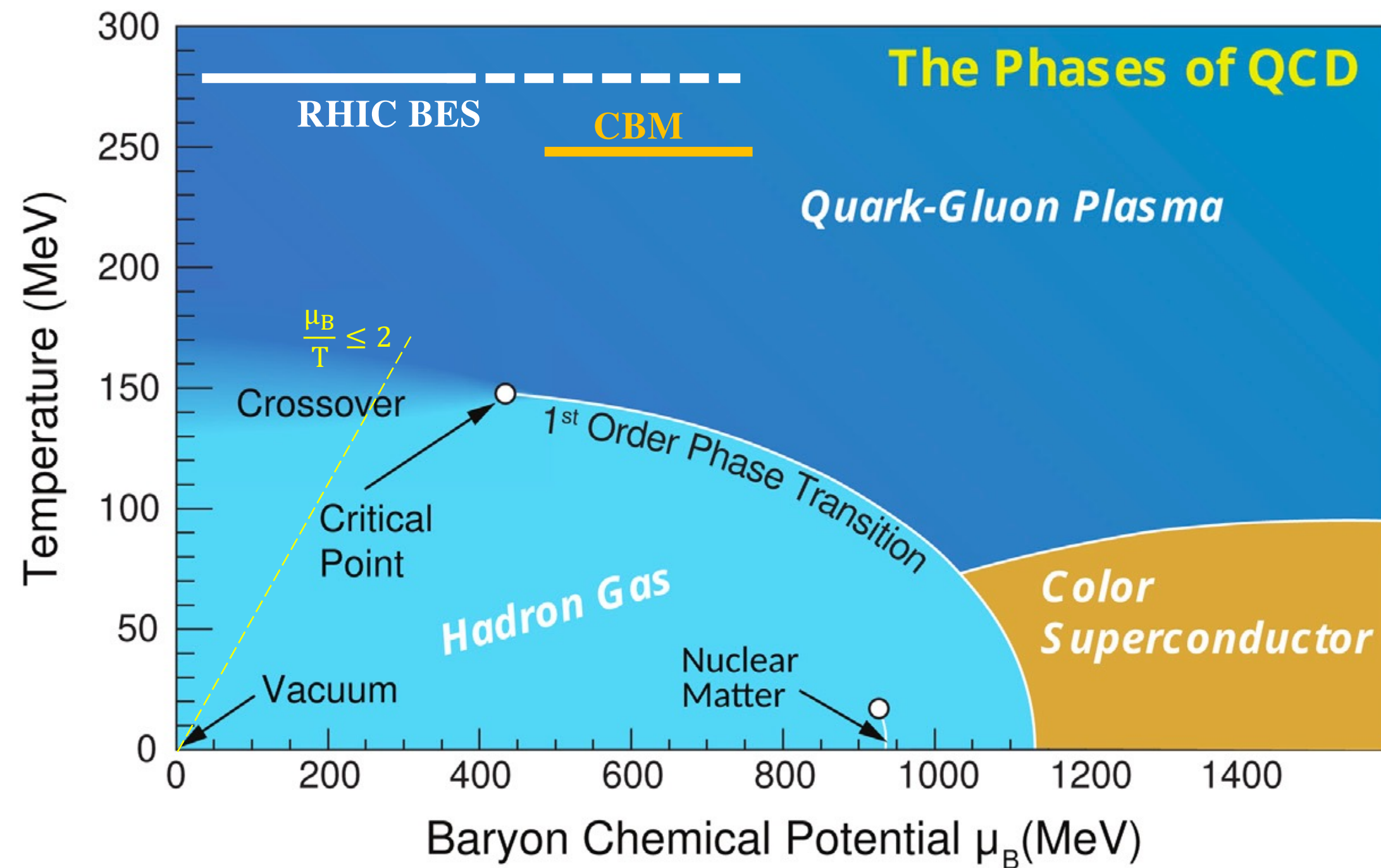
Heng-Tong Ding (丁亨通), Swagato Mukherjee, Peter Petreczky



Spicy Gluons 2024: Workshop for Young Scientists on the  
Quark-Gluon Matter in Extreme Conditions, May 15 - 19 @ Hefei

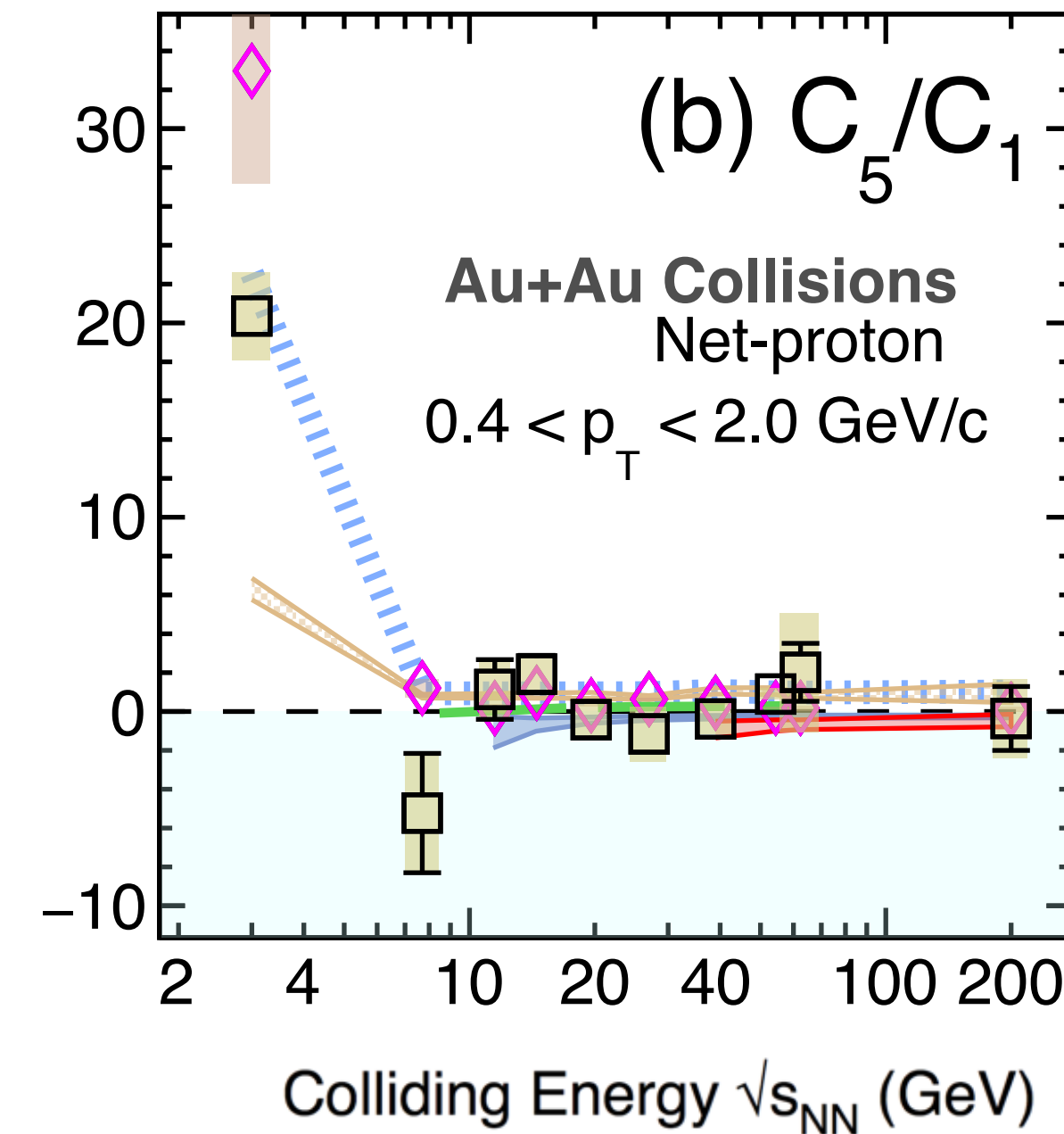
# Search for Criticality in QCD

Exploration of QCD phase diagram



D. Almaalol et al., arXiv:2209.05009

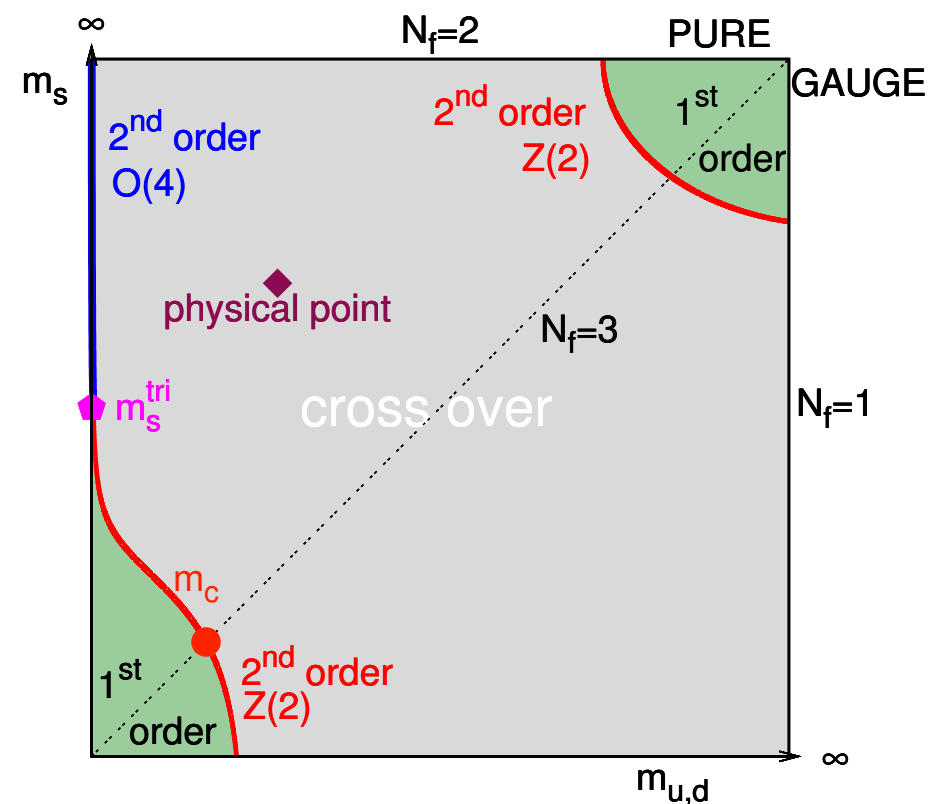
Searching for signatures of criticality in **Macroscopic** quantities



STAR, Phys. Rev. Lett. 130, 082301 (2023)

How criticality at **Macroscale** arises from **Microscopic** *d.o.f* of QCD ?

# Universal Scaling in QCD Chiral Transition



H.-T. Ding et al., arXiv:1504.05274

◆ Restoration of chiral symmetry: — Crossover at physical point;

— True chiral phase transition in limiting cases

A. Bazavov et al., [HotQCD], PLB 795 (2019) 15

H.-T. Ding et al., [HotQCD], PRL 123 (2019) 062002

◆ Role of  $U(1)_A$  symmetry in chiral phase transition:

Broken around  $T_c \Rightarrow$  2<sup>nd</sup> order O(4) phase transition

Pisarski & Wilczek, PRD 29 (1984) 338

Pelissetto & Vicari, PRD 88 (2013) 105018

## Universal O(2) scaling behaviors in *staggered* discretization scheme

Order parameter :

$$M(t, h) = h^{1/\delta} f_1(z) + f_{\text{reg}}(T, H)$$

Order parameter susceptibility :

$$\chi_M(t, h) = \partial M / \partial H = h_0^{-1} h^{1/\delta-1} f_2(z) + f'_{\text{reg}}$$

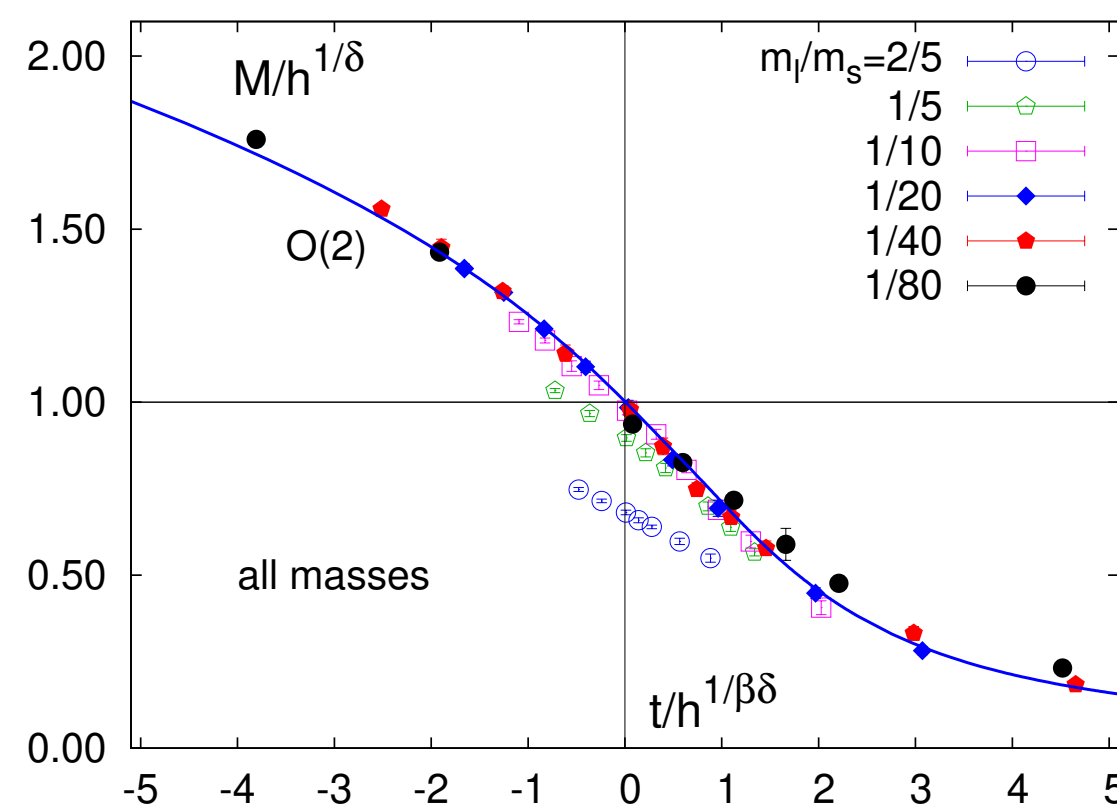
Higher order susceptibility :

$$\text{e.g., } \partial^2 \chi_M / \partial H^2 = h_0^{-2} h^{1/\delta-2} f_3(z) + f''_{\text{reg}}$$

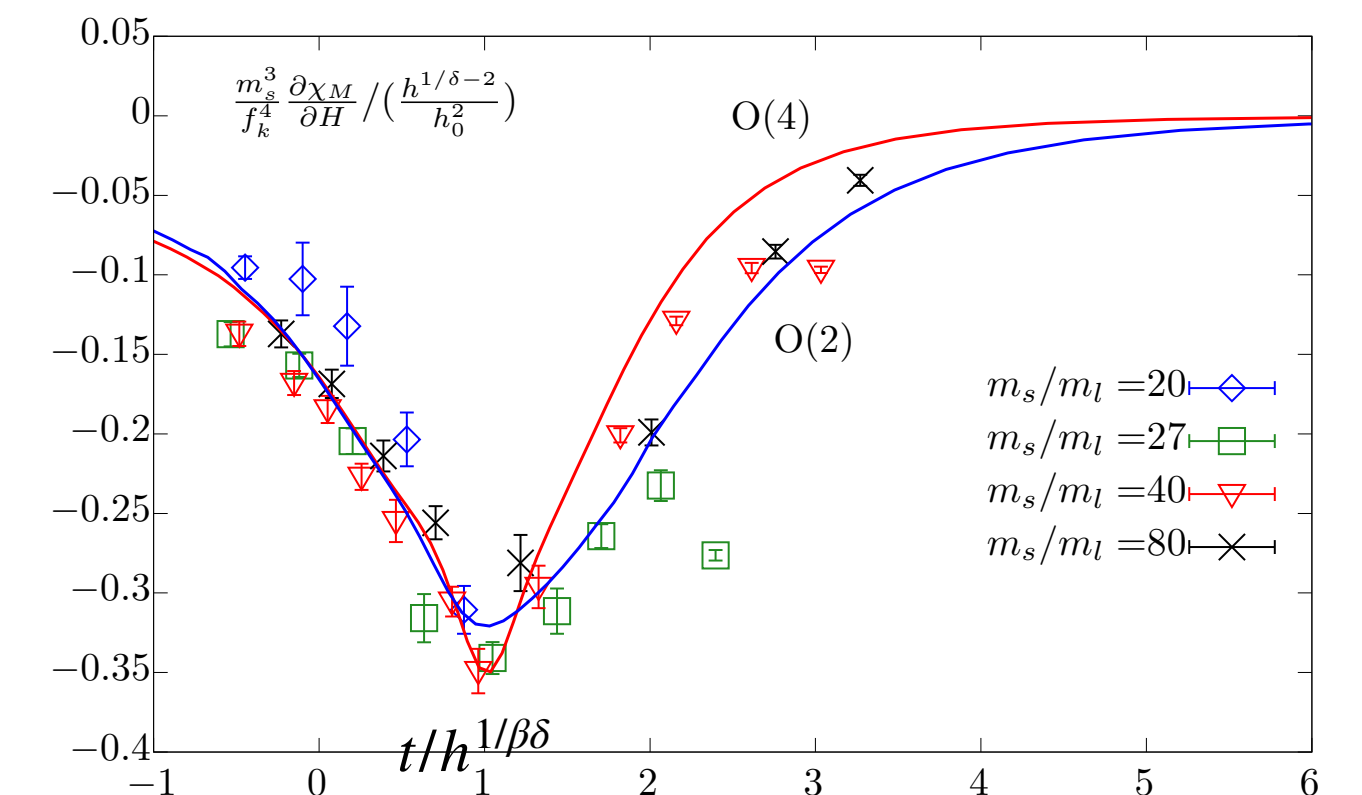
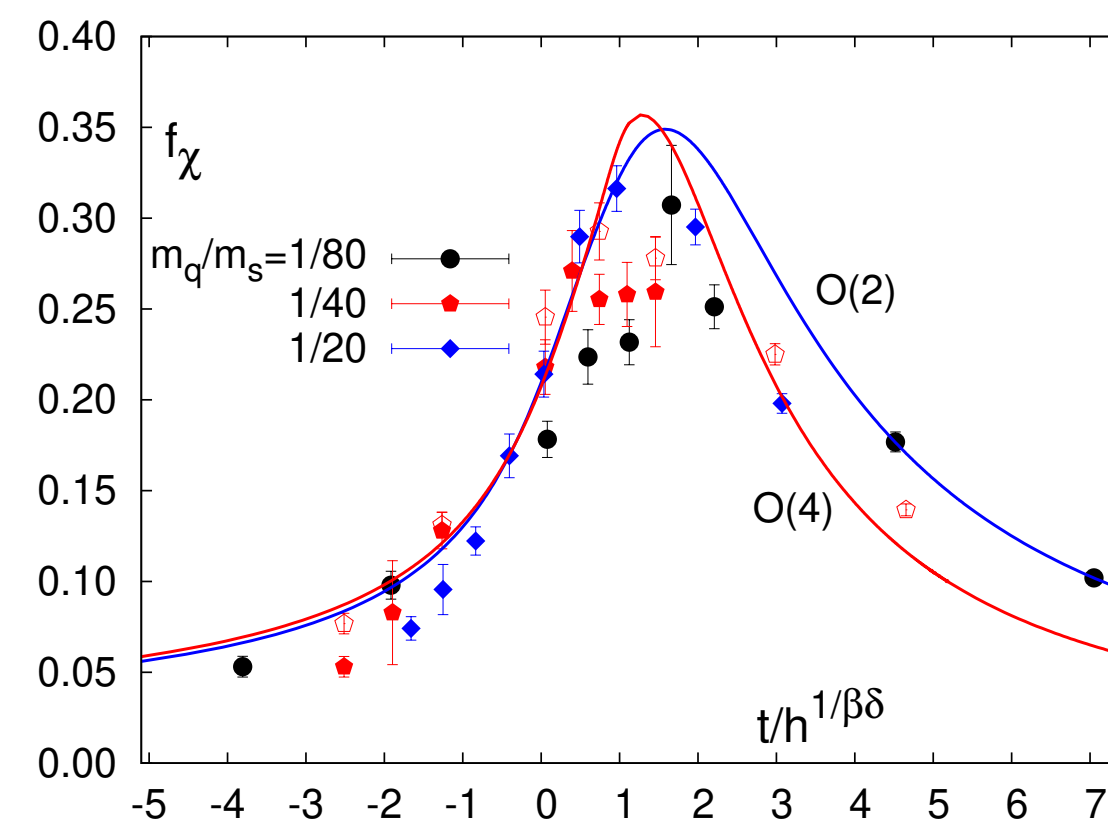
$$z = t/h^{1/\beta\delta}$$

$$t = \frac{T - T_c}{t_0 T_c}$$

$$h = \frac{m_l}{h_0 m_s}$$

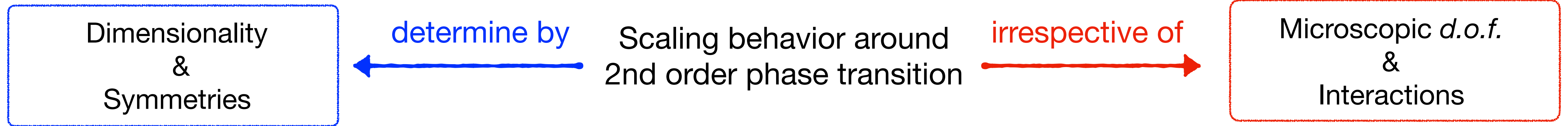


S. Ejiri et al., PRD 80, 094505 (2009)



H.-T. Ding, HWP et al., arXiv: 2112.00318

# Hard to Reach Microscopic Origin of Criticality



*E.g.:* Liquid to superfluid  $\lambda$ -transition in  $^4\text{He}$

*d.o.f.:* electrons & photons

Same dimensionality & symmetry  
Same 3-d  $O(2)$  scaling behaviors

Chiral phase transition in the chiral limit of light quark using staggered fermion discretization  
*d.o.f.:* quarks & gluons

**This talk:**  
**A first lattice QCD-based understanding of *Microscopic* Origin of Criticality in QCD !**

✓ *Able to study via effective theories*  
✗ *prevented by EFT to understand from micro level of d.o.f. !!*

# Banks-Casher Relation: Connect Macro to Micro in Chiral Limit

Chiral order parameter :  $\langle \bar{\psi}\psi \rangle = \int_0^\infty \frac{4m \rho(\lambda, m)}{\lambda^2 + m^2} d\lambda \xrightarrow{m \rightarrow 0} \pi \rho(\lambda = 0)$

Banks & Casher, NPB 169(1980) 103

$$\rho(\lambda, m) = \frac{T}{V} \langle \rho_U(\lambda) \rangle \equiv \frac{T}{V} \langle \sum_j \delta(\lambda - \lambda_j) \rangle \quad \text{with} \quad \mathbb{D}\psi_j = i\lambda_j \psi_j$$

**Calculable by Lattice QCD !!**

Connect **Macroscopic** quantity to **Microscopic d.o.f.** :  $\langle \bar{\psi}\psi \rangle \iff \rho_U(\lambda)$  : **energy spectra of massless quarks**

**Next:  
More generally...**

# From Quark Energy Spectra to its Cumulants

$$\langle \bar{\psi}\psi \rangle = \int_0^\infty \frac{4m \rho(\lambda, m)}{\lambda^2 + m^2} d\lambda = \frac{T}{V} \int_0^\infty \left\langle \frac{4m \rho_U(\lambda)}{\lambda^2 + m^2} \right\rangle d\lambda \quad \text{H.-T. Ding, HWP et al., PRL 131, 161903}$$

1st order cumulant :

$$\langle \bar{\psi}\psi \rangle = \frac{T}{V} \langle 2 \text{Tr}(\mathcal{D}[U] + m)^{-1} \rangle$$

1-point correlation of quark energy spectra  $\frac{4m \rho_U(\lambda)}{\lambda^2 + m^2}$

Generalization : from **generating functional** of cumulants

– **Generating functional** :  $\mathbb{G}(m; \epsilon) = \ln \left\langle \exp \left\{ -m \bar{\psi}\psi(\epsilon) \right\} \right\rangle_0 = \ln \left\langle \exp \left\{ -m \int_0^\infty P_U(\lambda; \epsilon) d\lambda \right\} \right\rangle_0$   $\langle \dots \rangle_0$  : average over QCD partition function in the chiral limit

Probe operator with valance quark mass  $\epsilon$  :  
 $\bar{\psi}\psi(\epsilon) \equiv 2 \text{Tr}(\mathcal{D}[U] + \epsilon)^{-1}$

Defining  $P_U(\lambda; \epsilon) \equiv \frac{4\epsilon \rho_U(\lambda)}{\lambda^2 + \epsilon^2}$

–  **$n$ -th order cumulant of  $\bar{\psi}\psi$**  :  $\mathbb{K}_n[\bar{\psi}\psi] = \frac{T}{V} (-1)^n \left. \frac{\partial^n \mathbb{G}(m; \epsilon)}{\partial m^n} \right|_{\epsilon=m}$

$$\mathbb{K}_n[\bar{\psi}\psi(m)] = \int_0^\infty K_1[P_U(\lambda_1; m), P_U(\lambda_2; m), \dots, P_U(\lambda_n; m)] \prod_{i=1}^n d\lambda_i$$

$K_1[X_1, X_2, \dots, X_n]$  denotes 1st order joint cumulant of  $n$ -variables

# Microscopic Encoding of Macroscopic Criticality

H.-T. Ding, HWP et al., PRL 131, 161903

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1; m), P_U(\lambda_2; m), \dots, P_U(\lambda_n; m)] \prod_{i=1}^n d\lambda_i \equiv \int_0^\infty P_n(\lambda) d\lambda$$

*n*-th order cumulant of the chiral order parameter

*n*-point correlation of the quark energy spectra

Chiral condensate :

$$\mathbb{K}_1[\bar{\psi}\psi] = \frac{T}{V} \langle \bar{\psi}\psi(m) \rangle = \int_0^\infty P_1(\lambda) d\lambda \xrightarrow{\text{Around } T_c} \sim m^{1/\delta} f_1(z)$$

$$z = z_0 \left( \frac{m_l}{m_s} \right)^{-\frac{1}{\beta\delta}} \frac{T - T_c}{T_c}$$

$$f_1(z) \equiv f_G(z)$$

$$f_2(z) \equiv f_\chi(z)$$

Disconnected susceptibility :

$$\mathbb{K}_2[\bar{\psi}\psi] = \frac{T}{V} \left\langle \left[ \bar{\psi}\psi(m) - \langle \bar{\psi}\psi(m) \rangle \right]^2 \right\rangle = \int_0^\infty P_2(\lambda) d\lambda \xrightarrow{\text{Around } T_c} \sim m^{1/\delta-1} f_2(z)$$

.....

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda) d\lambda \xrightarrow{\text{Around } T_c} \sim m_l^{1/\delta-n+1} f_n(z)$$

How criticality of  $\mathbb{K}_n[\bar{\psi}\psi]$  arises from  $P_n(\lambda)$  ?

# Microscopic Encoding of Macroscopic Criticality

Hints from the chiral limit :

$$P_U(\lambda; m) \equiv \frac{4m\rho_U(\lambda)}{\lambda^2 + m^2}$$

$$P_U(\lambda; m \rightarrow 0) = 2\pi\rho_U(\lambda)\delta(\lambda)$$

Generalized Banks-Casher relation :

$$\lim_{m \rightarrow 0} P_n(\lambda) = (2\pi)^n \underbrace{K_1[\rho_U(\lambda), \rho_U(0), \dots, \rho_U(0)]}_{(n-1) \text{ terms}} \delta(\lambda)$$

$$\implies \lim_{m \rightarrow 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)]$$

$n = 1$  back to Banks-Casher relation !

Criticality in  $\lim_{m \rightarrow 0} \mathbb{K}_n[\bar{\psi}\psi]$  must arise from universal behaviors of  $\lambda$ -**independent**  $\mathbb{K}_n[\rho_U(0)]$

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda) d\lambda \xrightarrow{\text{Around } T_c} \sim m_l^{1/\delta-n+1} f_n(z)$$

**Conjecture:**

$$P_n(\lambda) = m^{1/\delta-n+1} f_n(z) g_n(\lambda)$$

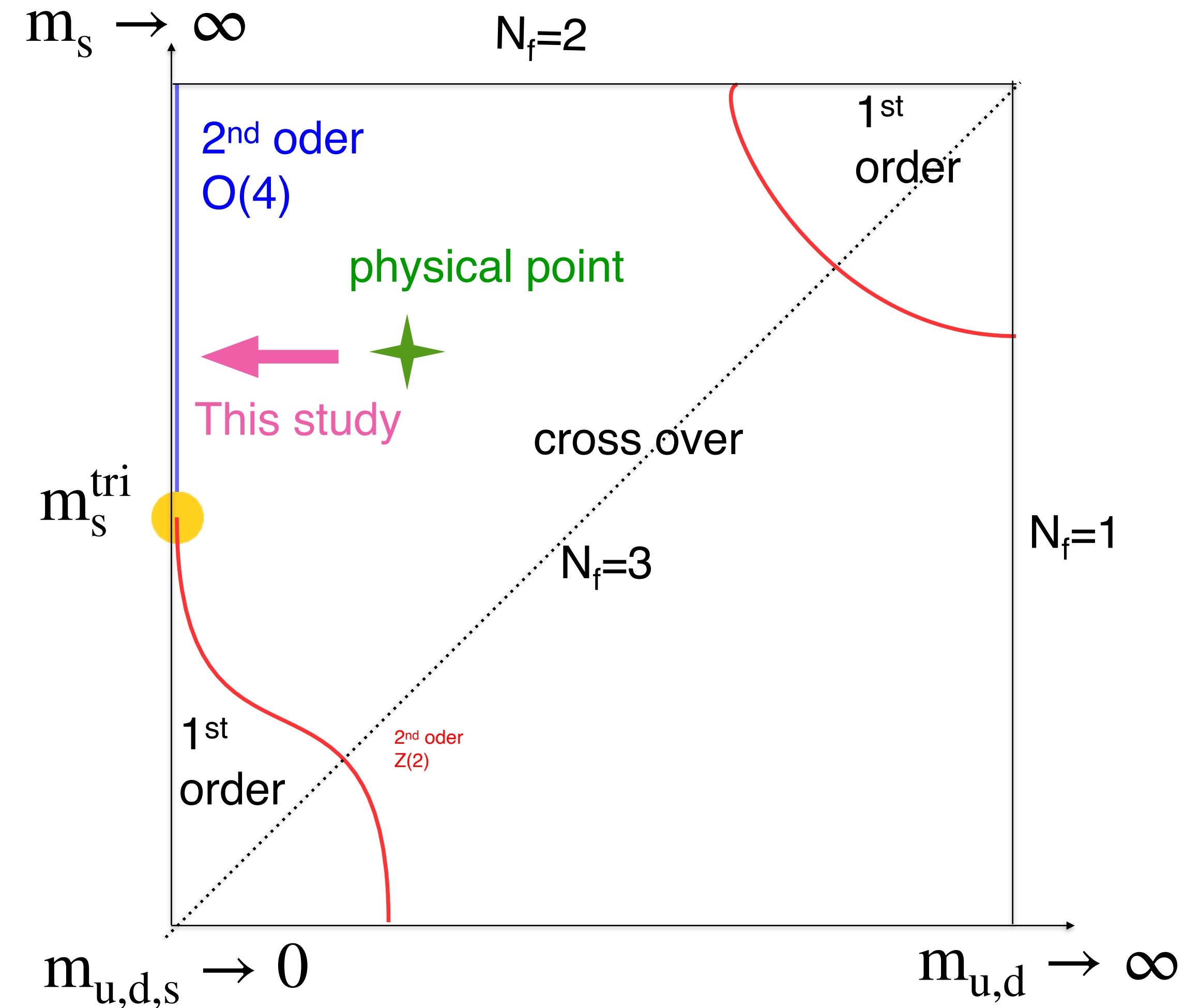
Scaling arise from  $P_n(\lambda)$  at **deep infrared**  $\lambda$  region

Include **all** system-specific  $\lambda$ -dependence



# Lattice Setup

- Actions: Highly improved staggered quarks and tree-level Symanzik gauge action
- Lattice size:  $N_\tau = 8, N_\sigma = 32, 40, 56$
- Quark mass:  $m_s^{\text{phy}}/m_l = 27, 40, 80, 160$   
( $m_\pi \approx 140, 110, 80, 55$  MeV)
- Temperatures:  $T \in (135, 176)$  MeV
- $\rho_U(\lambda)$  computed via Chebyshev filtering technique  
H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)
- HotQCD configurations; Measurements carried out on NSC<sup>3</sup> at CCNU & BNL

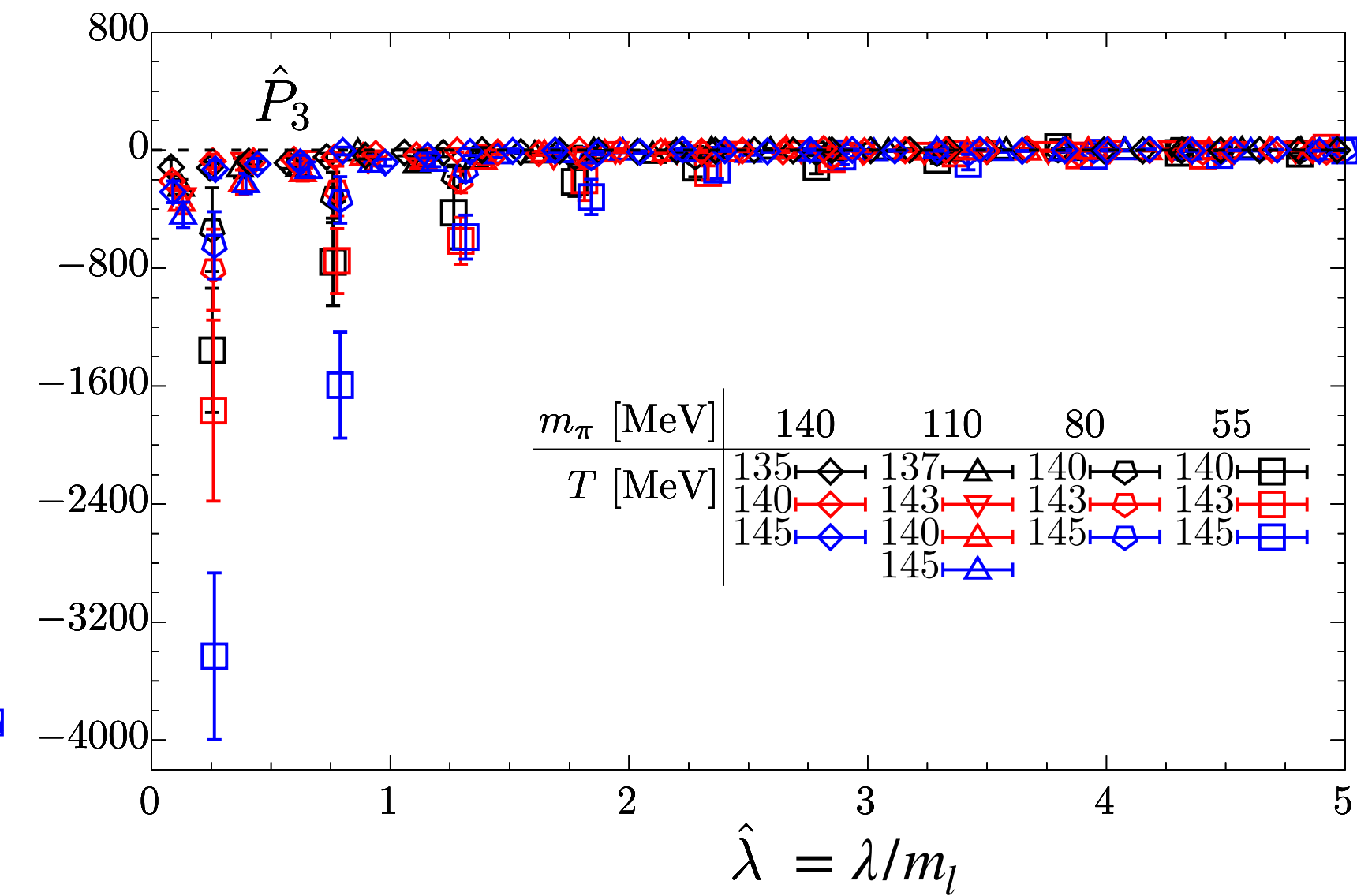
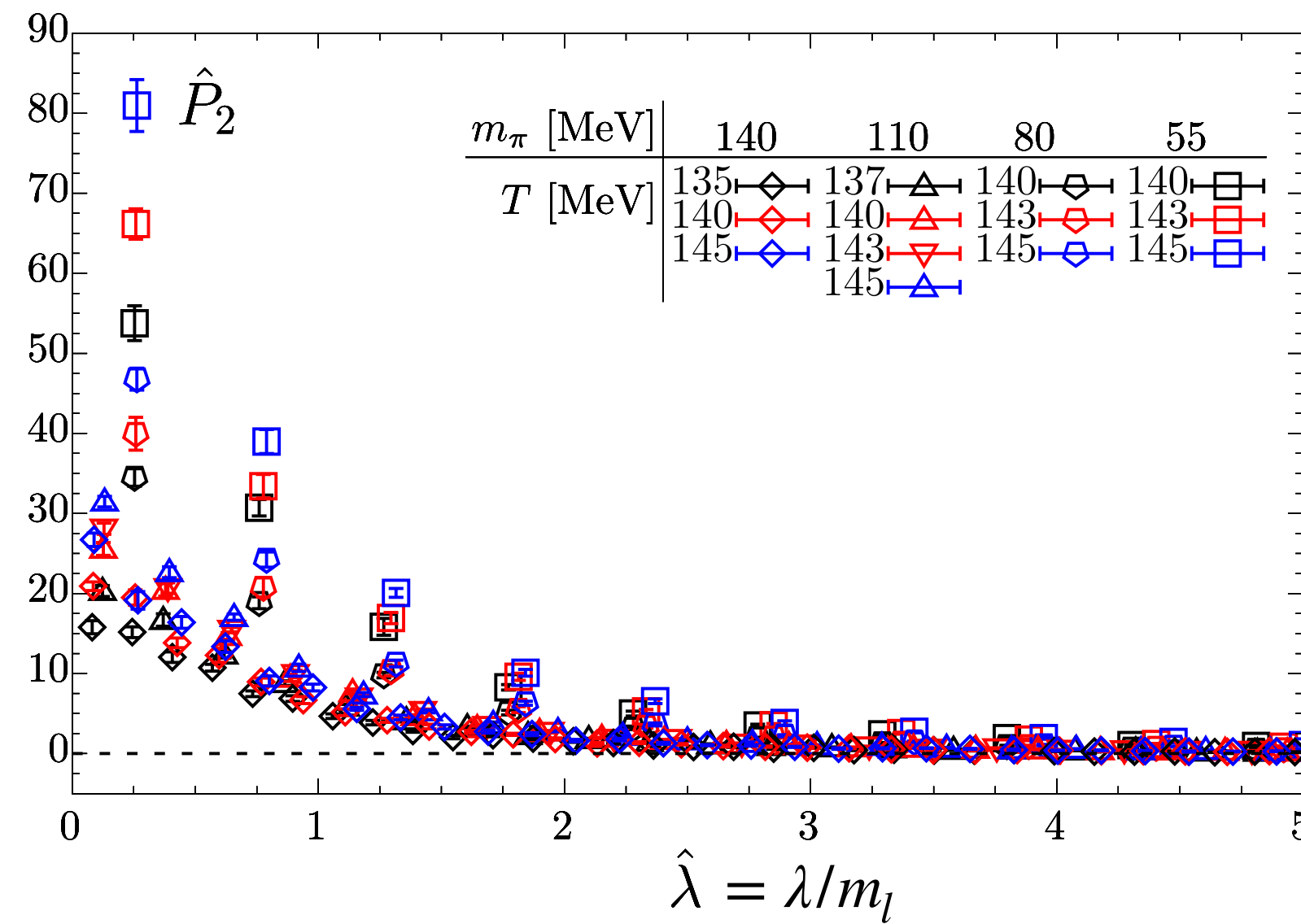
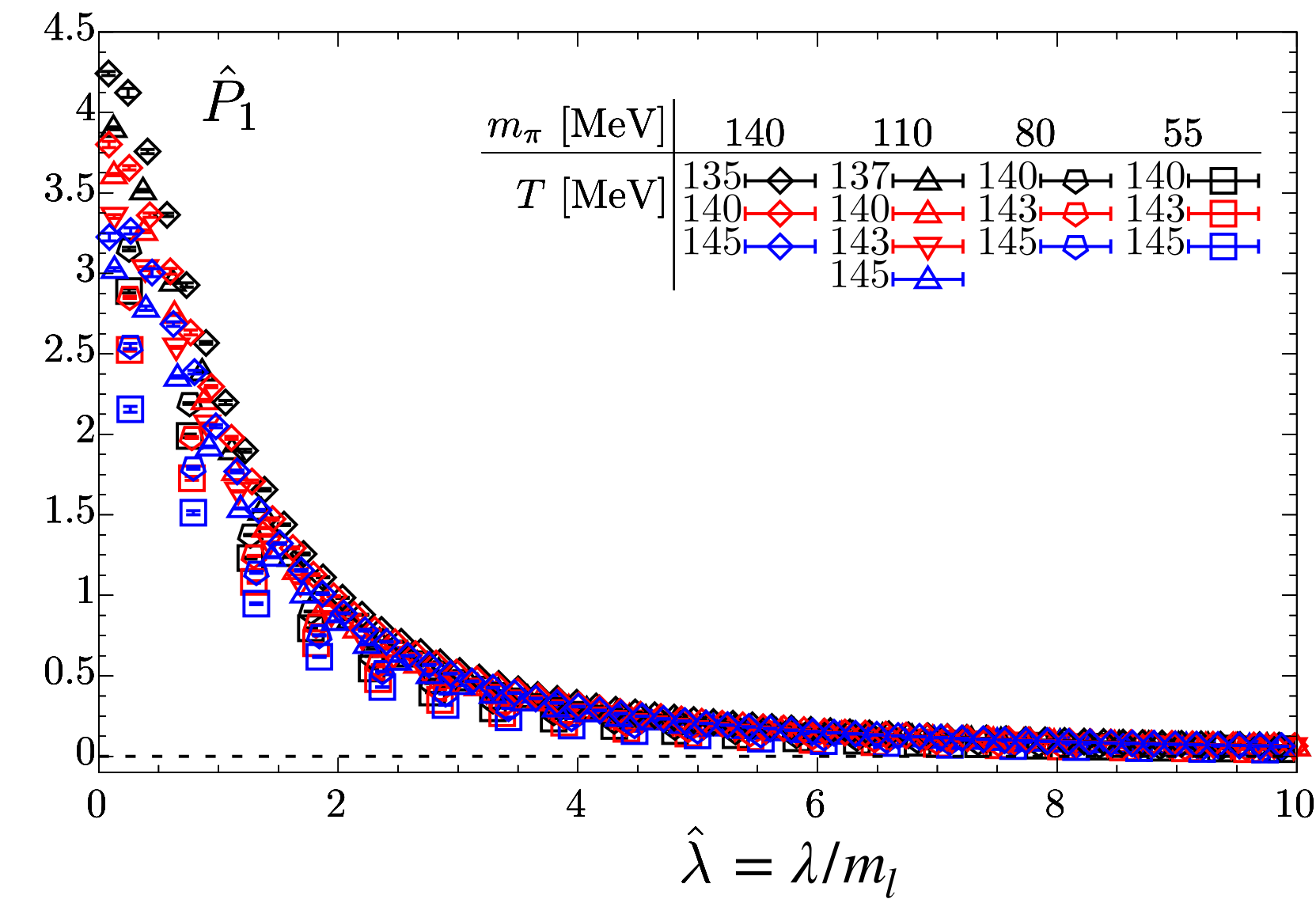


# $P_n(\lambda)$ around $T_c$

$$\hat{P}_1(\hat{\lambda}) = m_s^2(m_l/m_s)P_1(\lambda)/T_c^4$$

$$\hat{P}_2(\hat{\lambda}) = m_s^3(m_l/m_s)P_2(\lambda)/T_c^4$$

$$\hat{P}_3(\hat{\lambda}) = m_s^4(m_l/m_s)P_3(\lambda)/T_c^4$$



$$\hat{P}_1(\hat{\lambda}), \hat{P}_2(\hat{\lambda}) \text{ and } \hat{P}_3(\hat{\lambda})$$

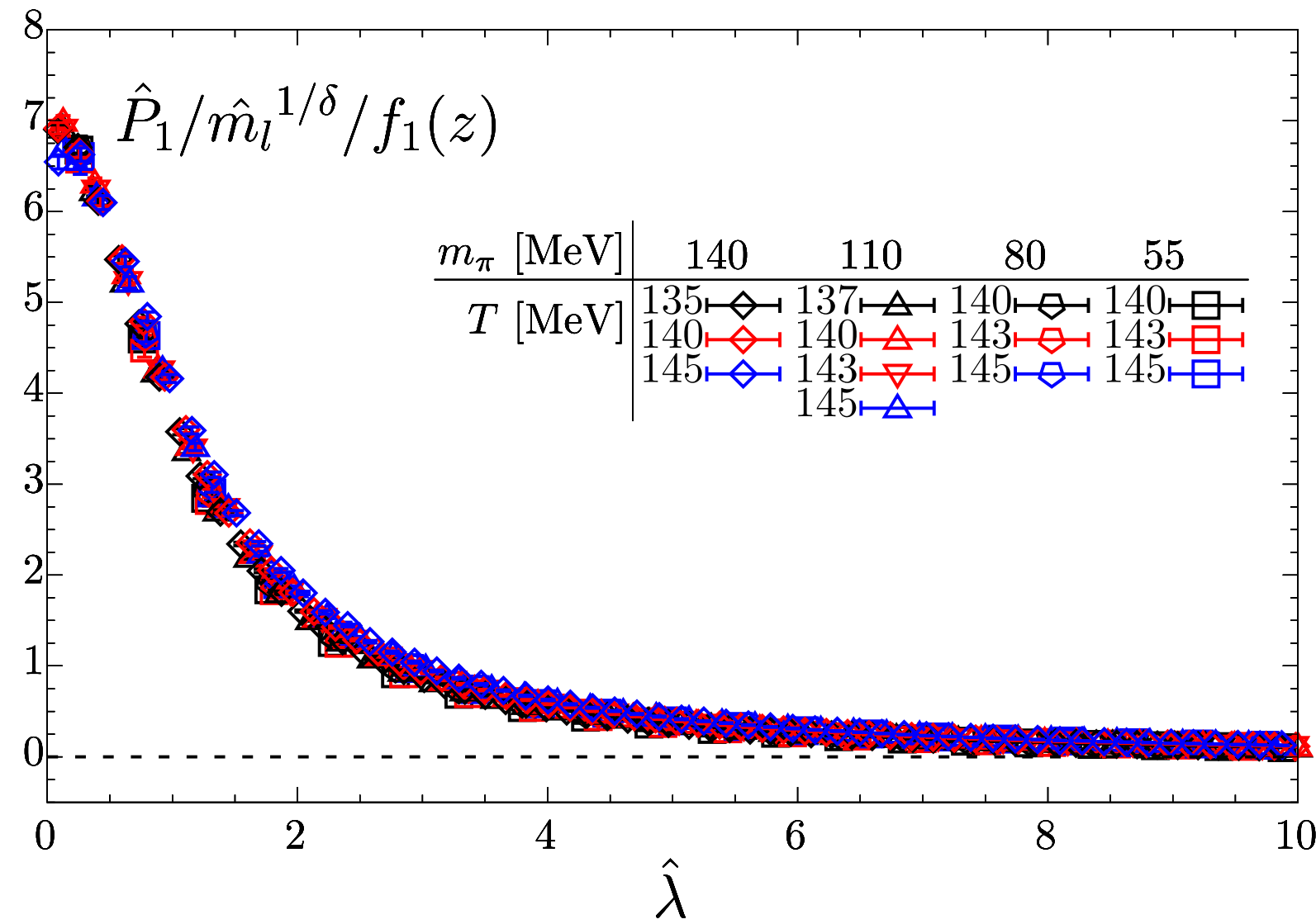
- Infrared lambda region dominates;
- Significant dependence on quark mass and temperature

$$\text{Conjecture: } \hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta-n+1} f_n(z) g_n(\hat{\lambda})$$

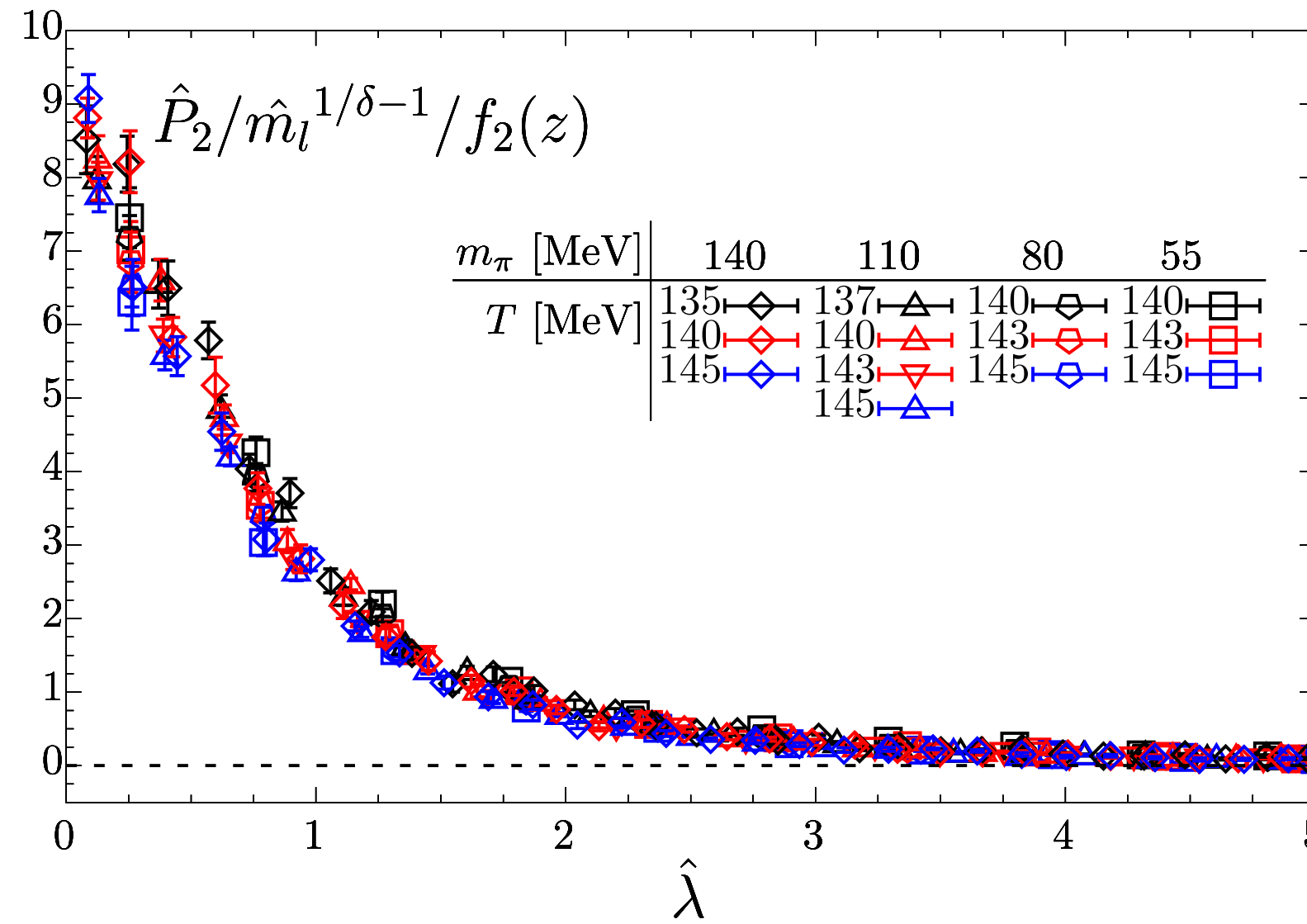
# Rescaled $P_n(\lambda)$ around $T_c$

Conjecture:  $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta-n+1} f_n(z) g_n(\hat{\lambda})$

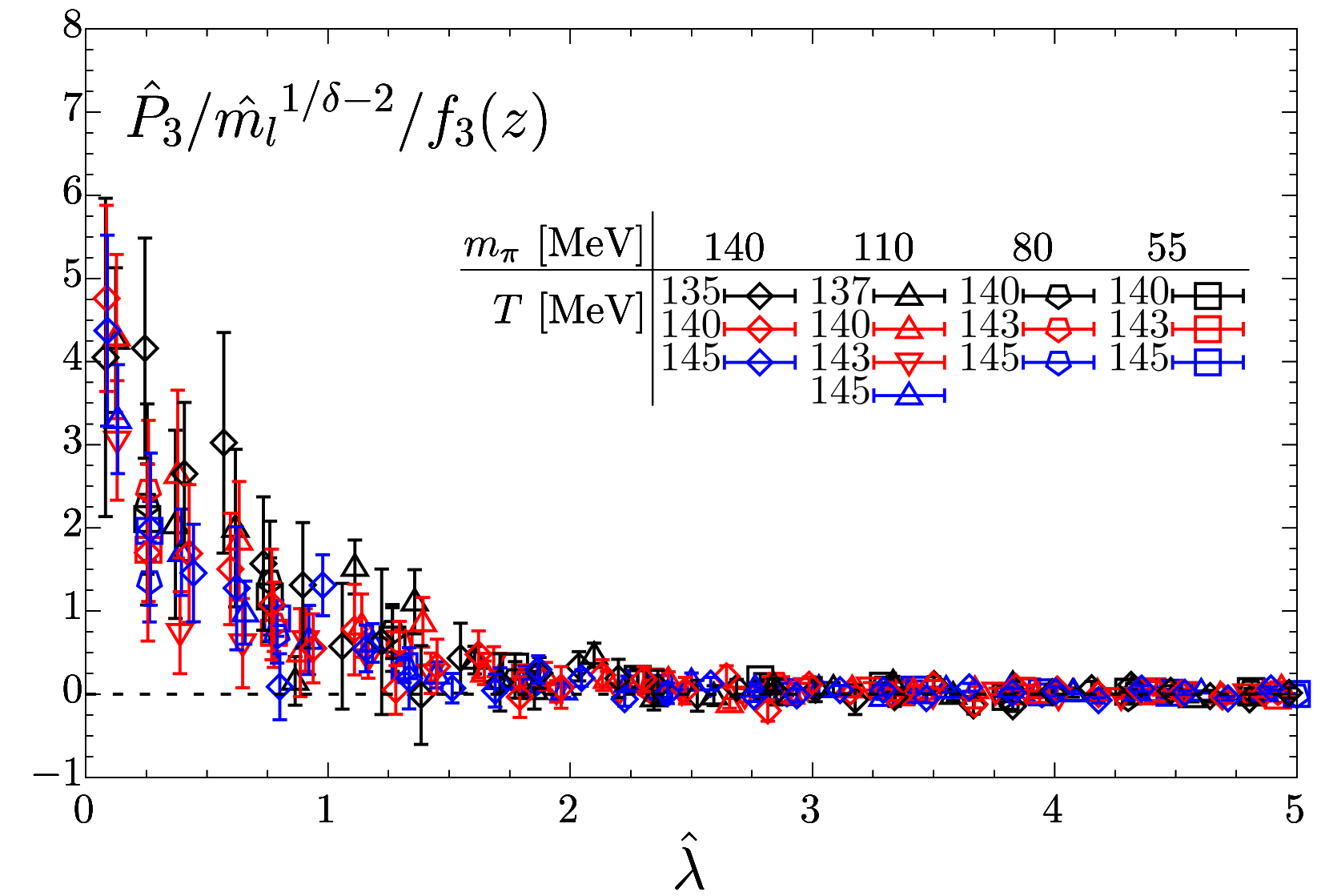
$$\hat{P}_1(\hat{\lambda}) / (m_l/m_s)^{1/\delta} / f_1(z)$$



$$\hat{P}_2(\hat{\lambda}) / (m_l/m_s)^{1/\delta-1} / f_2(z)$$



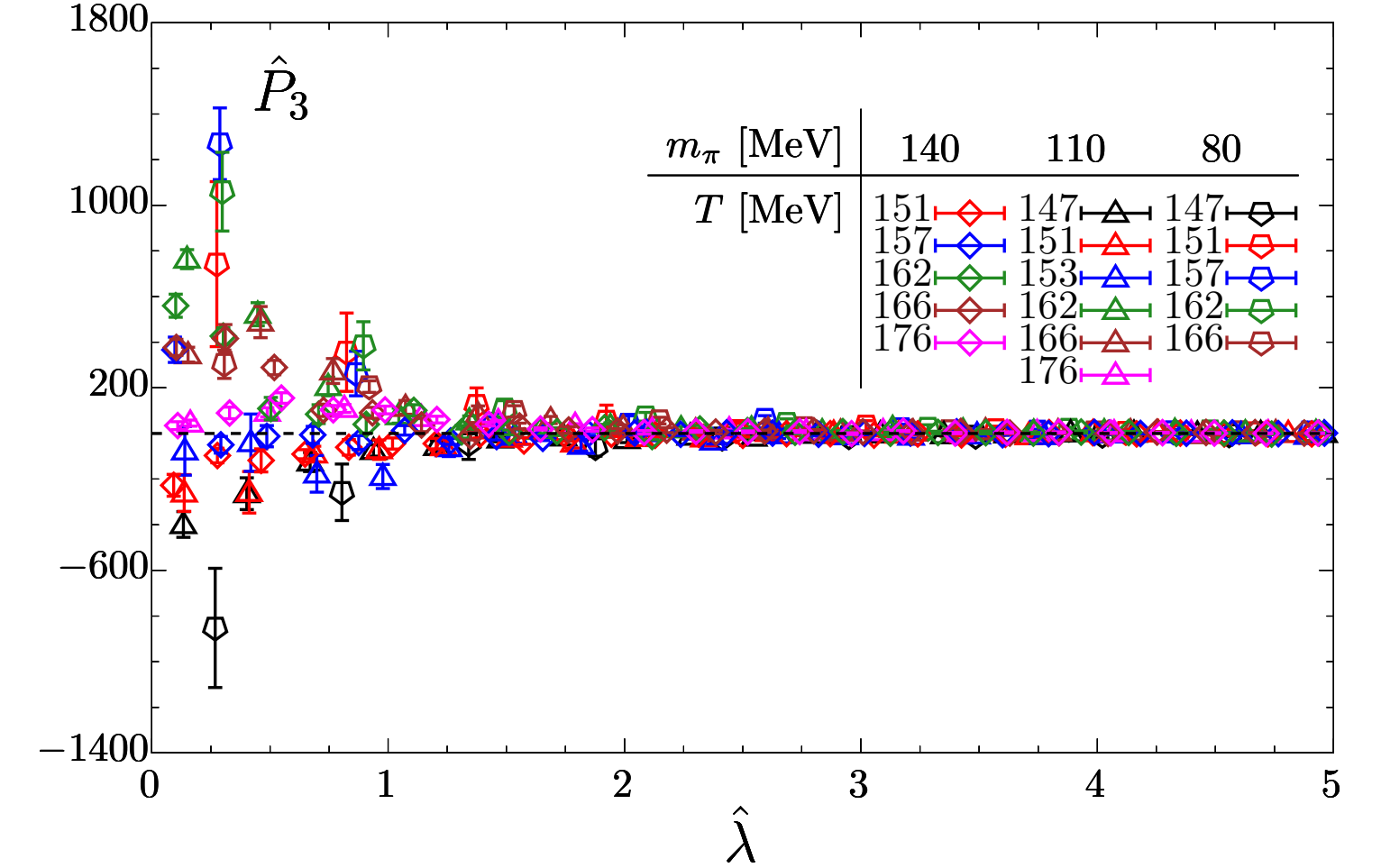
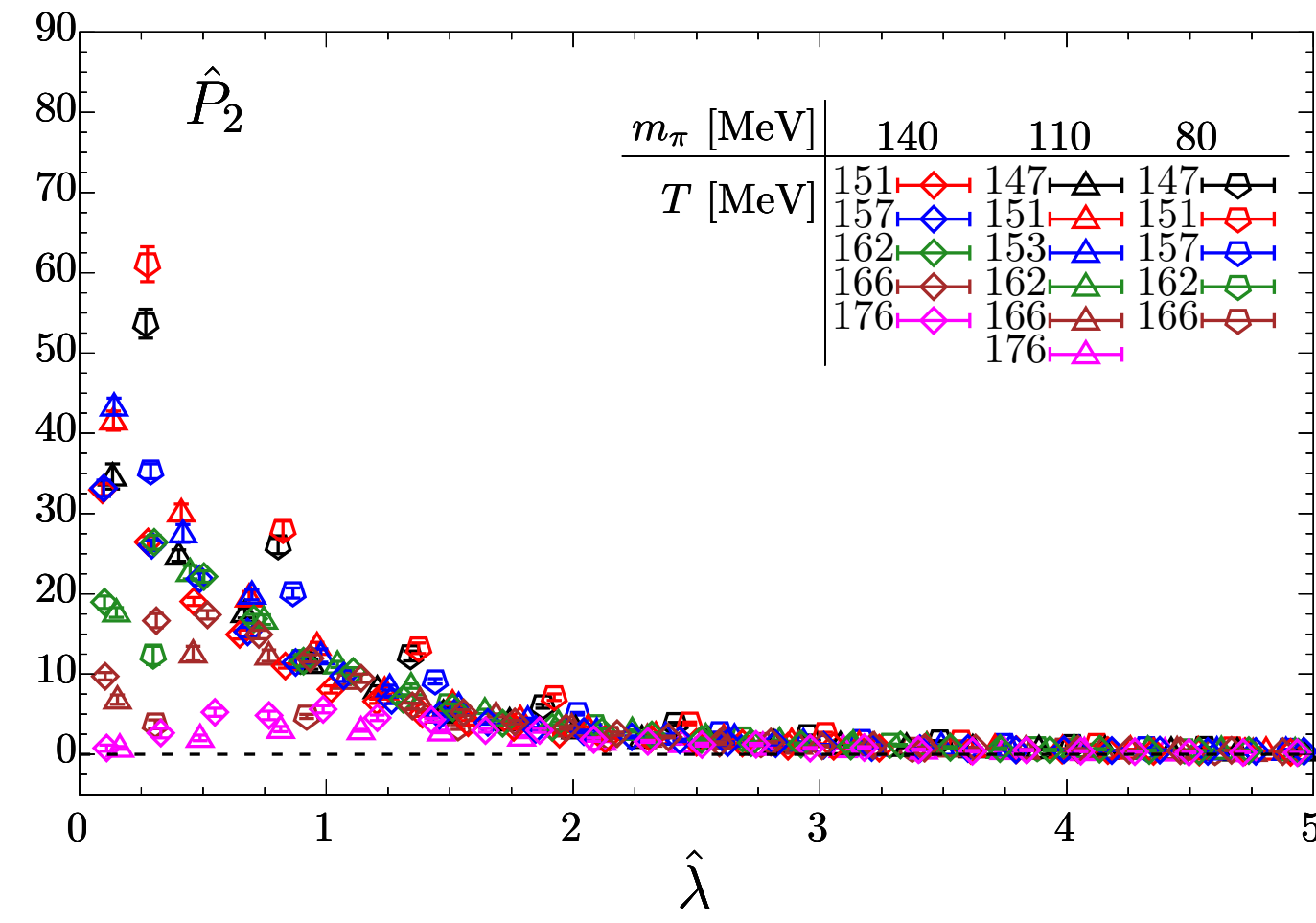
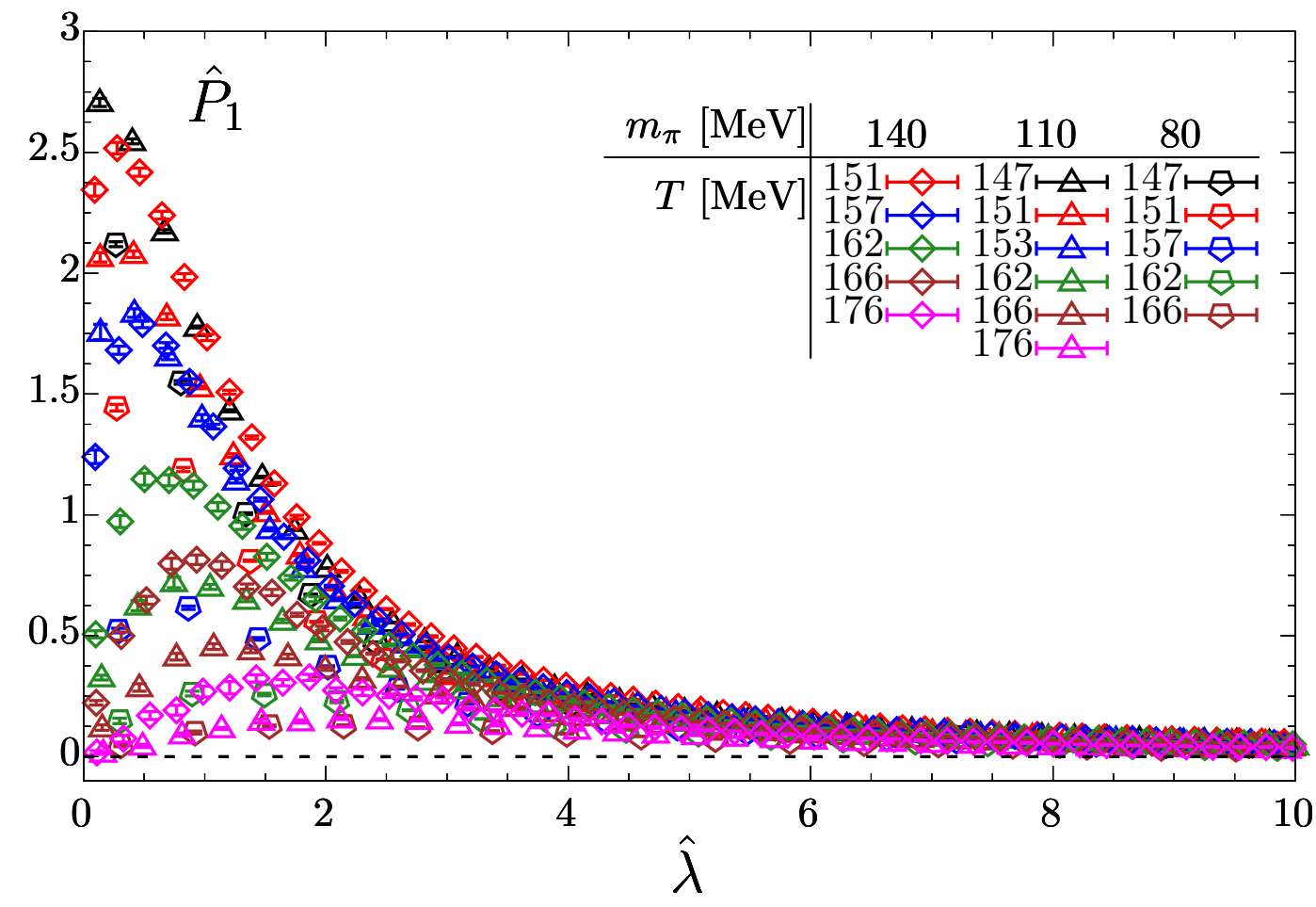
$$\hat{P}_3(\hat{\lambda}) / (m_l/m_s)^{1/\delta-2} / f_3(z)$$



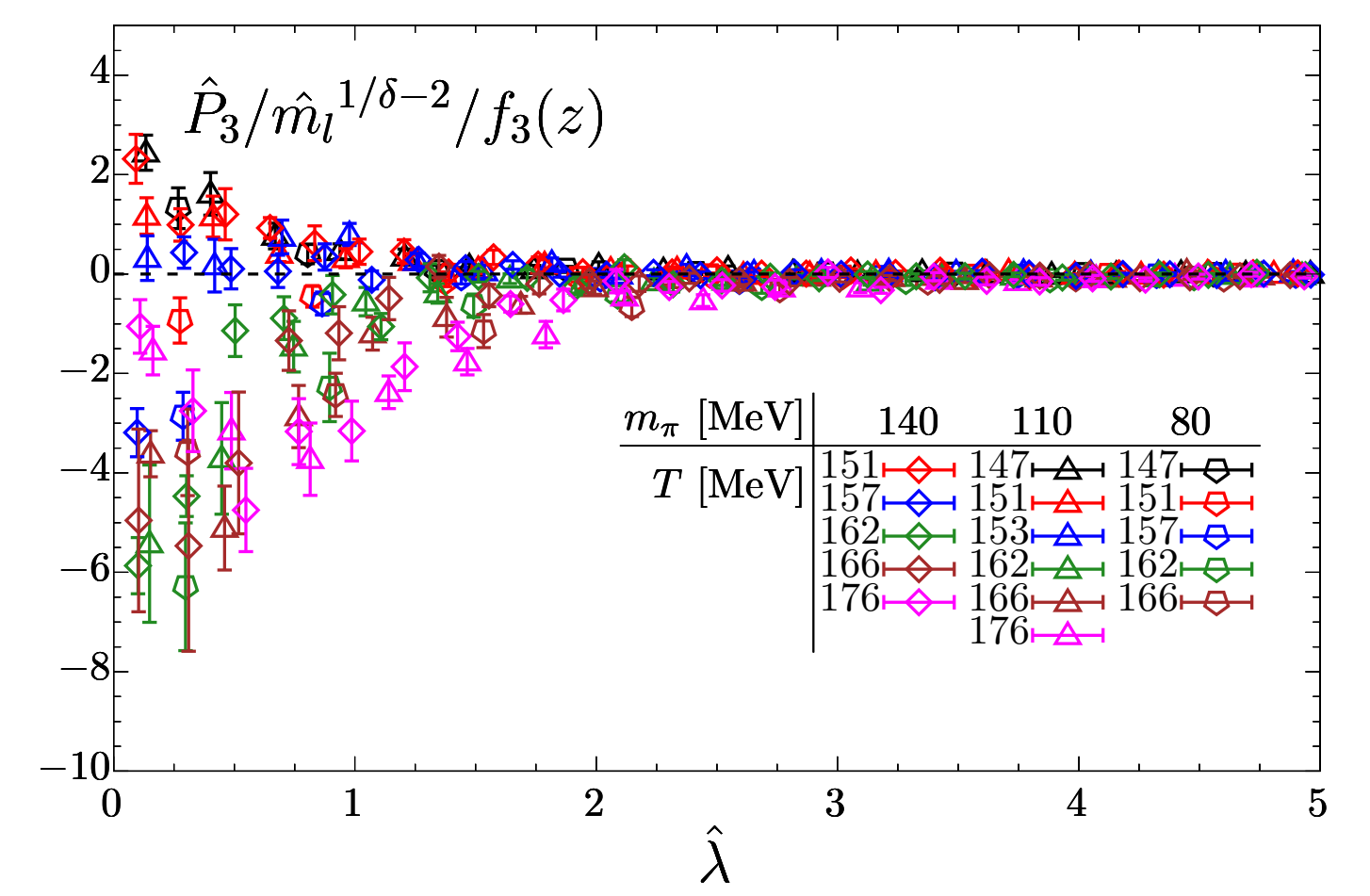
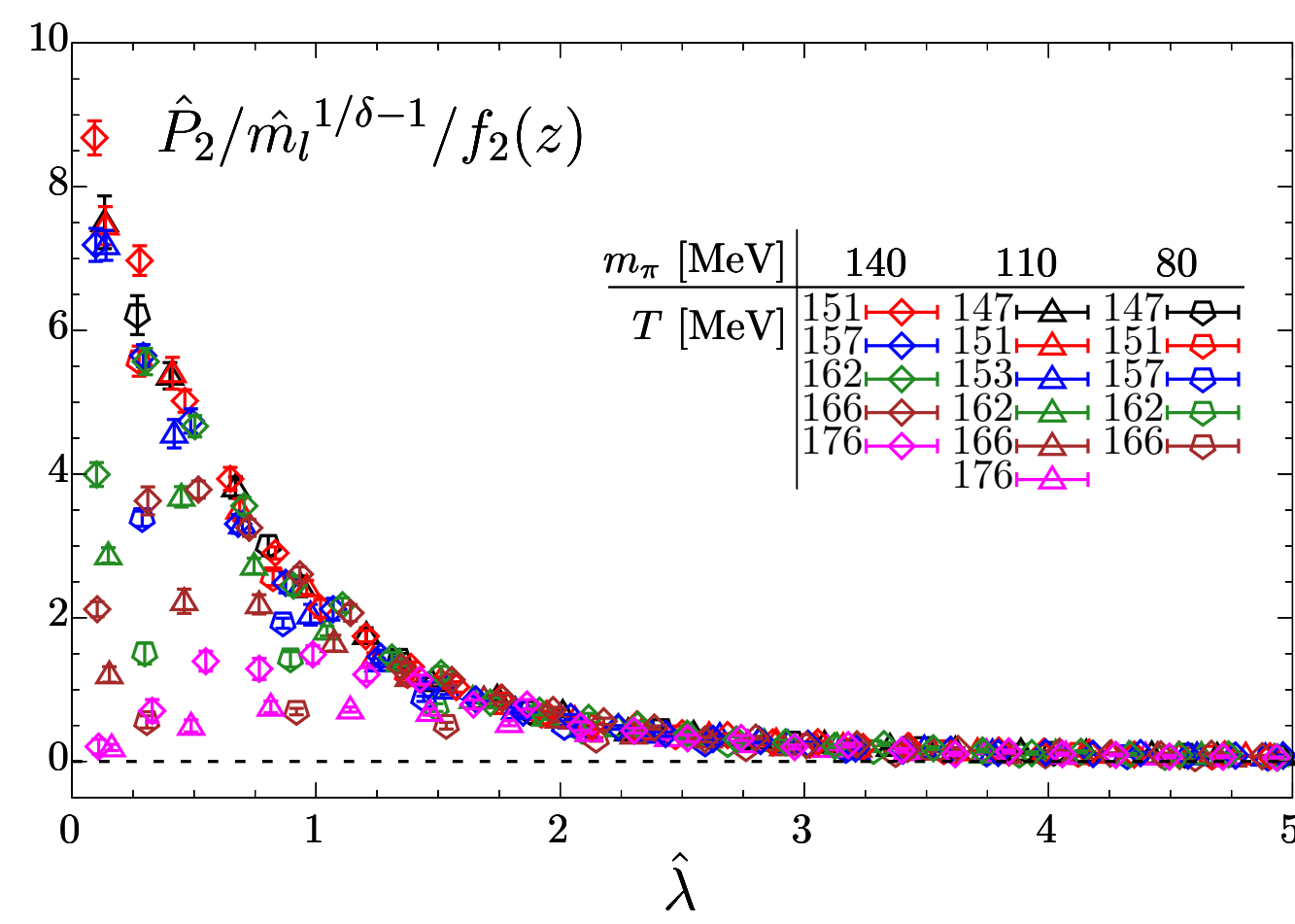
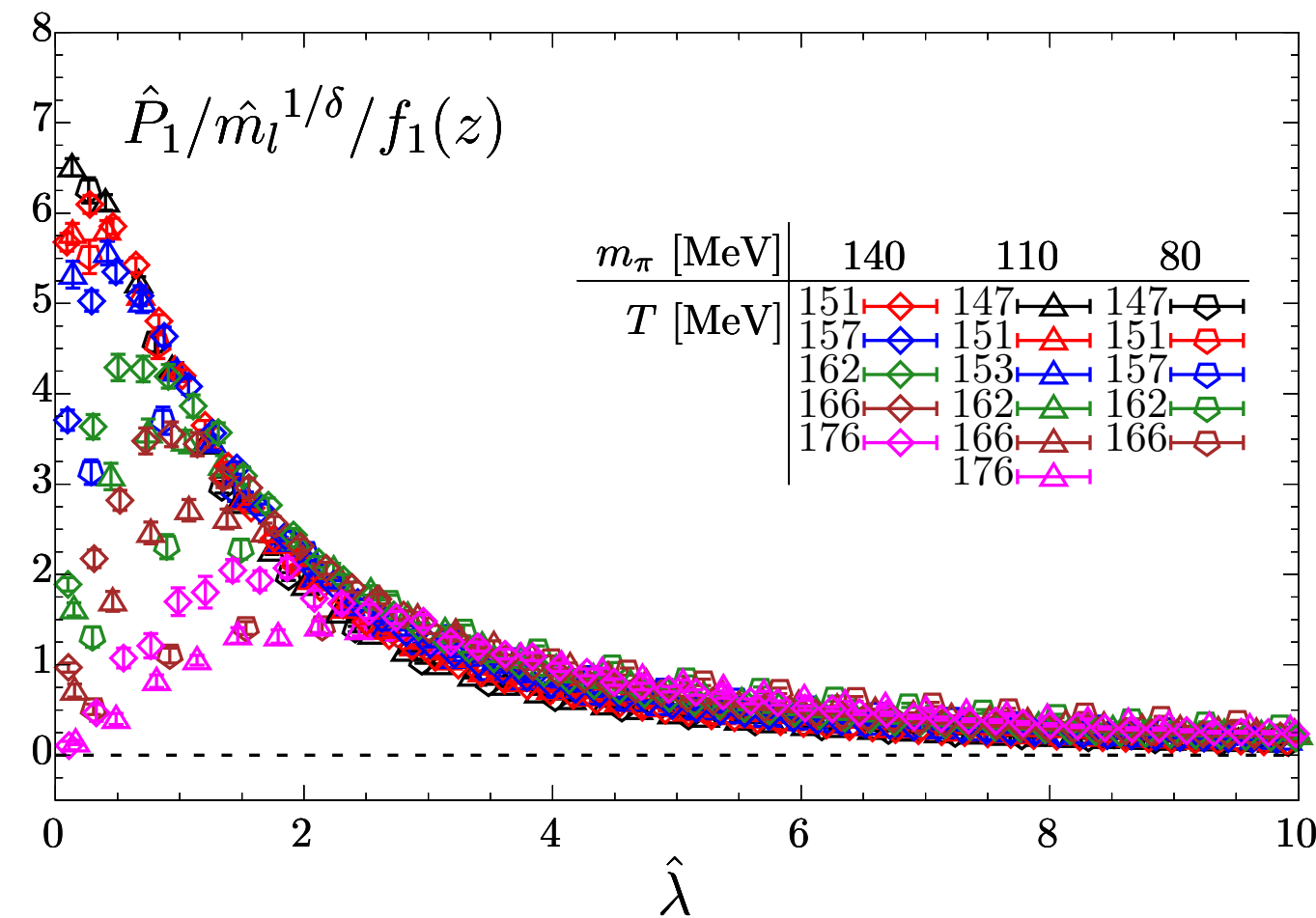
$z = z_0(m_l/m_s)^{-\frac{1}{\beta\delta}}(T - T_c)/T_c$ :  $O(2)$  scaling parameters adopted from [S. Ejiri et al., Phys. Rev. D 80, 094505 (2009); D. A. Clarke et al., Phys. Rev. D 103, L011501 (2021)]

- In the vicinity of  $T_c$ ,  $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta-n+1} f_n(z) g_n(\hat{\lambda})$
- Scaling behaviors in  $\hat{P}_n(\hat{\lambda})$  extend up to physical light quark mass

# $P_n(\lambda)$ and Rescaled $P_n(\lambda)$ away from $T_c$



Away from  $T_c$ , no scaling behaviors are observed in  $\hat{P}_n(\hat{\lambda})$



# Summary

- ✓ We establish a novel relation

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1; m), P_U(\lambda_2; m), \dots, P_U(\lambda_n; m)] \prod_{i=1}^n d\lambda_i \equiv \int_0^\infty P_n(\lambda) d\lambda$$

***$n$ -th order cumulant of the chiral condensate***

***$n$ -point correlation of the quark energy spectra***

- ✓ A generalization of the Banks-Casher relation is obtained:

$$\lim_{m \rightarrow 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)]$$

- ✓ Microscopic encoding of macroscopic criticality

$$P_n(\lambda) = m_l^{1/\delta - n + 1} f_n(z) g_n(\lambda)$$

- ✓ Universal behaviors manifested in microscopic energy levels of QCD extend up to physical light quark masses

Backup

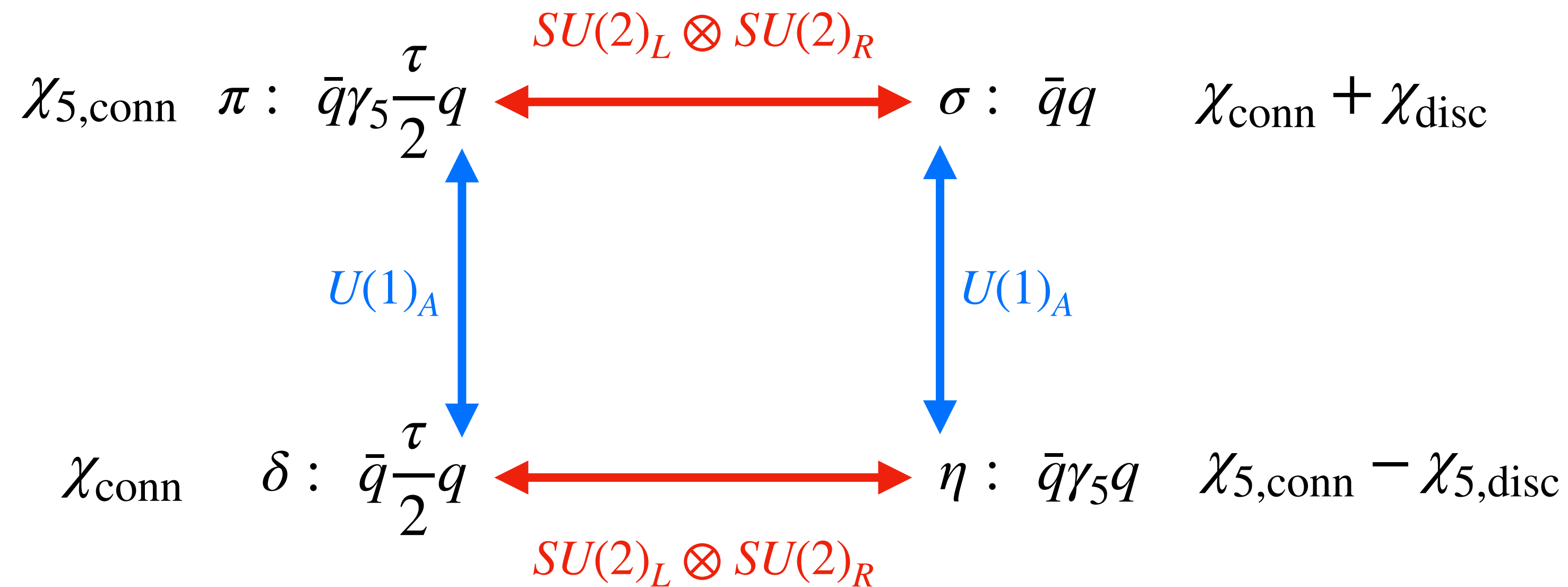
# Signatures of symmetry restorations

Susceptibilities defined as integrated two point correlation functions of quark bilinear  $J_M(x) = \bar{q}(x)\Gamma_M q(x)$

$$\chi_M = \int d^4x \left\langle J_M(x) J_M^\dagger(0) \right\rangle$$

A. Bazavov et al., [HotQCD], PRD 86 (2012) 094503

N. Carabba et al., [HotQCD], PRD 105 (2022) 5, 054034



Restoration of  $SU(2)_L \otimes SU(2)_R$

$$\begin{aligned} \chi_\pi - \chi_\sigma &= 0 \\ \chi_\delta - \chi_\eta &= 0 \end{aligned} \Rightarrow \chi_\pi - \chi_\delta = \chi_{\text{disc}} = \chi_{5,\text{disc}}$$

Effective restoration of  $U(1)_A$

$$\begin{aligned} \chi_\pi - \chi_\delta &= 0 \\ \chi_\sigma - \chi_\eta &= 0 \end{aligned} \Rightarrow \chi_\pi - \chi_\delta = \chi_{\text{disc}} = \chi_{5,\text{disc}} = 0$$

Related to Dirac eigenvalues:

$$\langle \bar{\psi}\psi \rangle = \int_0^\infty d\lambda \frac{4m_l \rho(\lambda, m_l)}{\lambda^2 + m_l^2} \quad \chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{8m_l^2 \rho(\lambda, m_l)}{(\lambda^2 + m_l^2)^2} \quad \chi_{\text{disc}} = \int_0^\infty d\lambda \frac{4m_l \cdot \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

# Microscopic origin in Dirac eigenvalues

$$\langle \bar{\psi}\psi \rangle_l = \int_0^\infty d\lambda \frac{4m_l \rho(\lambda, m_l)}{\lambda^2 + m_l^2} \quad \chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{8m_l^2 \rho(\lambda, m_l)}{(\lambda^2 + m_l^2)^2} \quad \chi_{\text{disc}} = \int_0^\infty d\lambda \frac{4m_l \cdot \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

- Restoration of  $SU(2)_L \otimes SU(2)_R$  symmetry:

$$\rho(0) = 0 \quad \text{from Banks-Casher formula} \quad \lim_{m_l \rightarrow 0} \lim_{V \rightarrow \infty} \langle \bar{\psi}\psi \rangle_l = \lim_{m_l \rightarrow 0} \lim_{V \rightarrow \infty} 2\pi\rho(0, m_l)$$

Banks and Casher, NPB 169 (1980) 103

- Effective restoration of  $U(1)_A$  symmetry:

A sizable gap in the near-zero mode

Cohen, arXiv:nucl-th/9801061

- Underlying structure of  $\rho(\lambda, m_l)$  responsible for symmetry restorations:

$$\rho(\lambda, m_l) = c_0 + c_1\lambda + c_2m_l^2\delta(\lambda) + c_3m_l + c_4m_l^2 + \dots$$

$$\Rightarrow \langle \bar{\psi}\psi \rangle = 2c_0\pi - 4c_1m_l \ln(m_l) + 2c_2m_l + 2c_3\pi + 2\pi c_4m_l^2$$

$$\chi_\pi - \chi_\delta = 2c_0\pi/m_l + 4c_1 + 4c_2 + 2c_3\pi + 2c_4\pi m_l$$

$c_0$  &  $c_1$  term: break both symmetries

Gross et al., RMP 81'

$c_2$  term: dilute instanton gas predicts

HotQCD, PRD86(2012)094503

$c_3$  term: break  $U(1)_A$  symmetry

Aoki et al., PRD86(2012)114512

$c_4$  term: make  $U(1)_A$  anomaly unmanifested in 2-pt correlators

Mass derivatives of  $\rho$  are needed to determine the microscopic origin



# Calculation of eigenvalue spectrum

- Commonly used method: Lanczos algorithm to calculate the individual low-lying eigenvalues
- Here we utilized the Chebyshev filtering technique combined with a stochastic estimate of the mode number

$$\text{Mode number : } n_{[s,t]} \approx \frac{1}{n_v} \sum_{k=1}^{n_v} \left[ \sum_{j=0}^p g_j^p \gamma_j v_k^T T_j(A) v_k \right]$$

$T_j$  : Chebyshev polynomial

$\gamma_j$  &  $g_j^p$  : expansion coefficients

$n_v$  : number of random vectors

$p$  : number of polynomial orders

$$\text{eigenvalue spectrum : } \rho_U(\lambda) = \frac{1}{4} \frac{n_{[\lambda-\delta/2, \lambda+\delta/2]}}{2\delta\lambda}$$

1/4 : Staggered Fermion Discretization Scheme

1/2 : positive and negative eigenvalue pairs

$\delta\lambda$  : bin-size

H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)

Yu Zhang, Lattice 19', arXiv: 2001.05217

Cossu et al., arXiv: 1601.00744

# $\partial^n \rho / \partial m_l^n$ and Quark Energy Spectra

Eigenvalue spectrum for (2+1)-flavor QCD:

$$\rho(\lambda, m_l) = \frac{T}{VZ[U]} \int D[U] e^{-S_G[U]} \det[\mathcal{D}[U] + m_s] \times \left( \det[\mathcal{D}[U] + m_l] \right)^2 \rho_U(\lambda)$$



Partition function:

$$Z[U] = \int D[U] e^{-S_G[U]} \det[\mathcal{D}[U] + m_s] \times \left( \det[\mathcal{D}[U] + m_l] \right)^2 = \exp \left( \int_0^\infty d\lambda \rho_U(\lambda) \ln [\lambda^2 + m_l^2] \right)$$



$m_l$  dependence enters  $\rho$ :

$$\det[\mathcal{D}[U] + m_l] = \prod_j \left( +i\lambda_j + m_l \right) \left( -i\lambda_j + m_l \right)$$



Eigenvalue spectrum for a given configuration:

$$\rho_U(\lambda) = \sum_j \delta(\lambda - \lambda_j)$$

Mass derivative of  $\rho(\lambda, m_l)$ :

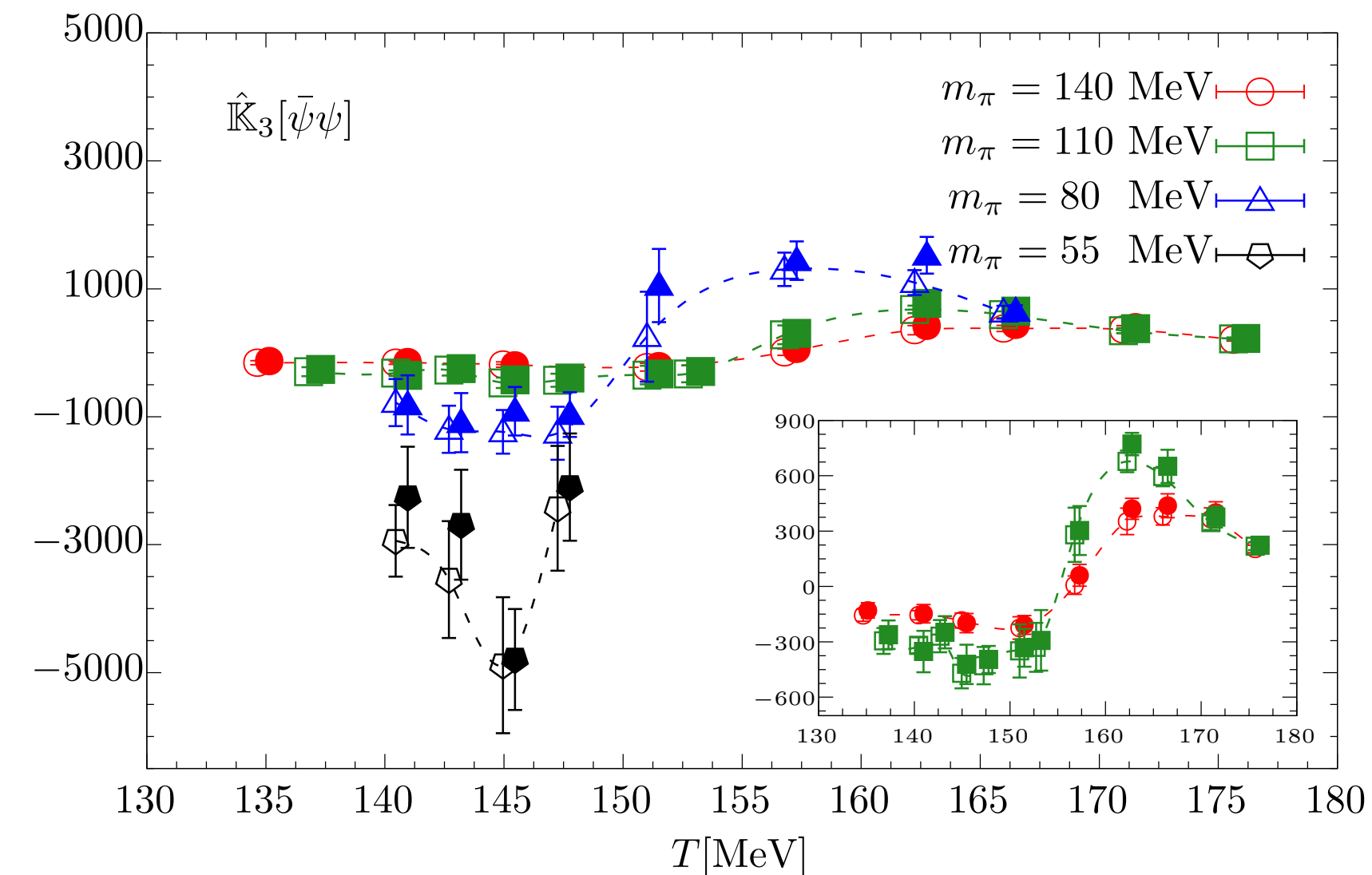
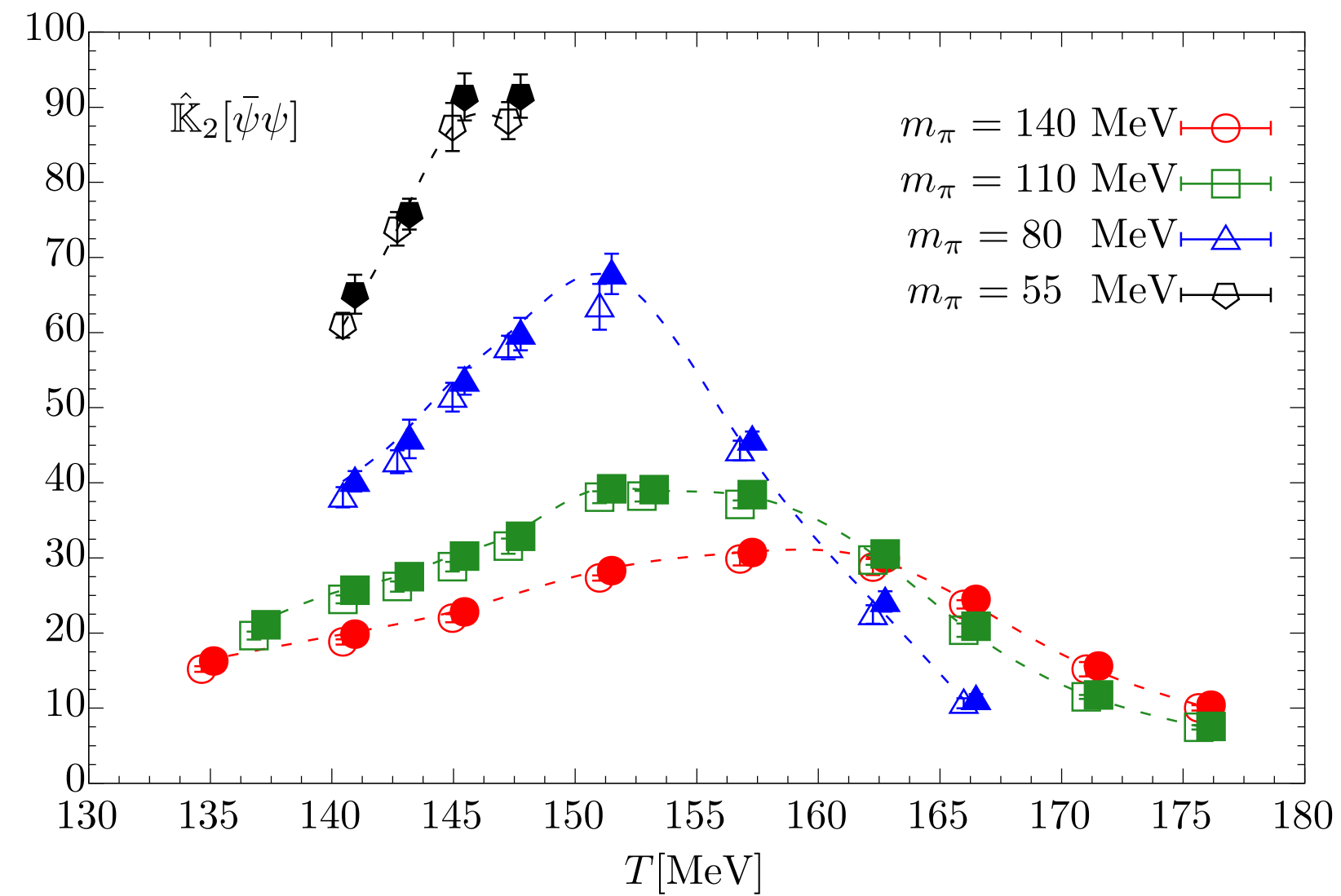
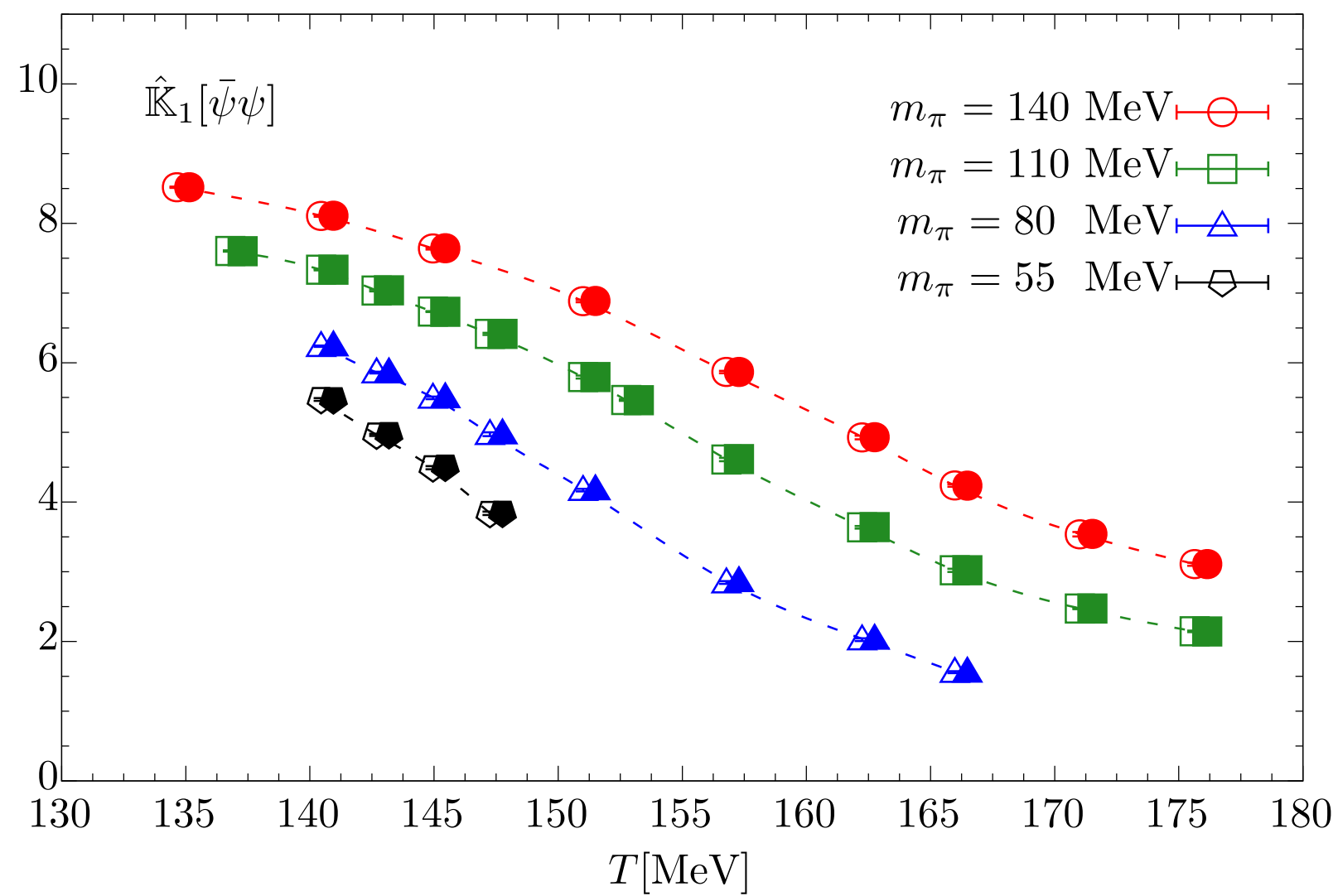
$$\frac{V}{T} \frac{\partial \rho(\lambda, m_l)}{\partial m_l} = \int_0^\infty d\lambda_2 \frac{4m_l C_2}{\lambda_2^2 + m_l^2} \quad C_2(\lambda, \lambda_2) = \langle \rho_U(\lambda) \rho_U(\lambda_2) \rangle - \langle \rho_U(\lambda) \rangle \langle \rho_U(\lambda_2) \rangle$$

# Reproduction of Cumulants $\mathbb{K}_n[\bar{\psi}\psi]$ via $P_n(\lambda)$

$$\mathbb{K}_1[\bar{\psi}\psi] = \frac{T}{V} \mathbb{K}_1[2 \text{Tr}M^{-1}] = \int_0^\infty P_1(\lambda) d\lambda$$

$$\mathbb{K}_2[\bar{\psi}\psi] = \frac{T}{V} \mathbb{K}_2[2 \text{Tr}M^{-1}] = \int_0^\infty P_2(\lambda) d\lambda$$

$$\mathbb{K}_3[\bar{\psi}\psi] = \frac{T}{V} \mathbb{K}_3[2 \text{Tr}M^{-1}] = \int_0^\infty P_3(\lambda) d\lambda$$

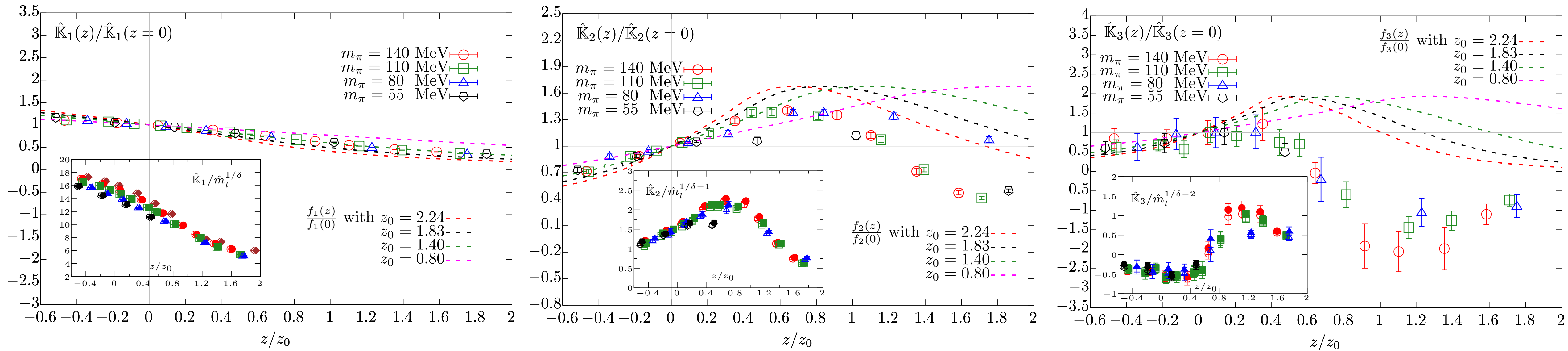


Open symbols: computation via inversions of the fermion matrix  $\text{Tr}M^{-1}$

Filled symbols: reconstructed from  $P_n(\lambda)$

Cumulants related to  $P_n(\lambda)$  can successfully reproduce their corresponding results from **inverse fermion matrix**

# Criticality in Macroscopic Cumulants $\mathbb{K}_n[\bar{\psi}\psi]$



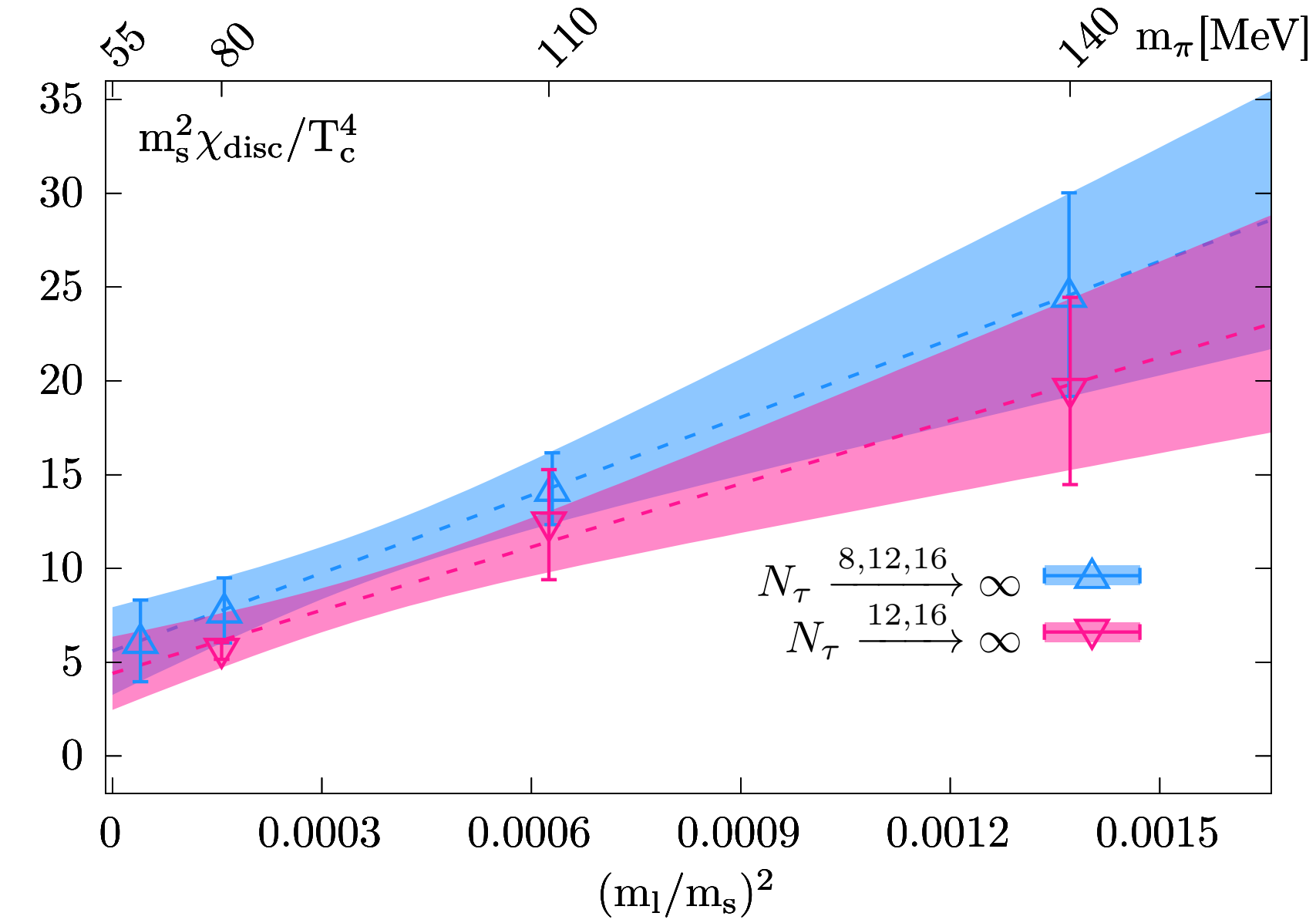
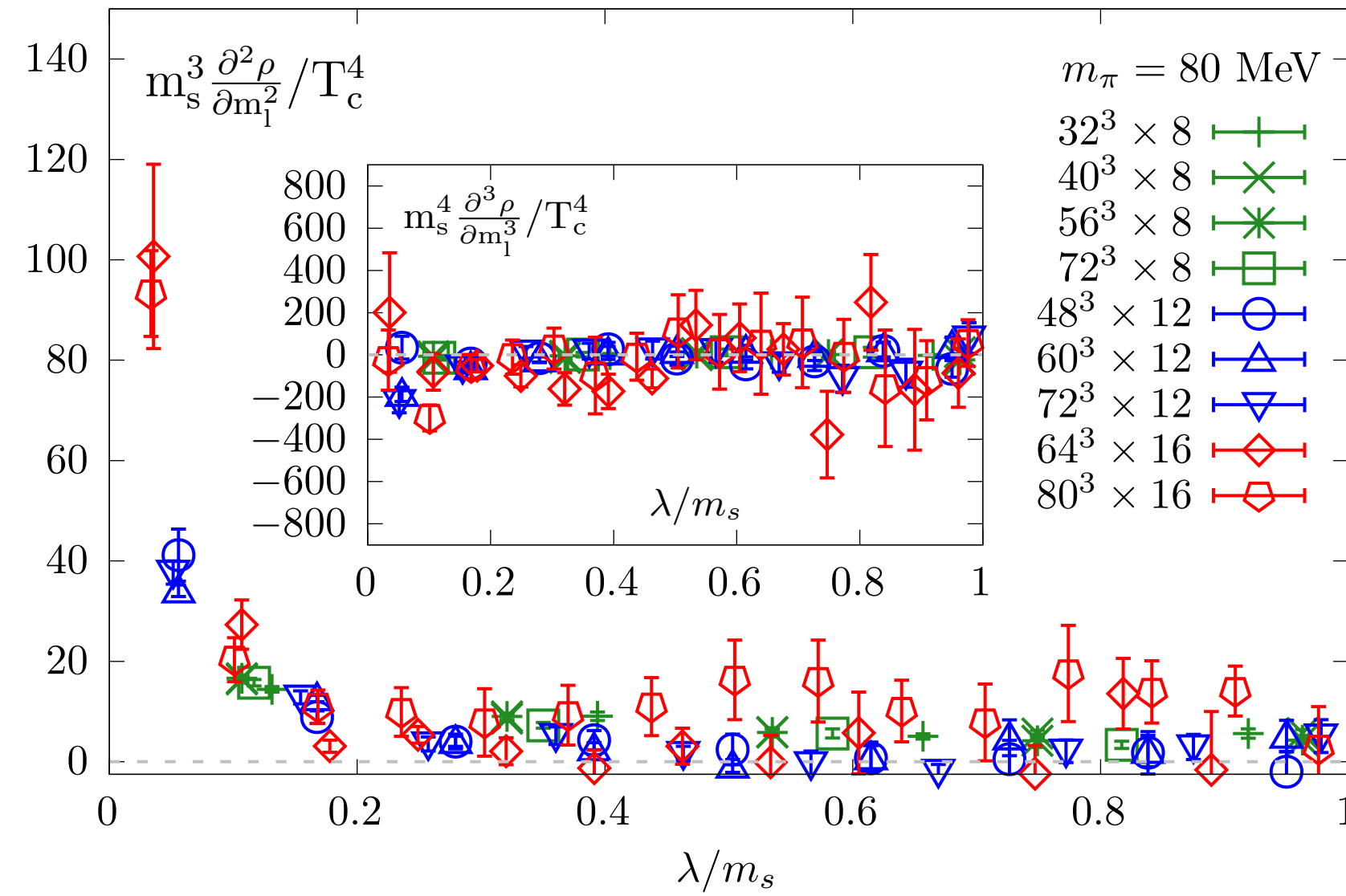
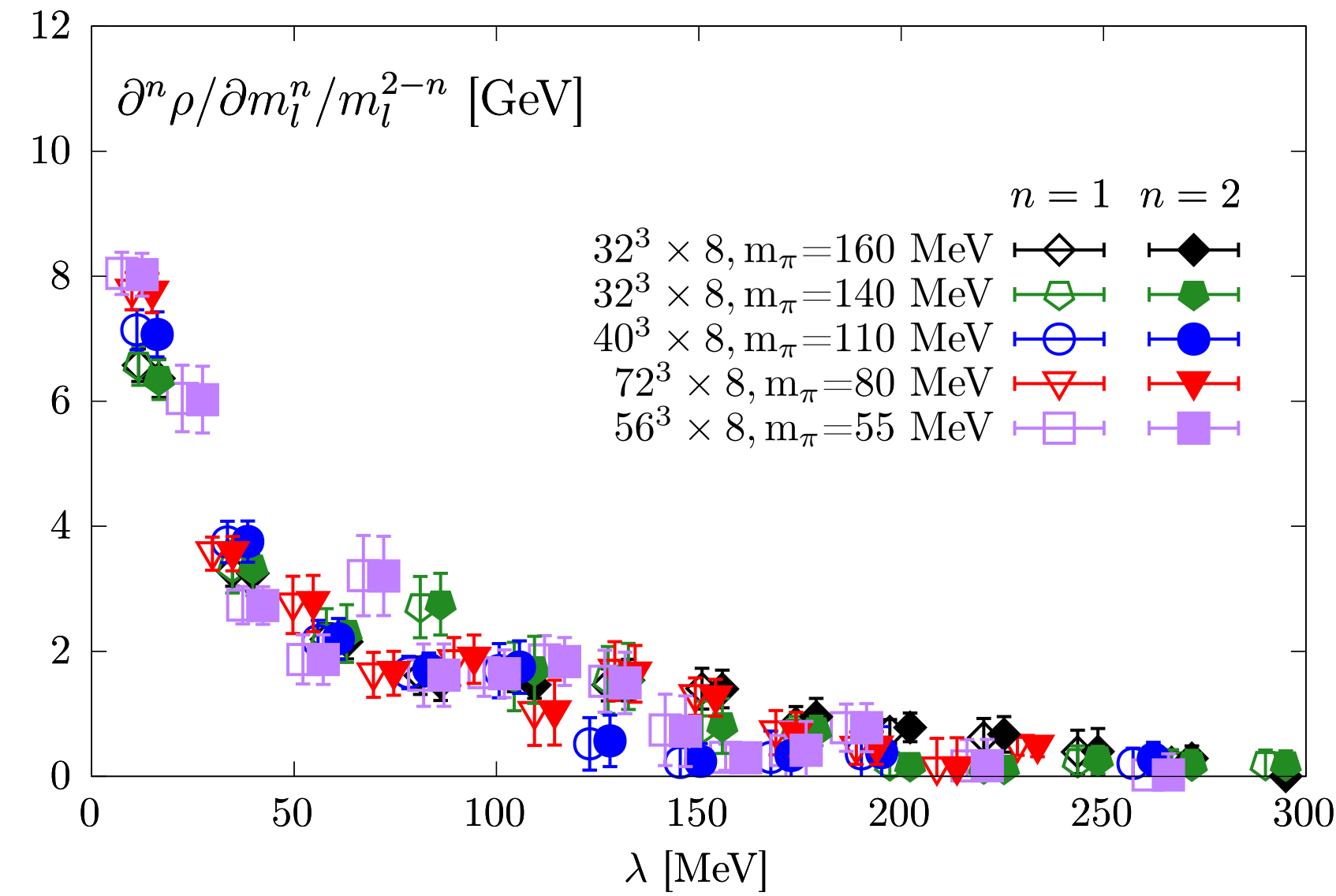
$O(2)$  scaling with  $\beta = 0.349$ ,  $\delta = 4.78$ ,  $z_0 = 1.83(9)$ ,  $T_c(N_\tau = 8) = 144.2(6)$  MeV

Parameters adopted from: S. Ejiri et al., Phys. Rev. D 80, 094505 (2009); D. A. Clarke et al., Phys. Rev. D 103, L011501 (2021)

- For  $|z/z_0| \lesssim 0.2$ ,  $K_n(z)/K_n(z=0)$  with  $n = 1, 2, 3$  can be well described by  $O(2)$  scaling function  $\frac{f_{n-1}(z)}{f_{n-1}(z=0)}$
- For  $|z/z_0| \lesssim 0.2$ ,  $K_n$  rescaled by  $H^{1/\delta-n+1}$  show small quark dependence

# Recent Results at high temperature

In (2+1)-flavor QCD at  $T \approx 1.6T_c$



$$\left. \begin{aligned} m_l^{-1} \partial \rho / \partial m_l &\approx \partial^2 \rho / \partial m_l^2 \\ \partial^3 \rho / \partial m_l^3 &\approx 0 \end{aligned} \right\} \Rightarrow \rho(\lambda \rightarrow 0, m_l \rightarrow 0) \propto m_l^2 \delta(\lambda)$$

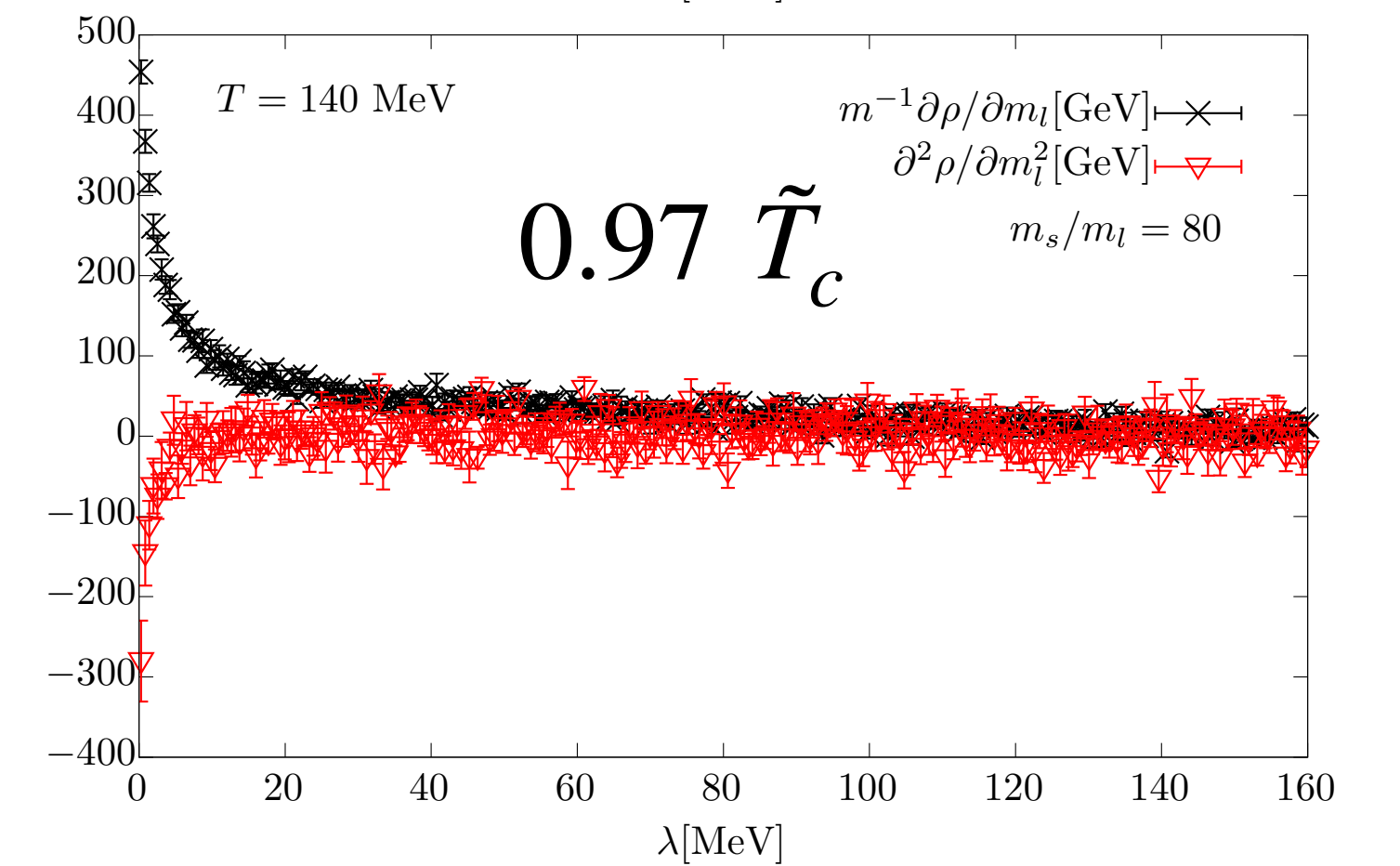
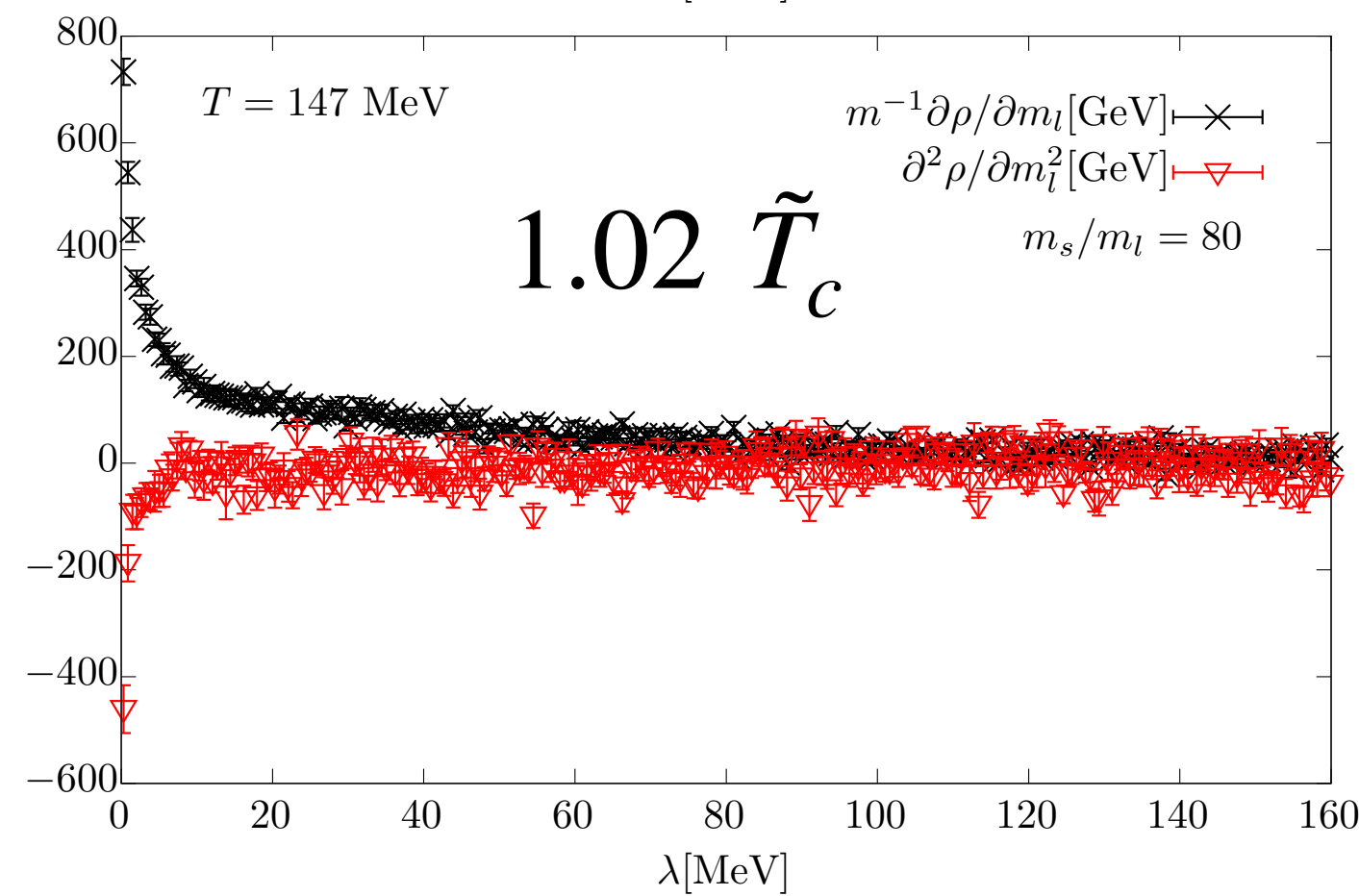
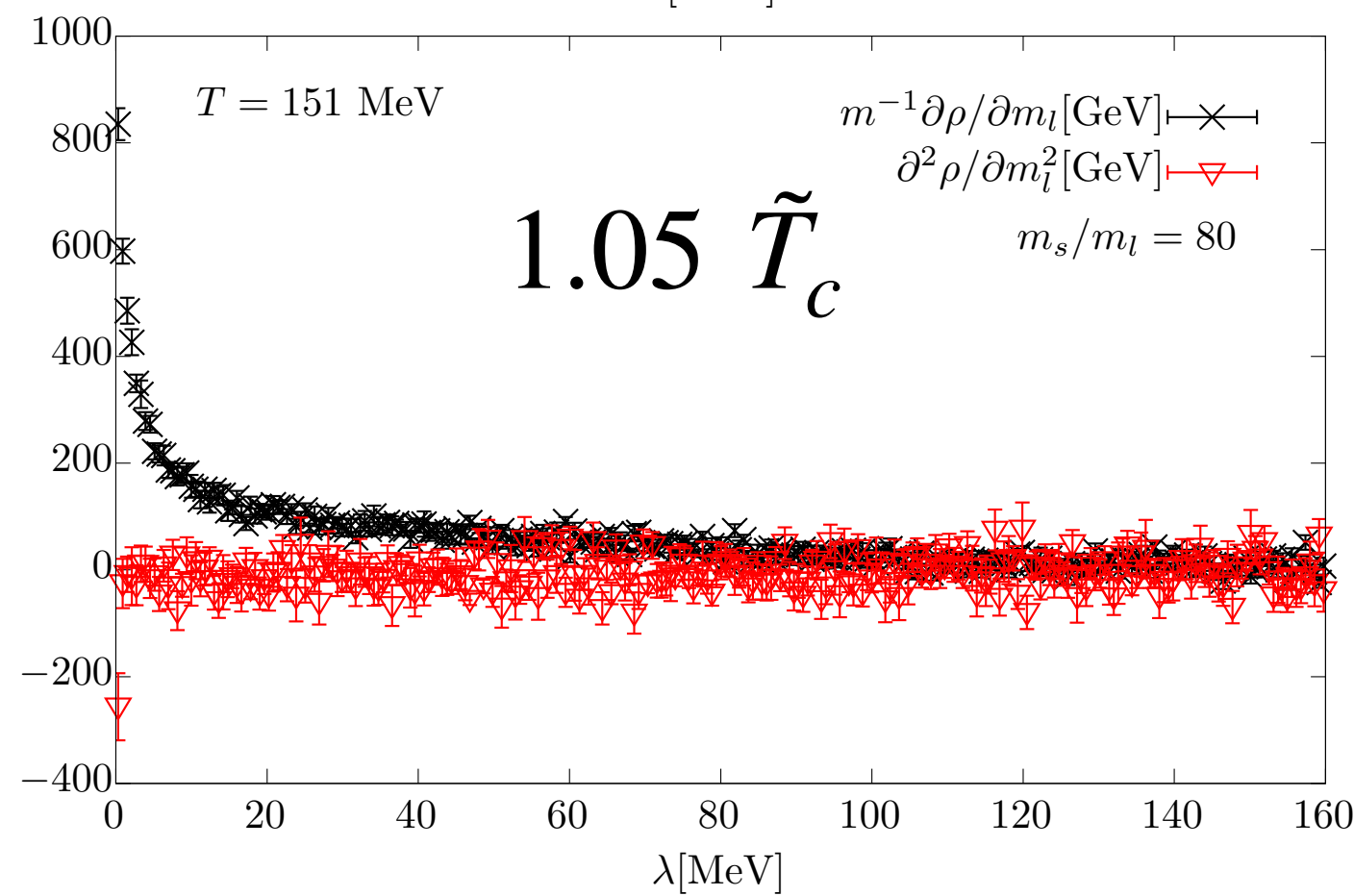
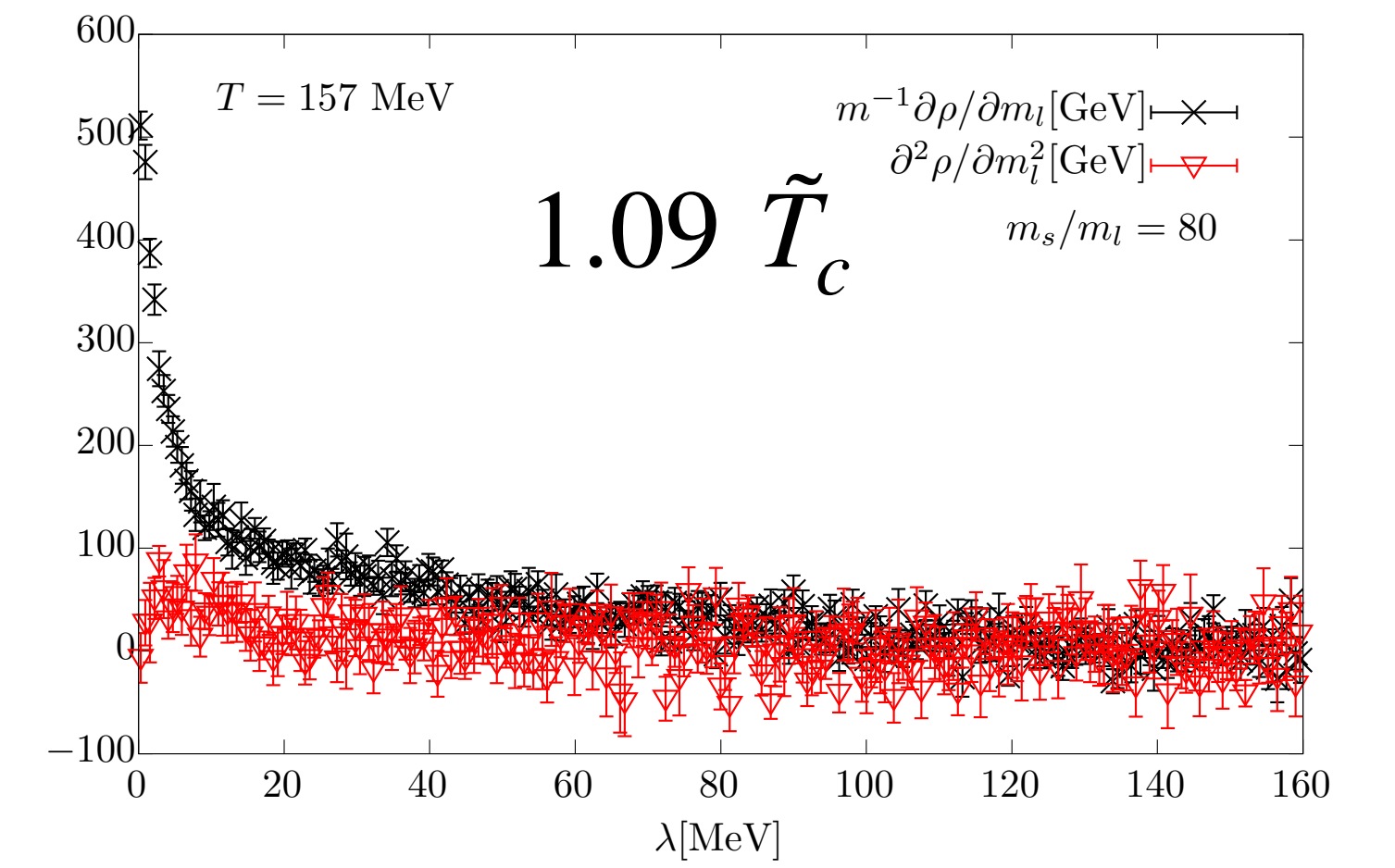
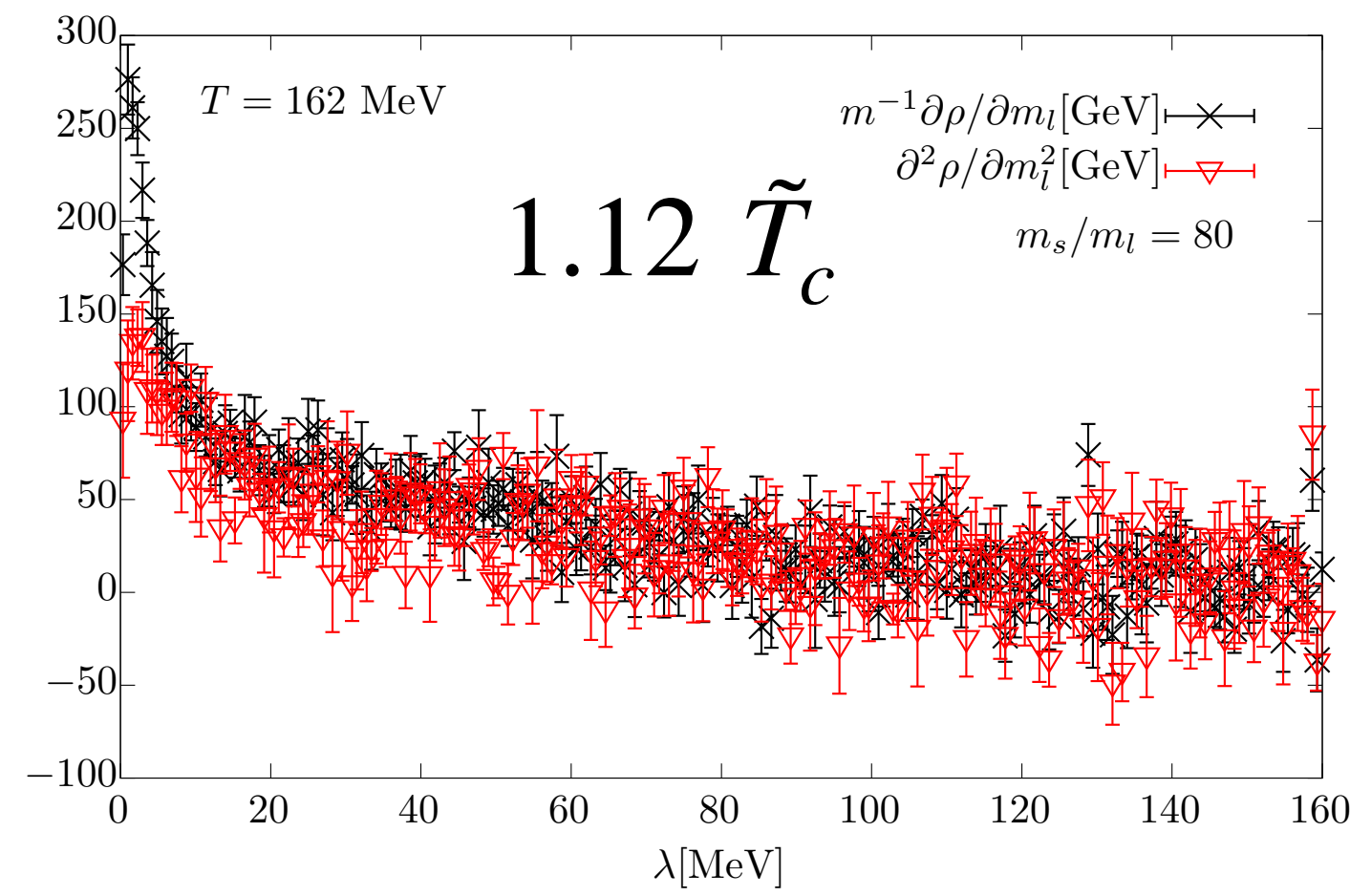
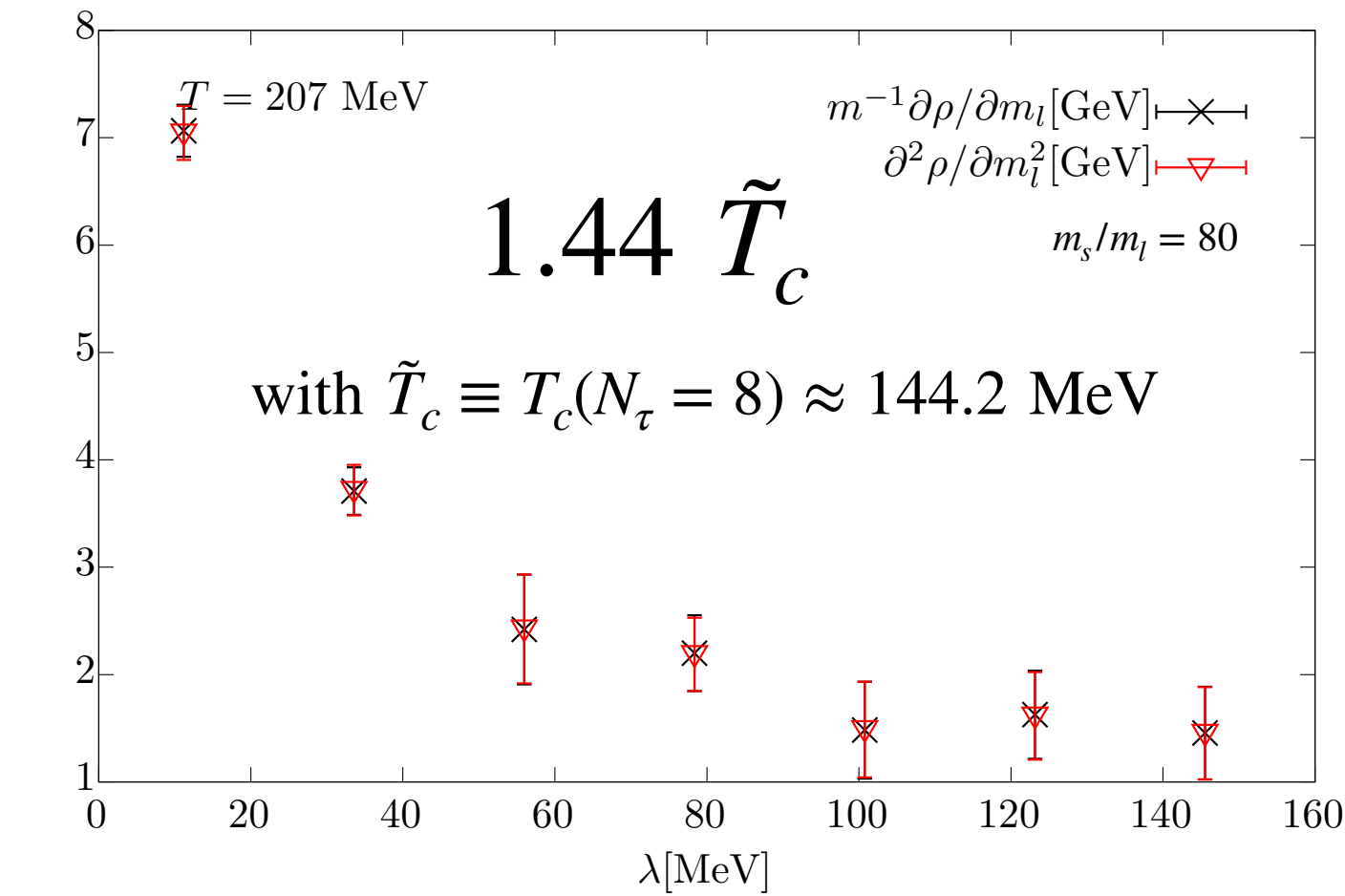
- Axial anomaly remains manifested in the  $U_A(1)$  measures at a 2-3 sigma level
- Chiral phase transition should be 2<sup>nd</sup> order belonging to O(4) universality class

# Temperature dependence of $\partial^n \rho / \partial m_l^n$ with $n = 1, 2$

HWP, Lattice 21', arXiv: 2112.00318

H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)

$$\frac{\partial \rho}{\partial m} = \frac{T}{V} \int_0^\infty \frac{4m K_1[\rho_U(\lambda), \rho_U(\lambda_2)]}{\lambda_2^2 + m^2} d\lambda_2, \quad \frac{\partial^2 \rho}{\partial m^2} = \frac{T}{V} \int_0^\infty \frac{(4m)^2 K_1[\rho_U(\lambda), \rho_U(\lambda_2), \rho_U(\lambda_3)]}{(\lambda_2^2 + m^2)(\lambda_3^2 + m^2)} d\lambda_2 d\lambda_3 + \frac{T}{V} \int_0^\infty \frac{4(\lambda_2^2 - m^2) K_1[\rho_U(\lambda), \rho_U(\lambda_2)]}{(\lambda_2^2 + m^2)^2} d\lambda_2$$



$\partial^2 \rho / \partial m^2 \neq m^{-1} \partial \rho / \partial m$ , Dilute Instanton Gas Approximation becomes invalid as getting closer to  $T_c$